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The calculus of multivectors on noncommutative jet spaces

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0. Introduction

Let \mathbb{F} be a free algebra over $\mathbb{k} := \mathbb{R}$ and suppose $a_1, \ldots, a_k \in \mathbb{F}$. Denote by \circ the associative multiplication in \mathbb{F} and by t the counterclockwise cyclic shift of co-multiples in the product $a_1 \circ \ldots \circ a_k$,

 $\mathfrak{t}(a_1 \circ \ldots \circ a_{k-1} \circ a_k) \stackrel{\text{def}}{=} a_k \circ a_1 \circ \ldots a_{k-1}.$

For the sake of definition, now assume that a given derivation $\partial : \mathbb{F} \to \mathbb{F}$ is such that its values at a_1, \ldots, a_k do not leave that set. By the Leibniz rule, the derivation is cyclic-shift invariant:

$$\partial (\mathfrak{t}(a_1 \circ \ldots \circ a_k)) = \mathfrak{t} (\partial (a_1 \circ \ldots \circ a_k)).$$

Indeed, both sides of the above equality are given by the sum

 $\partial(a_k) \circ a_1 \circ \ldots \circ a_{k-1} + a_k \circ \partial(a_1) \circ \ldots \circ a_k + a_k \circ a_1 \circ \ldots \circ \partial(a_{k-1}),$

up to a sequential order in which these *k* summands follow each other (see Fig. 1). This observation is generalised in an obvious way to the case where the elements of algebra \mathbb{F} are graded by some Abelian group, each element a_1, \ldots, a_k is homogeneous with respect to the grading, and $\partial : \mathbb{F} \to \mathbb{F}$ is a graded derivation (i.e. not necessarily preserving the set $\{a_1, \ldots, a_k\}$ at hand).

ABSTRACT

The Leibniz rule for derivations is invariant under cyclic permutations of co-multiples within the arguments of derivations. We explore the implications of this principle: in effect, we construct a class of noncommutative bundles in which the sheaves of algebras of walks along a tessellated affine manifold form the base, whereas the fibres are free associative algebras or, at a later stage, such algebras quotients over the linear relation of equivalence under cyclic shifts. The calculus of variations is developed on the infinite jet spaces over such noncommutative bundles.

In the frames of such field-theoretic extension of the Kontsevich formal noncommutative symplectic (super)geometry, we prove the main properties of the Batalin–Vilkovisky Laplacian and Schouten bracket. We show as by-product that the structures which arise in the classical variational Poisson geometry of infinite-dimensional integrable systems do actually not refer to the graded commutativity assumption.

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Fig. 1. The cyclic-shift invariance of derivations.

How much (graded-) commutativity is really needed to make the calculus of variations in the Lagrangian and Hamiltonian formalisms work, thus allowing for the Batalin–Vilkovisky technique for quantisation of gauge systems — and creating a cohomological approach to the complete integrability of infinite-dimensional KdV-type systems?¹

We claim that it is not the restrictive assumption of commutativity that shows through *arbitrary* permutations – but it is the linear equivalence $a \sim t(a)$ of words a, written in a given alphabet, with respect to the *cyclic* permutations t that is sufficient for the structures of the calculus of iterated variations to be well defined. Introduced in this cyclic-invariant setup, the Batalin–Vilkovisky Laplacian Δ and variational Schouten bracket [[,]] are proven to satisfy the main identities such as the cocycle condition $\Delta^2 = 0$, see (2a)–(2d). Both the definitions and assertions are then literally valid in the sub-class of graded-commutative geometries; the reason why is that the latter can be obtained from the former by using the postulated commutativity reduction at the end of the day when the proof is over.

The idea to establish the formal noncommutative symplectic geometry on the cyclic invariance, generalising the geometry of commutative symplectic manifolds, was introduced by Kontsevich in [20], cf. [21] and references therein. The quotient spaces of cyclic words were employed as target sets for maps from usual manifolds in [17] by Olver and Sokolov (cf. Model 1); several integrable equations of KdV-type were recovered in such noncommutative set-up.² Variations arise in the Poisson or Schouten brackets for integral functionals, their calculus was then pursued along the lines of [16]. The paper [17] initiated a classification and study of evolutionary ODE and PDE systems on associative algebras, which required the calculation of standard geometric structures for such models in jet spaces (e.g., see [22] in this context).

In this paper we further that approach to noncommutative jet spaces.³ Continuing the line of reasoning from [9,26,27] where the intrinsic regularisation of Batalin–Vilkovisky formalism is revealed, we verify the main identities for Δ and [[,]] in the variational noncommutative set-up of (homogeneous) local functionals *F*, *G*, *H*:

$$\Delta(F \times G) = \Delta F \times G + (-)^{|F|} \llbracket F, G \rrbracket + (-)^{|F|} F \times \Delta G,$$
(2a)

$$\llbracket F, G \times H \rrbracket = \llbracket F, G \rrbracket \times H + (-)^{(|F|-1) \cdot |G|} G \times \llbracket F, H \rrbracket,$$
(2b)

$$\Delta(\llbracket F, G \rrbracket) = \llbracket \Delta F, G \rrbracket + (-)^{|F|-1} \llbracket F, \Delta G \rrbracket,$$

$$Jacobi(\llbracket, \rrbracket) = 0 \quad \iff \quad \Delta^2 = 0.$$
(2d)

It is quite paradoxical that for a long time, these identities were proclaimed to be valid just formally [5,28]; for it was believed that the Batalin–Vilkovisky technique would necessarily contain some divergencies or "infinite constants", whereas their manual regularisation appealed to surreal principles like " $\delta(0) := 0$ " for the Dirac δ -function (see [9] and references therein for discussion on the history of the problem).

The notion of associative algebra structures itself has deserved much attention in the mathematical physics literature, e.g., in relation to the Yang–Baxter equation. Such structures arise naturally in the topological context; the calculus of cyclic

(2c)

¹ We refer to [1–8] or [9] and to [10–17] respectively (see also [18,19] in both contexts).

² Noncommutative extensions of classical infinite-dimensional systems can acquire new components that are invisible in the commutative world: e.g., there appear – often, through nonlocalities – the terms that contain the commutants $a_i \circ a_j - a_j \circ a_i$.

³ We note that the positive differential order calculus on infinite jet spaces lies far beyond the bare tensor calculus on usual commutative manifolds; for instance, compare [23] with [19] or contrast [24] vs [25] and [6] vs [9].

words serves the alphabet of homotopy group generators. Likewise, the multiplication in homology gives rise to the Gromov– Witten potential solving the WDVV equations, see [12] and [29,30], cf. [31]. Another construction, which will be discussed in Remark 2.10, stems from the calculation of matrix integrals in the Batalin–Vilkovisky framework [32,33]. Furthermore, associative but not necessarily commutative *-products are obtained – on finite-dimensional affine manifolds – by using the deformation quantisation procedure [34], cf. [35,36] and Model 2. Now we study the extent to which the differential calculus can be developed on the basis of associative algebra structures as input data.⁴

This paper consists of three parts. In Section 1 we introduce the static set-up of noncommutative infinite jet (super-) spaces. Based on the algorithmic construction of parity-odd Laplacian Δ and variational Schouten bracket [[,]], the calculus of iterated variations of local functionals – i.e., kinematics – is developed in Section 2. Such BV-geometry of local functionals is then contrasted in Section 3 with the noncommutative Poisson formalism, that is, the dynamics determined by variational multi-vectors.

The text is structured as follows. The commutative but not associative algebra \mathcal{A} of cyclic words written in the alphabet $\langle a^i \rangle$ of a free associative algebra is introduced in Section 1.1. The generators a^i themselves are viewed in Section 1.3 as words written in the alphabet $\langle \mathbf{x}_i^{\pm 1} \rangle$ of edges in the adjacency graph for a cell-complex tiling of the substrate manifold \mathcal{M}^n , which is introduced in Section 1.2. The alphabets $\langle \mathbf{x}_i^{\pm 1} \rangle$ and $\langle a^i \rangle$ provide the respective noncommutative analogues of base and fibre in the bundle π_{NC} : the base is the sheaf of [unital extensions of] free associative algebras generated by $\langle \mathbf{x}_i^{\pm 1} \rangle$ for a crystal tiling of \mathcal{M}^n , whereas the fibres of π_{NC} are [the unital extension of] the algebra \mathcal{A} of cyclic words written in the alphabet $\langle a^i \rangle$ (see the figure in Section 1.3). The jet space $J^{\infty}(\pi_{\text{NC}})$ of sections is built in Section 1.4; various elements of the jet-space language are then recovered. In particular, as soon as the notion of variational (co)vectors is available, we show why the Substitution Principle works for (non)commutative identities in total derivatives.

The second part begins with the definition of noncommutative analogue for the variational cotangent bundle over the infinite jet space $J^{\infty}(\pi_{\text{NC}})$, see Section 2.1. The sections target algebra alphabet $\langle a^i \rangle$ is doubled by using the canonical pairs $\langle a^i, a_i^{\dagger} \rangle$; sign convention (14) for the two ordered couplings of the virtual variations δa and δa^{\dagger} ensures the matching of signs in all the structures that are defined in what follows. In the meantime (see Section 2.3), the \mathbb{Z}_2 -parity reversion $\Pi : a_i^{\dagger} \rightleftharpoons b_i$ acts on the dual symbols a^{\dagger} , producing the parity-odd slots b. Now, the geometric approach of [9] to iterated variations works in the noncommutative set-up of evaluation maps $a = s(\mathbf{x}, \mathbf{x}^{\pm 1})$ and antimaps $a^{\dagger} = s^{\dagger}(\mathbf{x}, \mathbf{x}^{\pm 1})$ using the sheaf over M^n (see Fig. 4). Therefore, while giving the operational definition of BV-Laplacian Δ in Section 2.6, we focus on the unlock-and-join reconfigurations of cyclic words. The variational Schouten bracket [[,]] is a derivative structure, that is, it is determined by the parity-odd operator Δ via its action on products, as in (2a).⁵ Then we confirm that the variational Schouten bracket [[,]] is shifted-graded skew-symmetric and satisfies the Jacobi identity. The two structures Δ and [[,]] endow the ring of local functionals with the structure of differential graded Lie algebra.

The third part of this text narrates the noncommutative variational Poisson formalism. The notion of noncommutative variational multi-vectors is introduced in Section 3.1. We recall that not every grading-homogeneous integral functional over the infinite jet superspace $J^{\infty}(\pi_{\rm NC}^{(0|1)})$, canonically extended in Section 2, would be a well defined variational multi-vector containing the respective number of parity-odd slots **b**. Remark 2.5 is a key to that concept. Specifically, by viewing the variational multi-vectors as maps that take the respective tuples of – possibly, exact – variational covectors to the top-degree horizontal cohomology space of cyclic word-valued integral functionals, we analyse in Section 3.2 the geometry of iterated variational multivectors is the paradigm of steps and stops. Finally, we arrive at the definition of Poisson brackets. In Section 3.3 we study the geometry of differential forms that stands behind the criterion under which the variational noncommutative bi-vectors are Poisson, i.e. endow the space of noncommutative Hamiltonians with the variational Poisson brackets. (In particular, the Helmholtz lemma is proved in the setting of cyclic words.)

1. The nature of associative symbols

1.1. The algebra A of cyclic words

In this section we introduce the main object to consider in the future reasoning. Namely, by starting with a noncommutative free associative algebra, we define the commutative but not associative unital algebra A of cyclic words written

⁴ An alternative approach to noncommutativity suggests that manifolds – and derivative objects such as the fibre bundles – are determined as the spectra of associative noncommutative algebras. Provided that the algebras are 'smooth', they are viewed as the algebras of smooth functions on the objects which they determine. Nowadays, noncommutative geometry à la Connes [37] is a well-established domain. However, we keep the framework closer to the needs which one encounters in a class of path- and loop-based QFT models [38–40]. Let us therefore study the language of closed strings of symbols – written around the circles and encoding paths in the granulated space M^n (see Model 3).

⁵ In geometric terms, the bracket $[\![,]\!]$ of cyclic word-valued functionals is encoded by the standard topological pair of pants $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$ that links the cycles. In fact, this topological procedure also underlies each of the following structures and operations in the differential calculus under study: • multiplication × of cyclic words and word-valued function(al)s, • termwise action of derivations (e.g., in (10)), including • the commutation of vector fields, – and also • evaluation of multi-vectors at the tuples of covectors (see (39)): in particular, • the Poisson bracket of Hamiltonian functionals. Indeed, all of the above amounts to the detach-and-join picture $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$.

in the free algebra's alphabet. Note that for the sake of clarity, neither of these two algebras is graded; however, in what follows we shall extend the alphabet by using symbols of \mathbb{Z}_2 -valued parity. Throughout this text, the ground field \Bbbk is the field \mathbb{R} of real numbers.

Consider the free associative algebra Free (a^1, \ldots, a^m) with m generators a^1, \ldots, a^m ; let $m < \infty$ for definition. (One may presently think that the free algebra at hand is not necessarily unital.) Denote by \circ the multiplication in that algebra. By definition, put

$$\mathfrak{t}(a^{i}) = a^{i}, \qquad \mathfrak{t}(a^{i_{1}} \circ \ldots \circ a^{i_{\lambda}}) := a^{i_{\lambda}} \circ a^{i_{1}} \circ \ldots \circ a^{i_{\lambda-1}}, \quad \lambda > 1;$$

$$(3)$$

otherwise speaking, the operator t is the counterclockwise cyclic permutation of symbols in a homogeneous word of length $\lambda > 0$.

Introduce the linear equivalence relation \sim on Free (a^1, \ldots, a^m) by setting⁶

$$a \sim \mathfrak{t}(a),$$

where *a* is a homogeneous word as in (3), and then extending \sim onto the algebra by linearity: $a \sim a'$ and $b \sim b'$ imply $a + b \sim a' + b'$. For instance, one has that⁷

$$a^1 + a^2 \circ a^3 + a^1 \circ a^2 \circ a^3 \sim a^1 + a^3 \circ a^2 + a^3 \circ a^1 \circ a^2$$

Notice also that

$$a \sim \mathfrak{t}(a) \sim \, \ldots \, \sim \mathfrak{t}^{\lambda(a)-1}(a) \sim rac{1}{\lambda(a)} \sum_{i=1}^{\lambda(a)} \mathfrak{t}^{i-1}(a)$$

for any word *a* of length $\lambda(a) > 0$; by convention, a word of zero length is an element of the ground field k, see (6).

We denote by \mathcal{A} the quotient Free $(a^1, \ldots, a^m)/\sim$, that is, \mathcal{A} is the vector space of (formal sums of) cyclic words such that each homogeneous component $a^{i_1} \circ \ldots \circ a^{i_{\lambda}}$ can be read starting from any letter $a^{i_{\alpha}}$ for $1 \leq \alpha \leq \lambda$. Let us denote by $(a) \in \mathcal{A}$ the equivalence class of an element $a \in \text{Free}(a^1, \ldots, a^m)$ under cyclic permutations of symbols in all its homogeneous components (i.e. in all its "words" in proper sense).

Now we endow the vector space A of cyclic words with the algebra structure \times . Consider the equivalence classes (a_1) and (a_2) of two homogeneous elements $a_1, a_2 \in \text{Free}(a^1, \ldots, a^m)$ of positive lengths $\lambda(a_1)$ and $\lambda(a_2)$, respectively. Let their product be

$$(a_1) \times (a_2) \stackrel{\text{def}}{=} \frac{1}{\lambda(a_1) \cdot \lambda(a_2)} \Big(\sum_{i=1}^{\lambda(a_1)} \sum_{j=1}^{\lambda(a_2)} \mathfrak{t}^{i-1}(a_1) \circ \mathfrak{t}^{j-1}(a_2) \Big), \tag{4}$$

where the equivalence class on the right-hand side is normalised in such a way that the definition correlates with the commutative set-up (should it be recovered postfactum); now extend the product onto A by (bi-)linearity. The definition of operation \times says that, each homogeneous string of symbols in the first co-multiple read, time after time starting from every next letter, it is then pasted – time after time in its turn – in between every two consecutive letters occurring in each homogeneous string contained in the second co-multiple. Sure, this is the classical topological pair of pants $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$ in which every symbol in either of the factors has the right to be read first, see the figure.



Proposition 1. Multiplication (4) on A is commutative.

Proof. Notice that not only the necklace (a_1) is unlocked at all possible multiplication signs \circ and joined to (a_2) in between each pair of adjacent symbols in that word but, as one shifts the symbols in (a_2) around the circle, exactly the same is done with respect to the insertion of $t^{j-1}(a_2)$ into (a_1) . \Box

However, it is readily seen that the symbols in homogeneous strings in (a_1) and (a_2) always stay next to each other in the nested product $((a_1) \times (a_2)) \times (a_3)$, whereas they are separated by the symbols from (a_3) in at least one homogeneous

⁶ It is readily seen that $a^{i_1} \circ \ldots \circ a^{i_{\lambda}} = \mathfrak{t}^{\lambda-1}(\mathfrak{t}(a^{i_1} \circ \ldots \circ a^{i_{\lambda}}))$ so that $a \sim a$ and $\mathfrak{t}(a) \sim a$, whence the transitive relation \sim is reflexive and symmetric indeed.

⁷ We emphasise that the cyclic invariance itself does *not* imply the commutativity: even though $a^i \sim a^i$ and $a^i \circ a^j \sim a^j \circ a^i$ one has that $a^i \circ a^j \circ a^k \sim a^i \circ a^k \circ a^j$ unless some of the indexes coincide.



Fig. 2. The letters a^1 are (not) separated by the letters a^2 .

term in $(a_1) \times ((a_2) \times (a_3))$, provided that the alphabet contains at least two different letters and the length of the word a_3 is greater than one.⁸

Proposition 2. If $m \ge 2$ so that the letters a^1 and a^2 are distinct in the alphabet, multiplication (4) on A is not associative:

$$((a_1) \times (a_2)) \times (a_3) \nsim (a_1) \times ((a_2) \times (a_3)), \tag{5}$$

see the figure below.⁹



Counterexample 1.1 ("*abba*"). Let $a_1 := a^1$, $a_2 := a^1$, and $a_3 := a^2a^2$. Then $(a_1) \times (a_2) = (a^1 \circ a^1)$ so that these two copies of the letter a^1 always stay next to each other in any product of $(a_1) \times (a_2)$ with any other word. On the other hand (see Fig. 2), the word $(a_2) \times (a_3)$ is equal to $(a^2a^1a^2)$, whence the nested product $(a_1) \times ((a_2) \times (a_3))$ contains the term $\frac{1}{3}a^1a^2a^1a^2$, which is absent on the left-hand side of (5) for these a_1, a_2, a_3 .

Convention. By interpreting the ground field \Bbbk as the linear span of the zero-length word **1** and its equivalence class (**1**), we extend the commutative algebra of cyclic words to $\mathcal{A} \oplus \Bbbk \cdot (\mathbf{1})$, now endowed with the multiplication \times such that, in agreement with the vector space structure of \mathcal{A} , formula (**4**) is extended by

$$(k) \times (a) \stackrel{\text{def}}{=} k \cdot (a) \tag{6}$$

for any $k \in \mathbb{k}$ and all cyclic words (*a*). Allowing for the slightest abuse of notation, we continue denoting by A the unital algebra of cyclic words that contains such zero-length but non-empty strings of symbols.

Open problem 1 (*Prime Decomposition*). Is there a way to detect that a given sum $(a) \in A$ of several cyclic words is the product $(b) \times (c)$ of two shorter cyclic words $(b), (c) \in A$ of positive length?

Let us give several examples of natural constructions of the algebra A that contains nonnegative-length cyclic words written in an alphabet a^1, \ldots, a^m . By realising every such algebra as fibre in a bundle π_{NC} over a given manifold M^n (e.g., in the trivial bundle over a finite-dimensional affine real manifold, cf. Section 1.2), we shall proceed in Section 1.4 with the construction of the space $J^{\infty}(\pi_{NC})$ of infinite jets of sections for such bundles π_{NC} .

Model 1. Consider the algebra Mat(n, \mathbb{R}) of square matrices of size n × n with real entries. Roughly speaking, as n $\rightarrow +\infty$, the matrix multiplication \circ will never become commutative (yet it always stays associative). For definition, let $m := n^2$ be the

$$((\cdot \times \cdot) \times \cdot) : p^+ \sqcup p^+ \sqcup n^0 \longmapsto (p^+ \times p^+) \times n^0 = p^+ \sqcup p^+ \sqcup n^0 \longmapsto p^+ \sqcup D_2^1,$$

one obtains the input objects intact after the first interaction event. But the arrangement

$$(\cdot \times (\cdot \times \cdot))$$
: $p^+ \sqcup p^+ \sqcup n^0 \longmapsto p^+ \times (p^+ \times n^0) = p^+ \times D_2^1 = He_2^2$

produces helium-3 via deuterium. This fusion process is not associative.

 $^{^{8}}$ Obviously, the associativity equation for \times can be satisfied incidentally, for a special choice of the three co-multiples.

⁹ Let us recall that in Nature, not all processes are associative. For example, take a proton p^+ , another proton, and a neutron n^0 . Letting their strong interaction events be arranged using

dimension of entire matrix algebra and choose a basis a^1, \ldots, a^m in it. Although this \mathbb{R} -algebra is not free, we still introduce the linear equivalence relation \sim on the vector space of words written in the alphabet $\mathbf{a} = \langle a^1, \ldots, a^m \rangle$, which yields the cyclic word algebra \mathcal{A} .

Because the matrix multiplication is not commutative, the content of every cyclic word $(a) = (a^{i_1} \circ \cdots \circ a^{i_{\lambda}})$ of length $\lambda > 0$, viewed as the actual product of λ matrices going in a specified sequential order, can take up to λ different values, namely,

$$a^{i_1} \circ \cdots \circ a^{i_{\lambda}}, \quad \mathfrak{t}(a^{i_1} \circ \cdots \circ a^{i_{\lambda}}), \quad \dots, \quad \mathfrak{t}^{\lambda-1}(a^{i_1} \circ \cdots \circ a^{i_{\lambda}}).$$
 (7)

The value depends on the place where the multiplication is started along the orientation of the cycle (see the figure).



This effect – the value of a word (*a*) of length $\lambda > 0$ can co-exist in $s \leq \lambda$ realisations – will be natural to the other two models which we consider below. Reproduced verbatim by the star-product \star in Model 2, such value multiplicity can be suppressed ($1 \leq s \leq \lambda$ so that the first equality is attained and the last inequality is strict if $\lambda > 1$) in the model of walks, e.g., along closed contours a^i from a point to itself within a given manifold (see Section 1.2).

Now let M^n be a real manifold and $\pi_{NC} : M^n \times A \to M^n$ be the trivial bundle. By construction, sections of π_{NC} viewed as noncommutative bundle are obtained as follows. First, let $\mathbf{s} = (s^1(\mathbf{x}), \ldots, s^m(\mathbf{x}))$ be a tuple of functions from $C^{\infty}(M^n \to \mathbb{R})$ (e.g., compact-supported over M^n). A tuple \mathbf{s} chosen, over every $\mathbf{x} \in M^n$ the *i*th generator a^i of the matrix algebra $Mat(n, \mathbb{R})$ is taken with the real coefficient $s^i(\mathbf{x})$. Likewise, every product $a^{i_1} \circ \ldots \circ a^{i_\lambda}$ acquires the coefficient $s^{i_1} \cdot \ldots \cdot s^{i_\lambda}$. Finally, such coefficient is passed through \sim to the quotient A modulo the linear equivalence, pointwise over $\mathbf{x} \in M^n$. So, all cyclic words in A are weighted by smooth real coefficients, depending on points of M^n , in such a way that the multiplication of cyclic words is respected by those weights.

Model 2. Likewise, let M^n be a finite-dimensional affine real Poisson manifold and $\star = \cdot + \hbar \{ , \}_{\mathcal{P}} + \bar{o}(\hbar)$ be the arising associative non-commutative star-product in the unital algebra $C^{\infty}(M^n \to \mathbb{R})[[\hbar]]$ of formal power series (see [34]; an expansion \star mod $\bar{o}(\hbar^4)$ is given in [35]). Keeping in mind the linearity of \star over \hbar , suppose $a^1, \ldots, a^m \in C^{\infty}(M^n \to \mathbb{R})[[\hbar]]$. Using the addition and \star -product generate from this (in)finite alphabet and \hbar a unital subalgebra of nonnegative-length words 1, $\hbar, \ldots, a^i, \ldots, a^{i_1} \star \cdots \star a^{i_{\lambda}}, \ldots$, and pass to the quotient algebra \mathcal{A} of cyclic words. (Our earlier remark that every such homogeneous word $(a^{i_1} \star \cdots \star a^{i_{\lambda}})$ can co-exist in up to λ different values is still in order.) Now, the construction of the noncommutative bundle π_{NC} of cyclic-word algebras \mathcal{A} over the affine manifold M^n at hand is immediate; its section is a choice which function from $C^{\infty}(M^n \to \mathbb{R})[[\hbar]]$ each element a^i of the alphabet is equal to. Whenever all the elements of the alphabet are compact-supported over the base manifold M^n , so are all the cyclic words.

An outline of the third model is stretched over several sections; it will be concluded on Model 3 by comparing the result with the standard graded-commutative geometry of the Batalin–Vilkovisky (BV) superbundle $\zeta^{(0|1)}$. Let us specify at once that the sheaf $M_{\rm NC}^n$ of algebras of walks (introduced in Section 1.2) and realisation of sections in $\pi_{\rm NC}$ as the (cyclic) word algebra mappings in Section 1.3 are pertinent to this model. At the same time, the construction of the symplectic-dual variables a_i^{\dagger} in Section 2.1 and of their parity-odd neighbours b_i (see Section 2.3) is common to all the models.¹⁰

1.2. The sheaves of algebras of walks

In this section we motivate the construction of the algebra A that contains nonnegative-length cyclic words written in the alphabet a^1, \ldots, a^m . By introducing several new elements into the picture now, in Section 1.4 we shall recover the notion of space of infinite jets $J^{\infty}(\pi_{\rm NC})$ of sections of the noncommutative bundle $\pi_{\rm NC}$ in which the algebra A provides the fibres.

Let M^n be an oriented affine real manifold of positive dimension n. Suppose that a tiling of the manifold M^n is given, that is, M^n is realised by $M^n = \bigcup_{\alpha \in \mathcal{I}} \overline{\Delta}_{\alpha}$ via¹¹ the complex of cells Δ_{α} of dimension n, see Fig. 3(a). (Of course, the manifold M^n

¹⁰ This is why from Section 2.4 onwards, we shall assume that densities of integral functionals over the jet superspace $J^{\infty}(\pi_{NC}^{(0|1)})$ do not depend explicitly on the edge alphabet $\mathbf{\tilde{x}}^{\pm 1}$ of a tiling of the base manifold M^n underlying the noncommutative superbundle $\pi_{NC}^{(0|1)}$. Indeed, the availability of such edge alphabet is a feature of the third model, which we presently discuss.

¹¹ The closure $\overline{\Delta}_{\alpha}$ of each cell Δ_{α} is taken with respect to the Euclidean topology on the manifold M^n under study.



Fig. 3. A fragment of cell-complex tiling (a) and its adjacency graph (b).

can be topologically nontrivial: e.g., roll the plane to a cylinder, respecting – in one of the many possible ways – a given regular crystal structure on \mathbb{E}^2 .) We remark also that the choice of a tiling can be not unique for a given manifold M^n . Next, construct the tiling adjacency graph: each cell Δ_{α} represented by the vertex in the dual picture (see Fig. 3(b)), two vertices are connected by the edge iff the respective cells in the tiling are adjacent through a common face of lower dimension¹² (that of n - 1).

Definition 1. Two binary operations are defined for paths along the edges between adjacent cells in a tiling: namely, the formal addition + and multiplication \circ . Whenever a(connected component of a) path *b* continues a (connected part of a) path *a*, we write $a \rightarrow b$. Suppose onward that *a*, *b*, and *c* are connected paths. If $a \rightarrow b$, then $a \circ b$ is the connected path obtained by using the concatenation; otherwise, we set $a \circ b = a + b$.

The respective neutral elements for + and \circ are 0 and the null path $\bullet = 1$.

The addition +is commutative and associative; clearly, the multiplication of paths is not always commutative.

Lemma 3. At the same time, the multiplication of (sums of) paths is not associative.

(Of course, if $a \to b$ and $b \to c$, then $(a \circ b) \circ c = a \circ (b \circ c)$.)

Proof. Namely, if $a \to b$ and $a \to c$ but $b \not\rightarrow c$, then $(a \circ b) \circ c = a \circ b + c$, yet the associator right-hand side is different: $a \circ (b \circ c) = a \circ (b + c) = a \circ b + a \circ c$. \Box

Finally, let us inspect the distributivity law.

- If $a \to c$ and $b \to c$, then $(a + b) \circ c = a \circ c + b \circ c$.
- If $a \rightarrow c$ but $b \not\rightarrow c$, then $(a + b) \circ c = a \circ c + b + c = a \circ c + b \circ c$ as well.
- If $a \rightarrow c$ nor $b \rightarrow c$, then $(a + b) \circ c = a + c + b + c = a \circ c + b \circ c$.

And now, the other way round:

- If $a \to b$ and $a \to c$, then $a \circ (b + c) = a \circ c + a \circ c$.
- If $a \rightarrow b$ but $a \not\rightarrow c$, then $a \circ (b + c) = a \circ b + a + c = a \circ b + a \circ c$ as well.
- If $a \rightarrow b$ nor $a \rightarrow c$, then $a \circ (b + c) = a + b + a + c = a \circ b + a \circ c$.

The *unital algebra of walks* is the vector space of formal sums (with respect to the addition +) of paths that are multiplied by using the concatenation \circ ; both the operations are proclaimed k-linear.

Without any extra assumptions made about the tiling, the cells adjacency table and the portrait of edges in the dual graph are *local*. Indeed, a *quasi* crystal structure of the cell complex realisation of M^n could contain defects. Consequently, the larger an open domain $U \subseteq M^n$ is, the larger can be the alphabet of edges which are used to encode paths within U as words.

¹² The *discrete* adjacency table, *finite* for every vertex Δ_{α} in the dual complex, is the main profit that one gains by taking the tiling of space, however tiny be the diameter of each cell with respect to a given distance function on M^n . The property of base manifold M^m to be affine, that is, to admit a flat structure (consisting of an atlas of charts with affine transition maps) is natural in this context. Namely, affine reparametrisations within a tiling domain amount to a change of frame's reference to a point which marks that domain.

For the sake of definition, we assume that the substrate manifold's tiling is globally regular, so that the crystal structure $\{\Delta_{\alpha}\}$ is formed by (in)finite replication of a finite union of cells.

Definition 2. The edge alphabet is any minimal (i.e. without repetitions) subset of the set of edges such that every walk between cells in a given tiling of M^n can be expressed using that subset. Up to a permutation of edges and up to a choice – which direction of an *i*th edge is denoted by the symbol \vec{x}_i^{+1} and the other by \vec{x}_i^{-1} , – every alphabet $\vec{x}^{\pm 1}$ consists of

- (1) all the edges connecting the cells in their finite union which is replicated so that the tiling is made, and
- (2) the edges which interconnect that generating union of cells with all those replicas which are adjacent to that union (cf. [38,41]).

Example 1.2. Consider the honeycomb tiling of the plane, see the figure. The regularity assumption makes the alphabet $\vec{\mathbf{x}}^{\pm 1}$ finite even if the tiling of the (non)compact manifold M^n is infinite. We denote by N the cardinality of the set of generators, so that the chosen alphabet is $\vec{\mathbf{x}}^{\pm 1} = \langle \vec{\mathbf{x}}_1^{\pm 1}, \ldots, \vec{\mathbf{x}}_N^{\pm 1} \rangle$. Let us remember that the number N of elementary displacements $\vec{\mathbf{x}}_i$ depends on the choice of a tiling for the affine manifold M^n of dimension n. Now, the price that one pays is that the coding of edges can no longer be referred to any specific cell, hence a presence of irregular, non-periodic defects is no longer possible.



From now on, let an alphabet $\vec{\mathbf{x}}^{\pm 1} = \langle \vec{\mathbf{x}}_i^{\pm 1} \rangle$ be fixed for a given crystal tiling of the affine manifold M^n under study. For every value of the index *i*, the symbols $\vec{\mathbf{x}}_i^{\pm 1}$ and $\vec{\mathbf{x}}_i^{-1}$ denote the edges passed in the adjacency graph in either of the two directions.¹³

Over the substrate manifold M^n let us construct the almost constant sheaf (see [42,43]) of unital extensions $\mathbb{k} \cdot \mathbf{1} \oplus \text{Free}_{\mathbb{k}}(\vec{x}_1^{\pm 1}, \ldots, \vec{x}_N^{\pm 1})$ of free algebras generated by $\vec{x}_i^{\pm 1}$ and \vec{x}_i^{-1} . The sheaf is glued – from such unital algebras of walks – over open subsets $U \subseteq M^n$. For all pairs $U_j \subseteq U_i$ of non-empty open subsets of the set M^n with a chosen topology (e.g., the Euclidean one), the restriction homomorphisms are the identity mapping unless $U_j \subseteq \Delta_\alpha$ for some cell, marked by $\alpha \in \mathcal{I}$ in the tiling; in that case, the sheaf structure over U_j is the null path component $\mathbb{k} \cdot \mathbf{1}$ and the restriction mapping is the canonical projection. Over the empty subset of M^n , the sheaf structure is empty by definition.

Notation. This sheaf over M^n will be denoted by M_{NC}^n ; it remembers the topology on the substrate manifold and it carries the finite alphabet $\vec{\mathbf{x}}^{\pm 1}$ of the *N* edges that interconnect cells in (the replicas of) a fundamental domain in the tiling.

1.3. The formal fibre A of π_{NC}

We start building a noncommutative analogue π_{NC} of the variational cotangent bundle over M^n and then, using that noncommutative object, the analogue $\pi_{NC}^{(0|1)}$ of the Batalin–Vilkovisky superbundle over the space–time. Recalling from Section 1.1 the construction of the algebra A of cyclic words, we notice that whenever such algebra is realised as a fibre, it suffices to evaluate the generators a^i of the free algebra. Such "sections" (that is, the generator evaluation mappings) are then extended onto (the quotient of) the target space Free (a^1, \ldots, a^m) by using both the multiplication \circ and addition +. Indeed, consider the evaluation map $\mathbf{s}|_U$: Free $(a^1, \ldots, a^m) \rightarrow M_{NC}^n|_U$ which, at every point \mathbf{x} in a chart $U \subseteq M^n$ within the substrate manifold M^n , takes each generator a^i to a word of positive proper length¹⁴ – or to a formal sum of such words – written in the alphabet $\mathbf{\tilde{x}}^{\pm 1} = {\mathbf{\tilde{x}}_j^{\pm 1}, 1 \leq j \leq N}$:

$$a^i = s^i(\mathbf{x}, \vec{\mathbf{x}}^{\pm 1}), \qquad 1 \leqslant i \leqslant m,$$
(8)

¹³ A possibility to walk every edge, hence every path *backwards* – along the respective reverses $\vec{x}_i^{\pm 1}$, reading the words right to left, – is a forerunner of the introduction of canonical conjugate symbols a_j^{\dagger} , which are responsible for the dual, parity-odd part of the picture. This will be discussed in Sections 2.1 and 2.3, see Fig. 4 in particular.

¹⁴ Obviously, the case where $a^i = s^i(\mathbf{x})$ for some *i* would be somewhat special: the algebra \mathcal{A} of nonnegative-length cyclic words was unital by construction, but the above assignment would convert the generator a^i to the multiple of the neutral element at every \mathbf{x} in a chart. To exclude this situation from the study, let us technically assume that the lexicographic length of all the word(s) in each component s^i is strictly positive. Moreover, one should even require that the walk s^i along the edges $\vec{x}_i^{\pm 1}$ of the graph be more than a null path **1**, for it could be that the walk is contractable: e.g., $s^i = \vec{x}_j \circ \vec{x}_j^{-1} = \mathbf{1}$.

each word taken with a smooth coefficient from $C^{\infty}(M^n)$. Actually, formula (8) is a compact notation: its right-hand side evaluates at $\mathbf{x} \in M^n$ the infinitely many coefficients of $\vec{x}_i^{\pm 1}$, $\vec{x}_i^{\pm 1} \circ \vec{x}_j^{\pm 1}$, $\vec{x}_i^{\pm 1} \circ \vec{x}_j^{\pm 1}$, $\vec{x}_i^{\pm 1} \circ \vec{x}_j^{\pm 1}$, etc.



By construction, the value $a|_s$ of a homogeneous word a written in the alphabet $\mathbf{a} = \{a^i, 1 \le i \le m\}$ is the product of the map values at the consecutive letters of that word. For instance, we postulate that

 $(a^i \circ a^j)|_{\mathbf{s}}(\mathbf{x}, \vec{\mathbf{x}}^{\pm 1}) = s^i(\mathbf{x}, \vec{\mathbf{x}}^{\pm 1}) \circ s^j(\mathbf{x}, \vec{\mathbf{x}}^{\pm 1});$

the multiplication \circ on the right-hand side is the multiplication in the sheaf of free associative algebras, so that one proceeds recursively.

Convention (*Irreducibility*). Let the mapping in (8) be such that no positive-length word $a \in \text{Free}(a^1, \ldots, a^m)$ is evaluated to a zero proper length word $a|_s$ in the algebra of walks (i.e., the null path **1** with a nonzero coefficient from $C^{\infty}(M^n)$). (For example, we exclude the case where $a^1 := \vec{x}_1 \circ \vec{x}_2^{-1}$ and $a^2 := \vec{x}_2 \circ \vec{x}_1^{-1}$, so that $a^1 \circ a^2|_s = \mathbf{1}$ at all $x \in M^n$.)

The construction of sections (8) is furthered to the quotient $\mathcal{A} = \text{Free}(a^1, \ldots, a^m)/\sim$, which yields the evaluation mapping \mathbf{s} from the sheaf of algebras \mathcal{A} over the commutative manifold M^n to the sheaf of unital algebras $\mathfrak{X}(\mathbf{x}^{\pm 1})$ of cyclic words written in the edge alphabet for a given tiling of M^n . (The restriction maps r_V^U in that sheaf, for $V \subseteq U$ open in M^n , are the identity mapping of $\mathfrak{X}(\mathbf{x}^{\pm 1})$, except for the constant mapping to the null word (1) over $V \subseteq \Delta_{\alpha}$ at some $\alpha \in \mathcal{I}$; over $\emptyset \subset M^n$, the sheaf structure is empty.)

Remark 1.1. Let us remember that this evaluation mapping *s* is not a homomorphism of the cyclic word algebras A and $\chi(\vec{x}^{\pm 1})$, respectively. The inequality,

$$((a_1) \stackrel{\mathcal{A}}{\times} (a_2))|_{s}(\mathbf{x}) \neq (a_1)|_{s} \stackrel{\chi(\mathbf{x}^{\pm 1})}{\times} (a_2)|_{s}(\mathbf{x}),$$

can occur for some words $(a_1), (a_2) \in A$ and at some point $\mathbf{x} \in M^n$. Indeed, the multiplication \times in A unlocks the cyclic words in between the letters \mathbf{a} that will later be evaluated using (8), whereas the multiplication \times in $\mathfrak{X}(\mathbf{x}^{\pm 1})$ unlocks the cyclic words between *every* two consecutive symbols from the edge alphabet $\mathbf{x}^{\pm 1}$ (see also Remark 2.8).

Remark 1.2. Evaluation (8) of a word *a* from Free (*a*) paves the way (weighted by elements of $C^{\infty}(M^n)$) along the edges $\vec{x}_i^{\pm 1}$ of the graph which we started with. If the path $a|_s$ is closed, then it does not matter where one starts reading that cyclic word (now written in the alphabet $\vec{x}^{\pm 1}$); hence the value $(a)|_s(\vec{x}^{\pm 1})$ is uniquely defined. However, the cyclic invariance of the word (*a*) does *not* imply that the path $a|_s$ is closed.¹⁵ Strictly speaking, not every word written in the alphabet $\vec{x}_i^{\pm 1}$ encodes some path connecting cells in the tiling. (Still the converse is true: every path is encoded by the respective word and every closed path – written by using the alphabet *a* and map (8) – is described by the equivalence class of cyclic words.)

It is readily seen that for a word *a* of length $\lambda > 0$, the evaluation of (*a*) by using (8) can produce up to λ different elements in the space of cyclic words $\mathfrak{X}(\vec{\mathbf{x}}^{\pm 1})$. Such co-existence of the value $(a)|_{s}$ of a given cyclic word (*a*) in several states occurs due to the noncommutativity of the concatenation \circ of words in the edge alphabet $\vec{\mathbf{x}}^{\pm 1}$.

¹⁵ Alternatively, it could require some effort to make a given value a cyclic word indeed by contracting the graph between the path loose ends.

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Let us remember that the same multiple-value effect can also be produced (moreover, regardless of the availability of an edge alphabet) whenever the multiplication \cdot of coefficients in (8) – or in (11b) in what follows – is replaced by using a noncommutative associative star-product \star on the affine manifold M^n (see Model 2 and Remark 2.7).

Remark 1.3. The cyclic shift operation (3) on Free (a^1, \ldots, a^m) descends to the identity mapping $(a) \mapsto (a)$ on the algebra \mathcal{A} of cyclic words (a). In what follows – in particular, starting from the moment when the alphabet is \mathbb{Z}_2 -graded, $\mathfrak{t}(\gamma_1 \circ \ldots \circ \gamma_{\lambda}) = (-)^{|\gamma_1 \circ \ldots \circ \gamma_{\lambda-1}| \cdot |\gamma_{\lambda}|} \gamma_{\lambda} \circ \gamma_1 \circ \ldots \circ \gamma_{\lambda-1}$ for $\lambda \ge 2$, so that the restriction of \mathfrak{t} on $\mathcal{A}^{(0|1)}$ is not just the identity – we shall *not* attempt viewing the mapping \mathfrak{t} as conjugation $\gamma_{\lambda} \circ (\gamma_1 \circ \ldots \circ \gamma_{\lambda}) \circ \gamma_{\lambda}^{-1}$ whenever the rightmost comultiple is well defined. In terms of the algebra of walks on a lattice such conjugations would mean that the entire contour $\gamma_1 \circ \ldots \circ \gamma_{\lambda}$ is first displaced by γ_{λ} , then read in full, and followed by a step back. This is what one should avoid, especially on *irregular* lattices. Conversely, we shall always view the shift \mathfrak{t} as a replacement of the marker ∞ at which one begins reading a given cyclic word.

Remark 1.4 ($\mathbf{1}(\mathbf{x}) \in C^{\infty}(M^n)$). As soon as the unital algebra \mathcal{A} of cyclic words is placed over the "points" of M_{NC}^n – in earnest, over usual points $\mathbf{x} \in M^n$ of the substrate manifold – the zero-length words in \mathcal{A} are weighted pointwise over M^n by elements of the ring $C^{\infty}(M^n)$ that now plays the rôle of the ground field \Bbbk . This blow-up $\Bbbk \hookrightarrow C^{\infty}(M^n)$ is standard in the differential calculus on (jet) bundles in the commutative case (cf. [16,19,25,44]).

1.4. The geometry of jet space $J^{\infty}(\pi_{NC})$

Now we recall the standard construction of infinite jet space $J^{\infty}(\pi_{\text{NC}})$ towered over the substrate manifold M^n and the sheaf M_{NC}^n . We emphasise that this construction (local with respect to $\mathbf{x} \in U \subseteq M^n$) refers only to the affine structure on the domain set M^n and to the vector space organisation of objects over it.

Expansion (8) yields the infinite jet alphabet which consists of $a^i \equiv a^i_{\varnothing}$ and $a^i_{\chi^j}, a^i_{\chi^j \chi^k}, \ldots, a_\sigma$ for $|\sigma| \ge 0$ over a chart $U \subseteq M^n$ with local coordinates $\mathbf{x} = (x^1, \ldots, x^n)$; here σ is a multiindex. The evaluation mappings $\mathbf{a} = \mathbf{s}(\mathbf{x}, \mathbf{x}^{\pm 1})$ are extended for all $|\sigma| \ge 0$ by $\mathbf{a}_\sigma = (\frac{\partial^{|\sigma|}}{\partial \mathbf{x}^\sigma} \mathbf{s})(\mathbf{x}, \mathbf{x}^{\pm 1})$ using the jets jet^{∞}(\mathbf{s}). Under the assumption that the base manifold M^n be affine, the jet letters \mathbf{a}_σ are well behaved under a change $\mathbf{x} = \mathbf{x}(\mathbf{\tilde{x}})$ of local coordinates. Let us denote by $[\mathbf{a}]$ the differential dependence on letters $a^i, a^i_{\chi^j}, \ldots, a_\sigma$ up to some arbitrarily high but always finite order $|\sigma| < \infty$. The construction of the algebra $\mathcal{F}(\pi_{NC})$ of cyclic-word valued functions on $J^{\infty}(\pi_{NC})$ is standard: namely, it is the inductive limit of filtered algebras [16,44]. Likewise, the *total derivatives* $\frac{d}{dx^i}$, which we denote synonymically by D_{x^i} for $1 \le i \le n$ making no further distinction between $(\frac{d}{d\mathbf{x}})^{\sigma'}$ and $D^{\sigma}_{\mathbf{x}}$, are introduced by using the restrictions of elements $f \in \mathcal{F}(\pi_{NC})$ to 'graphs' of (8), i.e.

$$\frac{\mathrm{d}}{\mathrm{d}x^{i}}(f)\Big|_{\mathrm{jet}^{\infty}(\boldsymbol{a}=\boldsymbol{s}(\cdot,\vec{\mathbf{x}}^{\pm 1}))}(\boldsymbol{x}_{0}) \stackrel{\mathrm{def}}{=} \frac{\partial}{\partial x^{i}}\Big|_{\boldsymbol{x}_{0}}\left(f|_{\mathrm{jet}^{\infty}(\boldsymbol{a}=\boldsymbol{s}(\cdot,\vec{\mathbf{x}}^{\pm 1}))}\right).$$
(9)

This determines the usual coordinate expressions for $1 \le i \le n$,

$$\frac{\overrightarrow{d}}{dx^{i}} = \frac{\partial}{\partial x^{i}} + \sum_{j=1}^{m} \sum_{|\sigma| \ge 0} a^{j}_{\sigma \cup \{i\}} \frac{\overrightarrow{\partial}}{\partial a^{j}_{\sigma}}$$

which starts at ∞ and acts along the orientation of every cyclic word, and

$$\frac{\overleftarrow{\mathsf{d}}}{\mathsf{d}x^i} = \frac{\partial}{\partial x^i} + \sum_{j=1}^m \sum_{|\sigma| \ge 0} \frac{\overleftarrow{\partial}}{\partial a^j_{\sigma}} a^j_{\sigma \cup \{i\}}$$

which acts from ∞ clockwise. Both the operators \overleftarrow{D}_{x^i} and \overrightarrow{D}_{x^i} show up, first, through the substrate part $\mathbf{1} \cdot \partial/\partial x^i$ plus the *m* sums – formally, infinite – of cyclic words such that the derivations $\partial/\partial a^j_{\sigma}$ sit in their locks.

Remark 1.5. We have that

$$\frac{\mathrm{d}}{\mathrm{d}x^{i}} = 1 \underbrace{\bigcirc}_{\partial x^{i}} \frac{\partial}{\partial x^{i}} + \sum_{j,\sigma} a^{j}_{\sigma \cup \{i\}} \underbrace{\bigcirc}_{\partial \sigma} \frac{\partial}{\partial a^{j}_{\sigma}},$$

whence all the terms which are produced from the counterclockwise action of \overrightarrow{d}/dx^i via the Leibniz rule on a given cyclic word $f \in \mathcal{F}(\pi_{NC})$ have the shape¹⁶

$$\cdots + 1 \underbrace{\overrightarrow{\partial}_{\partial x^{i}} \cdot \cdot \cdot x^{i}}_{\partial \partial x^{i}} + \cdots + a^{j}_{\sigma \cup \{i\}} \underbrace{\overrightarrow{\partial}_{\partial a^{j}_{\sigma}} \cdot \cdot \cdot a^{j}_{\sigma}}_{\partial \partial a^{j}_{\sigma}} + \cdots$$
(10)

¹⁶ If the base coordinates x^k are not considered as symbols of any alphabet at hand, then the entire coefficient $\in C^{\infty}(M^n)$ of the cyclic word $f \in \mathcal{F}(\pi_{NC})$ can be placed at the lock ∞ .

(The derivations proceed along the orientation of the argument f, acting on the symbols in front of which the cyclic word f is disrupted.) This shows that the operation $d/dx^i \otimes f \mapsto d/dx^i(f)$ is again a topological pair of pants $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$.

2. Differential graded Lie algebra of noncommutative local functionals

2.1. The variational symplectic dual

We shall presently extend the alphabet a^1, \ldots, a^m of the associative algebra $Free_k(a^1, \ldots, a^m)$ which we started with. Namely, we introduce the new symbols $a_1^{\dagger}, \ldots, a_m^{\dagger}$ that *ought to be* the canonical conjugates of the respective variables a^1, \ldots, a^m ; let us explain what this means.

First, let us consider the free associative algebra standing alone, that is, *before* the evaluation of generators by (8) under a given map **s**. In this set-up, there still remain two ways to understand the nature of new generators a_i^{\dagger} , namely, the coarse and fine. The former is to proclaim that the vector space $V^{\dagger} := \operatorname{span}_{\Bbbk}(a_1^{\dagger}, \ldots, a_m^{\dagger})$ is dual to the linear span $V := \operatorname{span}_{\Bbbk}(a_1^{1}, \ldots, a_m^{\dagger})$ under the \Bbbk -valued coupling; by construction, the elements a_i^{\dagger} specify the basis dual to that of a^i in V. The new letters are then incorporated into the set of generators of (the unital extension of) the associative algebra $\Bbbk \cdot 1 \oplus \operatorname{Free}_{\Bbbk}(a_1^1, \ldots, a_m^{\dagger})$. This definition is sufficient (which is explained in Section 3) to make the noncommutative variational Poisson formalism work.

The fine approach is as follows; although less is required, it is still enough to construct the (non)commutative Batalin–Vilkovisky geometry. Suppose that the generators a^i of the free associative algebra undergo a shift by $\delta \boldsymbol{a} = \delta a^i \cdot \vec{e}_i$, where the *m* vectors \vec{e}_i constitute the adapted¹⁷ basis in $T_a V$, each of them pointing along the respective generator in the vector space $V = \operatorname{span}_{\Bbbk}(a^1, \ldots, a^m)$. Likewise, consider the adapted basis $\vec{e}^{\dagger,i}$ in the tangent space $T_{a^{\dagger}}V^{\dagger}$ at the point \boldsymbol{a}^{\dagger} of the vector space $V^{\dagger} = \operatorname{span}_{\Bbbk}(a_1^{\dagger}, \ldots, a_m^m)$. We require that the frame $\vec{e}^{\dagger,i}$ be \Bbbk -dual to the frame \vec{e}_i , $1 \leq i \leq m$, so that the variation $\delta \boldsymbol{a}^{\dagger} = \delta a_i^{\dagger} \cdot \vec{e}^{\dagger,i}$ is the canonical conjugate of the diagonal deformation $\delta \boldsymbol{a} = \delta a^i \cdot \vec{e}_i$, see (13) and (14).

Remark 2.1. In the second approach, we do not proclaim that the new symbols a_i^{\dagger} are the duals of the old generators a^i (or their inverses, or *reverses*, cf. (12)). In other words, we do not use the isomorphism between the vector space $V^{\dagger} = \operatorname{span}_{\Bbbk}(a_1^{\dagger}, \ldots, a_m^{\dagger})$ and the vector space $T_{a^{\dagger}}V^{\dagger}$ tangent to it at a point. Note that the left-hand side of the isomorphism $V^{\dagger} \simeq T_{a^{\dagger}}V^{\dagger}$ exploits the *global* vector-space organisation of V^{\dagger} whereas the right-hand side refers to its *local* portrait near the point a^{\dagger} . This is what the Batalin–Vilkovisky and Poisson formalisms really need.

So, we extend the set a^1, \ldots, a^m of generators by the symbols $a_1^{\dagger}, \ldots, a_m^{\dagger}$: at every *i*, the new symbol a_i^{\dagger} matches the respective generator a^i in the above sense. As before, we take the quotient of the free algebra Free $_{\Bbbk}(a^1, \ldots, a^m; a_1^{\dagger}, \ldots, a_m^{\dagger})$ over the linear relation \sim of equivalence under cyclic shifts. Thus we obtain the unital commutative non-associative algebra $(\Bbbk \cdot \mathbf{1} \oplus \text{Free}_{\Bbbk}(a^1, \ldots, a^m; a_1^{\dagger}, \ldots, a_m^{\dagger}))/\sim$ of cyclic words (written now in the double alphabet).¹⁸ We postulate that the resulting algebra of cyclic words becomes the fibre in the noncommutative bundle π_{NC} over the sheaf M_{NC}^n of algebras of walks along a given tiling of the substrate affine manifold M^n .

Remark 2.2. Let us examine how the noncommutative sections (8), which evaluate a^i to $s^i(\mathbf{x}, \mathbf{x}^{\pm 1})$ over $\mathbf{x} \in M^n$, can be extended to the double alphabet evaluation using sections $(\mathbf{s}, \mathbf{s}^{\dagger})$ of π_{NC} . The guiding principle that one must keep in mind is that in the Batalin–Vilkovisky (BV) formalism, the quantum action functional is constrained by the natural postulate $\langle 1 \rangle = 1$ for the averaging over sections of the BV superbundle. This condition implies that the objects in that formalism are effectively independent of a choice of sections by using which the new, dual variables could be evaluated at $\mathbf{x} \in M^n$. Hence the generators a_i^{\dagger} could acquire *whatever* values; indeed, no physics depends on them at the end of the day. (If so, leaving the respective components of the sections unspecified would be another option.)

However, we are also free to assign the values $\mathbf{a} = \hat{\mathbf{s}}(\mathbf{x}, \vec{\mathbf{x}}^{\pm 1})$ and $\mathbf{a}^{\dagger} = \mathbf{s}^{\dagger}(\mathbf{x}, \vec{\mathbf{x}}^{\pm 1})$ in a way we choose.¹⁹

Convention. For a given section \boldsymbol{s} of $\pi_{\rm NC}$,

$$a^{i} = s^{i}(\boldsymbol{x}, \vec{\boldsymbol{x}}^{\pm 1}) = \sum_{J} f^{iJ}(\boldsymbol{x}) \vec{\mathbf{x}}_{j_{1}}^{\alpha(1)} \circ \ldots \circ \vec{\mathbf{x}}_{j_{\lambda}}^{\alpha(\lambda)}, \qquad f^{iJ} \neq 0,$$
(11a)

¹⁷ In other words, only the *diagonal* deformations of the associative algebra generators are now allowed. This should be expected; for in the commutative BV-geometry, the variables a^i and $b_i = \Pi(a_i^{\dagger})$, see below, describe the conjugate field–antifield or ghost–antighost pairs that stem from the different generations of Noether's identities between the Euler–Lagrange equations of motion. Hence by construction, the variables a^i or b_i at different values of the index *i* are fibre coordinates in different vector bundles, merged later to their Whitney sum (see [44, §2, 6, 11] or [18] and references therein).

¹⁸ The space of free algebra generators is, strictly speaking, not the direct sum span_k $\langle a \rangle \oplus$ span_k $\langle a^{\dagger} \rangle$ because under a rescaling of the generators a, the dual letters a^{\dagger} can be rescaled inverse proportionally.

¹⁹ The *fourth* scenario is specific to the (non)commutative variational Poisson formalism, in the frames of which the symbols a^{\dagger} play the rôles of placeholders for the variational covectors that are not exact; but still, the isomorphism $V^{\dagger} \simeq T_{a^{\dagger}}V^{\dagger}$ is explicitly used in the assignment $a^{\dagger} := p$ (we shall discuss this in Section 3).

we set the respective components of s^{\dagger} equal to the sum of formal *reverses* for each nonzero, homogeneous word in s,

$$a_i^{\dagger} := s_i^{\dagger}(\boldsymbol{x}, \vec{\boldsymbol{x}}^{\pm 1}) = \sum_J \frac{1}{f^{iJ}(\boldsymbol{x})} \vec{x}_{j_{\lambda}}^{-\alpha(\lambda)} \circ \dots \circ \vec{x}_{j_1}^{-\alpha(1)},$$
(11b)

where, at every point $\mathbf{x} \in U \subseteq M^n$, the sum is taken over the indexes J such that the coefficients f^{iJ} do not vanish.²⁰

Example 2.1. If

$$a^{i} = \sum_{k \in \mathbb{Z}} (\text{loop})^{k}, \quad \text{then} \quad a^{\dagger}_{i} = \sum_{k \in \mathbb{Z}} (\text{loop})^{-k}, \tag{12}$$

that is, all the reiterations of a closed path are walked backwards.

Remark 2.3. Convention (11b) means that, whenever each component s^i of the map s is just a single word, the respective dual a_i^{\dagger} becomes the weighted *reverse* – and true *inverse* – of the path $a^i(\mathbf{x}, \mathbf{x}^{\pm 1})$.

Remark 2.4. When cyclic words (*a*) are evaluated using (11) at points $\mathbf{x} \in M^n$, each resulting cyclic word from the algebra $\mathfrak{X}(\mathbf{x}^{\pm 1})$ acquires an overall coefficient (which is supposed to be a smooth function on M^n). The associativity of multiplication \cdot of the coefficients $f_J(\mathbf{x})$ is used here. Note however that the commutativity of \cdot can be relaxed, yet if so, the result of evaluation $(a)|_{(\mathbf{s},\mathbf{s}^{\dagger})}(\mathbf{x}) \in \mathfrak{X}(\mathbf{x}^{\pm 1})$ would depend on a position of the lock ∞ between letters of the word a, see (7) and Remark 2.7.

2.2. Elementary (non)commutative variations

The precedence $\vec{e}_1 \prec \ldots \prec \vec{e}_m \prec \vec{e}^{\dagger,1} \prec \ldots \prec \vec{e}^{\dagger,m}$ of the basic vectors for virtual shifts endows the Cartesian sum $T_a \operatorname{span}(a^1, \ldots, a^m) \oplus T_{a^{\dagger}} \operatorname{span}(a^{\dagger}_1, \ldots, a^{\dagger}_m)$ of the dual spaces with an orientation; it fixes the signs in all the structures of (non)commutative symplectic geometry. The signs show up through the two couplings $T_a V \times T_{a^{\dagger}} V^{\dagger} \to \Bbbk$ and $T_{a^{\dagger}} V^{\dagger} \times T_a V \to \Bbbk$ (which we denote by \langle , \rangle in both cases, making no confusion; for the sequential order is essential). Namely, we have that

$$\langle \vec{e}_i, \vec{e}^{\dagger,j} \rangle = \delta_i^j \quad \text{and} \quad \langle \vec{e}^{\dagger,j}, \vec{e}_i \rangle = -\delta_i^j,$$
(13)

where δ_i^j is the Kronecker symbol that equals unit iff i = j and which is set equal to zero otherwise, see [9, §2.2].

Note that the virtual deformations $\delta \boldsymbol{a} = \delta a^i(\boldsymbol{x}) \cdot \vec{e}_i(\boldsymbol{x})$ and $\delta \boldsymbol{a}^{\dagger} = \delta a^{\dagger}_j(\boldsymbol{x}) \cdot \vec{e}^{\dagger,j}(\boldsymbol{x})$ can be dependent on $\boldsymbol{x} \in M^n$ – and they should be such. By construction, each pair ($\delta \boldsymbol{a}, \delta \boldsymbol{a}^{\dagger}$) of virtual shifts for the generators a^i and a^{\dagger}_i is a map belonging to the space Map $(M^n \to T_{(\boldsymbol{a}, \boldsymbol{a}^{\dagger})} \operatorname{span}_{\Bbbk}(\boldsymbol{a}; \boldsymbol{a}^{\dagger}))$. We let the shifts be *normalised* at all internal points $\boldsymbol{x} \in \operatorname{supp}(\delta a^i) \subseteq M^n$ by the constraint

$$\delta a^{i}(\mathbf{x}) \cdot \delta a^{\dagger}_{i}(\mathbf{x}) \equiv 1.$$
 (no summation!)

This is why the couplings of virtual deformations are invisible in the ready-to-use formulae. Indeed, it is enough to know the signs

$$\langle \delta a^{i}(\boldsymbol{x}) \cdot \underbrace{\vec{e}_{i}^{\text{first}}, \vec{e}^{\dagger,i}(\boldsymbol{y})}_{\boldsymbol{x}=\boldsymbol{y}} \cdot \delta a^{\dagger}_{i}(\boldsymbol{y}) \rangle \bigg|_{\boldsymbol{x}=\boldsymbol{y}} = +1$$
(14a)

and

$$\langle \delta a_i^{\dagger}(\boldsymbol{y}) \cdot \underbrace{\vec{e}}_{i,i}^{\text{first}}(\boldsymbol{y}), \quad \underbrace{\vec{e}}_i(\boldsymbol{x}) \\ \rightarrow \delta a^i(\boldsymbol{x}) \rangle \bigg|_{\boldsymbol{x}=\boldsymbol{y}} = -1,$$
(14b)

at all the internal points **x** of the support supp(δa^i), see [9,26,36] for illustrations.²¹

2.3. Parity-odd neighbours $\mathbf{b} = \Pi(\mathbf{a}^{\dagger})$

From now on, let the set-up be \mathbb{Z}_2 -graded by the function $|\cdot|$ that takes values in \mathbb{Z} and determines the parity $(-)^{|\cdot|}$. All the objects which have been considered in the preceding sections were parity-even, of proper grading 0. Let us relay the parity of symbols a_i^{\dagger} by postulating that the new parity-odd variables carry the grading +1 (or *minus* one, or any

 $^{^{20}}$ In view of Remark 2.2, the fact that the extension s^{\dagger} remains undefined at the zero locus of all these coefficients makes no harm.

²¹ The usefulness of carrying the coefficients $\delta a(\cdot)$ and $\delta a^{\dagger}(\cdot)$ all way long is revealed in the geometry of *iterated* variations; let us also remember that we shall not always indicate the reference of frames $\vec{e}_i(\cdot)$ and $\vec{e}^{\dagger,i}(\cdot)$ to points of the substrate manifold M^n . However, the fact that such reference is not impossible is crucial for the consistency of the formalism.



Fig. 4. The elementary displacements $\vec{x}^{\pm 1}$ in a tiling of M^n versus the gauge connection fields ϕ over the space–time M^n ; the canonical duality of diagonal variations for the opposite-parity halves of the alphabet versus the opposite-parity field–antifield and ghost–antighost pairs.

other (un)conventional odd integer number). To keep track of the reversed parity, let us denote these generators by $\boldsymbol{b} = (b_1, \ldots, b_m)$ so that $\Pi : a_i^{\dagger} \rightleftharpoons b_i$.

In the cyclic world, the concept of \mathbb{Z}_2 -grading works as follows:²²

$$\mathfrak{t}(\gamma_1 \circ \ldots \circ \gamma_{\lambda}) = (-)^{|\gamma_1 \circ \ldots \circ \gamma_{\lambda-1}| \cdot |\gamma_{\lambda}|} \gamma_{\lambda} \circ \gamma_1 \circ \ldots \circ \gamma_{\lambda-1}.$$

$$\tag{15}$$

We denote by $\mathcal{A}^{(0|1)}$ the graded commutative unital non-associative algebra of cyclic words written in the alphabet 1, $a^1, \ldots, a^m, b_1, \ldots, b_m$. By introducing the notation $\mathcal{A}^{(0|1)}$ we stress that the superdimension, equal to (m|m), is positive in both the parity-even and odd components of the generators space span_k(a; b).

Remark 2.5 ("(*abab*) = 0?"). The idea that cyclic words acquire and accumulate the extra sign factors, whenever a parity-odd symbol overtakes the rest of the word, creates the following subtlety.

Set m = 1 for definition and, omitting the symbols \circ of associative multiplication, first consider the cyclic word (*abaab*). The identical, parity-odd letters *b* contained in it can be distinguished nevertheless: one of them is *followed* by *aa* but preceded only by *a*, whereas the other is *preceded* by *aa* and followed by just a single copy of letter *a*; we have that (*abaab*) $\sim -(aabab)$.

On the other hand, the cyclic word (*abab*) does not contain any mechanism to distinguish between the two parity-odd entries *b*, yet ($\underline{ab}ab$) ~ $-(\underline{ab}\underline{ab})$ by construction. In fact, this word is synonymic to zero in the algebra of cyclic words which are written in the parity-extended alphabet.²³ Let us be aware of the existence of this class of synonyms for zero; the calculus of iterated variations which we presently develop is indifferent to these synonyms existence.

Model 3 (*The BV-geometry*). We take the algebra $\mathcal{A}^{(0|1)}$ as fibre²⁴ in the noncommutative superbundle $\pi_{NC}^{(0|1)}$ over the sheaf M_{NC}^n . This picture is summarised in Fig. 4(a), in which one easily recognises the noncommutative generalisation of the classical Batalin–Vilkovisky geometry (see Fig. 4(b)). The rôle of physical fields ϕ as sections of their bundle π is now played by the primitive displacements $\mathbf{x}^{\pm 1}$ in granulated space. The fibre algebra generated by the symbols a^i and b_i was known to us before as the Whitney sum of parity-even and odd components in the Batalin–Vilkovisky superbundle $\boldsymbol{\zeta}^{(0|1)}$, pulled back – by the projection π – over the total space of the bundle of physical fields. The symbols \boldsymbol{a} and $\boldsymbol{b} = \Pi(\boldsymbol{a}^{\dagger})$ of opposite parities form the noncommutative analogue of the BV-zoo $\boldsymbol{q}, \boldsymbol{q}^{\dagger}$ inhabited by the (anti)fields and (anti)ghosts. The rôle of the BV-bundle sections is granted to the two maps \boldsymbol{s} and \boldsymbol{s}^{\dagger} .²⁵

²² Let *F* be a homogeneous word of grading |F|, written by using the \mathbb{Z}_2 -graded alphabet. A full turn $F \mapsto t^{\lambda(F)}(F)$ along the orientation on the circle that carries the cyclic word *F* of length $\lambda(F)$ yields the sign factor $(-)^{|F|\cdot(|F|-1)} = (+)$; the equality is valid because the product of two consecutive integers standing in the exponent is always even. This argument shows also that, for a cyclic word to be rotated from a given configuration (determined by the position of the lock ∞ in between the word's letters) to another one, a choice to direct that rotation (counter)clockwise does not matter. Indeed, every clockwise rotation can be realised via one full turn clockwise (that would leave no effect by the above) followed by the appropriate shift backwards, in the counterclockwise direction.

²³ Analogous notions of zero non-oriented graphs equipped with edge ordering and of zero oriented graphs with an ordering of outgoing edges at every vertex are known from [45] and [34,45], respectively (cf. [35,46,47] for illustrations).

²⁴ Note that the parity reversion Π does not modify the topology of spaces, whence conventions (14) remain valid for the virtual variations $\delta \mathbf{b} = \delta b_i(\mathbf{x}) \cdot \vec{e}^{\dagger,i}(\mathbf{x})$. Note also that the presence of grading does not modify our earlier convention (11b) for the evaluation of symbols – as soon as a calculation governed by such graded arithmetic rule is over.

²⁵ We recall from [9] that the normalised variations δs and δs^{\dagger} were the dual components in sections of the tangent bundle $\mathbb{T}\zeta^{(0|1)}$; the vectors $\delta s(x, \phi(x), s(x, \phi(x)))$ and $\delta s^{\dagger}(x, \phi(x), s^{\dagger}(x, \phi(x)))$ were attached at points of graphs of sections for the BV-superbundle induced over π . The construction of these test shifts was laborious indeed in the graded-commutative world. On the other hand, the noncommutative target spaces contain nothing else but the basic letters *a* and *b* that undergo the virtual deformations, so that the picture is simplified considerably.

Remark 2.6. It will readily be seen that both the Batalin–Vilkovisky Laplacian of the integral functionals given by *zero* words – or the Schouten bracket taken for zero word functionals with any other cyclic-word functional – vanish identically.

Remark 2.7. Models 1 and 2, as well as Models 3 and 2 can be combined. For instance (for the latter pair), the (quasi)crystal tiling of an affine manifold M^n yields the alphabet $\mathbf{x}^{\pm 1}$ and concatenation \circ in the algebra of formal paths that show up in (11), whereas a given Poisson structure on that manifold M^n yields the associative \star -product which is used to multiply the coefficients $f_j(\mathbf{x}; \hbar) \in C^{\infty}(M^n)((\hbar))$ occurring in (11).

However, it is the graded-commutative model over the sheaves M_{NC}^n of algebras of walks along a tiling of M^n which will be the default set-up in the further study.

2.4. The ring of noncommutative local functionals

Let us proceed from *functions* on the space $J^{\infty}(\pi_{NC}^{(0|1)})$ of jets of sections (11) to the notion of *functionals* that take the evaluation mappings ($\mathbf{s}, \mathbf{s}^{\dagger}$) to formal cyclic words²⁶ written in the alphabet $\mathbf{x}^{\pm 1}$ of edges in the adjacency graph for a given crystal tiling of the substrate manifold M^n .

Convention. On the infinite jet space $J^{\infty}(\pi_{NC}^{(0|1)})$, every cyclic word (f) is a sum of its homogeneous components, each weighted by the coefficients that (can) depend on points \mathbf{x} of the substrate manifold M^n . For the sake of definition, let us assume that every such coefficient is C^{∞} -smooth on M^n ; their asymptotic behaviour must also be specified in advance so that the integration by parts makes sense. Specifically, if the manifold M^n is closed, then there is nothing to discuss: the empty boundary carries no boundary terms. However, should there be one, $\partial M^n \neq \emptyset$, or should the manifold M^n be non-compact (e.g., let $M^n = \mathbb{R}^n$ with the standard Euclidean topology), then we postulate that the coefficients decay rapidly towards the boundary ∂M^n or spatial infinity, respectively.

Likewise, we suppose that the supports supp δa^i of the $C^{\infty}(M^n)$ -smooth infinitesimal variations $\delta a^i(\cdot) \cdot \vec{e}_i(\cdot) : M^n \rightarrow T_a \operatorname{span}(a^1, \ldots, a^m)$ are compact and supp $\delta a^i \cap \partial M^n = \emptyset$.

The volume element $dvol(\mathbf{x})$ on M^n in the construction of integral functionals over the jet space $J^{\infty}(\pi_{NC}^{(0|1)})$ is another piece of external data.

Convention. We suppose that a volume element $dvol(\mathbf{x})$ is given at all points $\mathbf{x} \in M^n$ (possibly, in a way that depends on the tiling at hand). Also, we technically assume in this text that the volume element $dvol(\mathbf{x})$ may not depend on a choice of the mappings $(\mathbf{s}, \mathbf{s}^{\dagger})$ – that is, in a sense, on a configuration of noncommutative "fields" over the granulation M_{NC}^n of the physical space M^n .

One could think that the volume element $dvol(\cdot)$ is placed in the locks of cyclic words; this idea is practical because, whenever any such word is unlocked, it is converted at once into a singular linear integral operator supported on the diagonal; the volume element then disappears, giving way to the attachment points' congruence mechanism through the locality of couplings (13) in (14).

Convention. From now on we restrict the study to the class of functionals such that densities of the integral functionals $F = \int f(\mathbf{x}, [\mathbf{a}], [\mathbf{b}]) \circ \operatorname{dvol}(\mathbf{x})$ do not depend explicitly on the edge alphabet $\mathbf{x}^{\pm 1}$ (but can do so implicitly through a differential dependence of densities on \mathbf{a} or \mathbf{b} , which are evaluated at the jets $j_{\mathbf{x}}^{\infty}(\mathbf{s}, \mathbf{s}^{\dagger})$ of sections (11) for $\pi_{NC}^{(0|1)}$. (We recall that such vertical subtheory makes the full theory in Models 1 and 2, cf. footnote 10.)

Notation. The vector space of such integral functionals will be denoted by $\bar{H}^n(\pi_{NC}^{(0|1)})$.

Integral functionals $F_1, \ldots, F_\ell \in \bar{H}^n(\pi_{NC}^{(0|1)})$ are the building blocks in the *local* functionals such as $F_1 \times \cdots \times F_\ell \in \bar{H}^{n^{\otimes \ell}}(\pi_{NC}^{(0|1)})$.

Definition 3. Let $F_1 = \int f_1(\mathbf{x}_1, [\mathbf{a}], [\mathbf{b}]) \circ dvol(\mathbf{x}_1)$ and $F_2 = \int f_2(\mathbf{x}_2, [\mathbf{a}], [\mathbf{b}]) \circ dvol(\mathbf{x}_2)$ be two linear integral functionals the densities of which do not depend explicitly on any letters from the edge alphabet $\mathbf{x}^{\pm 1}$. The *product*

$$F_1 \times F_2 = \iint (f_1) \big|_{(\boldsymbol{x}_1, [\boldsymbol{a}], [\boldsymbol{b}])} \times (f_2) \big|_{(\boldsymbol{x}_2, [\boldsymbol{a}], [\boldsymbol{b}])} \circ \operatorname{dvol}(\boldsymbol{x}_1) \cdot \operatorname{dvol}(\boldsymbol{x}_2) \in \bar{H}^{n^{\otimes 2}}(\boldsymbol{\pi}_{\mathsf{NC}}^{(0|1)})$$

is the horizontal cohomology class of linear integral functionals over $(M^{n\otimes 2}, \operatorname{dvol}(\cdot)^{\otimes 2})$ such that their densities are equivalent to the product $(f_1) \times (f_2)$ in $\mathcal{A}^{(0|1)}$.

Setting $\bar{H}^{n\otimes 0}(\pi_{NC}^{(0|1)})$ equal to $\Bbbk \cdot (1)$ by definition, we extend the bi-linear operation \times recursively from pairs of integral functionals to the multiplication of products of any nonnegative number of functionals. Because the operation \times is not

²⁶ Such cyclic words are *formal* because (*i*) they could encode no realisable paths along the edges of the graph and (*ii*), although "cyclic" by construction, each homogeneous component of such words could not encode a *closed* walk, even if it did specify *some* walk along the edges.

associative, there are the respective Catalan number ways to arrange the multiplications in $F_1 \times \cdots \times F_\ell$ by inserting the $\ell - 1$ balanced pairs of parentheses. We let the *default ordering* be lexicographic: $(\cdots (F_1 \times F_2) \times \cdots \times F_{\ell-1}) \times F_\ell$.

Corollary 4. The multiplication \times of local functionals over $J^{\infty}(\pi_{NC}^{(0|1)})$ is graded-commutative: $F \times G = (-)^{|F| \cdot |G|} G \times F$ for F and G homogeneous.

Notation. Denote by

$$\overline{\mathfrak{M}}^{n}\left(\boldsymbol{\pi}_{\mathsf{NC}}^{(0|1)}\right) = \bigoplus_{\ell \geqslant 0} \bar{H}^{n^{\otimes \ell}}\left(\boldsymbol{\pi}_{\mathsf{NC}}^{(0|1)}\right) \tag{16}$$

the \mathbb{Z}_2 -graded commutative non-associative unital ring of local functionals in the noncommutative set-up under study.

To define the value of a local functional *F* at a section ($\mathbf{s}, \mathbf{s}^{\dagger}$), first let us consider the class of *integral* functionals such as $F = \int f(\mathbf{x}, [\mathbf{a}], [\mathbf{b}]) \circ \operatorname{dvol}(\mathbf{x})$, where the cyclic word ($f \circ \operatorname{dvol}(\mathbf{x})$) marks an equivalence class modulo integrations by parts (no boundary terms!).

Definition 4. The value of such integral functional at a given mapping $(\mathbf{s}, \mathbf{s}^{\dagger})$ is

$$F(\boldsymbol{s}, \boldsymbol{s}^{\dagger}) \stackrel{\text{def}}{=} \int_{M^{n}} f\left(\boldsymbol{x}, \operatorname{jet}_{\boldsymbol{x}}^{\infty}(\boldsymbol{s}), \operatorname{jet}_{\boldsymbol{x}}^{\infty}(\boldsymbol{s}^{\dagger})\right) \circ \operatorname{dvol}(\boldsymbol{x}) \in \mathfrak{X}(\vec{\boldsymbol{x}}^{\pm 1});$$
(17)

the integral makes sense due to our earlier assumptions on the global choice of alphabet $\vec{\mathbf{x}}^{\pm 1}$ on the entire M^n (that is, the tiling $M^n = \bigcup_{\alpha} \overline{\Delta}_{\alpha}$ is not *quasi* crystal) and on the class of functional coefficients depending on \mathbf{x} , so that the (im)proper integral converges.

The evaluation of products $F_1 \times \cdots \times F_\ell$ of functionals at a given mapping $(\mathbf{s}, \mathbf{s}^{\dagger})$ goes as follows; without loss of generality suppose $\ell = 2$. First, double $(\mathbf{s}, \mathbf{s}^{\dagger}) \mapsto (\mathbf{s}, \mathbf{s}^{\dagger})^{\otimes 2}$ for the $\ell = 2$ copies of the substrate manifold M^n , and then integrate over $M^{n\otimes 2}$ in the element of $\bar{H}^{n\otimes 2}(\pi_{NC}^{(0|1)})$.

Remark 2.8. Through the evaluation procedure, local functionals keep track of the fibre algebra $\mathcal{A}^{(0|1)}$ of cyclic words (even though neither the letters a^i nor b_j show up in the functionals' values that belong to the functionals value space $\mathfrak{X}(\vec{\mathbf{x}}^{\pm 1})$ of cyclic words written in the edge alphabet $\vec{\mathbf{x}}^{\pm 1}$).

Indeed, we recall from Remark 1.1 that generally speaking,

$$\left(F_1 \stackrel{\mathcal{A}^{(0|1)}}{\times} F_2\right)(\boldsymbol{s}, \boldsymbol{s}^{\dagger}) \neq F_1(\boldsymbol{s}, \boldsymbol{s}^{\dagger}) \stackrel{\boldsymbol{\mathfrak{X}}(\vec{\mathbf{x}}^{\pm 1})}{\times} F_2(\boldsymbol{s}, \boldsymbol{s}^{\dagger})$$

Moreover, although the multiplication \times in $\mathcal{A}^{(0|1)}$ is \mathbb{Z}_2 -graded commutative, that grading is lost in the course of functionals' evaluation at the mappings ($\mathbf{s}, \mathbf{s}^{\dagger}$); the multiplication \times in the non-graded algebra $\mathfrak{X}(\mathbf{x}^{\pm 1})$ is just commutative.

In the remaining part of this paper we reveal the structure of differential (shifted-) graded Lie algebra – more specifically, the BV algebra – on the \mathbb{Z}_2 -graded commutative non-associative unital ring $\overline{\mathfrak{M}}^n(\boldsymbol{\pi}_{NC}^{(0|1)})$ of local functionals. First we introduce some notation. Let us recall that the generators a^i and b_i are evaluated at sections ($\boldsymbol{s}, \boldsymbol{s}^{\dagger}$), whereas the generator virtual shifts ($\delta \boldsymbol{a}, \delta \boldsymbol{b}$) are taken from the space $\operatorname{Map}(M^n \to T_{(\boldsymbol{a},\boldsymbol{b})}\operatorname{span}_{\Bbbk}(\boldsymbol{a}; \boldsymbol{b}))$. To permit the iteration of variations, one has to deal with the space of local functionals such that densities of their integral building blocks can contain not only the generators but also their shifts (see footnote 28).

Notation. In order to avoid an agglomeration of formulae, let us denote by

$$\overline{\mathfrak{N}}^{n}\big(\mathbb{T}\boldsymbol{\pi}_{\mathsf{NC}}^{(0|1)}\big) = \bigoplus_{\ell \geqslant 0} \bar{H}^{n^{\otimes \ell}}\big(\mathbb{T}\boldsymbol{\pi}_{\mathsf{NC}}^{(0|1)}\big)$$

the vector space of such local functionals over the jet space $J^{\infty}(\pi_{\rm NC}^{(0|1)})$.

Remark 2.9. The multiplication of functionals, as part of the construction of space $\overline{\mathfrak{M}}^n(\pi_{NC}^{(0|1)})$, is provided by Definition 3. The BV Laplacian Δ (see Definition 5 and Definition 6) is a local variational operator on the space of local functionals, hence every argument of Δ is encoded by a cyclic word. This means that *first* such argument is formed (if necessary, by using the structure \times of algebra $\mathcal{A}^{(0|1)}$ whenever that input object is a product of several integral functionals; parentheses would specify the consecutive order of non-associative multiplications). Secondly, the BV algebra's differential operations Δ or $[\![,]\!]$ rework the input into an element of $\overline{\mathfrak{M}}^n(\pi_{NC}^{(0|1)})$. In particular, at no moment are any intermediate objects from $\overline{\mathfrak{N}}^n(\mathbb{T}\pi_{NC}^{(0|1)})$ multiplied anew by using the structure \times for (16).

For example, identity (24) frames an application of the differential structure $[\![,]\!]$ to the functional $F \times (G \times H) \in \overline{\mathfrak{M}}^n(\pi_{N^{(0)}}^{(0)})$, referred to at least three copies of the underlying manifold M^n . The same ordering – multiplication, then variation over M^n – applies to both sides of identity (28) where the BV Laplacian Δ works on the product $F \times G$ twice (in particular, via $[\![,]\!]$ to which the operator Δ is parent).

Therefore, let us remember that it is the ring $\overline{\mathfrak{M}}^n(\pi_{NC}^{(0|1)})$ but not the larger vector space $\overline{\mathfrak{N}}^n(\mathbb{T}\pi_{NC}^{(0|1)})$ on which the BV algebra structure is well defined. The reduction from $\overline{\mathfrak{N}}^n(\mathbb{T}\pi_{NC}^{(0|1)})$ to $\overline{\mathfrak{M}}^n(\pi_{NC}^{(0|1)})$ amounts to a perfect matching and then coupling of the (co)vectors \vec{e}_i and $\vec{e}^{\dagger,i}$ in all pairs of canonically dual components $\delta \boldsymbol{a}$ and $\delta \boldsymbol{b}$ of the variations. In the next section we recall the geometric mechanism of integration by parts; the way *how* the couplings are reconfigured itself is the algorithmic definition of the BV algebra structure (see Section 2.6).

2.5. Elements of the geometric theory of variations

The Gel'fand framework of singular integral distributions is known, e.g., from [48]. In our case, the space $\overline{\mathfrak{N}}^n(\mathbb{T}\pi_{NC}^{(0|1)})$ of local functionals over the tangent superbundle $\mathbb{T}\pi_{NC}^{(0|1)}$ is the space of basic objects on which the variations act by singular linear integral operators.

For consistency, let us outline key ideas in the geometry of iterated variations (introduced in [9,26] and illustrated in [27,36]); they are as follows.

- The unlinking of a cyclic word, together with an intention to paste the open string of symbols contained in it into another word as an uninterrupted fragment, converts the (procedure of) insertion of that string into a singular linear integral operator supported on the diagonal.
- Such operators are singular because the restriction to the diagonal over points in copies of the substrate manifold M^n is ensured by ordered couplings (13) which are *not defined* off the diagonal $\mathbf{x} = \mathbf{y}$ in (14).
- The definitions of the Batalin–Vilkovisky Laplacian △ and variational Schouten bracket [[,]] are operational, that is, every such definition is an algorithm for the on-the-diagonal reconfiguration of the couplings.
- The objects that are usually viewed in the calculus of variations as differential forms are either the volume element $dvol(\mathbf{x})$ on the substrate manifold M^n or the dual bases $\vec{e}_i, \vec{e}^{\dagger,i}$ in the tangent spaces attached at the point (\mathbf{a}, \mathbf{b}) of the fibre algebra (this is what its alphabet was doubled for). The *orientation* uniquely determines the signs of couplings (13) by ordering the tangent vectors. This explains why such differential 1-forms anticommute.

Convention. In the course of virtual variation of the symbols a^i_{σ} and $b_{i,\tau}$ by using²⁷

$$(\delta a^{i}) \left(\overleftarrow{\frac{\partial}{\partial \boldsymbol{x}}} \right)^{\sigma} (\boldsymbol{x}) \cdot \vec{\boldsymbol{e}}_{i}(\boldsymbol{x}) \quad \text{and} \quad (\delta b_{j}) \left(\overleftarrow{\frac{\partial}{\partial \boldsymbol{x}}} \right)^{\tau} (\boldsymbol{x}) \cdot \vec{\boldsymbol{e}}^{\dagger,j}(\boldsymbol{x}),$$
(18)

the responses of integral functionals are always expanded with respect to the dual bases $\vec{e}^{\dagger,i}$ and \vec{e}_i . For instance, we obtain the singular linear integral operators

$$\vec{\delta a}(\cdot) = \int_{M^n} d\mathbf{y} \sum_{i=1}^m \sum_{|\sigma| \ge 0} (\delta a^i) \left(\frac{\overleftarrow{\partial}}{\partial \mathbf{y}}\right)^{\sigma}(\mathbf{y}) \cdot \langle \underline{\vec{e}}^{\text{first}}_{+,i}(\cdot) \rangle \frac{\overrightarrow{\partial}}{\partial a^i_{\sigma}}$$
(19a)

and

$$\vec{\delta b}(\cdot) = \int_{M^n} \mathrm{d}\boldsymbol{z} \sum_{j=1}^m \sum_{|\tau| \ge 0} (\delta b_j) \left(\frac{\overleftarrow{\partial}}{\partial \boldsymbol{z}}\right)^{\tau} (\boldsymbol{z}) \cdot \left(\underbrace{(-\vec{e}^{\dagger,j})(\boldsymbol{z}), \stackrel{\text{second}}{\vec{e}_i(\cdot)}}_{\partial b_{j,\tau}}\right) \frac{\overrightarrow{\partial}}{\partial b_{j,\tau}} .$$
(19b)

This convention will be illustrated in the sequel.

• Given by its own singular integral operator, each variation brings a new copy of the integration domain M^n into the picture. In consequence, all the intermediate objects $Obj \in \overline{\mathfrak{N}}^n(\mathbb{T}\pi_{NC}^{(0|1)})$ that emerge in the course of calculations do retain a kind of memory of the way how they were obtained from the input data.²⁸ That is, no calculation can be interrupted along the way.

Lemma 5 (Integration by Parts). From the powers of partial derivatives $(\overleftarrow{\partial}/\partial \mathbf{y})^{\sigma}$ that act on the test shifts in (19) one obtains, due to the locality of couplings $\langle \cdot, \cdot \rangle$, the powers of minus total derivatives $(-\overrightarrow{d}/d\mathbf{x})^{\sigma}$ that act on densities of integral functionals.

Explanation (*See* [9]). Consider a point \boldsymbol{y} of the affine manifold M^n and denote by $\boldsymbol{y} + \delta \boldsymbol{y} \in M^n$ a near-by point with coordinates $y^i + \delta y^i$, here and immediately below $1 \leq i, \alpha \leq n$; the notation $\lim_{\delta \boldsymbol{y} \to 0}$ makes obvious sense. For the sake of

²⁷ It is readily seen that the congruence of multi-indexes σ in $(\partial/\partial \mathbf{x})^{\sigma}$ and a_{σ}^{i} (as well as in the partial derivative $\vec{\partial}/\partial a_{\sigma}^{i}$, see (19a)) refers to the definition of vector as an equivalence class of curves passing through a point.

²⁸ In the (graded-)commutative language of bundles this means that their products $\zeta^{(0|1)} \times \mathbb{T}\zeta^{(0|1)} \times \cdots \times \mathbb{T}\zeta^{(0|1)}$, standing over $M^n \times M^n \times \cdots \times M^n$, are taken, but not their Whitney sums $\zeta^{(0|1)} \times_{M^n} \mathbb{T}\zeta^{(0|1)} \times_{M^n} \cdots \times_{M^n} \mathbb{T}\zeta^{(0|1)}$ are fibred over a single copy of the base manifold M^n .

brevity, put $\sigma := \{x^{\alpha}\}$. We have that, due to the absence of boundary terms and then by definition (by Newton–Leibniz and in the last line, by S. Lie),

$$\begin{split} &\int \mathbf{d}\mathbf{y} \langle (\delta a^{i}) \frac{\partial}{\partial y^{\alpha}} (\mathbf{y}) \cdot \vec{e}_{i}(\mathbf{y}), \vec{e}^{\dagger,i}(\mathbf{x}) \left(\frac{\partial}{\partial a_{x^{\alpha}}^{i}} f(\mathbf{x}, [\mathbf{a}], [\mathbf{b}]) \right) \Big|_{j \in t^{\infty}_{\mathbf{x}}(\mathbf{s}, \mathbf{s}^{\dagger})} \rangle = \\ &= \int \mathbf{d}\mathbf{y} \ \delta a^{i}(\mathbf{y}) \left(-\frac{\partial}{\partial y^{\alpha}} \right) \langle \vec{e}_{i}(\mathbf{y}), \vec{e}^{\dagger,i}(\mathbf{x}) \left(\frac{\partial}{\partial a_{x^{\alpha}}^{i}} f(\mathbf{x}, [\mathbf{a}], [\mathbf{b}]) \right) \Big|_{j \in t^{\infty}_{\mathbf{x}}(\mathbf{s}, \mathbf{s}^{\dagger})} \rangle \\ &\stackrel{\text{def}}{=} - \int \mathbf{d}\mathbf{y} \ \delta a^{i}(\mathbf{y}) \lim_{\delta y^{\alpha} \to +0} \frac{1}{\delta y^{\alpha}} \begin{cases} \langle \vec{e}_{i}(\mathbf{y} + \delta y^{\alpha}), \vec{e}^{\dagger,i}(\mathbf{x}) \\ +1 \ i \ \mathbf{x} = \mathbf{y} + \delta y^{\alpha} \\ - \langle \vec{e}_{i}(\mathbf{y}), \vec{e}^{\dagger,i}(\mathbf{x}) \\ +1 \ i \ \mathbf{x} = \mathbf{y} \end{cases} \frac{\partial}{\partial a_{x^{\alpha}}^{i}} f(\mathbf{x}, [\mathbf{a}], [\mathbf{b}]) \Big|_{j \in t^{\infty}_{\mathbf{x}}(\mathbf{s}, \mathbf{s}^{\dagger})} \rangle \\ &\stackrel{\text{def}}{=} \int \mathbf{d}\mathbf{y} \ \delta a^{i}(\mathbf{y}) \langle \vec{e}_{i}(\mathbf{y}), \vec{e}^{\dagger,i}(\mathbf{x}) \\ \stackrel{\text{def}}{\longrightarrow} \left(-\frac{\partial}{\partial x^{\alpha}} \right) \left(\frac{\partial}{\partial a_{x^{\alpha}}^{i}} f(\mathbf{x}, [\mathbf{a}], [\mathbf{b}]) \Big|_{j \in t^{\infty}_{\mathbf{x}}(\mathbf{s}, \mathbf{s}^{\dagger})} \right) \rangle \\ &\stackrel{\text{def}}{=} \int \mathbf{d}\mathbf{y} \ \delta a^{i}(\mathbf{y}) \langle \vec{e}_{i}(\mathbf{y}), \vec{e}^{\dagger,i}(\mathbf{x}) \rangle \cdot \left(\left(-\frac{\partial}{\partial x^{\alpha}} \right) \frac{\partial}{\partial a_{x^{\alpha}}^{i}} f(\mathbf{x}, [\mathbf{a}], [\mathbf{b}]) \right) \Big|_{j \in t^{\infty}_{\mathbf{x}}(\mathbf{s}, \mathbf{s}^{\dagger})} . \end{cases}$$

For multi-indexes σ longer than $\{x^{\alpha}\}$ the powers of partial derivatives $(\overleftarrow{\partial}/\partial \mathbf{y})^{\sigma}$ are processed by repeated integrations by parts; this yields the powers of minus total derivatives $(-\overrightarrow{d}/d\mathbf{x})^{\sigma}$. In the course of derivation of densities with respect to not a_{σ}^{i} but $b_{j,\tau}$ and so, in the course of using the other of two (co)vectors couplings, all reasonings are still performed in exactly the same way. \Box

Convention. In every calculation, the integrations by parts are performed last, prior only to the reconfigurations of couplings and their evaluation by using (14). For instance, the derivative $(\overleftarrow{\partial}/\partial \mathbf{y})^{\sigma}$ in formula (19a) channels through $\vec{e}_i(\mathbf{y})$ and $\vec{e}^{\dagger,i}(\mathbf{x})$ on the diagonal $\mathbf{y} = \mathbf{x}$ (which is the locus where the coupling is defined); the derivative thus becomes $(-\vec{d}/d\mathbf{x})^{\sigma}$ that falls on (a derivative of) the argument's density at $\mathbf{x} \in M^n$.

This principle makes the variations (graded-)permutable.

Notation. To keep track where the total derivatives would come from after integration by parts and to emphasise that such integrations are performed at the end of a calculation, we embrace the (powers of) minus the total derivatives by using the delimiters $[\ldots]$. Likewise, in the notation for those total derivatives we preserve the base variables from singular linear integral operators. (We remember that couplings (14) wright the diagonal, hence the above convention refers to notation only.) In these terms, operators (19) can be realised by using the formulas

$$\vec{\delta a}(\cdot) = \int_{M^n} \mathrm{d} \boldsymbol{y} \sum_{i=1}^m \sum_{|\sigma| \ge 0} \delta a^i(\boldsymbol{y}) \cdot \langle \underbrace{\vec{e}_{i}(\boldsymbol{y}), \vec{e}^{\dagger, i}(\cdot)}_{\mathbf{q}, \mathbf{q}, \mathbf{q}, \mathbf{q}} \rangle^{\Gamma} \Big(-\frac{\vec{d}}{\mathbf{d} \boldsymbol{y}} \Big)^{\sigma} \Big] \frac{\vec{\partial}}{\partial a^i_{\sigma}}$$

and

$$\vec{\delta \boldsymbol{b}}(\cdot) = \int_{M^n} \mathrm{d}\boldsymbol{z} \sum_{j=1}^m \sum_{|\tau| \ge 0} \delta b_j(\boldsymbol{z}) \cdot \left(\underbrace{(-\vec{e}^{\dagger,j})}_{\text{first}}(\boldsymbol{z}), \, \vec{e}_i(\cdot) \right)^{\lceil} \left(-\frac{\vec{d}}{\mathrm{d}\boldsymbol{z}} \right)^{\tau \rceil} \frac{\vec{\partial}}{\partial b_{j,\tau}} \,,$$

respectively.

• By construction, *iterated* variations of a functional over a copy of M^n never spread from it to the fragments of other functionals in any composite object during multiple integrations by parts over M^n (e.g., see [26,27]).

Summarising, the BV calculus of iterated variations relies heavily on a reference of each object to the copy of manifold M^n over which that object was defined; the locality of couplings (14) provides a restriction to the diagonal over all such copies at the end of the day. The association with own bases is the mechanism that discriminates between the fibre letters from different words in the input. Indeed, integrations by parts over the words' substrates M^n act by total derivatives only on the letters from the respective words.

We refer to [9,26,27,36] for more details and illustrations of these guiding principles.

2.6. How the Batalin–Vilkovisky Laplacian determines the Schouten bracket

Now we are ready to outline the construction of parity-odd BV Laplacian Δ . On the space of local functionals over the jet space $J^{\infty}(\pi_{NC}^{(0|1)})$, it is the parent structure for the noncommutative variational Schouten bracket [[,]]. We establish the main properties of these structures, recalling further the relations between them.

$$\underbrace{\stackrel{\langle (1)}{\overset{\sigma}{\rightarrow}} \stackrel{(2)}{\overset{\varphi}{\rightarrow}}_{\vec{\delta a}}}_{\vec{\delta a}} \underbrace{\stackrel{\vec{\delta b}}{\overset{\langle (3)}{\overset{\varphi}{\rightarrow}} \stackrel{\langle 4)}{\overset{\varphi}{\rightarrow}}}_{\langle (3)} \xrightarrow{\overset{\langle 1)}{\overset{\varphi}{\rightarrow}}}_{\langle (2)} \underbrace{\stackrel{\langle 3)}{\overset{\varphi}{\rightarrow}}_{\langle (2)} \xrightarrow{\langle 4)}_{\vec{\sigma}}}_{\langle (2)}$$

Fig. 5. The on-the-diagonal reconfiguration of couplings is the operational definition of BV Laplacian Δ ; the variations are normalised by (14).

Definition 5. The Batalin–Vilkovisky Laplacian is the reconfiguration – shown in Fig. 5 – of (co)vector couplings in the second variation $\overrightarrow{\delta a}(\overrightarrow{\delta b}(\cdot))$ of a local functional on the jet space $J^{\infty}(\pi_{NC}^{(0|1)})$.

The analytic construction of BV Laplacian Δ **.** First, let us consider an *integral* functional $F = \int f(\mathbf{x}, [\mathbf{a}], [\mathbf{b}]) \circ \operatorname{dvol}(\mathbf{x}) \in \bar{H}^n(\pi_{\mathrm{NC}}^{(0|1)})$. Let $\delta a^{i_1}(\mathbf{y}_1) \cdot \vec{e}_{i_1}(\mathbf{y}_1)$ and $\delta b_{i_2}(\mathbf{y}_2) \cdot \vec{e}^{\dagger, i_2}(\mathbf{y}_2)$ be a pair of test shifts of the parity-even and odd letters in the fibre alphabet; assume normalisation (14). Construct the second variation²⁹

$$\vec{\delta a}(\vec{\delta b}(F)) = \iint_{M^n} d\mathbf{y}_1 d\mathbf{y}_2 \int \left\{ (\delta a^{i_1}) \left(\frac{\partial}{\partial \mathbf{y}_1} \right)^{\sigma_1} (\mathbf{y}_1) \cdot \left\langle \vec{e}_{i_1}(\mathbf{y}_1) \middle| \vec{e}^{\dagger,i_1}(\mathbf{x}) \right\rangle \frac{\partial}{\partial a_{\sigma_1}^{i_1}} \circ \\ \circ (\delta b_{i_2}) \left(\frac{\overleftarrow{\partial}}{\partial \mathbf{y}_2} \right)^{\sigma_2} (\mathbf{y}_2) \cdot \left\langle (-\vec{e}^{\dagger,i_2}) (\mathbf{y}_2) \middle| \vec{e}_{i_2}(\mathbf{x}) \right\rangle \frac{\overrightarrow{\partial}}{\partial b_{i_2,\sigma_2}} f(\mathbf{x}, [\mathbf{a}], [\mathbf{b}]) \right\} dvol(\mathbf{x})$$

At the end of a reasoning (of which the object ΔF could be only a small piece), the integrations by parts carry the derivatives off the virtual test shifts, which yields

$$\begin{split} &\iint_{M^{n}} \mathbf{d} \mathbf{y}_{1} \, \mathbf{d} \mathbf{y}_{2} \int \left\{ \delta a^{i_{1}}(\mathbf{y}_{1}) \cdot \left\langle \vec{e}_{i_{1}}(\mathbf{y}_{1}) \middle| \vec{e}^{\dagger,i_{1}}(\mathbf{x}) \right\rangle \cdot \\ & \cdot \, \delta b_{i_{2}}(\mathbf{y}_{2}) \cdot \left\langle (-\vec{e}^{\dagger,i_{2}})(\mathbf{y}_{2}) \middle| \vec{e}_{i_{2}}(\mathbf{x}) \right\rangle^{\lceil} \left(-\frac{\mathbf{d}}{\mathbf{d} \mathbf{x}} \right)^{\sigma_{1} \cup \sigma_{2} \rceil} \frac{\overrightarrow{\partial}^{2}}{\partial a^{i_{1}}_{\sigma_{1}} \partial b_{i_{2},\sigma_{2}}} f(\mathbf{x}, [\mathbf{a}], [\mathbf{b}]) \right\} \, \mathrm{dvol}(\mathbf{x}). \end{split}$$

Finally, the two pairs of couplings are reconfigured according to the scenario in Fig. 5, which gives the action of operator

$$\iint_{M^n} \mathrm{d}\boldsymbol{y}_1 \,\mathrm{d}\boldsymbol{y}_2 \begin{cases} \langle \delta a^{i_1}(\boldsymbol{y}_1) \vec{e}_{i_1}(\boldsymbol{y}_1) | & |\delta b_{i_2}(\boldsymbol{y}_2) \cdot (-\vec{e}^{\dagger,i_2})(\boldsymbol{y}_2) \rangle \\ \langle \vec{e}^{\dagger,i_1}(\boldsymbol{x}) | & |\vec{e}_{i_2}(\boldsymbol{x}) \rangle \end{cases}$$

on the basic (co)vectors over $\mathbf{x} \in M^n$. The couplings wright the diagonal $i_1 = i_2$ in the summation over the indexes. Normalisation (14) and the couplings values (13) make each line in the formula above equal to -1; their product equals unit. \Box

Corollary 6. In particular, this gives us the integrand of ΔF whenever this object is the endpoint of a reasoning; namely, we obtain

$$\sum_{i=1}^{m} \sum_{\substack{|\sigma_1| \ge 0 \\ |\sigma_2| \ge 0}} \left(-\frac{\overrightarrow{\mathbf{d}}}{\mathbf{d}\boldsymbol{x}} \right)^{\sigma_1 \cup \sigma_2} \left(\frac{\overrightarrow{\partial}^2}{\partial a_{\sigma_1}^i \partial b_{i,\sigma_2}} f \right) (\boldsymbol{x}, [\boldsymbol{a}], [\boldsymbol{b}]).$$

We emphasise that, should the object ΔF itself be a constituent element of a larger expression, other partial derivatives $\overrightarrow{\partial} / \partial a_{\tau_1}^{i_1}$ or $\overrightarrow{\partial} / \partial b_{j_2,\tau_2}$ could accumulate at the given density f of the functional F, whereas all the powers of minus the total derivatives would still gather outside those higher-order partial derivatives.

Lemma 7. The linear operator

$$\Delta: \bar{H}^{n(1+k)}(\mathbb{T}\boldsymbol{\pi}_{NC}^{(0|1)}) \longrightarrow \bar{H}^{n(2+k)}(\mathbb{T}\boldsymbol{\pi}_{NC}^{(0|1)})$$

is a differential for every $k \ge 0$.

Proof. The idea is as follows: if two normalised variations are interchanged in an integral functional within the image of Δ^2 , this yields an indistinguishable result of opposite sign.³⁰

²⁹ Summation over the (multi)indices i_{α} , σ , τ or the like is implicit in this formula and in what follows.

³⁰ It is readily seen that this mechanism establishes the property $\Delta^2 = 0$ of the BV Laplacian to be a differential whenever acting on any local, i.e. not only integral functional. Indeed, within the definition of Δ , both the derivations with respect to the generators work by the graded Leibniz rule along the argument's cyclic word; it is the integrations by parts over the manifold M^n which keep track of a possible composite structure $H = F_1 \times \cdots \times F_\ell$ of that cyclic word, should it be made from integral functionals $F_1, \ldots, F_\ell \in \tilde{H}^{n(1+k)}(\mathbb{T}\pi_{NC}^{(0)1})$.



Fig. 6. There remains only one cyclic word within the minimal scheme)(yet there appears a product of two cyclic (sub)words if the scheme \asymp is adopted.

Namely, let $\delta \mathbf{s}_1 = (\delta a_1^i, \delta b_{1,i})$ and $\delta \mathbf{s}_2 = (\delta a_2^j, \delta b_{2,j})$ be two normalised shifts of the generators \mathbf{a} and \mathbf{b} , and let $H = \int h(\mathbf{x}, [\mathbf{a}], [\mathbf{b}]) \operatorname{dvol}(\mathbf{x})$ be an integral functional over $J^{\infty}(\boldsymbol{\pi}_{NC}^{(0|1)})$. (It suffices to consider the minimal picture $H \in \bar{H}^n(\boldsymbol{\pi}_{NC}^{(0|1)})$ without any variations already built into H.) By definition, we have that

$$\Delta(\Delta H)(\mathbf{s}, \mathbf{s}^{\dagger}) = \int_{M} d\mathbf{z}_{1} \int_{M} d\mathbf{z}_{2} \int_{M} d\mathbf{y}_{1} \int_{M} d\mathbf{y}_{2} \int_{M} d\operatorname{vol}(\mathbf{x}) \cdot \left\{ \left\langle (\delta a_{1}^{\alpha}) \left(\frac{\overleftarrow{\partial}}{\partial \mathbf{z}_{1}}\right)^{\sigma_{1}}(\mathbf{z}_{1}) \ \overrightarrow{e_{\alpha}(\mathbf{z}_{1})}, (-\overrightarrow{e}^{\dagger \alpha})(\mathbf{z}_{2}) (\delta b_{1,\alpha}) \left(\frac{\overleftarrow{\partial}}{\partial \mathbf{z}_{2}}\right)^{\sigma_{2}}(\mathbf{z}_{2}) \right\rangle \underbrace{\langle \overrightarrow{e}^{\dagger \alpha}(\mathbf{x}), \overrightarrow{e}_{\alpha}(\mathbf{x}) \rangle}_{-1} \\ \left\langle (\delta a_{2}^{\beta}) \left(\frac{\overleftarrow{\partial}}{\partial \mathbf{y}_{1}}\right)^{\tau_{1}}(\mathbf{y}_{1}) \ \overrightarrow{e_{\beta}(\mathbf{y}_{1})}, (-\overrightarrow{e}^{\dagger \beta})(\mathbf{y}_{2}) (\delta b_{2,\beta}) \left(\frac{\overleftarrow{\partial}}{\partial \mathbf{y}_{2}}\right)^{\tau_{2}}(\mathbf{y}_{2}) \right\rangle \underbrace{\langle \overrightarrow{e}^{\dagger \beta}(\mathbf{x}), \overrightarrow{e}_{\beta}(\mathbf{x}) \rangle}_{-1} \\ \left. \overrightarrow{\partial}_{\alpha} \overrightarrow{a_{\sigma_{1}}} \ \overrightarrow{\partial}_{b_{\alpha,\sigma_{2}}} \ \overrightarrow{\partial}_{\alpha} \overrightarrow{a_{\tau_{1}}}^{\beta} \ \overrightarrow{\partial}_{b_{\beta,\tau_{2}}} h(\mathbf{x}, [\mathbf{a}], [\mathbf{b}]) \right\} \right|_{\operatorname{jet}_{\mathbf{x}}^{\infty}(\mathbf{s}, \mathbf{s}^{\dagger})}.$$

By exchanging the integrand's upper two lines and then relabelling $\alpha \rightleftharpoons \beta$, $\sigma \rightleftharpoons \tau$ so that $\delta a_1^{\alpha} \rightleftharpoons \delta a_2^{\beta}$ and $\delta b_{1,\alpha} \rightleftharpoons \delta b_{2,\beta}$, and by swapping the reference $\mathbf{y} \rightleftharpoons \mathbf{z}$ to copies of the base manifold M^n , we almost recover the initial expression (which should be the case), yet the order in which the parity-odd partial derivatives follow is inverse,

$$\frac{\overrightarrow{\partial}}{\partial b_{\alpha,\sigma_2}} \circ \frac{\overrightarrow{\partial}}{\partial b_{\beta,\tau_2}} \longmapsto \frac{\overrightarrow{\partial}}{\partial b_{\beta,\tau_2}} \circ \frac{\overrightarrow{\partial}}{\partial b_{\alpha,\sigma_2}} = -\frac{\overrightarrow{\partial}}{\partial b_{\alpha,\sigma_2}} \circ \frac{\overrightarrow{\partial}}{\partial b_{\beta,\tau_2}}$$

Therefore the integrand of functional $\Delta^2(H)$ vanishes, which proves the assertion. \Box

Remark 2.10 (*The Geometric Realisations of* Δ). There are at least two schemes to algorithmically define the BV Laplacian Δ in the noncommutative set-up: on the basis of a minimal model, which we denote by)(in Fig. 6, a larger construction \asymp can be built. Still both options reproduce the same structure Δ whenever the alphabet a^i , b_i is proclaimed graded-commutative. The minimal option)(suggests an orientation-preserving attachment $\downarrow \uparrow$ of the respective pairs of loose ends in the argument $F = \int f(\mathbf{x}, [\mathbf{a}], [\mathbf{b}]) \circ \text{dvol}(\mathbf{x})$ of Δ . Specifically, for a cyclic word $(f) = w(\mathbf{x}) \cdot (c_1 \circ \ldots \circ c_{\lambda})$ written using letters c_{α} from the alphabet a^i_{σ} , $b_{j,\tau}$ (and weighted by a smooth coefficient w depending on $\mathbf{x} \in M^n$), the BV Laplacian yields the sum (in a term portrayed here, without loss of generality w.r.t. the sequential order of the letters a and b), $\Delta(F) =$

$$\sum_{i=1}^{m} \sum_{\substack{|\sigma| \ge 0 \\ |\tau| \ge 0}} \int \left(-\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}} \right)^{\sigma \cup \tau} \frac{\partial^2}{\partial a_{\sigma}^i \partial b_{i,\tau}} w(\boldsymbol{x}) \cdot \left(c_1 \dots c_{\mu-1} a_{\sigma_0}^{i_0} c_{\mu+1} \dots c_{\nu-1} b_{i_0,\tau_0} c_{\nu+1} \dots c_{\lambda} \right) \cdot \mathrm{dvol}(\boldsymbol{x})$$

$$\stackrel{\mathcal{I}}{=} \sum_{\substack{i_0=1 \\ |\tau_0| \ge 0}}^{m} \sum_{\substack{|\sigma_0| \ge 0 \\ |\tau_0| \ge 0}} \pm \int \left(-\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}} \right)^{\sigma \cup \tau} w(\boldsymbol{x}) \cdot \left(c_1 \dots c_{\mu-1} \circ c_{\mu+1} \dots c_{\nu-1} \circ c_{\nu+1} \dots c_{\lambda} \right) \cdot \mathrm{dvol}(\boldsymbol{x}).$$

The sign \pm in front of each integral is $(-)^{|\partial/\partial b| \cdot |c_1...c_{\nu-1}|}$. The two pairs of adjacent loose ends, namely, $c_{\mu-1} \circ \widehat{a_{\sigma_0}^{i_0}} \circ c_{\mu+1}$ and $c_{\nu-1} \circ \widehat{b_{i_0,\tau_0}} \circ c_{\nu+1}$, link to $c_{\mu-1} \circ c_{\mu+1}$ and $c_{\nu-1} \circ c_{\nu+1}$ respectively, so that the integrand of every term in $\Delta(F)$ is a cyclic word in which two letters were erased but the cyclic ordering of all the remaining letters is preserved.

Conversely, according to the second scheme, which we denote by \approx , the cyclic word (*f*) is disrupted at both $a_{\sigma_0}^{i_0}$ and b_{i_0,τ_0} ; next, either of the strings of adjacent symbols $c_{\mu+1} \dots c_{\nu-1}$ and $c_{\nu+1} \dots c_{\lambda}c_1 \dots c_{\mu-1}$ is rolled into a cyclic word. In this way, the integrand of every term in such realisation of $\Delta(F)$ is the word which itself is the product of two cyclic words: $\Delta(F) \cong$

$$\sum_{i=1}^{m} \sum_{\substack{|\sigma| > 0 \\ |\tau| > 0}} \pm \int \left(-\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}} \right)^{\sigma \cup \tau} w(\boldsymbol{x}) \cdot \langle \operatorname{sign} \rangle \cdot \left(c_{\nu+1} \dots c_{\lambda} c_1 \dots c_{\mu-1} \right) \times \left(c_{\mu+1} \dots c_{\nu-1} \right) \cdot \operatorname{dvol}(\boldsymbol{x})$$

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In this formula, the overall sign $\pm = (-)^{|\partial/\partial b| \cdot |c_1 \dots c_{\nu-1}|}$ in front of the integral is the same as before. Yet now there are three ways to define the sign from the linking:

- (1) $\langle \text{sign} \rangle := (-)^{|c_{\nu+1}...c_{\lambda}| \cdot (|c_1...c_{\nu-1}|+1)}$, i.e. $b_{i,\tau}$ is involved,
- (2) $(sign) := (-)^{|c_{\nu+1}...c_{\lambda}| \cdot |c_1...c_{\nu-1}|}$, or

. .

(3) $(\operatorname{sign}) := (-)^{|c_{\nu+1}...c_{\lambda}| \cdot |c_1...c_{\mu-1}|}$ regardless of the other subword.

It is the second variant for which a unique term $(c_1 \dots c_{\mu-1} \circ c_{\mu+1} \dots c_{\nu-1} \circ c_{\nu+1} \dots c_{\lambda})$ from the scheme)(is reproduced – with proper sign – when the product $(c_{\nu+1} \dots c_{\lambda}c_1 \dots c_{\mu-1}) \times (c_{\mu+1} \dots c_{\nu-1})$ of two cyclic words is expanded. Let us remember however that for any choice of that sign within the scheme \asymp , other cyclic words can appear. Obviously, such would be the terms in which the original consecutive order of letters along (f) is broken by the graded extension of formula (4). Indeed, under the multiplication \times the content of the first co-multiple is pasted in between *every* pair of letters in the other co-multiple (and *vice versa*).

We use the minimal scheme)(throughout this text in the realisations of Definition 5. This choice is motivated in Remark 2.3: whenever a pair of mutually inverse paths s^i and s^{\dagger}_i is skipped out from a given closed contour $(f)|_{s,s^{\dagger}}$, the remaining disjoint parts are linked, orientation preserved, to a new closed contour.

On the other hand, the larger scheme \asymp is reminiscent of the matrix integral methods from string theory [32,33]. Let $n \gg 1$ and consider the algebra Mat($n \times n, k$) of square matrices with k-valued entries. Recall that the k-valued trace tr of a product of such matrices is insensitive to cyclic permutations of comultiples. Let $\dot{a} = Q$ be a polynomial (in a) vector field on the space of generators a of the matrix algebra. The divergence div Q is quadratic with respect to the traces of cyclic subwords that are formed by sub-strings of letters in the polynomial coefficients of Q: it is readily seen that div $Q = \sum tr(\bigcirc)^{k} tr(\bigcirc)$. Now in a larger setting, suppose that the vector field $Q = Q^{f} = \{f, \cdot\}$ itself is obtained using the parity-odd symplectic form $da \wedge db$ for the double alphabet a, b, that is, Q^{f} is produced by applying the skew gradient to a given Hamiltonian f. Then the calculation of div (grad f) goes in parallel with the construction of BV Laplacian Δ within the scheme \asymp . Still let us remember that in such framework of [33], it is the k-valued traces which are multiplied using the operation \cdot in the ground field k, but not the cyclic words themselves (which can be multiplied using \times in the k-algebra $\mathcal{A}^{(0|1)}$). In this respect the matrix integral formalism differs from our present study.

Definition 6 ($\Delta(F \times G)$). Let *F* and *G* be integral functionals on $J^{\infty}(\pi_{NC}^{(0|1)})$ and assume *F* homogeneous. Applied to the product $F \times G$ of two integral functionals (see Definitions 3 and 5), the BV Laplacian Δ is the parent structure for the (non)commutative variational *Schouten bracket* [[,]], or *antibracket*,

$$\Delta(F \times G) \stackrel{\text{def}}{=} \Delta(F) \times G + (-)^{|F|} \llbracket F, G \rrbracket + (-)^{|F|} F \times \Delta G.$$
(20)

In other words, the bracket [[,]] measures the deviation for \varDelta from being a graded derivation.

The definition of Δ acting on products $F \times G$ of (homogeneous) local functionals F and G over $J^{\infty}(\pi_{NC}^{(0|1)})$ is recursive; it extends by linearity to the entire space of local functionals.

Corollary 8. The (non)commutative variational Schouten bracket is shifted-graded skew-symmetric:

$$\llbracket F, G \rrbracket = -(-)^{||F|-1] \cdot (|G|-1)} \llbracket G, F \rrbracket$$
(21)

for any homogeneous local functionals F and G over $J^{\infty}(\pi_{NC}^{(0|1)})$.

The analytic construction of the Schouten bracket [[,]]. By the graded Leibniz rule, we have that

$$(\overrightarrow{\delta a} \circ \overrightarrow{\delta b})(F \times G) = \overrightarrow{\delta a} \left(\overrightarrow{\delta b}(F) \times G + (-)^{|F|}F \times \overrightarrow{\delta b}(G) \right) = = \left(\overrightarrow{\delta a} \circ \overrightarrow{\delta b} \right)(F) \times G + (-)^{|F|}\overrightarrow{\delta a}(F) \times \overrightarrow{\delta b}(G) + \overrightarrow{\delta b}(F) \times \overrightarrow{\delta a}(G) + (-)^{|F|}F \times \left(\overrightarrow{\delta a} \circ \overrightarrow{\delta b} \right)(G).$$

Using the lemma $\overrightarrow{\partial}/\partial \mathbf{b}(F) = (-)^{|F|-1}(F) \overleftarrow{\partial}/\partial \mathbf{b}$, let us reverse the direction in which the operators $\overrightarrow{\delta a}$ and $\overrightarrow{\delta b}$ act on F in the second and third terms of the formula above; this yields³¹

$$= \left(\overrightarrow{\delta a} \circ \overrightarrow{\delta b}\right)(F) \times G + (-)^{|F|} \left((F) \overleftarrow{\delta a} \times \overrightarrow{\delta b}(G) - (F) \overleftarrow{\delta b} \times \overrightarrow{\delta a}(G) \right) + (-)^{|F|} F \times \left(\overrightarrow{\delta a} \circ \overrightarrow{\delta b} \right)(G).$$

³¹ Further processing of the first and last terms in the formula at hand – that is, the on-the-diagonal reconfigurations of couplings and integrations by parts – is analogous to the algorithm for dealing with the second and third terms, see Definition 5. The result is (20).

Let us have a closer look at the difference of the second and third terms: for integral functionals F and G it is

$$\begin{split} &\iint \mathbf{dy}_{1} \operatorname{dvol}(\mathbf{x}_{1}) \left(f(\mathbf{x}_{1}, [\mathbf{a}], [\mathbf{b}]) \right) \underbrace{\overleftarrow{\partial}}_{\partial a_{\sigma_{1}}^{i_{1}}} \left\langle \underbrace{\vec{e}}_{+,i_{1}}^{\text{second}}(\mathbf{x}_{1}), \vec{e}_{i_{1}}(\mathbf{y}_{1}) \cdot \left(\overrightarrow{\partial} \right) \right\rangle^{\sigma_{1}} \left(\delta a^{i_{1}} \right) \left(\mathbf{y}_{1} \right) \right\rangle \times \\ &\times \iint \mathbf{dy}_{2} \left\langle (\delta b_{i_{2}}) \left(\underbrace{\overleftarrow{\partial}}_{\partial \mathbf{y}_{2}} \right)^{\sigma_{2}} (\mathbf{y}_{2}) \cdot \underbrace{(-\vec{e}}_{\dagger,i_{2}}^{\text{first}})(\mathbf{y}_{2}), \underbrace{\vec{e}}_{i_{2}}(\mathbf{x}_{2})}_{i_{2}} \right) \underbrace{\overrightarrow{\partial}}_{\partial b_{i_{2},\sigma_{2}}} \left(g(\mathbf{x}_{2}, [\mathbf{a}], [\mathbf{b}]) \right) \right) dvol(\mathbf{x}_{2}) - \\ &- \iint \mathbf{dy}_{1} \operatorname{dvol}(\mathbf{x}_{1}) \left(f(\mathbf{x}_{1}, [\mathbf{a}], [\mathbf{b}]) \right) \underbrace{\overleftarrow{\partial}}_{\partial b_{i_{1},\sigma_{1}}}_{i_{1},\sigma_{1}} \left(\underbrace{\vec{e}}_{i_{1}}(\mathbf{x}_{1}), (-\vec{e})_{i_{1}}^{\text{first}})(\mathbf{y}_{1}) \cdot \left(\underbrace{\overrightarrow{\partial}}_{\partial \mathbf{y}_{1}} \right)^{\sigma_{1}} (\delta b_{i_{1}})(\mathbf{y}_{1}) \right\rangle \\ &\times \iint \mathbf{dy}_{2} \left\langle (\delta a^{i_{2}}) \left(\underbrace{\overleftarrow{\partial}}_{\partial \mathbf{y}_{2}} \right)^{\sigma_{2}} (\mathbf{y}_{2}) \cdot \underbrace{\vec{e}}_{i_{2}}^{\text{first}} \left(\underbrace{\vec{e}}_{i_{1}}(\mathbf{x}_{2}), (-\vec{e})_{i_{1}}^{\text{first}} \right) \right\rangle \underbrace{\overrightarrow{\partial}}_{\partial d_{\sigma_{2}}^{i_{2}}} \left(g(\mathbf{x}_{2}, [\mathbf{q}], [\mathbf{q}^{\dagger}]) \right) dvol(\mathbf{x}_{2}), \end{aligned}$$

where the (co)vectors marked 'second' replace the respective letters in the already-built product of cyclic words f and g; let us remember that in the construction of $\Delta(F \times G)$, the multiplication \times is performed *ab initio* and let us bear in mind that the (co)vectors belonging to shifts (18), here marked 'first', do *not* become parts of that cyclic word. The conversion of two pairs of variations in $(F)\delta \vec{a} \times \delta \vec{b}(G) - (F)\delta \vec{b} \times \delta \vec{a}(G)$ into one integral object – via integrations by parts on the diagonal $\mathbf{x}_1 = \mathbf{y}_1 = \mathbf{y}_2 = \mathbf{x}_2$ through many consecutive reconfigurations of the couplings – determines the functional³²

$$\iiint d\mathbf{x}_{1}d\mathbf{y}_{1}d\mathbf{y}_{2} dvol(\mathbf{x}_{2}) \left\langle \delta a^{i_{1}}(\mathbf{y}_{1}) \cdot \frac{\vec{e}_{i_{1}}(\mathbf{y}_{1})}{\vec{e}_{i_{1}}(\mathbf{y}_{1})}, \underbrace{(-\vec{e}^{\dagger,i_{2}})(\mathbf{y}_{2})}_{(-\vec{e}^{\dagger,i_{2}})(\mathbf{y}_{2})} \cdot \delta b_{i_{2}}(\mathbf{y}_{2}) \right\rangle \cdot \left(f(\mathbf{x}_{1}, [\mathbf{a}], [\mathbf{b}])\right) \underbrace{\frac{\overleftarrow{\partial}}{\partial a_{\sigma_{1}}^{i_{1}}} \left(-\frac{\overleftarrow{d}}{d\mathbf{y}_{1}}\right)^{\sigma_{1}} \circ \left\langle \frac{\vec{e}^{\dagger,i_{1}}(\mathbf{x}_{1})}{(-\vec{e}^{\dagger,i_{1}})}\right| \times \underbrace{|\underbrace{\underline{e}_{i_{2}}(\mathbf{x}_{2})}_{(\mathbf{x}_{2}}|^{second}}_{(\mathbf{x}_{2}, [\mathbf{a}], [\mathbf{b}])} \right) - \int \iiint d\mathbf{x}_{1}d\mathbf{y}_{1}d\mathbf{y}_{2} dvol(\mathbf{x}_{2}) \left\langle \delta b_{i_{1}}(\mathbf{y}_{1})(-\vec{e}^{\dagger,i_{1}})(\mathbf{y}_{1}), \underbrace{\underline{e}_{i_{2}}(\mathbf{y}_{2})}_{(\mathbf{x}_{2}, [\mathbf{y}_{2}])} \cdot \delta a^{i_{2}}(\mathbf{y}_{2}) \right\rangle \cdot \left(f(\mathbf{x}_{1}, [\mathbf{a}], [\mathbf{b}])\right) \underbrace{\frac{\overleftarrow{\partial}}{\partial b_{i_{1},\sigma_{1}}} \left[\left(-\frac{\overleftarrow{d}}{d\mathbf{y}_{1}}\right)^{\sigma_{1}} \circ \left(\underline{e}^{\dagger,i_{1}}(\mathbf{x}_{1})\right)}_{(\mathbf{x}_{1}, [\mathbf{x}])}\right] \times \underbrace{|\underbrace{\underline{e}^{\dagger,i_{2}}(\mathbf{x}_{2})}_{(\mathbf{x}_{2}, [\mathbf{x}])}}_{(\mathbf{x}_{2}, [\mathbf{x}])} \right\rangle \circ \left[\left(-\frac{\overrightarrow{d}}{d\mathbf{y}_{2}}\right)^{\sigma_{2}} \frac{\overrightarrow{\partial}}{\partial a_{\sigma_{2}}^{i_{2}}} \left(g(\mathbf{x}_{2}, [\mathbf{a}], [\mathbf{b}])\right)\right).$$
(22)

Evaluating both couplings in the minuend, we obtain $(-1) \cdot (-1) = +1$; likewise, in the subtrahend we have that $(+1) \cdot (+1) = +1$; at every value of the indexes, the respective shift components contribute with $\delta a^{\bullet} \cdot \delta a_{\bullet}^{\dagger} = 1$. We emphasise that the expression [[*F*, *G*]], which has been constructed by following the couplings' re-attachment mechanism, itself can serve as a constituent part of a larger object. Because the reconfigurations of couplings and integrations by parts occur prior only to the restriction of output to the jet jet[∞]_x(**s**, **s**[†]) at the diagonal **x**₁ = **x**₂ = **y**₁ = **y**₂ =: **x** ∈ *M*ⁿ for the section (**s**, **s**[†]), this means that other partial derivatives can freely overtake the horizontal derivatives along the base *M*ⁿ. This is why the total derivatives were embraced by using [...] and why the shifts' own base variables **y**_i were used in (22) instead of the variables **x**_i from the functionals' densities. \Box

Remark 2.11. In effect, the only minus sign making the *difference* of two terms is determined by the precedence $a \prec b$ versus succedence $b \succ a$, that is, by the sequential order in which the parity-even and odd partial derivatives are distributed between the ordered pair $F \prec G$ of input objects.

Corollary 9. Suppose that the Schouten bracket of integral functionals F and G is the endpoint of a calculation, that is, the reasoning stops there and the object $[\![F,G]\!]$: $\Gamma(\pi_{NC}^{(0|1)}) \rightarrow \chi(\vec{\mathbf{x}}^{\pm 1})$ is used only for its evaluation at mappings $(\mathbf{s}, \mathbf{s}^{\dagger})$ but it is not contained in any larger formula. Should this be known in advance, then one re-derives the familiar expression,

$$\llbracket F, G \rrbracket = \int \left\{ \left((f) \underbrace{\frac{\overleftarrow{\partial}}{\partial a_{\sigma}^{i}} \left[\left(- \underbrace{\overleftarrow{d}}{d\mathbf{x}} \right)^{\sigma} \right]}_{\underbrace{\partial b_{i,\sigma}}{\sigma}} \times \underbrace{\left[\left(- \underbrace{\overrightarrow{d}}{d\mathbf{x}} \right)^{\tau} \right] \frac{\overrightarrow{\partial}}{\partial b_{i,\tau}}}_{\left[\left(- \underbrace{\overrightarrow{d}}{d\mathbf{x}} \right)^{\tau} \right] \frac{\overrightarrow{\partial}}{\partial a_{\tau}^{i}}} (g) \left(\mathbf{x}, [\mathbf{a}], [\mathbf{b}] \right) \right\} dvol(\mathbf{x}),$$
(23)

where, we remember, the multiplication $f \times g$ is performed ab initio to construct the object $F \times G$ over $M^n \times M^n$; the underbraced operators then proceed by the four Leibniz rules along the two comultiples, either of which is built into the product but exists over the respective copy of underlying manifold M^n .

³² The remaining volume element can be either $dvol(\mathbf{x}_1)$ or $dvol(\mathbf{x}_2)$; its final location is prescribed by either the right-to-left or left-to-right (which is the case here) direction of couplings in the output. From (13) it is clear that a simultaneous swap "first \Rightarrow second" in a pair of couplings would give the extra factor $(-1) \cdot (-1) = +1$, so that expression's overall sign does not change.

Remark 2.12 (*The Geometric Realisation of* [[,]]). The geometric construction of every term in the noncommutative variational Schouten bracket of integral functionals goes as follows. Without loss of generality suppose that either of the arguments *F* and *G* consists of just one cyclic word (otherwise, proceed by linearity).

For consistency let us first recall the geometric mechanism of left multiplication $(F \times) G$.



Namely, by using $t^{r_a+r_b}$ rotate the necklace F counterclockwise until $r_a \ge 0$ parity-even and $r_b \ge 0$ parity-odd symbols would have passed through the lock ∞_F ; when the $(r_a + r_b + 1)$ th symbol approaches ∞_F , open that lock. Likewise, using $t^{-(p_a+p_b)}$ rotate the ring G clockwise and, as soon as $p_a \ge 0$ parity-even and $p_b \ge 0$ parity-odd symbols would have passed through ∞_G , unlock G just before its $(p_a + p_b + 1)$ th symbol. Place the loose ends of the two open words next to each other, preserving the orientation of two strings of symbols, and join the facing ends of the two strings, forming the new cyclic word that inherits



the orientation.³³

From the old markers ∞_F and ∞_G where the reading of cyclic words *F* and *G* started, in opposite directions issue the derivations $\partial/\partial a_{\sigma}^i$ and $\partial/\partial b_{i,\tau}$ of opposite parities. Let one of them work against the orientation \bigcirc , i.e. *clockwise* over *F* and let the other act *counterclockwise*, i.e. along the orientation on *G*. (Each obeying the Leibniz rule, either of those derivations of course also reworks the $r_a + r_b$ – resp., $p_a + p_b$ – symbols which are found in the string of *F* – resp., in *G* with its ∞_G – behind the lock ∞_F with respect to the orientation of cyclic words. The calculation of grading and parity then involves negative integer numbers.) The antecedence $\partial/\partial a_{\sigma}^i|_F \prec \partial/\partial b_{i,\tau}|_G$ yields the plus sign, whereas the opposite sequential order of *F* vs *G* yields the minus sign in front of the corresponding term in the Leibniz rule expansions.³⁴ In every such term we integrate by parts in order to shake $|\sigma|$ and $|\tau|$ derivatives off the arguments a_{σ}^i and $b_{i,\tau}$ of two derivations. Recall that the emerging powers of minus the total derivatives now act in $F \times G$ over $M^n \times M^n$ only on the sub-strings from the words *F* or *G* where the symbols a_{σ}^i and $b_{i,\tau}$ initially belonged to, see (22).

Finally, rotate the letters around the new word counterclockwise so that the old location of ∞_G in between the symbols of *G* or after to the last symbol of *G* reaches the new linking $\infty_{[\![F,G]\!]}$ of strings, nearest to ∞_G in the positive direction. The terminal configuration is displayed here; it carries |F| + |G| - 1 parity-odd symbols, it preserves the orientation of both the



input words F and G, and it carries the sign factor determined by the ordered coupling of (co)vectors.

Corollary 10. For a given homogeneous integral functional $F \in \overline{H}^n(\pi_{NC}^{(0|1)})$ of grading |F|, the operator $[[F, \cdot]]$ proceeds over letters of its cyclic-word(s) argument by the graded Leibniz rule (and by linearity); this operator's proper grading $|[[F, \cdot]]|$ is |F| - 1.

³³ Note that by the above construction, the symbols from *F* preserve their consecutive order when forming a sub-string in the cyclic word $F \times G$, as well as the symbols from *G* do.

 $^{^{34}}$ One easily recognises the sign convention from (13) in the antecedence of derivations.

Corollary 11. The bi-linear (non)commutative variational Schouten bracket [[,]] itself is a shifted-graded derivation of the product \times in the algebra of local functionals:

$$\llbracket F, G \times H \rrbracket = \llbracket F, G \rrbracket \times H + (-)^{(|F|-1) \cdot |G|} G \times \llbracket F, H \rrbracket,$$
(24)

where *F* and *G* are assumed homogeneous and where both terms on the right-hand side are understood as applications of $[[F, \cdot]]$ to the cyclic word *G* × *H* within the BV Laplacian action $\Delta(F \times (G \times H))$ on the non-associative product of three comultiples.

Proof. It is clear that the terms in $\llbracket F, G \times H \rrbracket$ are grouped in two parts: those in which the parity-odd derivations $\overrightarrow{\partial} / \partial b_{i,\tau}$ act on *G* and those for *H*; the former do not contribute with any extra sign factors whereas the latter do – in a way which depends on the parity |G|. This means that $\llbracket F, G \times H \rrbracket = \llbracket F, G \rrbracket \times H + \cdots$ in terms of $\llbracket F, \cdot \rrbracket$ acting on the product $G \times H$. Proceeding by linearity if necessary, suppose also that *H* is also homogeneous. To grasp the sign in front of the term which has been omitted, let us swap the graded multiples *G* and *H*. We have that $G \times H = (-)^{|G| \cdot |H|} H \times G$, whence $\llbracket F, G \times H \rrbracket = (-)^{|G| \cdot |H|} \llbracket F, H \rrbracket \times G + \cdots$ in terms of $\llbracket F, \cdot \rrbracket$ acting on the product $H \times G$. By recalling that the grading $|\llbracket F, H \rrbracket|$ of the respective class of substrings in $\llbracket F, G \times H \rrbracket$ equals |F| + |H| - 1, we conclude that

$$\llbracket F, G \times H \rrbracket = \llbracket F, G \rrbracket \times H + (-)^{|G| \cdot |H|} (-)^{(|F| + |H| - 1) \cdot |G|} G \times \llbracket F, H \rrbracket,$$

which yields formula (24). \Box

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Remark 2.13. Shifted-graded skew-symmetry (21) of the noncommutative variational Schouten bracket for homogeneous *local* functionals $F, G \in \overline{\mathfrak{M}}^n(\pi_{NC}^{(0|1)})$ can now be re-derived, from Corollaries 8 and 11, by induction on the respective numbers ℓ', ℓ'' of building blocks in the arguments F and G.

Theorem 12. Let F, G, and H be homogeneous integral functionals on $J^{\infty}(\pi_{NC}^{(0|1)})$ so that their gradings are |F|, |G|, and |H| respectively. Then each of the following three tautologically equivalent statements is valid:

(i) The noncommutative variational Schouten bracket satisfies the shifted-graded Jacobi identity

$$(-)^{(|F|-1)\cdot(|H|-1)}\llbracket F, \llbracket G, H \rrbracket \rrbracket + (-)^{(|F|-1)\cdot(|G|-1)}\llbracket G, \llbracket H, F \rrbracket \rrbracket + (-)^{(|G|-1)\cdot(|H|-1)}\llbracket H, \llbracket F, G \rrbracket \rrbracket = 0.$$

(ii) The Jacobi identity for the bracket $[\![,]\!]$ is the graded Leibniz rule for the operator $[\![F, \cdot]\!]$ acting on $[\![G, H]\!]$, namely,

$$F, [G, H]] = [[F, G]], H] + (-)^{(|F|-1) \cdot (|G|-1)} [G, [F, H]].$$
(25)

(iii) The graded commutator of operators $[[F, \cdot]]$ and $[[G, \cdot]]$ is equal to the operator $[[[F, G]] \cdot]]$, that is,

$$\llbracket F, \llbracket G, \cdot \rrbracket \rrbracket (H) - (-)^{(|F|-1) \cdot (|G|-1)} \llbracket G, \llbracket F, \cdot \rrbracket \rrbracket (H) = \llbracket \llbracket F, G \rrbracket, \cdot \rrbracket (H).$$
(26)

The arrangement of parentheses in (26) is $(F \times G) \times H$; both the other variants (*i*-*ii*) are obtained from (26) using multiplication by sign factors.³⁵

Proven immediately below for the case of integral building blocks from $\bar{H}^n(\pi_{NC}^{(0|1)})$, assertion (*iii*) of Theorem 12 is then extended by induction to the space $\overline{\mathfrak{M}}^n(\pi_{NC}^{(0|1)})$ of noncommutative local functionals.

Proof. Consider the consecutive action of operators $[F \cdot]]$ and $[G \cdot]]$ of gradings |F| - 1 and |G| - 1, respectively, on an integral functional *H*. Each operator proceeds over letters in every cyclic word of *H* by the graded Leibniz rule. It is readily seen that by taking the graded *difference* of the two applications, as it stands on the left-hand side of (26), we cancel all the terms in which the strings of symbols from *F* and *G* are pasted into *H* not hitting each other (that is, rather staying next to each other or becoming separated by the argument's own letters). Therefore, both sides of (26) contain the *second* variation of *F* or *G* but only the *first* variation of *H*.

Note further that all the integrals by parts always involve only the letters that belong to (what remains of) the functional which is varied, see Section 2.5. Consequently, both sides of (26) contain the same configurations of powers of total derivatives that fall on the letters from the second or first, first or second, and first variations of *F*, *G*, and *H*, respectively. This shows that it is sufficient to inspect the matching of signs – as they occur on the left- and right-hand sides of (26) – in front of the insertions of symbols from *F* into *G*, and *vice versa*. Without loss of generality, let us suppose that each of the functionals *F* and *G* consists of just a single cyclic word.

³⁵ Each reading of the Jacobi identity for $[\![,]\!]$ is valid regardless of the sequential order of multiplications in $F \times G \times H$ after a reduction to the gradedcommutative set-up. From the first paragraph in the proof below it is seen why the parentheses configuration is $(F \times G) \times H$ in the non-associative setting. In the meantime, we conclude that the Jacobi identity for $[\![,]\!]$ renders the fact that the commutator of adjoint actions is the adjoint action by the bracket, cf. [19].

Every term in $[[G, \cdot]](H)$ is obtained from the cyclic words



as follows (see Remark 2.12). First, the ring *G* is rotated counterclockwise, transporting *p* odd symbols through ∞_G , which gives the sign $(-)^{p\cdot(|G|-1)}$, and then *G* is unlocked at ∞_G . At the same time, *H* is rotated clockwise and unlocked as soon as *q* odd letters would have passed the lock ∞_H . The word obtained from *G* is pasted, orientation preserved, into the similarly obtained fragments of *H*; the loose ends of the two strings are joined, making a new circle. Contracting one pair of variations $(\delta a, \delta b)$ destroys one parity-odd symbol in either *G* or *H*. Finally, the *q* parity-odd letters of *H* are pushed counterclockwise – so many of them that the old ∞_H coincides with $\infty_{[G,H]}$, placed at the moment of linking at the concatenation of strings' loose ends nearest to ∞_H in positive direction. The sign factor which is gained when the lock of *H* is restored on its proper place equals $(-)^{q\cdot([G]-1)}$; the *minus one* in the exponent counts the parity-odd letter destroyed by the coupling. The resulting necklace – a term in [[*G*, *H*]] – looks like this:



The total sign accumulated up to this moment is $(-)^{p \cdot (|G|-1)} \cdot (-)^{q \cdot (|G|-1)}$. Now the operator $[[F, \cdot]]$ approaches that ring from the left. Arguing as above, we rotate the cyclic word

counterclockwise, letting r parity-odd symbols pass through ∞_F (this yields $(-)^{r\cdot(|F|-1)}$). Having unlocked that ring at ∞_F , we carry this term in $[\![F, \cdot]\!]$ of grading |F| - 1 along the p + q parity-odd symbols in the pre-fabricated linking of G and H. By the time the loose ends of $[\![F, \cdot]\!]$ reach the former location of ∞_G in G, the sign factor $(-)^{(p+q)\cdot(|F|-1)}$ is accumulated, and the configuration is this:



By having realised the scenario which the first term on the left-hand side of (26) provides, we obtain the overall sign

$$(-)^{r\cdot(|F|-1)} \cdot (-)^{p\cdot(|G|-1)} \cdot (-)^{q\cdot(|G|-1)} \cdot (-)^{(p+q)\cdot(|F|-1)} = (-)^{r\cdot(|F|-1)} \cdot (-)^{(p+q)\cdot(|F|+|G|-2)}.$$
(27)

Moreover, now it is clear what the extra sign contribution to the formula above would be, should the insertion of the unlocked *F* start later – with respect to the cyclic order – than the starting point ∞_G of the turned-and-unlocked cyclic word *G*.



On the other hand, let us calculate the overall sign factor of the very same geometric configuration on the right-hand side of (26). So, we first produce the respective term in [[F, G]]. Let us recall from the above that the word



is unlocked straight after ∞_G , but

is first rotated counterclockwise by r parity-odd slots; this yields the sign $(-)^{r \cdot (|F|-1)}$ and gives the word



It contains |F| + |G| - 1 parity-odd letters; let us use it in the action of $[[[F, G]], \cdot]]$ on H. By rotating the word to-paste counterclockwise by p parity-odd symbols, we gain the sign $(-)^{p \cdot (|F|+|G|-2)}$; proceeding by the Leibniz rule over q parity-odd letters in H, we obtain another sign factor $(-)^{q \cdot (|F|+|G|-2)}$. In total, the overall sign that occurs on the right-hand side of (26) for the configuration that we knew before is

$$(-)^{r \cdot (|F|-1)} \cdot (-)^{p \cdot (|F|+|G|-2)} \cdot (-)^{q \cdot (|F|+|G|-2)}$$

This is exactly (27).

To process – in both sides of (26) – the configurations in which the symbols from *G* are pasted in between the letters of *F*, and those are already installed in *H*, let us first swap *F* and *G*. By Corollary 8, the right-hand side of (26) becomes

$$(-)^{(|F|-1)\cdot(|G|-1)}$$

Second, multiply both sides of (26) by the sign factor $-(-)^{(|F|-1)\cdot(|G|-1)}$; this gives

 $-(-)^{(|F|-1)\cdot(|G|-1)}\llbracket F, \llbracket G, \cdot \rrbracket \rrbracket (H) + \llbracket G, \llbracket F, \cdot \rrbracket \rrbracket (H) \text{ versus } \llbracket \llbracket G, F \rrbracket, \cdot \rrbracket (H).$

Finally, relabel $F \rightleftharpoons G$ back; by having thus recovered both sides of (26) in its authentic form, we convert the configurations to-consider into those which we did cope with. The proof is complete. \Box

Lemma 13. Let $F \in \overline{H}^{n(1+k)}(\mathbb{T}\pi_{NC}^{(0|1)})$ and $G \in \overline{H}^{n(1+\ell)}(\mathbb{T}\pi_{NC}^{(0|1)})$ be two integral functionals (here $k, \ell \ge 0$), and assume F is homogeneous. Then

$$\Delta(\llbracket F, G \rrbracket) = \llbracket \Delta F, G \rrbracket + (-)^{|F|-1} \llbracket F, \Delta G \rrbracket.$$
⁽²⁸⁾

This claim will be extended to all elements of the algebra of local functionals over $J^{\infty}(\pi_{NC}^{(0|1)})$; the inductive proof of Theorem 14 is based on this lemma.

Proof. The key idea is that the structures Δ and $[\![,]\!]$ yield equivalence classes of integral functionals which, after integration by parts at the end of the day, are *independent* of a choice of the built-in test shifts normalised by (14). Consequently, the composite structure $\Delta([\![\cdot,\cdot]\!])$ does not change under swapping $\delta a_2^{\alpha} \rightleftharpoons \delta a_2^{\beta}$, $\delta b_{1,\alpha} \rightleftharpoons \delta b_{2,\beta}$ of the respective variations δs_1 and δs_2 in Δ and in $[\![,]\!]$. Hence the terms which are skew-symmetric under such exchange necessarily vanish (cf. the proof of Lemma 7).

For the sake of clarity, let us assume that $F = \int f(\mathbf{x}_1, [\mathbf{a}], [\mathbf{b}]) \operatorname{dvol}(\mathbf{x}_1)$ and $G = \int g(\mathbf{x}_2, [\mathbf{a}], [\mathbf{b}]) \operatorname{dvol}(\mathbf{x}_2)$ are building blocks from the cohomology group $\overline{H}^n(\pi_{NC}^{(0|1)})$; this simplification is legitimate because new variations which come from Δ

and [[,]] do not interfere with any other test shifts if those are already absorbed by the densities f and g. Suppose that δs_1 and δs_2 are two normalised variations of the generators a^i and b_i . By definition, we have that 36

$$\begin{split} \overrightarrow{\delta a} \ \overrightarrow{\delta b} \ (\llbracket F, G \rrbracket) &= \int_{M} d\mathbf{z}_{1} \int_{M} d\mathbf{z}_{2} \int_{M} d\mathbf{y}_{1} \int_{M} d\mathbf{y}_{2} \int_{M} d\mathbf{x}_{1} \int_{M} dvol(\mathbf{x}_{2}) \\ \left\{ \langle (\delta d_{2}^{j_{1}}) \left(\frac{\overleftarrow{\partial}}{\partial \mathbf{z}_{1}} \right)^{\tau_{1}} (\mathbf{z}_{1}) \cdot \overrightarrow{\mathbf{e}}_{j_{1}} (\mathbf{z}_{1}), \overrightarrow{\mathbf{e}}^{\dagger + j_{1}} (\cdot) \right\rangle \frac{\overrightarrow{\partial}}{\partial d_{\tau_{1}}^{j_{1}}} \circ \langle (\delta b_{2,j_{2}}) \left(\frac{\overleftarrow{\partial}}{\partial \mathbf{z}_{2}} \right)^{\tau_{2}} (\mathbf{z}_{2}) \cdot (-\overrightarrow{\mathbf{e}}^{\dagger + j_{2}}) (\mathbf{z}_{2}), \overrightarrow{\mathbf{e}}_{j_{2}} (\cdot) \right\rangle \frac{\overrightarrow{\partial}}{\partial b_{j_{2},\tau_{2}}} \right\} \\ \left[\langle \delta a_{1}^{i_{1}} (\mathbf{y}_{1}) \overrightarrow{\mathbf{e}}_{i_{1}} (\mathbf{y}_{1}), (-\overrightarrow{\mathbf{e}}^{\dagger + i_{2}}) (\mathbf{y}_{2}) \delta b_{1,i_{2}} (\mathbf{y}_{2}) \rangle \right] \\ \left(f(\mathbf{x}_{1}, [\mathbf{a}], [\mathbf{b}]) \right) \underbrace{\overleftarrow{\partial}}_{\partial d_{\sigma_{1}}^{i_{1}}} \left[\left(-\frac{\overleftarrow{d}}{d\mathbf{y}_{1}} \right)^{\sigma_{1}} \left] \langle \overrightarrow{\mathbf{e}}^{\dagger + i_{1}} (\mathbf{x}_{1}) \right] \times |\overrightarrow{\mathbf{e}}_{i_{2}} (\mathbf{x}_{2}) \rangle} \left[\left(-\overrightarrow{\mathbf{d}}_{\mathbf{y}_{2}} \right)^{\sigma_{2}} \left] \frac{\overrightarrow{\partial}}{\partial b_{i_{2},\sigma_{2}}} \left(g(\mathbf{x}_{2}, [\mathbf{a}], [\mathbf{b}]) \right) - \left(\langle \delta b_{1,i_{1}} (\mathbf{y}_{1}) (-\overrightarrow{\mathbf{e}}^{\dagger + i_{1}}) (\mathbf{y}_{1}), \overrightarrow{\mathbf{e}}_{i_{2}} (\mathbf{y}_{2}) \right) \delta a_{1}^{i_{2}} (\mathbf{y}_{2}) \right] \\ \left(f(\mathbf{x}_{1}, [\mathbf{a}], [\mathbf{b}]) \right) \underbrace{\overleftarrow{\partial}}_{\partial b_{i_{1},\sigma_{1}}} \left[\left(-\frac{\overleftarrow{\mathbf{d}}}{\mathbf{dy}_{1}} \right)^{\sigma_{1}} \left] \langle \overrightarrow{\mathbf{e}}_{i_{1}} (\mathbf{x}_{1}) \right] \times \left| \overrightarrow{\mathbf{e}}_{i_{1}} (\mathbf{x}_{2}) \right\rangle} \left[\left(-\overrightarrow{\mathbf{d}}_{\mathbf{dy}_{2}} \right)^{\sigma_{2}} \left] \frac{\overrightarrow{\partial}}{\partial b_{i_{2},\sigma_{2}}} \left(g(\mathbf{x}_{2}, [\mathbf{a}], [\mathbf{b}]) \right) \right] \right] \end{aligned}$$

The partial derivatives $\overrightarrow{\partial}/\partial a_{\tau_1}^{j_1} \circ \overrightarrow{\partial}/\partial b_{j_2,\tau_2}$ are distributed between the arguments f and g by the graded Leibniz rule. Whenever *none* of the two operators overtakes the density of F, the reconfiguration yields $[\![\Delta F, G]\!]$. Likewise, if *both* derivatives indexed by j overtake F and then also overtake an old derivative that fell on g, we obtain $(-)^{|F|-1}[\![F, \Delta G]\!]$, which is the second term on the right-hand side of (28).

We claim that the remaining four terms cancel out by virtue of independence – of both Δ and [[,]] – of a choice of normalised virtual shifts.

The two mixed terms can informally be visualised using

$$\frac{\overrightarrow{\delta}}{\delta b}(f)\frac{\overleftarrow{\delta}}{\delta a}\times\frac{\overrightarrow{\delta}}{\delta a}\frac{\overrightarrow{\delta}}{\delta b}(g)\pm\frac{\overrightarrow{\delta}}{\delta a}(f)\frac{\overleftarrow{\delta}}{\delta b}\times\frac{\overrightarrow{\delta}}{\delta b}\frac{\overrightarrow{\delta}}{\delta a}(g).$$

They contribute to the integrand with the difference of equal terms,

$$\begin{cases} \langle \delta a_{1}^{i_{1}}(\boldsymbol{y}_{1}) \vec{e}_{i_{1}}(\boldsymbol{y}_{1}), (-\vec{e}^{\dagger,i_{2}})(\boldsymbol{y}_{2}) \delta b_{1,i_{2}}(\boldsymbol{y}_{2}) \rangle \cdot \langle \delta b_{2,j_{2}}(\boldsymbol{z}_{2}) (-\vec{e}^{\dagger,j_{2}})(\boldsymbol{z}_{2}), \vec{e}_{j_{1}}(\boldsymbol{z}_{1}) \delta a_{2}^{j_{1}}(\boldsymbol{z}_{1}) \rangle \cdot \\ \\ (f(\boldsymbol{x}_{1}, [\boldsymbol{a}], [\boldsymbol{b}])) \begin{cases} \overleftarrow{\partial}_{\partial a_{\sigma_{1}}^{i_{1}}}^{\sigma_{1}} \left(\left(-\frac{\overleftarrow{d}}{d\boldsymbol{y}_{1}} \right)^{\sigma_{1}} \langle \vec{e}^{\dagger,i_{1}}(\boldsymbol{x}_{1}) \right) \times |\vec{e}_{i_{2}}(\boldsymbol{x}_{2})\rangle \left[\left(-\frac{\overrightarrow{d}}{d\boldsymbol{y}_{2}} \right)^{\sigma_{2}} | \overrightarrow{\partial}_{\partial b_{i_{2},\sigma_{2}}} \right] \\ (-)^{|F|-1} \overleftarrow{\partial}_{\partial b_{j_{2},\tau_{2}}}^{\sigma_{2}} \left[\left(-\frac{\overleftarrow{d}}{d\boldsymbol{z}_{2}} \right)^{\tau_{2}} \langle \vec{e}_{j_{2}}(\boldsymbol{x}_{1}) \right] \times |\vec{e}^{\dagger,j_{1}}(\boldsymbol{x}_{2})\rangle \left[\left(-\frac{\overrightarrow{d}}{d\boldsymbol{z}_{1}} \right)^{\tau_{1}} | \overrightarrow{\partial}_{\partial d_{\tau_{1}}^{j_{1}}} \right] \\ - \langle \delta b_{1,i_{1}}(\boldsymbol{y}_{1}) (-\vec{e}^{\dagger,i_{1}})(\boldsymbol{y}_{1}), \vec{e}_{i_{2}}(\boldsymbol{y}_{2}) \delta a_{1}^{i_{2}}(\boldsymbol{y}_{2})\rangle \cdot \langle \delta a_{2}^{j_{1}}(\boldsymbol{z}_{1}) \vec{e}_{j_{1}}(\boldsymbol{z}_{1}), (-\vec{e}^{\dagger,j_{2}})(\boldsymbol{z}_{2}) \delta b_{2,j_{2}}(\boldsymbol{z}_{2})\rangle \cdot \\ \\ \left(f(\boldsymbol{x}_{1}, [\boldsymbol{a}], [\boldsymbol{b}])) \right) \begin{cases} \overleftarrow{\partial}_{\partial b_{i_{1},\sigma_{1}}} \left[\left(-\frac{\overleftarrow{d}}{d\boldsymbol{y}_{1}} \right)^{\sigma_{1}} \langle \vec{e}_{i_{1}}(\boldsymbol{x}_{1}) \right] \times |\vec{e}^{\dagger,i_{2}}(\boldsymbol{x}_{2})\rangle} \left[\left(-\vec{e}^{\dagger,i_{2}}(\boldsymbol{x}_{2}) \right)^{\sigma_{2}} \langle \vec{\partial}_{\partial a_{\sigma_{2}}} \\ \frac{\partial_{\partial a_{\sigma_{1}}}^{i_{1}}}{\partial a_{\sigma_{1}}^{i_{1}}} \left[\left(-\frac{\overleftarrow{d}}{d\boldsymbol{y}_{1}} \right)^{\sigma_{1}} \langle \vec{e}_{i_{1}}(\boldsymbol{x}_{1}) \right] \times |\vec{e}^{\dagger,i_{2}}(\boldsymbol{x}_{2})\rangle} \left[\left(-\vec{e}^{\dagger,i_{2}}(\boldsymbol{x}_{2}) \right)^{\sigma_{2}} \langle \vec{\partial}_{\partial a_{\sigma_{2}}} \\ \frac{\partial_{\partial a_{\sigma_{2}}}^{i_{1}}}{\partial a_{\sigma_{1}}^{i_{1}}}} \right] \end{cases} \end{cases} \left(g(\boldsymbol{x}_{2}, [\boldsymbol{a}], [\boldsymbol{b}]) \right),$$

which yields zero after summation over all the (multi)indices.

To prove that each of the remaining two terms,³⁷

$$\frac{\overrightarrow{\delta}}{\delta \mathbf{b}}(f)\frac{\overleftarrow{\delta}}{\delta \mathbf{b}} \times \frac{\overrightarrow{\delta}}{\delta \mathbf{a}}\frac{\overrightarrow{\delta}}{\delta \mathbf{a}}(g) \quad \text{and} \quad \frac{\overrightarrow{\delta}}{\delta \mathbf{a}}(f)\frac{\overleftarrow{\delta}}{\delta \mathbf{a}} \times \frac{\overrightarrow{\delta}}{\delta \mathbf{b}}\frac{\overrightarrow{\delta}}{\delta \mathbf{b}}(g)$$

cancels by itself, let us inspect its behaviour under a swap $\delta s_1 \rightleftharpoons \delta s_2$ of coefficients in the normalised test shifts.³⁸

³⁶ To keep track of their origin, we preserve the notation for base variables y_{μ} and z_{ν} in the minus total derivatives acting at the end of the day on densities of the functionals *F* and *G*.

³⁷ Note that $\overrightarrow{\delta} / \delta \boldsymbol{b}((f) \overleftarrow{\delta} / \delta \boldsymbol{b}) = (\overrightarrow{\delta} / \delta \boldsymbol{b}(f)) \overleftarrow{\delta} / \delta \boldsymbol{b}.$

³⁸ This mechanism has already been implemented in the short proof of Lemma 7.

Namely, the integrand of the third term is $(-)^{|F|-1}$ times

$$\left\langle \delta b_{2,j_2}(\boldsymbol{z}_2) \underbrace{(-\vec{e}^{\dagger,j_2})(\boldsymbol{z}_2), \vec{e}_{j_1}(\boldsymbol{z}_1)}_{(\boldsymbol{z}_1)} \delta a_2^{j_1}(\boldsymbol{z}_1) \right\rangle \cdot \left\langle \delta b_{1,i_2}(\boldsymbol{y}_2) \underbrace{(-\vec{e}^{\dagger,i_2})(\boldsymbol{y}_2), \vec{e}_{i_1}(\boldsymbol{y}_1)}_{(\boldsymbol{z}_1)} \delta a_1^{i_1}(\boldsymbol{y}_1) \right\rangle \cdot \right\rangle \cdot \left\langle \delta b_{1,i_2}(\boldsymbol{y}_2) \underbrace{(-\vec{e}^{\dagger,i_2})(\boldsymbol{y}_2), \vec{e}_{i_1}(\boldsymbol{y}_1)}_{(\boldsymbol{z}_1)} \delta a_1^{i_1}(\boldsymbol{y}_1) \right\rangle \cdot \right\rangle \cdot \left\langle \delta b_{1,i_2}(\boldsymbol{y}_2) \cdot \left[\left(-\vec{e}^{\dagger,i_2})\right)^{\tau_1} \left(-\vec{e}^{\dagger,i_2}\right) \right] \right\rangle \cdot \left\langle \delta b_{1,i_2}(\boldsymbol{y}_2) \cdot \left[\left(-\vec{e}^{\dagger,i_2}\right)\right)^{\tau_1} \right] \right\rangle \cdot \left\langle \delta b_{1,i_2}(\boldsymbol{y}_2) \cdot \left[\left(-\vec{e}^{\dagger,i_2}\right)\right)^{\tau_1} \left(-\vec{e}^{\dagger,i_1}\right)^{\tau_1} \right] \right\rangle \cdot \left\langle \delta b_{1,i_2}(\boldsymbol{y}_2) \cdot \left[\left(-\vec{e}^{\dagger,i_2}\right)\right)^{\tau_1} \left(-\vec{e}^{\dagger,i_1}\right)^{\tau_1} \right) \right\rangle \cdot \left\langle \delta b_{1,i_2}(\boldsymbol{y}_2) \cdot \left[\left(-\vec{e}^{\dagger,i_2}\right)\right)^{\tau_1} \left(-\vec{e}^{\dagger,i_1}\right)^{\tau_1} \right) \right\rangle \cdot \left\langle \delta b_{1,i_2}(\boldsymbol{y}_2) \cdot \left[\left(-\vec{e}^{\dagger,i_2}\right)\right)^{\tau_1} \left(-\vec{e}^{\dagger,i_2}\right)^{\tau_1} \right\rangle \cdot \left\langle \delta b_{1,i_2}(\boldsymbol{y}_2) \cdot \left(-\vec{e}^{\dagger,i_2}\right)^{\tau$$

By construction, the lower-line derivations – from $[\![,]\!]$ – act first on f and g, and then the (graded-)derivations from the upper line – from Δ – work on the respective arguments. Now let the (multi)indexes be relabelled as above: $i \rightleftharpoons j$, $\sigma \rightleftharpoons \tau$, and $\delta a_1^i \rightleftharpoons \delta a_2^j$, $\delta b_{1,i} \rightleftharpoons \delta b_{2,i}$ on top of $\mathbf{y} \rightleftharpoons \mathbf{z}$. On the one hand, no relabelling of summation indices would affect any sum. On the other hand, such relabelling swaps the two lines between f and g, producing the minus sign factor due to the interchange of two parity-odd derivatives that fall on the first argument f. Consequently, the entire sum vanishes.

The only remaining term is processed analogously; the same relabelling of (multi)indices swaps the parity-odd derivations that act on the second argument g. Equal to minus itself, the fourth term vanishes. This concludes the proof. \Box

Theorem 14. Let F and G be two noncommutative local functionals over the infinite jet space $J^{\infty}(\pi_{NC}^{(0|1)})$; suppose F is homogeneous. The Batalin–Vilkovisky Laplacian Δ satisfies the relation

$$\Delta(\llbracket F, G \rrbracket) = \llbracket \Delta F, G \rrbracket + (-)^{|F|-1} \llbracket F, \Delta G \rrbracket.$$
⁽²⁸⁾

In other words, the operator Δ is a graded derivation of the noncommutative variational Schouten bracket [[,]].

Proof. We prove this by induction over the number of building blocks in each argument of the Schouten bracket on the left hand side of (28). To assert the claim in full, one reduces the set-up to integral functionals *F*, swaps the arguments $F \rightleftharpoons G$ of the Schouten bracket [[,]] by using formula (21), and repeats the reasoning.³⁹

If *F* and *G* both belong to $\overline{H^*}(\mathbb{T}\pi_{NC}^{(0|1)})$, then Lemma 13 states the assertion, which is the base of induction. To make an inductive step, without loss of generality let us assume that the first argument of $[\![,]\!]$ in (28) is a product of *two* elements from $\overline{\mathfrak{N}}^n(\mathbb{T}\pi_{NC}^{(0|1)})$, each of them containing fewer multiples from $\overline{H^*}(\mathbb{T}\pi_{NC}^{(0|1)})$ than the product. Denote such factors by *F* and *G* and suppose for definition that either of them, as well as the second argument *H* of the Schouten bracket, is homogeneous. Using Corollaries 8 and 11, we expand $(F \times G)[\overline{\Gamma, H}]$ and deduce that

$$[[F \times G, H]] = F \times [[G, H]] + (-)^{[G] \cdot ([H] - 1)} [[F, H]] \times G.$$
⁽²⁹⁾

Therefore, recalling **Definition 6** of the Schouten bracket, we have that

$$\begin{split} &\Delta(\llbracket F \times G, H \rrbracket) = \Delta F \times \llbracket G, H \rrbracket + (-)^{|F|} \llbracket F, \llbracket G, H \rrbracket \rrbracket + (-)^{|F|} F \times \Delta(\llbracket G, H \rrbracket) \\ &+ (-)^{|G| \cdot (|H| - 1)} \left\{ \Delta(\llbracket F, H \rrbracket) \times G + (-)^{|F| + |H| - 1} \llbracket \llbracket F, H \rrbracket, G \rrbracket + (-)^{|F| + |H| - 1} \llbracket F, H \rrbracket \times \Delta G \right\}. \end{split}$$

Using the inductive hypothesis in the third and fourth terms of the right-hand side in the above formula, we continue the equality and obtain

$$\Delta F \times [\![G, H]\!] + (-)^{|F|} [\![F, [\![G, H]\!]]\!] + (-)^{|F|} \Big\{ F \times [\![\Delta G, H]\!] + (-)^{|G|-1} F \times [\![G, \Delta H]\!] \Big\} + (-)^{|G| \cdot (|H|-1)} \Big\{ [\![\Delta F, H]\!] \times G + (-)^{|F|-1} [\![F, \Delta H]\!] \times G + (-)^{|F|+|H|-1} \Big[[\![I]\![F, H]\!], G]\!] + [\![F, H]\!] \times \Delta G \Big] \Big\}.$$
(30)

On the other hand, let us expand the right-hand side of (28), which now is

 $\llbracket \Delta(F \times G), H \rrbracket + (-)^{|F| + |G| - 1} \llbracket F \times G, \Delta H \rrbracket;$

we recall the definition of [[,]] and we then use (29). We obtain

$$\begin{split} \llbracket \Delta F \times G + (-)^{|F|} \llbracket F, G \rrbracket + (-)^{|F|} F \times \Delta G, H \rrbracket + (-)^{|F|+|G|-1} \llbracket F \times G, \Delta H \rrbracket \\ &= (-)^{|G| \cdot (|H|-1)} \llbracket \Delta F, H \rrbracket \times G + \Delta F \times \llbracket G, H \rrbracket + (-)^{|F|} \llbracket \llbracket F, G \rrbracket, H \rrbracket \\ &+ (-)^{|F|} \Big\{ (-)^{(|G|-1) \cdot (|H|-1)} \llbracket F, H \rrbracket \times \Delta G + F \times \llbracket \Delta G, H \rrbracket \Big\} \\ &+ (-)^{|F|+|G|-1} \Big\{ (-)^{|G| \cdot |H|} \llbracket F, \Delta H \rrbracket \times G + F \times \llbracket G, \Delta H \rrbracket \Big\}. \end{split}$$
(31)

³⁹ This is essential because Jacobi identity (26), which will be used explicitly in the proof below, requires the arrangement of parentheses (($\cdot \times \cdot$) × \cdot) but not ($\cdot \times (\cdot \times \cdot)$) in the course of multiplication of the three functionals *F*, *G*, and *H* in (29).

Comparing (31) with (30), which was derived from the inductive hypothesis, we see that all the terms match except for

$$(-)^{|F|} \left\{ \llbracket F, \llbracket G, H \rrbracket \rrbracket - (-)^{(|F|-1) \cdot (|G|-1)} \llbracket G, \llbracket F, H \rrbracket \rrbracket \right\}$$

from (30) versus

 $(-)^{|F|}$ [[[F, G]], H]]

from (31). These three terms constitute $(-)^{|F|}$ times the left- vs right-hand sides of Jacobi identity (26) for the noncommutative variational Schouten bracket. The balance of (30) and (31) completes the inductive step and concludes the proof.

Theorem 15. The Batalin–Vilkovisky Laplacian Δ is a differential on the space of local functionals over $J^{\infty}(\pi_{NC}^{(0|1)})$,

$$\Delta^2 = 0.$$

Summarising, the space $\overline{\mathfrak{M}}^n(\pi_{NC}^{(0|1)})$ of cyclic word-valued local functionals is a (non)associative graded-commutative BV algebra.

Proof. We prove Theorem 15 by induction over the number of building blocks from $\bar{H}^*(\mathbb{T}\pi_{NC}^{(0|1)})$ in the argument $H \in \overline{\mathfrak{N}}^n(\mathbb{T}\pi_{NC}^{(0|1)})$ of Δ^2 . If $H \in \overline{H}^*(\pi_{NC}^{(0|1)})$ itself is an integral functional, then by Lemma 7 there remains nothing to prove. Suppose now that $H = F \times G$ for some $F, G \in \overline{\mathfrak{N}}^n(\mathbb{T}\pi_{NC}^{(0|1)})$ and assume that the functional F is homogeneous. Then Definition 6 yields that

$$\Delta^{2}(F \times G) = \Delta \left(\Delta F \times G + (-)^{|F|} \llbracket F, G \rrbracket + (-)^{|F|} F \times \Delta G \right).$$

Using Definition 6 again and also Theorem 14, we continue the equality:

$$\begin{split} &= \Delta^2 F \times G + (-)^{|\Delta F|} \llbracket \Delta F, G \rrbracket + (-)^{|\Delta F|} \Delta F \times \Delta G \\ &+ (-)^{|F|} \llbracket \Delta F, G \rrbracket + (-)^{|F|} (-)^{|F|-1} \llbracket F, \Delta G \rrbracket \\ &+ (-)^{|F|} \Delta F \times \Delta G + (-)^{|F|} (-)^{|F|} \llbracket F, \Delta G \rrbracket + (-)^{|F|} (-)^{|F|} F \times \Delta^2 G \end{split}$$

By the inductive hypothesis, the first and last terms in the above formula vanish; taking into account that $|\Delta F| = |F| - 1$ in \mathbb{Z}_2 , the terms with $\Delta F \times \Delta G$ cancel against each other, as do the terms containing $[\![\Delta F, G]\!]$ and $[\![F, \Delta G]\!]$. The proof is complete. \Box

Remark 2.14. In the BV context, the non-associativity of the algebra of cyclic words is a property still not a burden. To establish that the BV Laplacian Δ is a differential on the algebra of local functionals, we de facto proved that for any three such functionals *F*, *G*, and *H* one has that $\Delta^2(F \times G \times H) = 0$. In view of the non-associativity of the product ×, the parentheses were arranged in the lexicographic order ($F \times G \times H$) = 0. In view of the non-associativity of the product ×, the parentheses were arranged in the lexicographic order ($F \times G \times H$). This was essential for a verification of Jacobi identity (25), see footnote 35. Yet because the multiplication × is graded-commutative so that $F \times (G \times H) = (-)^{|F| \cdot (|G| + |H|)} (G \times H) \times F$, the arrangement ($\cdot \times (\cdot \times \cdot)$) is transformed into (($\cdot \times \cdot) \times \cdot$), which was considered before. Now relabelling the *arbitrary* functionals via $F \leftarrow G \leftarrow H \leftarrow F$, we deduce that the non-associativity of operation × in the argument of $\Delta^2(F \times G \times H)$ is not restrictive.⁴⁰

Remark 2.15. We conclude that the proof of all these assertions about the Batalin–Vilkovisky Laplacian and variational Schouten bracket remains literally valid in the graded-commutative set-up. Indeed, when the proof is over, it suffices to let N := 0 (so that there are no generators $\vec{x}_i^{\pm 1}$) and proclaim that the letters a_{σ}^i and $b_{j,\tau}$ are graded-permutable; the proof itself does not require that assumption.

Likewise, the formalism developed in Section 2 survives arbitrary changes of cell decomposition for manifolds $(M^n, \text{dvol}(\cdot))$, even though the tilings of newly produced spaces, whenever irregular, would make the alphabet $\vec{\mathbf{x}}^{\pm 1}$ point-dependent.

We also conclude that by shrinking the substrate manifold M^n to a point, so that n = 0 and N = 0, we recover the standard properties of the parity-odd differential $\Delta_0 = \overrightarrow{\partial}^2 / \partial a^i \partial b_i$ and parity-odd Poisson bracket in the (formal non)commutative geometry of symplectic supermanifolds of superdimension (m|m). The locality of couplings (13) still in force, our reasoning explains why the differentials of *two* Hamiltonians and the Poisson bi-vector are referred to *the same* point when the Poisson bracket is constructed.

3. Noncommutative variational Poisson formalism

The noncommutative variational cotangent superspace, which we built in Section 2 for the bundle π_{NC} from Section 1, and the calculus of local functionals on jet spaces $J^{\infty}(\pi_{NC}^{(0|1)})$, see Section 2, refer to the canonical *symplectic* structure encoded by (13). Let us now introduce a more narrow (sic!) class of variational noncommutative geometries in which the *Poisson* structures are defined.

⁴⁰ In the weight factor $\exp(\frac{i}{\hbar}S^{\hbar})$ of the Feynman path integral, the comultiples are copies of the (quantum BV-)action functional S^{\hbar} , whence the nominal non-associativity of structure × is all the more negligible.

3.1. Noncommutative variational multivectors

Let us recall that the notion of space of integral functionals $\bar{H}^n(\pi_{NC}^{(0|1)})$ was based in Section 2 on an obvious analytic idea to integrate the sections $\mathbf{s} \in \Gamma(\pi_{NC})$ over $dvol(\mathbf{x})$ on the substrate manifold M^n ; the integrals take every such evaluation mapping to the cyclic word(s) written in the edge alphabet $\mathbf{x}^{\pm 1}$ (see (17)). When the \mathbb{Z}_2 -valued parity function was introduced, the parity-odd symbols \mathbf{b} and extension \mathbf{s}^{\dagger} of \mathbf{s} to maps defined on $\mathcal{A}^{(0|1)}$ were felt as the objects that make everything go much better as soon as one gets rid of them; we refer to Remark 2.5 in particular.

Taking this into account, let us describe a very different geometric approach to the use of \mathbb{Z}_2 -parity graded noncommutative integral functionals. Namely, we shall view the parity-odd symbols **b** and their derivatives as *placeholders* for (non)commutative variational covectors; such placeholders appear in the fully skew-symmetric poly-linear maps on the space $\bar{H}^n(\pi_{NC}^{(0|1)})$ of purely even Hamiltonian functionals. By making this construction precise, which forces us to narrow the class of graded-homogeneous functionals under study, we resolve the difficulty which is known from Remark 2.5.

The key idea is that – unlike it is the case for cyclic-word integral functionals of generic nature – the (non)commutative variational *multivectors* are organised in precisely the same way with respect to each parity-odd entry \boldsymbol{b} , as long as the shifts t around the circle and integrations by parts are allowed.

Let $P \in \bar{H}^n(\pi_{NC}^{(0|1)})$ be a homogeneous functional of grading $|P| = k \ge 0$. If k = 0, none of the cyclic words in P contains any parity-odd symbols $b_{i,\tau}$. If k = 1, then there is the noncommutative linear total differential operator A (that is, an operator which is polynomial in the total derivatives and the coefficients of which are operators of left and right multiplication by functions of \mathbf{x} or by parity-even symbols $\mathbf{x}^{\pm 1}$ or a_{σ}^i from the alphabet on $J^{\infty}(\pi_{NC}^{(0|1)})$ such that

$$P = (A(\boldsymbol{b})).$$

Clearly, there remains nothing more to do; for the above key idea is already realised.

Suppose now k = 2; pick *one* parity-odd letter in every cyclic word of *P* and throw all the derivations off every such letter by using a suitable number of integrations by parts; then, if necessary, transport the letters around the circle so that those $b_{i,\emptyset}$ stand immediately after the lock ∞ in the positive, counterclockwise direction. This brings *P* to the normal shape

$$P \cong \frac{1}{2} \left(\boldsymbol{b} \circ \boldsymbol{A}(\boldsymbol{b}) \right) ; \tag{32}$$

by construction, A is the arising $(m \times m)$ -size matrix linear noncommutative total differential operator of one argument.

Arguing as above and picking *some* parity-odd letter in every word of a given integral functional *P* of grading *k*, we transform it to the sum of cyclic words, each starting with $b_{j,\emptyset}$ for $1 \le j \le m$,

$$P \cong \frac{1}{k!} \left(\boldsymbol{b} \circ A(\underbrace{\boldsymbol{b}, \ldots, \boldsymbol{b}}_{k-1 \text{ slots}}) \right), \tag{33}$$

where the noncommutative total differential operator A is poly-linear in its k - 1 arguments.⁴¹

To make the construction of operator A independent of our initial choice of *some* parity-odd entries, let us analyse the properties such an operator must have. We consider the case k = 2 because it will be essential in what follows. Through the chain of integrations by parts and by carrying the parity-odd letters around the circle,

$$P = \frac{1}{2} (\boldsymbol{b} \circ A(\boldsymbol{b})) \cong \frac{1}{2} \left((\boldsymbol{b}) \overleftarrow{A}^{\dagger} \circ \boldsymbol{b} \right) \sim -\frac{1}{2} \left(\boldsymbol{b} \circ (\boldsymbol{b}) \overleftarrow{A}^{\dagger} \right) \stackrel{\text{def}}{=} -\frac{1}{2} \left(\boldsymbol{b} \circ A^{\dagger}(\boldsymbol{b}) \right),$$
(34)

one defines the *adjoint operator* A[†] that acts on its argument in the left-to-right direction.⁴² The starting objects P and the resulting functional are identically the same if we require that

$$A = -A^{\dagger}.$$
(35)

For example, let n = 1, m = 1 and consider $P = \frac{1}{2}(b \circ b_x)$ with $A = \vec{d}/dx$, see [22].

The requirements which the poly-linear operator *A* of k - 1 arguments must satisfy are imposed for all $k \ge 3$ in the same way as in (34).

In what follows, we shall consider only the grading-homogeneous functionals on $J^{\infty}(\pi_{NC}^{(0|1)})$ for which the poly-linear operators *A* are well defined, so that normalisation (33) can be attained by starting from any parity-odd entry in every cyclic word of the functional at hand.

⁴¹ Of course, the notation for A acting on the *m*-tuples **b** is symbolic; in reality, every cyclic word of P carries k parity-odd entries $b_{i_1,\varnothing}, b_{i_2,\sigma_2^{i_2}}, \ldots, b_{i_k,\sigma_k^{i_k}}$

where $1 \leq i_{\alpha} \leq m$ and the multi-indexes are word-dependent. It is often the case that $|\sigma_{\alpha}^{i}| \neq |\sigma_{\alpha}^{j}|$ for $i \neq j$ at some α ; for instance, recall the differential order of entries in the matrix operator for the second Poisson structure of the renowned Boussinesq hierarchy.

⁴² Note that the *left* multiplications in *A* become the right multiplications in $\overleftarrow{A}^{\dagger}$, and *vice versa*. At the same time, the total derivative operators are reshaped by $(\overrightarrow{d}/d\mathbf{x})^{\sigma} \mapsto \circ(-\overrightarrow{d}/d\mathbf{x})^{\sigma} \mapsto (-\overrightarrow{d}/d\mathbf{x})^{\sigma} \circ$, e.g., the adjoint to $(aa\circ)\overrightarrow{D}_{x}(\cdot)(\circ a)$ is $(-\overrightarrow{D}_{x}) \circ ((a\circ)(\cdot)(\circ aa))$. Thirdly, the operator's matrix is transposed: $(A^{\dagger})^{ij} = (A^{ij})^{\dagger}$, where the rightmost symbol \dagger denotes the adjoint of a scalar differential operator.

Definition 7. Homogeneous integral functionals $P \in \bar{H}^n(\pi_{NC}^{(0|1)})$ of grading $k \ge 0$ and such that either $k \le 1$ or normalisation (33) is well defined are called *noncommutative variational k-vectors*. Let us denote by $\bar{H}^n_k(\pi_{NC}^{(0|1)}) \subsetneq \bar{H}^n(\pi_{NC}^{(0|1)})$ the vector space of noncommutative variational *k*-vectors on $J^\infty(\pi_{NC}^{(0|1)})$.

Note that by Remark 2.5, the subspaces $\bar{H}_k^n(\pi_{NC}^{(0|1)})$ do not exhaust the homogeneous components of grading k in $\bar{H}^n(\pi_{NC}^{(0|1)})$ for $k \ge 2$.

Remark 3.1. We claim that the vector space $\bigoplus_{k \ge 0} \bar{H}_k^n(\pi_{NC}^{(0|1)})$ of all noncommutative variational multivectors is closed under [[,]], which endows it with the structure of Gerstenhaber algebra with respect to the noncommutative variational Schouten bracket.

Definition 7 is constructive but implicit. It is instructive to see why the Schouten bracket [F, G] of a k-vector F and ℓ -vector G is a $(k + \ell - 1)$ -vector: this fact relies on a very distinguished structure – of the local variational differential operators $[[F, \cdot]]$ or $[[\cdot, G]]$ – whose normalisation (33) provides for the geometric model of [[,]] in Remark 2.12.

Remark 3.2. The price that one pays for the (non)commutative variational multivectors' realisation – uniform with respect to every parity-odd entry \mathbf{b} under integration by parts and cyclic shifts – is precisely having that legal possibility to integrate by parts. Yet we remember from Section 2.5 that all of such integration is postponed until the ultimate end of every object's construction in the frames of the geometry of iterated variations. Therefore, the variational calculus of (non)commutative variational multivectors is *step-by-step* indeed; every intermediate object is let to exist as a well-defined notion.

For instance, Poisson bi-vectors \mathcal{P} first take the Hamiltonians F to the respective one-vectors X_F , which are also known to us under the name of Hamiltonian evolution equations (e.g., of (non)commutative Korteweg-de Vries type). In turn, the well-defined one-vector X_F acts by the Schouten bracket $[X_F, \cdot]$ on a given 0-vector H, which defines the Poisson bracket $\{F, G\}_{\mathcal{P}}$, see Section 3.3.

Notice that no multiplication of copies of the substrate manifold M^n can be seen from this way of reasoning; in fact, the on-the-diagonal restriction in the last phase of construction of the Schouten bracket becomes the immediate next to the first step. This is why the Poisson framework of (non)commutative variational multivectors was not capable of providing the intrinsic self-regularisation of the Batalin-Vilkovisky formalism with generic local functionals.

3.2. Derived brackets

Let $P \in \bar{H}_k^n(\pi_{NC}^{(0|1)})$ be a noncommutative variational *k*-vector. Consider *k* integral functionals $H_1, \ldots, H_k \in \bar{H}_0^n(\pi_{NC}^{(0|1)})$ of grading zero (that is, a *k*-tuple of 0-vectors).

Definition 8. The *k*-linear bracket $\{\cdot, \ldots, \cdot\}_P$: $(\bar{H}_0^n \times \cdots \times \bar{H}_0^n)(\pi_{NC}^{(0|1)}) \rightarrow \bar{H}_0^n(\pi_{NC}^{(0|1)})$ is defined by the noncommutative variational *k*-vector *P* as the derived bracket,⁴³

$$\{H_1, \ldots, H_k\}_P \stackrel{\text{def}}{=} (-)^k \llbracket \ldots \llbracket P, H_1 \rrbracket, \ldots, H_k \rrbracket.$$
(36)

The nested Schouten brackets are underlined in order to emphasise that each of them produces an object, i.e. the noncommutative variational multivector with one parity-odd entry less than the two arguments had together. In consequence, the integrations by parts are legitimate at every such step. This makes the Poisson formalism on jet spaces a science of steps and stops.

Example 3.1. If k = 1 and the noncommutative variational one-vector is the cyclic word P = (A(b)) for some total differential operator A (i.e. for a linear operator that is polynomial in the total derivatives), then

$${H_1}_P = -[[P, H_1]] = (A(\delta H_1 / \delta a))$$

Likewise, if k = 2 and, after a suitable number of integrations by parts, the noncommutative variational bi-vector is represented by the cyclic word(s) $P = \frac{1}{2}(\boldsymbol{b} \circ A(\boldsymbol{b}))$, then it is readily seen that⁴⁴

$$\{H_1, H_2\}_P = \underline{\llbracket \underline{\llbracket H_1, P \rrbracket}, H_2 \rrbracket} \cong \left(A\left(\frac{\delta H_1}{\delta \boldsymbol{a}}\right) \circ \frac{\delta H_2}{\delta \boldsymbol{a}} \right) \sim \left(\frac{\delta H_2}{\delta \boldsymbol{a}^i} \circ A^{ij}\left(\frac{\delta H_1}{\delta \boldsymbol{a}^j}\right) \right).$$
(37)

Let us comment on every step in this construction. First, the variational one-vector X_{H_1} is produced from P and H_1 ; consider

$$\llbracket H_1, \frac{1}{2}(\boldsymbol{b} \circ A(\boldsymbol{b})) \rrbracket = \left(\frac{\delta H_1}{\delta \boldsymbol{a}} \circ \frac{1}{2} \sum_{|\tau|} \left(-\frac{\overrightarrow{\mathbf{d}}}{\mathbf{d}\boldsymbol{x}} \right)^{\tau} \frac{\overrightarrow{\partial}}{\partial \boldsymbol{b}_{\tau}} (\boldsymbol{b} \circ A(\boldsymbol{b})) \right).$$

 $^{^{43}}$ We refer to [49] for a review of the concept of derived brackets in the geometry of usual manifolds. An algebraic classification of N-ary brackets is obtained in [50]; by analysing the jet-bundle geometry in this context, in the paper [51] we developed the notion of Wronskian determinants for functions in many variables. In particular, we proved that every such structure W encodes a differential $\mathbf{d}_{W}^{2} = 0$.

⁴⁴ The first equality tells us that the bracket $\{\cdot, \cdot\}_P$ which the bi-vector P determines is a bracket between its arguments indeed.

When $P = \frac{1}{2} (\mathbf{b} \circ A(\mathbf{b}))$ is varied with respect to \mathbf{b} , the partial derivatives $\overrightarrow{\partial} / \partial b_{j,\tau}$ reach the first occurrence \mathbf{b}_{\varnothing} with $\tau = \varnothing$ at once; before they reach the argument \mathbf{b} of skew-adjoint operator A, let us integrate by parts: $\frac{1}{2} (\mathbf{b} \circ A(\mathbf{b})) \cong \frac{1}{2} (-A(\mathbf{b}) \circ \mathbf{b}) \sim \frac{1}{2} (\mathbf{b} \circ A(\mathbf{b}))$. This shows that due to the particular structure of bi-vectors – if compared with generic functionals of grading two, – the second term doubles and absorbs $\frac{1}{2}$. We get the one-vector $(\delta H_1 / \delta \mathbf{a} \circ A(\mathbf{b}))$; integrating by parts once again and using (35), we obtain the object

$$X_{H_1} = \left(-A\left(\frac{\delta H_1}{\delta \boldsymbol{a}}\right) \circ \boldsymbol{b}\right).$$

Now the construction of the outer Schouten bracket in (37) is elementary.

Lemma 16. Derived bracket (36) is totally antisymmetric under permutations of its arguments:

$$\{H_{\omega(1)}, \ldots, H_{\omega(k)}\}_P = (-)^{\omega} \{H_1, \ldots, H_k\}_P$$

for any $\omega \in S_k$ and any $H_1, \ldots, H_k \in \overline{H}_0^n(\pi_{NC}^{(0|1)})$.

Remark 3.3. The total skew-symmetry of object (36) produced in k separate steps – with integration by parts and full stop after each step – does not follow from the Jacobi identity for [[,]], which was established in Section 2. Rather, this is a manifestation of the noncommutative variational k-vectors' intrinsic property to be structurally identical with respect to every two graded entries **b**.

Sketch of the proof. It suffices to show that the derived bracket $\{\cdot, \ldots, \cdot\}_p$ changes its sign under a swap of two consecutive arguments H_i and H_{i+1} :

 $\ldots \llbracket \llbracket Q, H_i \rrbracket, H_{i+1} \rrbracket \ldots \cong - \cdots \llbracket \llbracket Q, H_{i+1} \rrbracket, H_i \rrbracket \ldots$

Consider the noncommutative variational multivector's necklace Q and mark, by using \otimes and \oplus , two parity-odd entries **b** (e.g., the two *consecutive* ones for the sake of clarity), see the figure on the facing page.



This object's inner Schouten bracket with H_i does basically the following: normalisation (33) throws all the derivatives off the entry \otimes and implants $\delta H_i/\delta a$ in its stead (the normalisation does exactly the same with every other entry b by the definition of multivector, but let us focus on the term such that the variation $\delta H_i/\delta a$ hits \otimes). Now reshape this output by making \oplus free of derivatives falling on it. Note that this session of integrations by parts again amounts to bringing the multivector to normalised shape (33), — only the neighbouring entry \otimes is occupied now by $\delta H_i/\delta a$, not by b. The outer Schouten bracket installs $\delta H_{i+1}/\delta a$ at \oplus (or at any other parity-odd entry; we consider just one term, for definition).

On the other hand, consider the very same scenario of putting $\delta H_i/\delta a$ for \otimes and $\delta H_{i+1}/\delta a$ for \oplus , done in the reverse order. To reach \oplus first in the construction of (now, inner) Schouten bracket, the derivation $\partial/\partial b$ has to overtake \otimes currently occupied by the parity-odd placeholder **b**; this overtaking yields the sought-for minus sign. The variation $\delta H_{i+1}/\delta a$ pasted for \oplus , we cast all the derivatives off the still-unused slot \otimes , leave $\delta H_i/\delta a$ there, and integrate by parts back, to isolate $\delta H_{i+1}/\delta a$ in the socket \oplus . It is readily seen that the two algorithms produce the identical portraits of letters and derivatives, yet those two differ by the sign factor. \Box

Remark 3.4. Continuing this line of reasoning, we conclude that for a given noncommutative variational *k*-vector *P*, the value $\{H_1, \ldots, H_k\}_P$ of derived bracket (36) at *k* arguments $H_1, \ldots, H_k \in \bar{H}_0^n(\pi_{NC}^{(0|1)})$ is equivalent, up to integration by parts, to the 0-vector

$$(-)^{\frac{k(k-1)}{2}} \cdot \frac{1}{k!} \sum_{\omega \in S_k} (-)^{\omega} \left(\frac{\delta H_{\omega(1)}}{\delta \boldsymbol{a}} \circ A \left(\frac{\delta H_{\omega(2)}}{\delta \boldsymbol{a}}, \dots, \frac{\delta H_{\omega(k)}}{\delta \boldsymbol{a}} \right) \right) \cong \{H_1, \dots, H_k\}_P,$$
(38)

where the alternating sum runs through the entire permutation group S_k ; note that it is the parity-even arguments H_i but not the slots for them which are shuffled.

Observation (38) allows us to extend the *mapping P* from the geometry of exact (non)commutative variational covectors $\delta H_i / \delta a$,

$$P(\delta H_1/\delta \boldsymbol{a}, \ldots, \delta H_k/\delta \boldsymbol{a}) \stackrel{\text{def}}{=} \{H_1, \ldots, H_k\}_P,$$

to *k*-tuples of arbitrary variational covectors $\mathbf{p}_i = (p_{i,\alpha} \circ \delta a^{\alpha})$. (Let us think of the variational covectors $(\mathbf{p} \circ \delta \mathbf{a}) = (p_{\alpha}(\mathbf{x}, \mathbf{x}^{\pm 1}, [\mathbf{a}]) \circ \delta a^{\alpha})$ on $J^{\infty}(\pi_{\text{NC}})$ as of (the formal sums of) necklaces equipped with the extra earrings δa^{α} , by which those cyclic words are handled.) The case k = 1 with $P(\mathbf{p}_1) := (A(\mathbf{p}_1))$ is elementary; for $k \ge 2$, we put⁴⁵

$$P(\boldsymbol{p}_{1}, \ldots, \boldsymbol{p}_{k}) := (-)^{\frac{k(k-1)}{2}} \cdot \frac{1}{k!} \sum_{\omega \in S_{k}} (-)^{\omega} \left(\boldsymbol{p}_{\omega(1)} \circ A(\boldsymbol{p}_{\omega(2)}, \ldots, \boldsymbol{p}_{\omega(k)}) \right).$$
(39)

However, generic variational covectors, not necessarily exact, will not be studied in particular in what follows – rather, the converse can be assumed in view of the Substitution Principle.

Theorem 17 (The Substitution Principle). Suppose that a tuple of identities

$$I((\boldsymbol{x}, \vec{\mathbf{x}}^{\pm 1}), [\boldsymbol{a}], [\boldsymbol{p}_1(\boldsymbol{x}, \vec{\mathbf{x}}^{\pm 1})], \dots, [\boldsymbol{p}_k(\boldsymbol{x}, \vec{\mathbf{x}}^{\pm 1})]) \equiv 0$$

holds on $J^{\infty}(\pi_{NC})$ for every k-tuple of noncommutative variational (co)vectors the coefficients $p_{i,\alpha}(\mathbf{x}, \mathbf{x}^{\pm 1})$ of which depend only on points $\mathbf{x} \in M^n$ and letters from the edge alphabet $\mathbf{x}^{\pm 1}$. Then the identities in total derivatives,

$$I((\boldsymbol{x}, \vec{\mathbf{x}}^{\pm 1}), [\boldsymbol{a}], [\boldsymbol{p}_1(\boldsymbol{x}, \vec{\mathbf{x}}^{\pm 1}, [\boldsymbol{a}])], \dots, [\boldsymbol{p}_k(\boldsymbol{x}, \vec{\mathbf{x}}^{\pm 1}, [\boldsymbol{a}])]) \equiv 0,$$

viewed as identities with respect to \mathbf{p}_i , are valid on $J^{\infty}(\pi_{NC})$ for all (co)vectors \mathbf{p}_i depending not only on \mathbf{x} and $\mathbf{x}^{\pm 1}$ but also admitting arbitrary, finite differential order dependence on the jet letters \mathbf{a}_{σ} , $|\sigma| < \infty$.

Remark 3.5. At this moment it is legitimate to view the variational (co)vectors $\mathbf{p}_i = (p_{i,\alpha} \circ \delta a^{\alpha})$ as bare collections of their indexed open-word components $p_{i,\alpha}$ that are already built into the identities \mathbf{I} . We emphasise that, unlike it is the case studied in Section 1.1 – the cyclic words in A do not carry any marked point, – the earrings $\partial/\partial \mathbf{a}_{\alpha}$ and $\delta \mathbf{a}$ are the only places where the (co)vectors can be unlocked.

Corollary 18. If, under the assumptions of Theorem 17, the identities in total derivatives $I(\mathbf{x}, \mathbf{x}^{\pm 1}, [\mathbf{a}], [\mathbf{p}_i]) \equiv 0$ with respect to $\mathbf{p}_1, \ldots, \mathbf{p}_k$ hold on $J^{\infty}(\pi_{NC})$ for every k-tuple of exact variational covectors $\mathbf{p}_i = (\delta \mathcal{H}_i / \delta \mathbf{a} \circ \delta \mathbf{a})$ which are obtained by variation of arbitrary linear integral functionals $\mathcal{H} \in H^n(\pi_{NC})$, then these identities hold for all covectors \mathbf{p}_i , i.e. not necessarily exact.

Indeed, it is always possible to represent locally an $(\mathbf{x}, \mathbf{x}^{\pm 1})$ -dependent cyclic word $\sum_{\alpha=1}^{m} (p_{i,\alpha}(\mathbf{x}, \mathbf{x}^{\pm 1}) \circ \delta a^{\alpha})$ as the variation $\delta \mathcal{H}$ of the functional $\sum_{j=1}^{m} \int (p_{i,\alpha}(\mathbf{x}, \mathbf{x}^{\pm 1}) \circ a^{\alpha} \operatorname{dvol}(\mathbf{x}))$ and then apply Theorem 17.

Proof of Theorem 17. For the sake of brevity, let each variational noncommutative covector \mathbf{p}_i consist of just one word written in the alphabet of $J^{\infty}(\pi_{\text{NC}})$. The crucial idea is that the position of the locks $\delta \mathbf{a}$ is fixed on the circles which carry the words \mathbf{p}_i . This means that, whenever one declares an arbitrary differential dependence of \mathbf{p}_i on \mathbf{a} , the words \mathbf{I} in principle lengthen but still, in the course of multiplications \times within the identities, each \mathbf{p}_i is never torn in between any consecutive pair of letters \mathbf{a} . Namely, during the evaluation of \mathbf{I} at the words \mathbf{p}_i those are unlocked, the letters and the words' overall coefficients depending on \mathbf{x} are then stretched to open strings (ordered counterclockwise). These strings of symbols are pasted into \mathbf{I} without splitting, i.e., the adjacent letters of \mathbf{p}_i never become separated by any other symbols.⁴⁶ Total derivatives (9) then work according to their definition: under a restriction of \mathbf{I} (hence of all \mathbf{p}_i) to the jet of a mapping $\mathbf{a} = \mathbf{s}(\mathbf{x}, \mathbf{x}^{\pm})$, each symbol a^i is replaced with the respective sum of open strings $s^i(\mathbf{x}, \mathbf{x}^{\pm 1})$ so that derivations (9) which act on \mathbf{a}_{σ} occurring anywhere (either in \mathbf{p}_i or in \mathbf{I} if the identities explicitly depend on $[\mathbf{a}]$) then reduce to the derivations $\partial/\partial x^i$ of real-valued functions defined at $\mathbf{x} \in U \subseteq M^n$. By the initial assumption of the theorem, its assertion is valid for all strings written in the basic alphabet ($\mathbf{x}, \mathbf{x}^{\pm 1}$) that replace the entries \mathbf{p}_i in \mathbf{I} . We conclude that the identities $\mathbf{I} = 0$ hold on $J^{\infty}(\pi_{\text{NC}})$ for the full set of arguments of the (co)vectors.⁴⁷

Remark 3.6. The proof remains literally valid in the case of (evolutionary) vector fields instead of variational *covectors*. This would be important for the description of variational noncommutative symplectic structures. At the same time, the proof reveals *why* this noncommutative phrasing of the Substitution Principle does *not* hold for arbitrary cyclic words $p_i(\mathbf{x}, \mathbf{x}^{\pm 1}, [\mathbf{a}])$ of unspecified nature.

Remark 3.7. Attempts to define the (non)commutative variational Schouten bracket of multivectors via a recursive procedure that involves the use of the two arguments' values at test covectors are sometimes practised in the literature (see discussion in [52] and references therein).

Open problem 2. Is there a way to detect that a given (non)commutative variational 0-vector $H \in \bar{H}_0^n(\pi_{NC}^{(0|1)})$ is the value of a (non)commutative variational *k*-vector at *k* zero-vectors?

⁴⁵ The isomorphism $V^{\dagger} \simeq T_{a^{\dagger}} V^{\dagger}$ is used here to convert the placeholders **b** for **p**_i into the virtual offsets $\sum_{\alpha=1}^{m} 1 \cdot \vec{e}^{\dagger,\alpha}$. The absorption of each argument **p**_i then goes closely to the lines of geometric construction of the Schouten bracket, see Remark 2.12.

⁴⁶ This scenario is realised irrespectively of presence or absence of letters \boldsymbol{a} 's on the necklaces \boldsymbol{p}_i , which is in contrast with formula (4).

⁴⁷ One does not even have to postulate that the mappings $\mathbf{a} = \mathbf{s}(\mathbf{x}, \mathbf{x}^{\pm 1})$ inserted in the explicit dependence of \mathbf{I} on $[\mathbf{a}]$ coincide with the mappings now standing for \mathbf{a} in the implicit dependence $[\mathbf{p}_i(\mathbf{x}, \mathbf{x}^{\pm 1}, [\mathbf{a}])]$.

3.3. Noncommutative variational Poisson structures

Now we analyse the construction of noncommutative variational Poisson brackets, recalling and re-proving several important facts — here, under the coarse assumption of cyclic invariance (e.g., the Helmholtz lemma reveals yet another mechanism for the differentials to anticommute).

Remark 3.8. Although the formalism is based on the noncommutative variational *symplectic* geometry from Section 2, the presence of differential operators *A* in the definition of the Poisson bracket { , }_{*P*} as derived with respect to a given Poisson bi-vector *P*, see (36), usually makes such brackets degenerate. Their Casimirs, forming the zeroth Poisson cohomology group with respect to $\partial_{P_1} = [[P_1, \cdot]]$, start the Magri scheme for systems possessing the bi-Hamiltonian structures (P_1, P_2), see Section 3.3.4 and [10,11].

3.3.1. The definition of Poisson bracket

Consider a noncommutative variational bi-vector \mathcal{P} and let H_1 , H_2 , $H_3 \in \overline{H}_0^n(\pi_{NC}^{(0|1)})$ be any three noncommutative variational 0-vectors.

Definition 9. Bi-linear, skew-symmetric derived bracket (37),

$$\{H_i, H_j\}_{\mathcal{P}} = \llbracket\llbracket H_i, \mathcal{P}\rrbracket, H_j\rrbracket, \qquad 1 \leq i < j \leq 3$$

is called the noncommutative variational Poisson bracket if it satisfies Jacobi identity,

$$\{\{H_1, H_2\}_{\mathcal{P}}, H_3\}_{\mathcal{P}} + \{\{H_2, H_3\}_{\mathcal{P}}, H_1\}_{\mathcal{P}} + \{\{H_3, H_1\}_{\mathcal{P}}, H_2\}_{\mathcal{P}} \cong 0$$

$$\tag{40}$$

for all H_1 , H_2 , $H_3 \in \overline{H}_0^n(\pi_{\text{NC}}^{(0|1)})$, which are then called the *Hamiltonians*.

If identity (40) holds, the noncommutative variational bi-vector $\mathcal{P} = \frac{1}{2} (\boldsymbol{b} \circ A(\boldsymbol{b}))$ is called *Poisson*; the skew-adjoint noncommutative linear operator A in total derivatives is then called a *Hamiltonian operator*, and the noncommutative variational one-vectors $X_{H_i} \stackrel{\text{def}}{=} [\mathcal{P}, H_i]$ are the *Hamiltonian one-vectors* (or *one-vector fields*) specified by their Hamiltonians H_i and the Poisson bi-vector \mathcal{P} .

Criterion 19. A noncommutative variational bi-vector \mathcal{P} is Poisson (i.e. the derived bracket $\{, \}_{\mathcal{P}}$ satisfies Jacobi identity (40)) if the bi-vector \mathcal{P} satisfies the classical master-equation

$$\llbracket \mathcal{P}, \mathcal{P} \rrbracket \cong 0 \in \bar{H}_{3}^{n} \left(\pi_{NC}^{(0|1)} \right). \tag{41}$$

The bi-vector \mathcal{P} is Poisson only if the value of $[\mathcal{P}, \mathcal{P}]$ at any triple H_1, H_2, H_3 of Hamiltonians is cohomologically trivial:

$$\llbracket \mathcal{P}, \mathcal{P} \rrbracket (H_1, H_2, H_3) \cong 0 \in \overline{H}_0^n (\pi_{NC}^{(0|1)}).$$

The assertion is aimed to emphasise that the Poisson bi-vectors are the primary objects, whereas the Poisson brackets are the derived structures.

Lemma 20. If a noncommutative variational k-vector \mathcal{Q} represents the class of zero in $\bar{H}_k^n(\pi_{NC}^{(0|1)})$, then, \mathcal{Q} viewed as the map $(\bar{H}_0^n \times \cdots \times \bar{H}_0^n)(\pi_{NC}^{(0|1)}) \rightarrow \bar{H}_0^n(\pi_{NC}^{(0|1)})$, its value $\mathcal{Q}(\delta H_1/\delta \mathbf{a}, \ldots, \delta H_k/\delta \mathbf{a}) = \{H_1, \ldots, H_k\}_{\mathcal{Q}}$ is cohomologically trivial for every k-tuple of the arguments $H_1, \ldots, H_k \in \bar{H}_0^n(\pi_{NC}^{(0|1)})$.

Sketch of the proof. Indeed, whenever the cyclic word $\mathcal{Q} = d_h \mathcal{R}(\boldsymbol{b}, \dots, \boldsymbol{b})$ carrying k parity-odd entries \boldsymbol{b} is exact with respect to the lift d_h of the de Rham differential for M^n onto $J^{\infty}(\pi_{NC}^{(0|1)})$, so is every term – in the sum over the $|S_k| = k!$ ways to permute the arguments H_1, \dots, H_k by using $\omega \in S_k$ – obtained by pasting whatever open string $\delta H_{\omega(i)}/\delta a^j$ of parity-even symbols instead of the *i*th copy of the symbol b_j . \Box

Remark 3.9. The gap between necessity,

• a variational bi-vector \mathcal{P} is Poisson only if all the values of the variational tri-vector $[\![\mathcal{P}, \mathcal{P}]\!]$ are trivial in $\bar{H}_0^n(\pi_{NC}^{(0|1)})$,

and sufficience,

• a variational bi-vector \mathcal{P} is Poisson if the variational tri-vector $[\![\mathcal{P}, \mathcal{P}]\!]$ itself is trivial in the respective horizontal cohomology group $\bar{H}_3^n(\boldsymbol{\pi}_{NC}^{(0|1)}) \neq \bar{H}_0^n(\boldsymbol{\pi}_{NC}^{(0|1)})$,

is the statement that, whenever the value $\mathcal{Q}(\delta H_1/\delta \boldsymbol{a}, \ldots, \delta H_k/\delta \boldsymbol{a})$ of a (non)commutative variational k-vector \mathcal{Q} at every k-tuple of exact variational covectors $\delta H_i/\delta \boldsymbol{a}$ is cohomologically trivial in $\bar{H}_0^n(\boldsymbol{\pi}_{NC}^{(0|1)})$, the k-vector \mathcal{Q} itself is cohomologically trivial in $\bar{H}_k^n(\boldsymbol{\pi}_{NC}^{(0|1)})$. This claim proven, Criterion 19 (and Lemma 20) would convert into an equivalence.

Lemma 21. In fact, this is true,

$$\mathcal{P}$$
 Poisson $\iff [\![\mathcal{P}, \mathcal{P}]\!] \cong 0,$

over topologically trivial, star-shaped domains $\subseteq M^n$.

Indeed, under the trivial topology assumption, the homotopy procedure (e.g., see [16] or [44]) in the constructive proof of the Poincaré lemma works both on the base, which we denote still by M^n , and in the topologically trivial fibres of the Whitney sum of the (non)commutative bundle π_{NC} and k copies of its dual $\hat{\pi}_{NC}$.

Sketch of the proof. Consider not the bundle π_{NC} such that $\mathcal{Q}(\delta H_1/\delta a, \ldots, \delta H_k/\delta a) \in \overline{H}^n(\pi_{NC})$ but introduce the Whitney sum $\pi_{NC} \times_{M_{NC}^n} \widehat{\pi}_{NC} \times_{M_{NC}^n} \widehat{\pi}_{NC}$ with *k* copies of the dual bundle (with the respective fibre variables p_1, \ldots, p_k that imitate the variational covectors). Now we have that the *n*th degree horizontal cohomology classes $\mathcal{Q}(p_1, \ldots, p_k)$ are *k*-linear and totally skew-symmetric w.r.t. the new covector variables p_{α} . All these classes are known to be trivial by our initial assumption. The homotopy procedure then yields a *k*-linear w.r.t. p_1, \ldots, p_k , totally skew-symmetric horizontal (n-1)-form \mathcal{R} such that $\mathcal{Q} = d_h(\mathcal{R})$ for all sections $p_{\alpha}(\mathbf{x}, \mathbf{x}^{\pm 1})$ of $\widehat{\pi}_{NC}$. The Substitution Principle now works. Finally, replacing the *k*-linear skew terms over the Whitney sum by the variational *k*-vectors (with *k* copies of the parity-odd **b**) over the superbundle $\pi_{NC}^{(0|1)}$ is technical. \Box

3.3.2. Noncommutative differential forms

To approach the proof of Criterion 19, let us recall several classical structures that appear on the infinite jet spaces $J^{\infty}(\pi_{\text{NC}})$: in particular, in the context of the Vinogradov *C*-spectral sequence [53].

By definition, put

$$\vec{\partial}_{\varphi(\boldsymbol{x},\vec{\mathbf{x}}^{\pm 1},[\boldsymbol{a}])}^{(\boldsymbol{a})} = \sum_{i=1}^{m} \sum_{|\sigma| \ge 0} \left((\varphi^{i}) \left(\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}} \right)^{\sigma} \right) \left(\boldsymbol{x}, \vec{\mathbf{x}}^{\pm 1}, [\boldsymbol{a}] \right) \circ \frac{\overrightarrow{\partial}}{\partial a_{\sigma}^{i}}$$

It is readily seen that these *evolutionary derivations* commute with the total derivatives on $J^{\infty}(\pi_{\rm NC})$:

 $\left[\vec{\partial}_{\alpha}^{(\boldsymbol{a})}, \vec{d}/dx^{k}\right] = 0$ for all $k = 1, \dots, n$.

Consequently, for any operator A in total derivatives we have that

 $\vec{\partial}_{\omega}^{(a)}(A(\boldsymbol{p})) = \left(\vec{\partial}_{\omega}^{(a)}(A)\right)(\boldsymbol{p}) + A\left(\vec{\partial}_{\omega}^{(a)}(\boldsymbol{p})\right).$

Next, define the *linearisation* $\ell_p^{(a)}$ of an object **p** over $J^{\infty}(\pi_{\rm NC})$ by setting

$$(\varphi) \overleftarrow{\ell}_{p}^{(a)} = \overrightarrow{\partial}_{\varphi}^{(a)}(p)$$

whenever the right-hand side is well defined.

Thirdly, for each value of the index *i* running from 1 to *m* and for every multi-index σ let us introduce the symbol da^i_{σ} . Now define the *Cartan differential* $d_{\mathcal{C}} : a^i_{\sigma} \mapsto da^i_{\sigma}, da^i_{\sigma} \mapsto 0$, also setting its action equal to zero on **x** and $\mathbf{x}^{\pm 1}$ and postulating that $d_{\mathcal{C}}$ is a graded derivation. By construction, let the differential $d_{\mathcal{C}}$ be correlated with other structures on $J^{\infty}(\pi_{NC})$ in the standard way: e.g., set $\vec{D}_{s^k}(da^i_{\sigma}) = da^i_{\sigma \cup \{k\}}$.

Let us explain what it means that the symbols da^i_{σ} and da^j_{τ} "anticommute". The key idea is that the precedence–succedence relation of such symbols in a given cyclic word manifests that circle's orientation, which is provided by construction.

Consider a cyclic word that carries *one* symbol da_{σ}^{i} ; the word thus acquires a marked point. The derivation d_{C} acts on (the rest of) the word by starting at da_{σ}^{i} and processing the letters a_{τ}^{i} by going in the positive direction. We say that all the symbols da_{τ}^{i} , newly produced by d_{C} from such a_{τ}^{i} are *succedent* with respect to the mark da_{σ}^{i} ; in turn, the old symbol da_{σ}^{i} is *precedent* for each new object da_{τ}^{i} . To change this precedence–succedence relation $da_{\sigma}^{i} \prec da_{\tau}^{i}$ but still let the circle's orientation stay intact, the object da_{τ}^{i} is proclaimed the new marked point – so that da_{σ}^{i} now *succeeds* it with respect to the positive order of letters written along the oriented circle. By definition, such involution of the relative order \prec of the two symbols, da_{σ}^{i} and da_{τ}^{j} , produces the factor -1 in front of the cyclic word that carries both of them. Clearly, $d_{C}^{2} = 0$.

Lemma 22 (Helmholtz). The linearisation $\vec{\ell}_{\delta H/\delta a}^{(a)}$ of an element in the image of variational derivative $\delta/\delta a$ is self-adjoint:

$$\vec{\ell}_{\delta H/\delta a}^{(a)} = \vec{\ell}_{\delta H/\delta a}^{(a)\dagger}.$$
(42)

Note that this half of Helmholtz' criterion does not refer to the topology of the set-up.

Proof. Let *H* be a noncommutative variational 0-vector. Up to an integration by parts, we have that $d_{\mathcal{C}}H \cong (\mathbf{d} \boldsymbol{a} \circ \delta H / \delta \boldsymbol{a})$. By the above,

$$0 = d_{\mathcal{C}}^{2}(H) \cong \left(d\boldsymbol{a} \circ \overrightarrow{\ell}_{\delta H/\delta \boldsymbol{a}}^{(\boldsymbol{a})} \right) \cong \left((d\boldsymbol{a}) \overleftarrow{\ell}_{\delta H/\delta \boldsymbol{a}}^{(\boldsymbol{a})\dagger} \circ \underline{d\boldsymbol{a}} \right) \sim -\left(\underline{d\boldsymbol{a}} \circ \overrightarrow{\ell}_{\delta H/\delta \boldsymbol{a}}^{(\boldsymbol{a})\dagger} (d\boldsymbol{a}) \right),$$

whence (42). \Box

3.3.3. Proof of Criterion 19

First, let us recall the renowned cancellation mechanism on the left-hand side of Jacobi identity (40). By definition, put $\mathbf{p}_i = \delta H_i / \delta \mathbf{a}$ for the three Hamiltonians. Integrating by parts in the inner and outer Poisson brackets in (40) and using formula (37), we get

$$\vec{\partial}_{A(\mathbf{p}_{1})}^{(\mathbf{a})}(\mathbf{p}_{2} \circ A(\mathbf{p}_{3})) + \vec{\partial}_{A(\mathbf{p}_{2})}^{(\mathbf{a})}(\mathbf{p}_{3} \circ A(\mathbf{p}_{1})) + \vec{\partial}_{A(\mathbf{p}_{3})}^{(\mathbf{a})}(\mathbf{p}_{1} \circ A(\mathbf{p}_{2}))
= \left(\vec{\partial}_{A(\mathbf{p}_{1})}^{(\mathbf{a})}(\mathbf{p}_{2}) \circ A(\mathbf{p}_{3})\right) + \left(\mathbf{p}_{2} \circ \vec{\partial}_{A(\mathbf{p}_{1})}^{(\mathbf{a})}(A)(\mathbf{p}_{3})\right) - \left(A(\mathbf{p}_{2}) \circ \vec{\partial}_{A(\mathbf{p}_{1})}^{(\mathbf{a})}(\mathbf{p}_{3})\right)
+ \left(\vec{\partial}_{A(\mathbf{p}_{2})}^{(\mathbf{a})}(\mathbf{p}_{3}) \circ A(\mathbf{p}_{1})\right) + \left(\mathbf{p}_{3} \circ \vec{\partial}_{A(\mathbf{p}_{2})}^{(\mathbf{a})}(A)(\mathbf{p}_{1})\right) - \left(A(\mathbf{p}_{3}) \circ \vec{\partial}_{A(\mathbf{p}_{2})}^{(\mathbf{a})}(\mathbf{p}_{1})\right)
+ \left(\vec{\partial}_{A(\mathbf{p}_{3})}^{(\mathbf{a})}(\mathbf{p}_{1}) \circ A(\mathbf{p}_{2})\right) + \left(\mathbf{p}_{1} \circ \vec{\partial}_{A(\mathbf{p}_{3})}^{(\mathbf{a})}(A)(\mathbf{p}_{2})\right) - \left(A(\mathbf{p}_{1}) \circ \vec{\partial}_{A(\mathbf{p}_{3})}^{(\mathbf{a})}(\mathbf{p}_{2})\right).$$
(43)

Applying Lemma 22 to the variational covectors $\mathbf{p}_i = \delta H_i / \delta \mathbf{a}$ as follows,

$$\begin{pmatrix} \vec{\partial}_{A(\mathbf{p}_1)}^{(\mathbf{a})}(\mathbf{p}_2) \circ A(\mathbf{p}_3) \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} \vec{\ell}_{\mathbf{p}_2}^{(\mathbf{a})}(A(\mathbf{p}_1)) \circ A(\mathbf{p}_3) \end{pmatrix} = \begin{pmatrix} \vec{\ell}_{\mathbf{p}_2}^{(\mathbf{a})\dagger}(A(\mathbf{p}_1)) \circ A(\mathbf{p}_3) \end{pmatrix}$$
$$\cong \begin{pmatrix} A(\mathbf{p}_1) \circ \vec{\ell}_{\mathbf{p}_2}^{(\mathbf{a})}(A(\mathbf{p}_3)) \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} A(\mathbf{p}_1) \circ \vec{\partial}_{A(\mathbf{p}_3)}^{(\mathbf{a})}(\mathbf{p}_2) \end{pmatrix},$$

we conclude that it is only the second column which survives the cancellation in (43). The left-hand side of Jacobi identity thus equals

$$\left(\frac{\delta H_1}{\delta a} \circ \vec{\partial}^{(a)}_{A(\delta H_3/\delta a)}(A) \left(\frac{\delta H_2}{\delta a}\right)\right) + \text{cyclic permutations.}$$
(44)

On the other hand, consider the bi-vector $\boldsymbol{\mathcal{P}} = \frac{1}{2} (\boldsymbol{b} \circ A(\boldsymbol{b}))$ and construct

$$\llbracket \boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{P}} \rrbracket \cong \Big(\big(\boldsymbol{b} \circ \boldsymbol{A}(\boldsymbol{b}) \big) \Big(\frac{\overleftarrow{\partial}}{\partial \boldsymbol{a}_{\sigma}} \circ \Big(\frac{\overrightarrow{\mathbf{d}}}{\mathbf{d}\boldsymbol{x}} \Big)^{\sigma} \big(\boldsymbol{A}(\boldsymbol{b}) \big) \Big);$$

the right-hand side contains, for every multi-index σ , the derivation that pastes its coefficient for each d_{σ}^{i} occurring in the coefficients of operator A within $(\mathbf{b} \circ A(\mathbf{b}))$.

The only thing which the evaluation of $[\![\mathcal{P}, \mathcal{P}]\!]$ at H_1, H_2 , and H_3 does,

$$\llbracket \boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{P}} \rrbracket \left(\delta H_1 / \delta \boldsymbol{a}, \delta H_2 / \delta \boldsymbol{a}, \delta H_3 / \delta \boldsymbol{a} \right) = (-)^3 \llbracket \llbracket \llbracket \llbracket \llbracket \underline{\llbracket} \boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{P}} \rrbracket, H_1 \rrbracket, H_2 \rrbracket, H_3 \rrbracket,$$

is the spreading of variational derivatives $\delta H_i/\delta a$ over the three slots **b** in the tri-vector $[\mathcal{P}, \mathcal{P}]$. In view of evaluation's total skew-symmetry (see Lemma 16), it is enough to sum up over the cyclic (hence, even) permutations in the group S_3 , and then double. This yields the three terms

$$\left(\frac{\delta H_1}{\delta a} \circ \left((A) \overleftarrow{\partial}_{A(\delta H_3/\delta a)}^{(a)}\right) \left(\frac{\delta H_2}{\delta a}\right)\right) + \text{cyclic permutations.}$$
(45)

Uniting the two parts of the reasoning, we conclude that the left-hand side (44) of Jacobi identity (40) for the bracket $\{,\}_{\mathcal{P}}$ and the value of tri-vector $[\![\mathcal{P}, \mathcal{P}]\!]$ at the same Hamiltonians H_1, H_2 , and H_3 as in (40) are equal, hence simultaneously (non)trivial, as elements of the cohomology group $\bar{H}_0^n(\pi_{NC}^{(0|1)})$. Referring to Remark 3.9 and Lemma 21 and setting $\mathcal{Q} = [\mathcal{P}, \mathcal{P}]$ there, we conclude that over star-shaped domains $\subseteq M^n$,

the bracket $\{,\}_{\mathcal{P}}$ is Poisson if and only if the classical master-equation $[\mathcal{P}, \mathcal{P}] \cong 0$ holds for \mathcal{P} .

3.3.4. Complete integrability

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In the final section we address the cohomological structures of (non)commutative variational Poisson theory. We recall how the differential $\partial_{\mathcal{P}} = [\![\mathcal{P}, \cdot]\!]$ specified by a given Poisson bi-vector \mathcal{P} owes its property $\partial_{\mathcal{P}}^2 = 0$ to a weak variant of the Jacobi identity for the variational Schouten bracket $[\![\cdot,\cdot]\!]$. (We remember that the (non)commutative variational Poisson formalism is a science of steps and stops, so that calculations involving $[\cdot, \cdot]$ can be interrupted at every moment, to make legitimate the integrations by parts within every object. This makes the weak variant of Jacobi identity for $[\cdot, \cdot]$ different from (26).)

Proposition 23. Let $F, G, H \in \bar{H}^n_*(\pi_{NC}^{(0|1)})$ be (non)commutative variational multivectors; suppose that F and G are homogeneous. Then the weak variant of Jacobi identity,

$$\underline{\llbracket F, \llbracket G, H \rrbracket \rrbracket} - (-)^{(|F|-1) \cdot (|G|-1)} \underline{\llbracket G, \llbracket F, H \rrbracket \rrbracket} \cong \underline{\llbracket \llbracket F, G \rrbracket, H \rrbracket},$$
(46)

holds modulo integrations by parts in every Schouten bracket.

• Equivalently, for every homogeneous (non)commutative variational multivector Z define the shifted (by -1) graded evolutionary vector field \mathbf{Q}^{Z} on the jet space $J^{\infty}(\boldsymbol{\pi}_{NC}^{(0|1)})$: by definition, let $[\![Z, \mathcal{H}]\!] \cong \mathbf{Q}^{Z}(\mathcal{H})$ for all $\mathcal{H} \in H_{0}^{n}(\boldsymbol{\pi}_{NC}^{(0|1)})$. In these terms, Jacobi identity (46) is

$$\begin{bmatrix} \vec{\boldsymbol{Q}}^F, \vec{\boldsymbol{Q}}^G \end{bmatrix} \cong \vec{\boldsymbol{Q}}^{\llbracket F, G \rrbracket},$$

that is, the graded commutator of adjoint actions $[[F, \cdot]]$ and $[[G, \cdot]]$ is equivalent, modulo integrations by parts, to the adjoint action of the object [[F, G]].

Corollary 24. By satisfying the master-equation $[\![\mathcal{P}, \mathcal{P}]\!] \cong 0$, each (non)commutative variational Poisson bi-vector \mathcal{P} determines the Poisson differential $\partial_{\mathcal{P}} = [\![\mathcal{P}, \cdot]\!]$.

Indeed, Jacobi identity (46) then reads $\partial_{\mathcal{P}}^2(\cdot) = \llbracket \mathcal{P}, \llbracket \mathcal{P}, \cdot \rrbracket \rrbracket \cong \frac{1}{2} \llbracket \llbracket \mathcal{P}, \mathcal{P} \rrbracket, \cdot \rrbracket = 0.$

Sketch of the proof (of Proposition 23). The graded derivation $\overleftarrow{\mathbf{Q}}^H \cong \llbracket \cdot, H \rrbracket$ which acts clockwise (i.e. against the orientation) along the cyclic word [[F, G]] is permutable with the graded derivations $\vec{\mathbf{Q}}^F$ and $\vec{\mathbf{Q}}^G$ which act counterclockwise on G and, respectively, on $-(-)^{(|F|-1)\cdot(|G|-1)}F$ in the object [[F, G]]. Depending on the origin – from either G or F – of an argument of $\mathbf{\hat{Q}}^{H}$ on the right-hand side of (46), the respective term in that Leibniz rule expansion is realised by using either

$$\left(\overrightarrow{\mathbf{Q}}^{F}(G)\right)\overleftarrow{\mathbf{Q}}^{H}=\overrightarrow{\mathbf{Q}}^{F}\left(\left(G\right)\overleftarrow{\mathbf{Q}}^{H}\right)$$

or

$$-(-)^{(|F|-1)\cdot(|G|-1)}\left(\overrightarrow{\boldsymbol{\varrho}}^{G}(F)\right)\overleftarrow{\boldsymbol{\varrho}}^{H}=-(-)^{(|F|-1)\cdot(|G|-1)}\overrightarrow{\boldsymbol{\varrho}}^{G}\left((F)\overleftarrow{\boldsymbol{\varrho}}^{H}\right),$$

so that all terms (and only those terms) on the left-hand side of (46) are recovered. \Box

For every \mathcal{P} , the Poisson differential $\partial_{\mathcal{P}}$ gives rise to the Poisson(-Lichnerowicz) cohomology groups $H_{\mathcal{P}}^k, k \ge 0$.

- The group H⁰_P is composed of the Casimirs H₀ ∈ H
 ⁿ(π^(0|1)_{NC}) such that [[P, H₀]] ≅ 0.
 The first Poisson cohomology group H¹_P consists of cocycle variational one-vectors X without Hamiltonians: [[P, X]] ≅
- 0 but X ≠ [[P, H]] for any H ∈ Hⁿ(π^(0|1)_{NC}).
 The second group H²_P contains nontrivial deformations of the Poisson bi-vector P, i.e. those shifts P ↦ P+ε·Q+ō(ε) infinitesimally preserving the classical master-equation [[P, P]] = 0 which are not generated by the bi-vector P itself: $\mathcal{Q} \neq [\![\mathcal{P}, X]\!]$ for any one-vector X.
- The third group $H^3_{\mathcal{P}}$ contains obstructions to the integrability of infinitesimal shifts $\mathcal{P} \mapsto \mathcal{P} + \varepsilon \cdot \mathcal{Q} + \overline{o}(\varepsilon)$ to genuine deformations $\mathcal{P} \mapsto \mathcal{P}(\varepsilon)$ at $\varepsilon > 0$.

These interpretations are standard [54]; we also refer to [46] for an illustration of classical Poisson deformation theory in the commutative set-up (in this context, see Open problem 3 at the end of this paper).

Likewise, the vanishing of some extra cohomological obstructions implies the existence of infinitely many Hamiltonians in involution and the presence of hierarchies of commuting flows. This is the renowned (Lenard-)Magri scheme [15].

Theorem 25 (The Magri Scheme). Let \mathcal{P}_1 and \mathcal{P}_2 be two (non)commutative variational Poisson bi-vectors on the jet space $\int_{\mathcal{P}_{n}}^{\infty}(\pi_{NC}^{(0|1)})$. Suppose they are compatible: $[[\mathcal{P}_{1}, \mathcal{P}_{2}]] \cong 0$, and assume that the first Poisson–Lichnerowicz cohomology group $H^{1}_{\mathcal{P}_{n}}$. with respect to the differential $\partial_{\mathcal{P}_1} = [\![\mathcal{P}_1, \cdot]\!]$ vanishes. Let $\mathcal{H}_0 \in \mathrm{H}^0_{\mathcal{P}_1} \subseteq \bar{H}^n(\pi_{\mathrm{NC}}^{(0|1)})$ be a Casimir of \mathcal{P}_1 . Then for any integer k > 0 there is a Hamiltonian functional $\mathcal{H}_k \in \bar{H}^n(\pi_{\mathrm{NC}}^{(0|1)})$ such that

$$\llbracket \boldsymbol{\mathcal{P}}_2, \mathcal{H}_{k-1} \rrbracket = \llbracket \boldsymbol{\mathcal{P}}_1, \mathcal{H}_k \rrbracket.$$

$$\tag{47}$$

Moreover, let $\mathcal{H}_{0}^{(\alpha)}$ and $\mathcal{H}_{0}^{(\beta)}$ be any two distinct Casimirs for the bi-vector \mathcal{P}_{1} and construct the two infinite sequences of the functionals $\mathcal{H}_{i}^{(\alpha)}$ and $\mathcal{H}_{i}^{(\beta)}$ by using (47), here $i, j \ge 0$. Let $\varphi_{i}^{(\alpha)} := \llbracket \mathcal{P}_{1}, \mathcal{H}_{i}^{(\alpha)} \rrbracket$ and similarly, $\varphi_{i}^{(\beta)} := \llbracket \mathcal{P}_{1}, \mathcal{H}_{i}^{(\beta)} \rrbracket$. Then for all i, jand α . β .

- the Hamiltonians H_i^(α) and H_j^(β) Poisson-commute with respect to either of the Poisson brackets, {, }_{P1} and {, }_{P2};
 the one-vectors φ_i^(α) and φ_j^(β) commute;
 the density of H_i^(α) is conserved, [[H_i^(α), φ_j^(β)]] ≅ 0, by virtue of each one-vector φ_j^(β).

Existence Proof. Main homological equality (47) is established by induction on k. Consider the bi-vectors \mathcal{P}_1 and \mathcal{P}_2 and a Hamiltonian \mathcal{H}_0 . The steps-and-stops variant of Jacobi identity, see (46), acquires the form

$$\llbracket \boldsymbol{\mathcal{P}}_1, \underline{\llbracket \boldsymbol{\mathcal{P}}_2, \mathcal{H}_0 \rrbracket} \rrbracket + \llbracket \boldsymbol{\mathcal{P}}_2, \underline{\llbracket \boldsymbol{\mathcal{P}}_1, \mathcal{H}_0 \rrbracket} \rrbracket \cong \llbracket \underline{\llbracket \boldsymbol{\mathcal{P}}_1, \boldsymbol{\mathcal{P}}_2 \rrbracket}, \mathcal{H}_0 \rrbracket.$$
(48)

Hence by starting with a Casimir for a given Poisson bi-vector \mathcal{P}_1 , we obtain that

$$0 = \llbracket \boldsymbol{\mathcal{P}}_2, 0 \rrbracket \cong \llbracket \boldsymbol{\mathcal{P}}_2, \llbracket \boldsymbol{\mathcal{P}}_1, \mathcal{H}_0 \rrbracket \rrbracket \cong - \llbracket \boldsymbol{\mathcal{P}}_1, \llbracket \boldsymbol{\mathcal{P}}_2, \mathcal{H}_0 \rrbracket \rrbracket \mod \llbracket \boldsymbol{\mathcal{P}}_1, \boldsymbol{\mathcal{P}}_2 \rrbracket \cong 0,$$

using Jacobi identity (48). The first Poisson cohomology $H^1_{\mathcal{P}_1} = 0$ is trivial by an assumption of the theorem, hence the closed element $[\![\mathcal{P}_2, \mathcal{H}_0]\!]$ in the kernel of $[\![\mathcal{P}_1, \cdot]\!]$ is exact: $[\![\mathcal{P}_2, \mathcal{H}_0]\!] \cong [\![\mathcal{P}_1, \mathcal{H}_1]\!]$ for some \mathcal{H}_1 . For $k \ge 1$ we have that

$$\llbracket \boldsymbol{\mathcal{P}}_1, \llbracket \boldsymbol{\mathcal{P}}_2, \mathcal{H}_k \rrbracket \rrbracket \cong -\llbracket \boldsymbol{\mathcal{P}}_2, \llbracket \boldsymbol{\mathcal{P}}_1, \mathcal{H}_k \rrbracket \rrbracket \cong -\llbracket \boldsymbol{\mathcal{P}}_2, \llbracket \boldsymbol{\mathcal{P}}_2, \mathcal{H}_{k-1} \rrbracket \rrbracket \cong 0$$

using (48) and by $[\![\mathcal{P}_2, \mathcal{P}_2]\!] \cong 0$. Consequently, by $H^1_{\mathcal{P}_1} = 0$ we have that $[\![\mathcal{P}_2, \mathcal{H}_k]\!] \cong [\![\mathcal{P}_1, \mathcal{H}_{k+1}]\!]$, and we thus proceed indefinitely. \Box

Definition 10. Bi-Hamiltonian evolutionary differential equations which satisfy the hypotheses of Theorem 25 and possess as many non-extendable sequences of local Hamiltonians in involution as the number of the unknowns are called the (infinite-dimensional) *completely integrable systems*.

The (non)commutative Korteweg–de Vries equation [15,17] is the best-known example of an infinite-dimensional completely integrable system.

Remark 3.10. The inductive step, that is, the existence of the next, (k + 1)th Hamiltonian functional in involution with all the preceding ones, is possible if and only if the seed \mathcal{H}_0 is a Casimir,⁴⁸ and therefore the Hamiltonian operators A_i in the bi-vectors $\mathcal{P}_i = \frac{1}{2} \langle \mathbf{b}, A_i(\mathbf{b}) \rangle$ are restricted onto the linear subspace which is spanned in the space of variational covectors by the Euler derivatives of the descendants of \mathcal{H}_0 , i.e. of the Hamiltonians of the hierarchy. We note that the image under A_2 of a generic element from the domain of operators A_1 and A_2 cannot be resolved w.r.t. A_1 by (47).

a generic element from the domain of operators A_1 and A_2 cannot be resolved w.r.t. A_1 by (47). For example, the image im A_2^{KdV} of the second Hamiltonian operator for the purely commutative Korteweg–de Vries equation is not entirely contained in the image of the first structure for the generic values of the arguments. But on the linear subspace of descendants \mathcal{H}_k of the Casimir $\int a \, dx \, \text{for } A_1^{KdV}$, the inclusion im $A_2^{KdV} \subseteq \text{im} A_1^{KdV}$ is attained.

Open problem 3 (*The Kontsevich Tetrahedral Flows*). Does the construction from [34,45] and [46] of the quartic-nonlinear flow $\dot{\mathcal{P}} = \mathcal{Q}_{1:\frac{6}{2}}([\mathcal{P}])$ on spaces of Poisson bi-vectors \mathcal{P} over affine *m*-dimensional manifolds N^m extend – in the frames of

cyclic word calculus – to the finite-dimensional⁴⁹ formal noncommutative Poisson geometry?

Is such cyclic-word generalisation also possible for the flow of nonlinearity degree six which is built in [55] from the pentagon-wheel cocycle in the graph complex?

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⁴⁸ The Magri scheme starts from any two Hamiltonians \mathcal{H}_{k-1} , $\mathcal{H}_k \in \overline{H}^n(\pi_{NC})$ that satisfy (47), but we operate with the maximal subspaces of the space of functionals such that the sequence $\{\mathcal{H}_k\}$ cannot be extended with any local quantities at k < 0.

⁴⁹ The construction of tetrahedral flow is known [46] to have no universal extension to the purely commutative variational set-up: the flows $\dot{\mathcal{P}} = \mathcal{Q}_{1:\frac{6}{2}}([\mathcal{P}])$ do not always preserve – even infinitesimally – the property of Cauchy data \mathcal{P} to be variational Poisson bi-vectors. Consequently, a search for noncommutative *and* variational generalisation for the existing flow $\dot{\mathcal{P}} = \mathcal{Q}_{1:\frac{6}{2}}([\mathcal{P}])$ is not in order.

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