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Persistent Flows in Deterministic Chains

Weiguo Xia , Guodong Shi , Ziyang Meng , Ming Cao , and Karl Henrik Johansson , *Fellow, IEEE*

Abstract—This paper studies, the role of persistent flows in the convergence of infinite backward products of stochastic matrices of deterministic chains over networks with nonreciprocal interactions between agents. An arc describing the interaction strength between two agents is said to be persistent if its weight function has an infinite l_1 norm; convergence of the infinite backward products to a rank-one matrix of a deterministic chain of stochastic matrices is equivalent to achieving consensus at the node states. We discuss two balance conditions on the interactions between agents, which generalize the arc-balance and cut-balance conditions in the literature, respectively. The proposed conditions require that such a balance should be satisfied over each time window of a fixed length instead of at each time instant. We prove that in both cases global consensus is reached if and only if the persistent graph, which consists of all the persistent arcs, contains a directed spanning tree. The convergence rates of the system to consensus are also provided in terms of the interactions between agents having taken place. The results are obtained under a weak condition without assuming the existence of a positive lower bound of all the nonzero weights of arcs and are compared with the existing results. Illustrative examples are provided to validate the results and show the critical importance of the nontrivial lower boundedness of the self-confidence of the agents.

Index Terms—Consensus, multiagent systems, persistent graphs, products of stochastic matrices.

I. INTRODUCTION

THE study of backward products of a chain of stochastic matrices can be associated with consensus seeking in multiagent systems, where the underlying update matrices are taken as the stochastic matrices. Note that convergence of the infinite backward products to a rank-one matrix of a deterministic chain of stochastic matrices is equivalent to reaching consensus at the node states. The study of consensus-seeking systems is motivated by opinion forming in social networks [3], [7], flocking behaviors in animal groups [16], [25], data fusion in engineered systems [5], and so on. Ample results on the convergence and convergence rate of the consensus system have been reported. Typical conditions involve the connectivity of the network topology and the interaction strengths between agents for both continuous time [4], [9], [12], [17], [18], [20], [27] and discrete-time systems [6], [11], [14], [15], [18], [20], [26].

In the literature, several types of balance conditions on the interaction weights are considered, among which the cut-balance condition [9], [12] and the arc-balance condition [20] are typical ones. The cut-balance condition requires that at each time instant, if a group of agents in the network influences the remaining ones then it is also influenced by the remaining ones bounded by a constant proportional amount. This type of conditions characterizes a reciprocal interaction relationship among the agents, which covers the symmetric interaction and type-symmetric interaction as special cases [9]. It was proved in [9] that under the cut-balance condition, the state of the consensus system converges; in addition, if two agents belong to the same strongly connected component in the unbounded interaction graph (called a persistent graph in the present paper), then they converge to the same limit. The convergence rate was provided in [12] for the system where the ratio of the reciprocal interaction weights is even allowed to take a slow diverging value instead of a constant value. In [4], a notion of balance condition called balanced asymmetry was proposed, which is stronger than the cut-balance condition, while the balanced asymmetric system includes the cut-balanced system, in which every agent has a positive self-weight, as a special case. The convergence of the system with the balanced asymmetry property is proved under the absolute infinite flow property [23], [24] for deterministic iterations [4].

The arc-balance condition requires that at each time instant the weight of each arc is bounded by a proportional amount of any other arc in the persistent graph. Under this condition, it was proved that the multiagent system reaches consensus under

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the condition that the persistent graph contains a directed spanning tree [20]. This persistent graph property behaves as forms of network Borel–Cantelli lemmas for consensus algorithms over random graphs [19]. If the persistent graph is strongly connected, the arc balance assumption is a special case of the cut-balance condition imposed on the persistent graph, while in the general case, these two conditions do not cover each other.

Note that the results for the discrete-time consensus system under the cut-balance condition in [9] and the arc-balance condition in [20] should be satisfied at each time instant. In applications, the interactions among agents may not be reciprocal instantaneously. For example, in a robotic network, the robots may take measurements and interact with other robots intermittently and asynchronously, inducing cases when a robot is influenced by another robot but may not influence this robot at the same time while the influence may happen at a later time. We will relax these assumptions by allowing that the total amount of the interaction weights over each time window of a fixed length satisfies such a condition. Therefore, the instantaneous arc-balance and cut-balanced conditions are relaxed to noninstantaneous balance conditions and specifically the cut-balance condition is relaxed to the requirement of noninstantaneous reciprocal interactions. We prove that in both cases global consensus is reached if and only if the persistent graph contains a directed spanning tree. In addition, the convergence rate of the system to consensus in both cases are also established in terms of the interactions between agents that have taken place. The technique to prove the result in the cut-balance case is inspired by that used to deal with consensus systems with balanced asymmetry property in [4] and the cut-balance property with slow divergence of reciprocal weights in [12]. It is worth noting that it is only assumed that the self-weight of each agent is bounded by a positive constant from below while the weights between agents can be arbitrary time-varying functions, which relaxes the existing assumptions [6], [9], [15], [18]. The critical assumption on the boundedness of the self-weight of each agent is also discussed and an illustrative example is provided. Some preliminary results have appeared at the IEEE Conference on Decision and Control in 2017 [29], but this paper provides a more comprehensive treatment of the work.

The rest of the paper is organized as follows. In Section II, the global consensus problem is formulated and two main results on the convergence and convergence rates making use of two different balance conditions are given. Sections III and IV present the proofs of the two results, respectively. Section V validates the results and gives an example to illustrate some critical conditions. The conclusion is drawn in Section VI.

II. PROBLEM DEFINITION AND MAIN RESULTS

A. Problem Definition

Consider a network with the node set $\mathcal{V} = \{1, \dots, N\}$, $N \geq 2$. Each node i holds a state $x_i(t) \in \mathbb{R}$. The initial time is $t_0 \geq 0$. The evolution of $x_i(t)$ is given by

$$x_i(t+1) = \sum_{j=1}^N a_{ij}(t)x_j(t) \quad (1)$$

where $a_{ij}(t) \geq 0$ stands for the influence of node j on node i at time t and $a_{ii}(t)$ represents the self-confidence of each node. If $a_{ij}(t) > 0$, $j \neq i$, at time t , then it is considered as the weight of arc (j, i) of the graph $\mathbb{G}(t) = (\mathcal{V}, \mathcal{E}(t))$, where $\mathcal{E}(t) \subseteq \mathcal{V} \times \mathcal{V}$. (j, i) is an incoming arc of node i and is an outgoing arc of node j .

For the time-varying arc weights $a_{ij}(t)$, we impose the following condition as our standing assumption throughout the paper.

Assumption 1: For all $i, j \in \mathcal{V}$ and $t \geq 0$, (i) $a_{ij}(t) \geq 0$; (ii) $\sum_{j=1}^N a_{ij}(t) = 1$; (iii) there exists a constant $0 < \eta < 1$ such that $a_{ii}(t) \geq \eta$.

Denote $x(t) = [x_1(t), \dots, x_N(t)]^T$ and $A(t) = [a_{ij}(t)]_{N \times N}$. We know that $A(t)$ is a stochastic matrix from Assumption 1. System (1) can be rewritten into the following compact form with a deterministic chain of stochastic matrices $\{A(t)\}$:

$$x(t+1) = A(t)x(t). \quad (2)$$

For $t > s \geq 0$, let $A(t, s) \triangleq A(t)A(t-1) \cdots A(s)$ denote the backward product. Note that convergence of the product $A(t, s)$ to a rank-one matrix of the chain $\{A(t)\}$ as t goes to infinity for all $s \geq 0$ is equivalent to achieving consensus at the node states along the system (1). Hence, in the sequel of the paper, we will focus on the consensus problem of system (1).

Remark 1: In Assumption 1, we only assume that the diagonal elements of $A(t)$ are lower bounded by η , but not requiring all nonzero elements of $A(t)$ to be lower bounded by η , a condition often imposed in the literature [2], [6], [18]. It will be seen in later discussions that it will bring many differences and require further efforts for the analysis of the system. The condition that the diagonal elements of $A(t)$ are lower bounded by η is critical for the consensus reaching of system (1) and its importance will be further illustrated by an example in Section V. Such a relaxation on the assumption on the elements of $A(t)$ may be applicable in the study of opinion dynamics models in social networks, where $a_{ii}(t)$ represents the self-weight of agent i and $a_{ij}(t)$ represents the weight that agent i assigns to agent j . The situation corresponds to the case when agent i is always confident with self-weight at least η while assigns weights with no lower bound to its neighbors.

We continue to introduce the following definition [20].

Definition 1: An arc (j, i) is called a persistent arc if

$$\sum_{t=0}^{\infty} a_{ij}(t) = \infty. \quad (3)$$

The set of all persistent arcs is denoted as \mathcal{E}_p and we call the digraph $\mathbb{G}_p = (\mathcal{V}, \mathcal{E}_p)$ the persistent graph.

The weight function of each arc in the persistent graph has an infinite l_1 norm as can be seen from (3). The notions of persistent arcs and persistent graph have also been considered in [4], [9], [12], and [13] for studying the consensus problem of discrete-time and continuous-time systems. In [4], the persistent graph \mathbb{G}_p is called an unbounded interactions graph. The following section shows that the connectivity of the persistent graph is fundamental for deciding consensus, while those edges whose time-varying interaction weights summing up to a finite

number is not critical. To be more precise, the consensus problem considered in this paper is defined as follows.

Definition 2: Global consensus is achieved for the considered network if for any initial time $t_0 \geq 0$, and for any initial value $x(t_0)$, there exists $x_* \in \mathbb{R}$ such that $\lim_{t \rightarrow \infty} x_i(t) = x_*$ for all $i \in \mathcal{V}$.

In addition, we not only derive conditions under which global consensus can be reached, but also characterize the convergence speed in terms of how much interaction among the nodes has happened in the network.

B. Balance Conditions

A central aim of this paper is to derive conditions under which the convergence to consensus of system (1) can be guaranteed by imposing merely the connectivity of the persistent graph. In this case, some balance conditions among the arc weights become essential [9], [20]. We introduce the following two balance conditions.

Assumption 2 (Balance Condition I): There exist an integer $L \geq 1$ and a constant $K \geq 1$ such that for any two distinct arcs $(j, i), (l, k) \in \mathcal{E}_p$, we have

$$\sum_{t=s}^{s+L-1} a_{ki}(t) \leq K \sum_{t=s}^{s+L-1} a_{ij}(t) \quad (4)$$

for all $s \geq 0$.

Assumption 3 (Balance Condition II): There exist an integer $L \geq 1$ and a constant $K \geq 1$ such that for any nonempty proper subset S of \mathcal{V} , we have

$$\sum_{t=s}^{s+L-1} \sum_{i \notin S, j \in S} a_{ij}(t) \leq K \sum_{t=s}^{s+L-1} \sum_{i \in S, j \notin S} a_{ij}(t) \quad (5)$$

for all $s \geq 0$.

Remark 2: The Balance Condition I is a generalized version of the arc-balance condition introduced in [20] where $L = 1$. The Balance Condition II is a generalized version of the cut-balance condition introduced in [9] where $L = 1$. These conditions require either the balance between the weights of different persistent arcs or the balance between the amounts of interactions between one group and its remaining part over each time window of a fixed length. When Assumptions 2 or 3 hold for $L = 1$, (4) or (5) imposes a restriction on such a balance condition that should be satisfied instantaneously. A relatively large L gives more flexibility on the interaction weights and allows possible noninstantaneous reciprocal interactions between agents. To determine if Assumption 2 holds and identify the length of the time window L over which the persistence graph must maintain the balance condition, some global property of the time-varying weight function $a_{ij}(t)$ is necessary.

C. Main Results

In this section, we first give some basic observations of the state evolution of system (1) and then present the main results.

Let

$$H(t) \doteq \max_{i \in \mathcal{V}} \{x_i(t)\}, \quad h(t) \doteq \min_{i \in \mathcal{V}} \{x_i(t)\}$$

be the maximum and minimum state value at time t , respectively. Denote $\Psi(t) \doteq H(t) - h(t)$, which serves as a metric of consensus. Note that $\Psi(t)$ measures the maximum difference among the states of the nodes.

Since Assumption 1 holds, the following lemma is easy to prove.

Lemma 1: Assume that Assumption 1 holds. $H(t)$ is nonincreasing, $h(t)$ is nondecreasing, and $\Psi(t)$ is nonincreasing.

Apparently reaching a consensus of system (1) implies that $\lim_{t \rightarrow \infty} \Psi(t) = 0$. In fact, the contrary is also true. In view of the above-mentioned lemma, for any initial time $t_0 \geq 0$ and any initial value $x^0 = x(t_0)$, there exist $H_*, h_* \in \mathbb{R}$ such that

$$\lim_{t \rightarrow \infty} H(t) = H_*; \quad \lim_{t \rightarrow \infty} h(t) = h_*.$$

If $\lim_{t \rightarrow \infty} \Psi(t) = 0$, we obtain $H_* = h_*$, which implies that $\lim_{t \rightarrow \infty} x_i(t) = H_*$ for all $i \in \mathcal{V}$.

Let $\lceil a \rceil$ represent the smallest integer that is no less than a , and $\lfloor a \rfloor$ represent the largest integer that is no greater than a . We present the following two main results, for the two types of balance conditions, respectively.

Theorem 1: Assume that Assumptions 1 and 2 hold.

- i) Global consensus is achieved for system (1) if and only if the persistent graph \mathbb{G}_p has a directed spanning tree.
- ii) If the persistent graph \mathbb{G}_p has a directed spanning tree, then for any initial time $t_0 \geq 0$, $\epsilon > 0$, and $\nu > 0$, we have

$$\Psi(t) \leq \epsilon \Psi(t_0), \quad \text{for all } t \geq T_\nu + t^* \quad (6)$$

where $T_\nu \geq t_0$ such that $\sum_{i=T_\nu}^{\infty} a_{ij}(t) \leq \nu$ for all $(j, i) \in \mathcal{E} \setminus \mathcal{E}_p$

$$t^* \doteq \inf \left\{ t \geq 1 : \sum_{k=0}^{t-1} \sum_{j=1, j \neq i, (j,i) \in \mathcal{E}_p}^N a_{ij}(T_\nu + k) \geq \omega_1 d_0 (\delta + 1) \right\} \quad (7)$$

$\delta > L(N-1)(1-\eta)$ is a constant, d_0 is the diameter of \mathbb{G}_p , $\omega_1 \doteq \lceil \frac{\log \epsilon^{-1}}{\log(1 - \frac{1}{2} \mathbf{Q}^{2d_0} \mathbf{R}^{d_0})^{-1}} \rceil$ with $\mathbf{R} \doteq K^{-1} \lfloor \frac{\delta}{N-1} - L(1-\eta) \rfloor$, $\mathbf{Q} \doteq e^{-\frac{(N-1)(K(1-\eta+\delta)+L(1-\eta)+\nu) \ln \eta}{\eta-1}}$.

Theorem 2: Assume that Assumptions 1 and 3 hold.

- i) Global consensus is achieved for system (1) if and only if the persistent graph \mathbb{G}_p has a directed spanning tree.
- ii) If the persistent graph \mathbb{G}_p has a directed spanning tree, then for any initial time $t_0 \geq 0$ and $\epsilon > 0$, we have

$$\Psi(t) \leq \epsilon \Psi(t_0), \quad \text{for all } t \geq k^* L + t_0 \quad (8)$$

where

$$k^* \doteq \inf \left\{ t \geq 1 : \min_{|S(0)|=\dots=|S(t-1)|} \sum_{k=0}^{t-1} \sum_{\substack{i \notin S(k+1) \\ j \in S(k)}}^{L-1} a_{ij}(kL + u + t_0) \geq \omega_2 \left\lfloor \frac{N}{2} \right\rfloor (\eta^L + 1) \right\} \quad (9)$$

with $W = \frac{\eta^L}{(N-1)L}$, $\omega_2 = \left\lceil \frac{\log \epsilon^{-1}}{\log(1 - K_*^{-1} \lfloor \frac{N}{2} \rfloor / (8N^2)^{\lfloor \frac{N}{2} \rfloor - 1})} \right\rceil$, $K_* = \max\{\frac{(N-1)K}{\eta^{L-1}}, \frac{N-1}{\eta^L}\}$, and $S(k)$, $k \geq 0$, being nonempty proper subsets of \mathcal{V} with the same cardinality.

Remark 3: For both cases, conclusions (ii) establish the convergence rates of system (1) to consensus in terms of the interactions between agents having taken place. t^* in Theorem 1 dictates the time taken for the weights of any persistent arc accumulating to some constant $\omega_1 d_0(\delta + 1)$. Similarly, k^*L in Theorem 2 dictates the time taken for the weights of the arcs between agents in some specific sequence of subsets exceeding some constant $\omega_2 \lfloor \frac{N}{2} \rfloor (\eta^L + 1)$. In each case, if the time needed to exceed multiples of the respective constant grows linearly, then global consensus is reached exponentially fast. The convergence to consensus may not be exponential if the time needed to exceed multiples of these constants does not grow linearly as illustrated by an example in Section V.

In the following sections, we prove these two theorems.

III. PROOF OF THEOREM 1

In this section, we first establish two key technical lemmas, and then present the proofs of Theorem 1. The main idea to prove the sufficiency part of Theorem 1 (i) is as follows. Starting from the time T_ν given in Theorem 1, an upper bound on the state of some root¹ i_0 of the persistent graph \mathbb{G}_p is first established for the time interval $[T_\nu, T_\nu + t_1]$, where t_1 is a positive integer defined later (Step 1 in Section III-B). Then, an upper bound on the states of the nodes, which have incoming arcs from i_0 in \mathbb{G}_p for the time interval $[T_\nu, T_\nu + t_1]$ is provided (Step 2 in Section III-B). Such an estimation (for a longer time interval) can be carried on for the states of nodes that can be reached by i_0 in two hops and so on, and finally for the states of all the nodes (Steps 3 and 4 in Section III-B). Therefore, the contraction of $\Psi(t)$ can be characterized and the convergence to consensus is proved and the conclusion in Theorem 1 (ii) readily follows.

A. Key Lemmas

First, we present the following lemma establishing a lower bound for the product of a finite sequence of real numbers.

Lemma 2: Let $b_k, k = 1, \dots, m$ be a sequence of real numbers of length m satisfying $b_k \in [\eta, 1], m \geq 0$, where $0 < \eta < 1$

is a given constant. Then, we have $\prod_{k=1}^m b_k \geq e^{-\frac{\zeta \ln \eta}{\eta-1}}$ if $\sum_{k=1}^m (1 - b_k) \leq \zeta$.

Proof: Noticing that $\ln y$ is a concave function on $(0, \infty)$, we obtain

$$\ln y = \ln \left[\frac{y-1}{\eta-1} \cdot \eta + \left(1 - \frac{y-1}{\eta-1}\right) \cdot 1 \right] \geq \frac{y-1}{\eta-1} \cdot \ln \eta$$

for all $y \in [\eta, 1]$. Therefore, we conclude that

$$\begin{aligned} \prod_{k=1}^m b_k &= e^{\sum_{k=1}^m \ln b_k} \\ &\geq e^{\sum_{k=1}^m \frac{b_k-1}{\eta-1} \cdot \ln \eta} = e^{-\frac{\ln \eta}{\eta-1} \sum_{k=1}^m (1-b_k)} \geq e^{-\frac{\zeta \ln \eta}{\eta-1}}. \end{aligned}$$

□

As shown in the following discussions, the fact that the lower bound $e^{-\frac{\zeta \ln \eta}{\eta-1}}$ is independent on m plays a key role in analyzing the node state evolution.

Next, we establish another lemma on the node state evolution.

Lemma 3: For system (1), suppose Assumption 1 holds and $x_i(s) \leq \mu h(s) + (1 - \mu)H(s)$ for some $s \geq t_0$ and $0 \leq \mu < 1$. Then, we have

$$\begin{aligned} x_i(s + \tau) &\leq \mu \prod_{k=0}^{T-1} a_{ii}(s+k) \cdot h(s) \\ &\quad + \left(1 - \mu \prod_{k=0}^{T-1} a_{ii}(s+k)\right) \cdot H(s) \end{aligned} \quad (10)$$

for all $\tau \leq T$ and $T = 0, 1, \dots$

Proof: First we have

$$\begin{aligned} x_i(s+1) &= \sum_{j=1}^N a_{ij}(s)x_j(s) \\ &= a_{ii}(s)x_i(s) + \sum_{j=1, j \neq i}^N a_{ij}(s)x_j(s) \\ &\leq a_{ii}(s) \left[\mu h(s) + (1 - \mu)H(s) \right] + (1 - a_{ii}(s))H(s) \\ &= \mu a_{ii}(s)h(s) + (1 - \mu a_{ii}(s))H(s). \end{aligned}$$

Recall that $H(t)$ is nonincreasing for all t . Thus, iteratively, we obtain

$$\begin{aligned} x_i(s+2) &= \sum_{j=1}^N a_{ij}(s+1)x_j(s+1) \\ &\leq a_{ii}(s+1)x_i(s+1) + (1 - a_{ii}(s+1))H(s+1) \\ &\leq a_{ii}(s+1) \left[\mu a_{ii}(s)h(s) + (1 - \mu a_{ii}(s))H(s) \right] \\ &\quad + (1 - a_{ii}(s+1))H(s) \\ &= \mu \prod_{k=0}^1 a_{ii}(s+k)h(s) + \left(1 - \mu \prod_{k=0}^1 a_{ii}(s+k)\right)H(s) \end{aligned}$$

¹A node is a root of a directed graph if there is a directed path from this node to every other node.

$a_{ii}(t) \in [0, 1]$ for $t \geq 0$, implies that $\mu a_{ii}(s) \geq \mu a_{ii}(s) a_{ii}(s + 1)$. Since $h(s) \leq H(s)$, we have

$$\begin{aligned} x_i(s+1) &\leq \mu a_{ii}(s) h(s) + (1 - \mu a_{ii}(s)) H(s) \\ &\leq \mu \prod_{k=0}^1 a_{ii}(s+k) h(s) + \left(1 - \mu \prod_{k=0}^1 a_{ii}(s+k)\right) H(s). \end{aligned}$$

Proceeding the analysis it is straightforward to see that the desired conclusion holds. \blacksquare

B. Proof of Theorem 1 (i)

Sufficiency: We introduce

$$\mathbf{A}_i(t) = \sum_{j=1, j \neq i, (j,i) \in \mathcal{E}_p}^N a_{ij}(t)$$

for each node $i \in \mathcal{V}$ and $t \geq 0$. According to the definition of the persistent graph, for any initial time t_0 and any $\nu > 0$, there exists an integer $T_\nu \geq t_0$ such that $\sum_{t=T_\nu}^\infty a_{ij}(t) \leq \nu$ for all $(j, i) \in \mathcal{E} \setminus \mathcal{E}_p$.

We divide the rest of the proof into four steps.

Step 1: Take $T_0 = T_\nu$ and $\delta > L(N-1)(1-\eta)$, where η is the constant in Assumption 1 and L is the integer in Assumption 2. Let i_0 be a root of the persistent graph \mathbb{G}_p . Let $\mathcal{V}_0 = \{i_0\}$, $\mathcal{V}_1 = \{i : (i_0, i) \in \mathcal{E}_p\}$, and \mathcal{V}_i be a subset of $\mathcal{V} \setminus (\cup_{j=0}^{i-1} \mathcal{V}_j)$ and consist of all the nodes each of which has a neighbor in $\cup_{j=0}^{i-1} \mathcal{V}_j$ in \mathbb{G}_p for $2 \leq i \leq d_0$, where d_0 is the diameter of \mathbb{G}_p . It is easy to see that the root i_0 can be selected such that $\cup_{i=0}^{d_0} \mathcal{V}_i = \mathcal{V}$. These sets are well defined since \mathbb{G}_p contains a directed spanning tree. Let i_1 be some node in \mathcal{V}_1 . Define

$$t_1 \doteq \inf \left\{ t \geq 1 : \sum_{k=0}^{t-1} \mathbf{A}_{i_1}(T_0 + k) \geq \delta \right\}.$$

Note that t_1 is finite since (i_0, i_1) is a persistent arc in \mathbb{G}_p . Assume that

$$x_{i_0}(T_0) \leq \frac{1}{2} h(T_0) + \frac{1}{2} H(T_0).$$

In this step, we establish a bound for $x_{i_0}(T_0 + \tau)$, $\tau = 0, \dots, t_1$.

Let s be the integer satisfying that $(s-1)L \leq t_1 < sL$. Since $\delta > L(N-1)(1-\eta)$, one has $s \geq N$. Since $a_{ii}(t) \geq \eta$, $i \in \mathcal{V}$, $t \geq 0$, by Assumption 1, one has $\mathbf{A}_i(t) \leq 1 - \eta$. Then by the definition of t_1 , it is easy to derive that $\sum_{k=0}^{t_1-1} \mathbf{A}_{i_1}(T_0 + k) \leq 1 - \eta + \delta$.

Since $a_{i_1 i_1}(T_0 + k) = 1 - \sum_{j=1, j \neq i_1}^N a_{i_1 j}(T_0 + k)$ and based on Assumption 1, we have

i) $a_{i_1 i_1}(T_0 + k) \in [\eta, 1]$ for all $k = 0, \dots, t_1 - 1$;

$$\begin{aligned} \text{ii) } \sum_{k=0}^{t_1-1} (1 - a_{i_1 i_1}(T_0 + k)) &= \sum_{k=0}^{t_1-1} \mathbf{A}_{i_1}(T_0 + k) \\ &+ \sum_{k=0}^{t_1-1} \sum_{j=1, j \neq i_1, (j, i_1) \notin \mathcal{E}_p}^N a_{i_1 j}(T_0 + k) \\ &\leq 1 - \eta + \delta + \nu(N-1). \end{aligned}$$

Therefore, we conclude from Lemma 2 that

$$\prod_{k=0}^{t_1-1} a_{i_1 i_1}(T_0 + k) \geq e^{-\frac{(1-\eta+\delta+\nu(N-1)) \ln \eta}{\eta-1}} \doteq \mathbf{S}. \quad (11)$$

It is clear from the definition of $\mathbf{A}_i(t)$ and the fact $(s-1)L \leq t_1 < sL$ that

$$\begin{aligned} &\sum_{k=0}^{(s-1)L-1} a_{i_1 i_r}(T_0 + k) \\ &\leq \sum_{k=0}^{t_1-1} a_{i_1 i_r}(T_0 + k) \leq \sum_{k=0}^{t_1-1} \mathbf{A}_{i_1}(T_0 + k) \leq 1 - \eta + \delta \end{aligned}$$

for all $(i_r, i_1) \in \mathcal{E}_p$. Since $t_1 \leq sL$, from Assumption 2 one has that for any $(j, i) \in \mathcal{E}_p$

$$\begin{aligned} &\sum_{k=0}^{t_1-1} a_{ij}(T_0 + k) \\ &= \sum_{k=0}^{(s-1)L-1} a_{ij}(T_0 + k) + \sum_{k=(s-1)L}^{t_1-1} a_{ij}(T_0 + k) \\ &\leq K \sum_{k=0}^{(s-1)L-1} a_{i_1 i_r}(T_0 + k) + \sum_{k=(s-1)L}^{sL-1} a_{ij}(T_0 + k) \\ &\leq K(1 - \eta + \delta) + L(1 - \eta) \end{aligned}$$

where the last inequality makes use of the fact that

$$\sum_{k=(s-1)L}^{sL-1} a_{ij}(T_0 + k) \leq \sum_{k=(s-1)L}^{sL-1} (1 - a_{ii}(T_0 + k)) \leq L(1 - \eta).$$

For any $i \neq i_1$, it is true that

$$\begin{aligned} &\sum_{k=0}^{t_1-1} (1 - a_{ii}(T_0 + k)) \\ &= \sum_{k=0}^{t_1-1} \mathbf{A}_i(T_0 + k) + \sum_{k=0}^{t_1-1} \sum_{j=1, j \neq i, (j,i) \notin \mathcal{E}_p}^N a_{ij}(t) \\ &\leq (N-1)(K(1 - \eta + \delta) + L(1 - \eta) + \nu). \end{aligned}$$

Thus, in view of Lemma 2, we have that

$$\prod_{k=0}^{t_1-1} a_{ii}(T_0 + k) \geq e^{-\frac{(N-1)(K(1-\eta+\delta)+L(1-\eta)+\nu) \ln \eta}{\eta-1}} = \mathbf{Q} \quad (12)$$

for $i \neq i_1$. Note that $\mathbf{Q} < \mathbf{S}$.

Since $x_{i_0}(T_0) \leq \frac{1}{2} h(T_0) + \frac{1}{2} H(T_0)$, based on Lemma 3 we obtain

$$\begin{aligned} x_{i_0}(T_0 + \tau) &\leq \frac{1}{2} \prod_{k=0}^{t_1-1} a_{i_0 i_0}(T_0 + k) \cdot h(T_0) \\ &+ \left(1 - \frac{1}{2} \prod_{k=0}^{t_1-1} a_{i_0 i_0}(T_0 + k)\right) \cdot H(T_0) \quad (13) \end{aligned}$$

for all $\tau = 0, \dots, t_1$. Then, (12) and (13) further imply

$$x_{i_0}(T_0 + \tau) \leq \frac{\mathbf{Q}}{2}h(T_0) + \left(1 - \frac{\mathbf{Q}}{2}\right)H(T_0). \quad (14)$$

for all $\tau = 0, \dots, t_1$.

Step 2: In this step, we establish a bound for $x_i(T_0 + t_1)$, $i \in \mathcal{V}_1$. Since $\sum_{k=0}^{t_1-1} \mathbf{A}_{i_1}(T_0 + k) \geq \delta$, there must exist a node i_r such that $(i_r, i_1) \in \mathcal{E}_p$ and

$$\sum_{k=0}^{t_1-1} a_{i_1 i_r}(T_0 + k) \geq \frac{\delta}{N-1}.$$

It follows that:

$$\begin{aligned} \sum_{k=0}^{(s-1)L-1} a_{i_1 i_r}(T_0 + k) &\geq \frac{\delta}{N-1} - \sum_{(s-1)L}^{t_1-1} a_{i_1 i_r}(T_0 + k) \\ &\geq \frac{\delta}{N-1} - \sum_{(s-1)L}^{sL-1} a_{i_1 i_r}(T_0 + k) \\ &\geq \frac{\delta}{N-1} - L(1-\eta) \\ &> 0 \end{aligned}$$

where the last inequality is true since $\delta > L(N-1)(1-\eta)$. From Assumption 2, for any arc $(j, i) \in \mathcal{E}_p$, one has that

$$\begin{aligned} \sum_{k=0}^{t_1-1} a_{ij}(T_0 + k) &\geq \sum_{k=0}^{(s-1)L-1} a_{ij}(T_0 + k) \\ &\geq K^{-1} \sum_{k=0}^{(s-1)L-1} a_{i_1 i_r}(T_0 + k) \\ &\geq K^{-1} \left[\frac{\delta}{N-1} - L(1-\eta) \right] \\ &= \mathbf{R} \end{aligned} \quad (15)$$

where \mathbf{R} is defined in Theorem 1. The above inequality also holds for the arc (i_0, i_1) since $(i_0, i_1) \in \mathcal{E}_p$.

First according to (14), we have

$$\begin{aligned} x_{i_1}(T_0 + 1) &= \sum_{j=1}^N a_{i_1 j}(T_0)x_j(T_0) \\ &\leq a_{i_1 i_0}(T_0)x_{i_0}(T_0) + (1 - a_{i_1 i_0}(T_0))H(T_0) \\ &\leq a_{i_1 i_0}(T_0) \left[\frac{\mathbf{Q}}{2}h(T_0) + \left(1 - \frac{\mathbf{Q}}{2}\right)H(T_0) \right] \\ &\quad + (1 - a_{i_1 i_0}(T_0))H(T_0) \\ &= \frac{\mathbf{Q}}{2}a_{i_1 i_0}(T_0)h(T_0) + \left(1 - \frac{\mathbf{Q}}{2}a_{i_1 i_0}(T_0)\right)H(T_0). \end{aligned}$$

Then for $T_0 + 2$, we have

$$\begin{aligned} x_{i_1}(T_0 + 2) &= \sum_{j=1}^N a_{i_1 j}(T_0 + 1)x_j(T_0 + 1) \\ &\leq a_{i_1 i_0}(T_0 + 1)x_{i_0}(T_0 + 1) + a_{i_1 i_1}(T_0 + 1)x_{i_1}(T_0 + 1) \\ &\quad + (1 - a_{i_1 i_0}(T_0 + 1) - a_{i_1 i_1}(T_0 + 1))H(T_0 + 1) \\ &\leq a_{i_1 i_0}(T_0 + 1) \left[\frac{\mathbf{Q}}{2}h(T_0) + \left(1 - \frac{\mathbf{Q}}{2}\right)H(T_0) \right] \\ &\quad + a_{i_1 i_1}(T_0 + 1) \left[\frac{\mathbf{Q}}{2}a_{i_1 i_0}(T_0)h(T_0) + \left(1 - \frac{\mathbf{Q}}{2}a_{i_1 i_0}(T_0)\right)H(T_0) \right] \\ &\quad + (1 - a_{i_1 i_0}(T_0 + 1) - a_{i_1 i_1}(T_0 + 1))H(T_0) \\ &= \frac{\mathbf{Q}}{2} \left[a_{i_1 i_0}(T_0 + 1) + a_{i_1 i_1}(T_0 + 1)a_{i_1 i_0}(T_0) \right] h(T_0) \\ &\quad + \left[1 - \frac{\mathbf{Q}}{2} \left[a_{i_1 i_0}(T_0 + 1) + a_{i_1 i_0}(T_0 + 1)a_{i_1 i_0}(T_0) \right] \right] H(T_0). \end{aligned}$$

By induction it is straight forward to find that

$$\begin{aligned} x_{i_1}(T_0 + t_1) &\leq \frac{\mathbf{Q}}{2} \left[\sum_{\tau=0}^{t_1-1} \prod_{k=\tau+1}^{t_1-1} a_{i_1 i_1}(T_0 + k)a_{i_1 i_0}(T_0 + \tau) \right] h(T_0) \\ &\quad + \left[1 - \frac{\mathbf{Q}}{2} \left[\sum_{\tau=0}^{t_1-1} \prod_{k=\tau+1}^{t_1-1} a_{i_1 i_1}(T_0 + k)a_{i_1 i_0}(T_0 + \tau) \right] \right] H(T_0) \\ &\leq \frac{\mathbf{Q}}{2} \left(\prod_{k=0}^{t_1-1} a_{i_1 i_1}(T_0 + k) \right) \left(\sum_{k=0}^{t_1-1} a_{i_1 i_0}(T_0 + k) \right) h(T_0) \\ &\quad + \left[1 - \frac{\mathbf{Q}}{2} \left(\prod_{k=0}^{t_1-1} a_{i_1 i_1}(T_0 + k) \right) \left(\sum_{k=0}^{t_1-1} a_{i_1 i_0}(T_0 + k) \right) \right] H(T_0) \\ &\leq \frac{1}{2} \mathbf{SQR}h(T_0) + \left(1 - \frac{1}{2} \mathbf{SQR}\right) H(T_0) \end{aligned} \quad (16)$$

where the last inequality is due to (11) and (15).

It is obvious from (16) and in view of (12) that for any $i \in \mathcal{V}_1$

$$x_i(T_0 + t_1) \leq \frac{1}{2} \mathbf{Q}^2 \mathbf{R} h(T_0) + \left(1 - \frac{1}{2} \mathbf{Q}^2 \mathbf{R}\right) H(T_0).$$

Step 3: We continue to define

$$t_2 \doteq \inf \left\{ t \geq t_1 + 1 : \sum_{k=t_1}^{t-1} \mathbf{A}_{i_1}(T_0 + k) \geq \delta \right\}.$$

Similarly, one can find an integer s such that $(s-1)L \leq t_2 - t_1 < sL$. In this step, we will give an upper bound for $x_i(T_0 + t_2)$ for $i \in \mathcal{V}_0 \cup \mathcal{V}_1 \cup \mathcal{V}_2$.

Similar to the calculations of (11) and (12) in Step 1, one can derive that

$$\prod_{k=t_1}^{t_2-1} a_{i_1 i_1}(T_0 + k) \geq \mathbf{S} \quad (17)$$

and

$$\prod_{k=t_1}^{t_2-1} a_{ii}(T_0 + k) \geq \mathbf{Q}, \quad i \neq i_1. \quad (18)$$

Using Lemma 3 and noting that $\mathbf{S} > \mathbf{Q}$, we obtain

$$x_i(T_0 + t_1 + \tau) \leq \frac{1}{2} \mathbf{Q}^3 \mathbf{R} h(T_0) + \left(1 - \frac{1}{2} \mathbf{Q}^3 \mathbf{R}\right) H(T_0)$$

for $i \in \mathcal{V}_0 \cup \mathcal{V}_1$, and $\tau = 0, \dots, t_2 - t_1$.

For any $i_2 \in \mathcal{V}_2$, there is an arc $(i, i_2) \in \mathcal{E}_p$ for some $i \in \mathcal{V}_0 \cup \mathcal{V}_1$. Similar to (15), Assumption 2 implies that

$$\sum_{k=t_1}^{t_2-1} a_{i_2 i}(T_0 + k) \geq K^{-1} \sum_{k=t_1}^{t_1 + (s-1)L-1} a_{i_1 i_r}(T_0 + k) \geq \mathbf{R} \quad (19)$$

for some $(i_r, i_1) \in \mathcal{E}_p$. Following similar calculations of $x_{i_1}(T_0 + t_1)$ in Step 2, we obtain

$$\begin{aligned} x_{i_2}(T_0 + t_2) &\leq \frac{1}{2} \mathbf{Q}^3 \mathbf{R} \left(\prod_{k=t_1}^{t_2-1} a_{i_2 i_2}(T_0 + k) \right) \left(\sum_{k=t_1}^{t_2-1} a_{i_2 i}(T_0 + k) \right) h(T_0) \\ &+ \left[1 - \frac{1}{2} \mathbf{Q}^3 \mathbf{R} \left(\prod_{k=t_1}^{t_2-1} a_{i_2 i_2}(T_0 + k) \right) \left(\sum_{k=t_1}^{t_2-1} a_{i_2 i}(T_0 + k) \right) \right] \\ &H(T_0) \leq \frac{1}{2} \mathbf{Q}^4 \mathbf{R}^2 h(T_0) + \left(1 - \frac{1}{2} \mathbf{Q}^4 \mathbf{R}^2\right) H(T_0). \quad (20) \end{aligned}$$

Step 4: Continuing this process, a time sequence t_1, \dots, t_{d_0} can be defined as

$$t_r \doteq \inf \left\{ t \geq t_{r-1} + 1 : \sum_{k=t_{r-1}}^{t-1} \mathbf{A}_{i_1}(T_0 + k) \geq \delta \right\}$$

for $r = 1, 2, \dots, d_0$, with $t_0 = 0$. The bound for $x_i(T_0 + t_{d_0})$ can be established as

$$x_i(T_0 + t_{d_0}) \leq \frac{1}{2} \mathbf{Q}^{2d_0} \mathbf{R}^{d_0} h(T_0) + \left(1 - \frac{1}{2} \mathbf{Q}^{2d_0} \mathbf{R}^{d_0}\right) H(T_0) \quad (21)$$

for all $i = 1, \dots, N$. A bound for $\Psi(T_0 + t_{d_0})$ is, thus, derived

$$\Psi(T_0 + t_{d_0}) \leq \left(1 - \frac{1}{2} \mathbf{Q}^{2d_0} \mathbf{R}^{d_0}\right) \Psi(T_0).$$

When $x_{i_0}(T_0) > \frac{1}{2} h(T_0) + \frac{1}{2} H(T_0)$, one can establish a lower bound for $x_i(T_0 + t_{d_0})$ by a symmetric argument and derive the same inequality for $\Psi(T_0 + t_{d_0})$ as above.

Repeating the aforementioned estimate, one can find an infinite increasing time sequence $t_1, \dots, t_{d_0}, t_{d_0+1}, \dots, t_{2d_0}, \dots$, defined by

$$t_r \doteq \inf \left\{ t \geq t_{r-1} + 1 : \sum_{k=t_{r-1}}^{t-1} \mathbf{A}_{i_1}(T_0 + k) \geq \delta \right\} \quad (22)$$

and we have

$$\Psi(T_0 + t_{rd_0}) \leq \left(1 - \frac{1}{2} \mathbf{Q}^{2d_0} \mathbf{R}^{d_0}\right)^r \Psi(T_0) \quad (23)$$

for $r = 1, 2, \dots$. It implies that the sequence $\Psi(T_0 + t_{rd_0})$, $r = 1, 2, \dots$, converges to 0 as r goes to infinity. Since $\Psi(T_0 + t_{rd_0})$ is a subsequence of a nonincreasing sequence $\Psi(t)$, $t \geq 0$, $\Psi(t)$ converges to 0 as t goes to infinity as well, which completes the proof.

Necessity: The proof of the necessity part is similar to that of [20, Th. 3.1] and is, thus, omitted here.

C. Proof of Theorem 1 (ii)

Note that from the definition of t_r in (22) and the definition of \mathbf{A}_{i_1} , one knows that for any $r \geq 1$

$$\sum_{k=t_{r-1}}^{t_r-1} \mathbf{A}_{i_1}(T_\nu + k) \leq 1 + \delta.$$

It follows that:

$$\sum_{k=0}^{t_{\omega_1}-1} \mathbf{A}_{i_1}(T_\nu + k) \leq \omega_1 d_0 (1 + \delta).$$

By the definition of t^* in (7), $t^* \geq t_{\omega_1} d_0$. For $t \geq T_\nu + t^*$, applying (23) we have

$$\begin{aligned} \Psi(t) &\leq \Psi(T_\nu + t^*) \leq \Psi(T_\nu + t_{\omega_1} d_0) \\ &\leq \left(1 - \frac{1}{2} \mathbf{Q}^{2d_0} \mathbf{R}^{d_0}\right)^{\omega_1} \Psi(T_\nu) \\ &\leq \epsilon \Psi(T_\nu). \end{aligned}$$

■

IV. PROOF OF THEOREM 2

In this section, we first establish some technical preliminaries, and then establish the convergence statement Theorem 2 (i) and the contraction rate of $\Psi(t)$ claimed in Theorem 2 (ii). The main idea to prove Theorem 2 (i) is as follows. System (1) is transformed to a new y -system (24) given below and the global consensus of system (1) is established by analyzing system (24). System (24) is shown to satisfy the cut-balance condition (see Lemma 6) and the balanced asymmetry condition as well (see Remark 5) and a theorem in [13] (Lemma 7) implies the convergence to consensus of system (1). To prove Theorem 2 (ii), the convergence rate of system (24) to consensus is first given in Proposition 2 and then making use of the relationship between systems (1) and (24) establishes the contraction rate of $\Psi(t)$. The proof of Proposition 2 is achieved by transforming the system to system (39) with a sorted state vector which preserves the balanced asymmetry condition and employing the techniques in [12].

A. Technical Preliminaries

Consider system (1) with the initial time t_0 . Let $y(t) = x(tL + t_0)$ and $B(t) = A((t+1)L - 1 + t_0) \cdots A(tL + 1 + t_0)A(tL + t_0)$. Then, the dynamics of y -system is given by

$$y(t+1) = B(t)y(t). \quad (24)$$

Letting $\Phi(t) \doteq \max_{i \in \mathcal{V}} y_i(t) - \min_{i \in \mathcal{V}} y_i(t)$, one has that $\Phi(t) = \Psi(tL + t_0)$. One can conclude that $\lim_{t \rightarrow \infty} \Psi(t) = 0$

if and only if $\lim_{t \rightarrow \infty} \Phi(t) = 0$ since $\Psi(t)$ is a nonincreasing function of t . Hence, we establish the global consensus of system (1) by studying the property of the y -system (24).

We first establish two technical lemmas.

Lemma 4: Let A_1, A_2, \dots, A_m be stochastic matrices and for each A_i , $1 \leq i \leq m$, assume that all the diagonal elements are no less than η , $0 < \eta < 1$. Let $B_m = A_1 A_2 \cdots A_m$ and $C_m = A_1 + \cdots + A_m$. Then, we have

$$\sum_{i \in S, j \notin S} (B_m)_{ij} \geq \eta^{m-1} \sum_{i \in S, j \notin S} (C_m)_{ij} \quad (25)$$

where S is an arbitrary nonempty proper subset of \mathcal{V} and $(B_m)_{ij}$ is the ij th element of B_m .

Proof: We show by induction that

$$\sum_{i \in S, j \in \bar{S}} (B_l)_{ij} \geq \eta^{l-1} \sum_{i \in S, j \in \bar{S}} (C_l)_{ij} \quad (26)$$

for $1 \leq l \leq m$. For the matrix $B_2 = A_1 A_2$, one has

$$\begin{aligned} \sum_{i \in S, j \in \bar{S}} (B_2)_{ij} &= \sum_{i \in S, j \in \bar{S}} \sum_{k=1}^N (A_1)_{ik} (A_2)_{kj} \\ &= \sum_{k \in S} \sum_{i \in S, j \in \bar{S}} (A_1)_{ik} (A_2)_{kj} \\ &\quad + \sum_{k \in \bar{S}} \sum_{i \in S, j \in \bar{S}} (A_1)_{ik} (A_2)_{kj} \\ &\geq \sum_{k \in S} \sum_{j \in \bar{S}} (A_1)_{ii} (A_2)_{kj} + \sum_{k \in \bar{S}} \sum_{i \in S} (A_1)_{ik} (A_2)_{jj} \\ &\geq \eta \sum_{k \in S, j \in \bar{S}} (A_2)_{kj} + \eta \sum_{k \in \bar{S}, i \in S} (A_1)_{ik} \\ &= \eta \sum_{i \in S, j \in \bar{S}} (C_2)_{ij}. \end{aligned} \quad (27)$$

Thus, (26) holds for $l = 2$. If $m = 2$, then the proof is complete.

Suppose that $m > 2$. Assume that (26) is true for $l \in \{2, \dots, s\}$, where $s \in \{2, \dots, m-1\}$. Since the diagonal elements of A_i are at least η , one has $(B_s)_{ii} \geq \eta^s$ for $1 \leq i \leq N$. Noting that $B_{s+1} = B_s A_{s+1}$ and $C_{s+1} = C_s + A_{s+1}$, we have

$$\begin{aligned} \sum_{i \in S, j \in \bar{S}} (B_{s+1})_{ij} &= \sum_{k \in S} \sum_{i \in S, j \in \bar{S}} (B_s)_{ik} (A_{s+1})_{kj} \\ &\quad + \sum_{k \in \bar{S}} \sum_{i \in S, j \in \bar{S}} (B_s)_{ik} (A_{s+1})_{kj} \\ &\geq \eta^s \sum_{k \in S, j \in \bar{S}} (A_{s+1})_{kj} + \eta \sum_{k \in \bar{S}, i \in S} (B_s)_{ik} \\ &\geq \eta^s \sum_{k \in S, j \in \bar{S}} (A_{s+1})_{kj} + \eta^s \sum_{k \in \bar{S}, i \in S} (C_s)_{ik} \\ &= \eta^s \sum_{i \in S, j \in \bar{S}} (C_{s+1})_{ij}. \end{aligned} \quad (28)$$

Hence, (26) holds for $l = s + 1$. Therefore, (26) holds for $1 \leq l \leq m$ by induction. \blacksquare

Lemma 5: Let A_1, A_2, \dots, A_m be $N \times N$ stochastic matrices, $B_m = A_1 A_2 \cdots A_m$ and $C_m = A_1 + \cdots + A_m$. Then, we have

$$\sum_{i \in S, j \notin S} (B_m)_{ij} \leq (N-1) \sum_{i \in S, j \notin S} (C_m)_{ij} \quad (29)$$

where S is an arbitrary nonempty proper subset of \mathcal{V} .

Proof: It will be shown by induction that

$$\sum_{i \in S, j \in \bar{S}} (B_l)_{ij} \leq (N-1) \sum_{i \in S, j \in \bar{S}} (C_l)_{ij} \quad (30)$$

for $1 \leq l \leq m$. In view of (27) and the fact that A_1 and A_2 are stochastic matrices, one has

$$\begin{aligned} \sum_{i \in S, j \in \bar{S}} (B_2)_{ij} &= \sum_{k \in S} \sum_{i \in S, j \in \bar{S}} (A_1)_{ik} (A_2)_{kj} \\ &\quad + \sum_{k \in \bar{S}} \sum_{i \in S, j \in \bar{S}} (A_1)_{ik} (A_2)_{kj} \\ &\leq |S| \sum_{k \in S} \sum_{j \in \bar{S}} (A_2)_{kj} + \sum_{k \in \bar{S}} \sum_{i \in S} (A_1)_{ik} \\ &\leq (N-1) \sum_{i \in S, j \in \bar{S}} (A_2)_{ij} + \sum_{i \in S, j \in \bar{S}} (A_1)_{ij} \\ &\leq (N-1) \sum_{i \in S, j \in \bar{S}} (C_2)_{ij} \end{aligned}$$

which implies that (30) holds for $l = 2$.

Suppose that $m > 2$. Assume that (30) holds for $l \in \{2, \dots, s\}$, where $s \in \{2, \dots, m-1\}$. Noting that $B_{s+1} = B_s A_{s+1}$ and B_s is a stochastic matrix, one has

$$\begin{aligned} \sum_{i \in S, j \in \bar{S}} (B_{s+1})_{ij} &= \sum_{k \in S} \sum_{i \in S, j \in \bar{S}} (B_s)_{ik} (A_{s+1})_{kj} + \sum_{k \in \bar{S}} \sum_{i \in S, j \in \bar{S}} (B_s)_{ik} (A_{s+1})_{kj} \\ &\leq (N-1) \sum_{k \in S, j \in \bar{S}} (A_{s+1})_{kj} + \sum_{k \in \bar{S}, i \in S} (B_s)_{ik} \\ &\leq (N-1) \sum_{k \in S, j \in \bar{S}} (A_{s+1})_{kj} + (N-1) \sum_{k \in \bar{S}, i \in S} (C_s)_{ik} \\ &\leq \eta^s \sum_{i \in S, j \in \bar{S}} (C_{s+1})_{ij}. \end{aligned} \quad (31)$$

Hence, (26) holds for $l = s + 1$. Therefore, (26) holds for $1 \leq l \leq m$ by induction. \blacksquare

We derive some useful properties of the system matrix $B(t)$ in (24) based on Assumption 3 in the following lemma.

Lemma 6: If Assumptions 1 and 3 hold, then each matrix $B(t)$, $t \geq 0$, has positive diagonals lower bounded by η^L and satisfies the cut-balance condition

$$\sum_{i \notin S, j \in S} b_{ij}(t) \leq M_* \sum_{i \in S, j \notin S} b_{ij}(t) \quad (32)$$

for any nonempty proper subset S of \mathcal{V} with $M_* = (N-1)K\eta^{-L+1}$. Let $\mathbb{G}'_p = (\mathcal{V}, \mathcal{E}'_p)$ be a directed graph where $(j, i) \in$

\mathcal{E}'_p if and only if $\sum_{t=0}^{\infty} b_{ij}(t) = \infty$. The persistent graph \mathbb{G}_p contains a directed spanning tree if and only if \mathbb{G}'_p contains a directed spanning tree.

Proof: Since $a_{ii}(t) \geq \eta$ for all $i \in \mathcal{V}, t \geq 0$, it is obvious that $b_{ii}(t) \geq \eta^L$ for all $t \geq 0$. Applying Lemmas 4 and 5 to system matrices $A(t)$ and $B(t)$ in (1) and (24) and in view of Assumption 3, one has

$$\begin{aligned} \sum_{i \notin S, j \in S} b_{ij}(t) &\leq (N-1) \sum_{i \notin S, j \in S} \sum_{u=0}^{L-1} a_{ij}(tL+u+t_0) \\ &\leq (N-1)K \sum_{i \in S, j \notin S} \sum_{u=0}^{L-1} a_{ij}(tL+u+t_0) \\ &\leq (N-1)K\eta^{-L+1} \sum_{i \in S, j \notin S} b_{ij}(t). \end{aligned}$$

Hence, (32) holds.

Sufficiency: Suppose that (j, i) is an arc of \mathbb{G}_p . By the definition of a persistent arc, $\sum_{t=0}^{\infty} a_{ij}(t) = \infty$. There must exist a time sequence t_{k_1}, t_{k_2}, \dots , diverging to infinity with nonnegative integers $k_1 < k_2 < \dots$ such that $k_s L \leq t_{k_s} \leq (k_s + 1)L - 1$, $s \geq 1$ and $\sum_{s=1}^{\infty} a_{ij}(t_{k_s} + t_0) = \infty$. One has that

$$\begin{aligned} b_{ij}(k_s) &\geq a_{ii}((k_s + 1)L - 1) \cdots a_{ij}(t_{k_s} + t_0) \cdots a_{jj}(k_s L + t_0) \\ &\geq \eta^{L-1} a_{ij}(t_{k_s} + t_0), \quad s \geq 1. \end{aligned}$$

It follows that $\sum_{t=0}^{\infty} b_{ij}(t) \geq \eta^{L-1} \sum_{s=1}^{\infty} a_{ij}(t_{k_s} + t_0) = \infty$, implying that (j, i) is a persistent arc of \mathbb{G}'_p . \mathbb{G}'_p contains a directed spanning tree since \mathbb{G}_p does.

Necessity: Suppose that (j, i) is an arc of \mathbb{G}'_p . Note that $\sum_{t=0}^{\infty} b_{ij}(t) = \infty$ and

$$\begin{aligned} b_{ij}(t) &= \sum_{k_1, \dots, k_{L-1} \in \mathcal{V}} a_{ik_{L-1}}((t+1)L-1) \\ &\quad \cdots a_{k_2 k_1}(tL+1+t_0) a_{k_1 j}(tL+t_0). \end{aligned}$$

It follows that there exist integers $k_1, \dots, k_{L-1} \in \mathcal{V}$ such that $\sum_{t=0}^{\infty} a_{ik_{L-1}}((t+1)L-1+t_0) \cdots a_{k_2 k_1}(tL+1+t_0) a_{k_1 j}(tL+t_0) = \infty$. Since $a_{ij}(t) \leq 1$ for all $i, j \in \mathcal{V}, t \geq 0$, one has that $\sum_{t=0}^{\infty} a_{k_{s+1} k_s}(tL+s+t_0) = \infty$ for all $0 \leq s \leq L-1$ with $k_0 = j$ and $k_L = i$, implying that $(k_s, k_{s+1}) \in \mathcal{E}_p$. This implies that there exists a directed path from node j to i in \mathbb{G}_p . Hence, if \mathbb{G}'_p contains a directed spanning tree, so does \mathbb{G}_p . ■

Remark 4: It has been proved in [9] that under the cut-balance condition (32) if \mathbb{G}'_p contains a directed spanning tree then it is strongly connected. Following a similar argument, one can show that when Assumption 3 holds, if \mathbb{G}_p contains a directed spanning tree then it is strongly connected.

Consider the system

$$y(t+1) = B(t)y(t) \quad (33)$$

where $B(t) = [b_{ij}(t)] \in \mathbb{R}^{N \times N}$, $b_{ij}(t) \geq 0$, and $\sum_{j=1}^N b_{ij}(t) = 1$. The following lemma is a convergence result of the cut-balanced system.

Lemma 7 ([13]): For system (33), suppose that the following assumptions hold.

- 1) There exists a $\gamma > 0$ such that $b_{ii}(t) \geq \gamma$ for all $i \in \mathcal{V}, t \geq 0$.
- 2) There exists a constant M_* such that for every t and nonempty proper subset S of \mathcal{V} , there holds

$$\sum_{i \notin S, j \in S} b_{ij}(t) \leq M_* \sum_{i \in S, j \notin S} b_{ij}(t). \quad (34)$$

Then, $\lim_{t \rightarrow \infty} y_i(t)$ exists for every i . Let $\mathbb{G}'_p = (\mathcal{V}, \mathcal{E}'_p)$ be the persistent graph where $(j, i) \in \mathcal{E}'_p$ if $\sum_{k=0}^{\infty} b_{ij}(t) = \infty$. If \mathbb{G}'_p contains a directed spanning tree, then global consensus is reached.

Lemma 7 is a special case of [22, Th. 1] restricted to deterministic systems. The result has also been proved in [4, Th. 2] for balanced asymmetric systems that include the system in Lemma 7 as a special case and we will introduce in the following section. In Lemma 7, the condition that \mathbb{G}'_p contains a directed spanning tree is also necessary for the global consensus of system (33), which has been proved in [4, Th. 2].

B. Proof of Theorem 2 (i)

Lemma 6 shows that the y -system (24) satisfies the assumptions of Lemma 7. One concludes that \mathbb{G}'_p defined in Lemma 6 contains a directed spanning tree if and only if global consensus of system (24) is reached. Combining with Lemma 6, the conclusion of Theorem 2 (i) immediately follows.

C. Proof of Theorem 2 (ii)

In this section, we provide a contraction rate of $\Phi(t)$ and, hence, a corresponding contraction rate of $\Psi(t)$ can be obtained. We have seen that system (24) satisfies the cut-balanced condition (34). Instead of considering the cut-balanced system, we consider a system with $B(t)$ satisfying the balanced asymmetric condition. As will be seen shortly, the balanced asymmetric condition includes the cut-balanced condition with $b_{ii}(t)$ lower bounded by a positive constant as a special case.

Assumption 4 (Balanced Asymmetry [4]): There exists a constant $M \geq 1$ such that for any two nonempty proper subsets S_1, S_2 of \mathcal{V} with the same cardinality, the matrices $B(t)$, $t \geq 0$, satisfy that

$$\sum_{i \notin S_1, j \in S_2} b_{ij}(t) \leq M \sum_{i \in S_1, j \notin S_2} b_{ij}(t). \quad (35)$$

Remark 5: As pointed out in [4, Remark 1], the balanced asymmetry condition is stronger than the cut-balance condition (34). But if $B(t)$ has positive diagonal elements lower bounded by a positive constant γ and satisfies (34), then it satisfies the balanced asymmetry condition with $M = \max\{M_*, \frac{N-1}{\gamma}\}$.

In the following, we consider system (33) and assume that the matrices $B(t)$, $t \geq 0$, satisfy the balanced asymmetry condition. We first establish the convergence rate of $\Phi(t) = \max_{i \in \mathcal{V}} y_i(t) - \min_{i \in \mathcal{V}} y_i(t)$ and then apply the result to the cut-balanced system (24). We introduce the notion of absolute infinite flow property [4], [23], which has a close relationship with the connectivity of persistent graphs.

Definition 3: The sequence of matrices $B(t)$, $t \geq 0$ is said to have the absolute infinite flow property if the following holds:

$$\sum_{t=0}^{\infty} \left(\sum_{\substack{i \notin S(t+1) \\ j \in S(t)}} b_{ij}(t) + \sum_{\substack{i \in S(t+1) \\ j \notin S(t)}} b_{ij}(t) \right) = \infty \quad (36)$$

for every sequence $S(t)$, $t \geq 0$, of nonempty proper subsets of \mathcal{V} with the same cardinality.

If the matrix sequence $B(t)$, $t \geq 0$, has the absolute infinite flow property and satisfies the balanced asymmetry condition, we can define an infinite time sequence t_0, t_1, t_2, \dots based on (36). Let $t_0^0 = t_0$ and define a finite time sequence $t_p^0, t_p^1, \dots, t_p^{\lfloor \frac{N}{2} \rfloor}$, $p \geq 0$. t_p^{q+1} is defined by

$$t_p^{q+1} \doteq \inf \left\{ t \geq t_p^q + 1 : \min_{|S(t_p^q)| = \dots = |S(t-1)|} \sum_{k=t_p^q}^{t-1} \sum_{\substack{i \notin S(k+1) \\ j \in S(k)}} b_{ij}(k) \geq 1 \right\} \quad (37)$$

where $|S|$ denotes the cardinality of a set S . Let $t_{p+1} = t_p^{\lfloor \frac{N}{2} \rfloor}$ and $t_{p+1}^0 = t_{p+1}$. We derive an infinite time sequence t_0, t_1, t_2, \dots . Since (35) holds, one has that for every sequence $S(t)$, $t \geq 0$, of nonempty proper subsets of \mathcal{V} with the same cardinality

$$\sum_{t=0}^{\infty} \sum_{\substack{i \notin S(t+1) \\ j \in S(t)}} b_{ij}(t) = \infty$$

from which it is clear that (37) is well defined.

Proposition 1: For system (33), assume that the sequence of matrices $B(t)$, $t \geq 0$, satisfies Assumption 4. If it has the absolute infinite flow property, then

$$\Phi(t_{p+1}) \leq \left(1 - M^{-\lfloor \frac{N}{2} \rfloor} / (8N^2)^{\lfloor \frac{N}{2} \rfloor} \right) \Phi(t_p) \quad (38)$$

and global consensus of system (33) is reached.

We introduce some new notations and lemmas for the proof of Proposition 1. For $t \geq 0$, let σ_t be a permutation of \mathcal{V} such that for $i < j$, either $y_{\sigma_t(i)}(t) < y_{\sigma_t(j)}(t)$ or $y_{\sigma_t(i)}(t) = y_{\sigma_t(j)}(t)$ and $\sigma_t(i) < \sigma_t(j)$ holds. Define $z_i(t) \doteq y_{\sigma_t(i)}(t)$, $t \geq 0$. From the definition of the permutation σ_t , one knows that for all $t \geq 0$, if $i < j$, then $z_i(t) \leq z_j(t)$. Hence, $z(t) = [z_1(t), \dots, z_N(t)]^T$ is a sorted state vector.

Remark 6: Inequality (38) for the contraction rate of $\Psi(t)$ takes the same form as that in [12, Proposition 2], which deals with a continuous-time system under persistent connectivity. We will employ similar ideas to derive (38). The dynamics of the continuous-time system considered in [9] and [12] is

$$\dot{y}_i(t) = \sum_{j=1}^N b_{ij}(t)(y_j(t) - y_i(t)).$$

The solution to the system is a locally absolutely continuous function y that satisfies the integral equation

$$y_i(t) = y_i(0) + \int_0^t \sum_{j=1}^N b_{ij}(s)(y_j(s) - y_i(s)) ds.$$

A key property of the continuous-time system proved in [10] is that the sorted state $z(t)$ satisfies an equation of the same form as the state $y(t)$

$$z_i(t) = z_i(0) + \int_0^t \sum_{j=1}^N c'_{ij}(s)(z_j(s) - z_i(s)) ds$$

where $c'_{ij}(t) \doteq b_{\sigma_t(i), \sigma_t(j)}(t)$. In addition, if $B(t)$, $t \geq 0$, satisfy the cut-balance condition, then $C'(t) = [c'_{ij}(t)]_{N \times N}$, $t \geq 0$, satisfy the cut-balance condition as well [12]. However, for the discrete-time system (33), $z(t)$ does not satisfy the equation of the same form as $y(t)$ with the above notations. We modify the definition of $c'_{ij}(t)$ such that this still holds.

Define $c_{ij}(t) \doteq b_{\sigma_{t+1}(i), \sigma_t(j)}(t)$. It is obvious that $\sum_{j=1}^N c_{ij}(t) = 1$ for all $i \in \mathcal{V}$, $t \geq 0$. In view of the definition $z_i(t) = y_{\sigma_t(i)}(t)$ for $t \geq 0$, one has

$$\begin{aligned} z_i(t+1) &= y_{\sigma_{t+1}(i)}(t+1) = \sum_{j=1}^n b_{\sigma_{t+1}(i), j}(t) y_j(t) \\ &= \sum_{j=1}^n b_{\sigma_{t+1}(i), \sigma_t(j)}(t) y_{\sigma_t(j)}(t) = \sum_{j=1}^N c_{ij}(t) z_j(t). \end{aligned} \quad (39)$$

In addition, the interaction weights $c_{ij}(t)$ have the following property.

Lemma 8: Assume that $B(t)$, $t \geq 0$ satisfy Assumption 4. For any nonempty proper subsets S_1 and S_2 of \mathcal{V} with the same cardinality, $c_{ij}(t)$ satisfies

$$\sum_{i \notin S_1, j \in S_2} c_{ij}(t) \leq M \sum_{i \in S_1, j \notin S_2} c_{ij}(t). \quad (40)$$

Proof: For a fixed t , let $S_3 = \{\sigma_{t+1}(i) : i \in S_1\}$ and $S_4 = \{\sigma_t(j) : j \in S_2\}$. Since σ_t, σ_{t+1} are permutations of \mathcal{V} , S_3 , and S_4 have the same cardinality and using (35), one has

$$\begin{aligned} \sum_{i \notin S_1, j \in S_2} c_{ij}(t) &= \sum_{i \notin S_1, j \in S_2} b_{\sigma_{t+1}(i), \sigma_t(j)}(t) \\ &= \sum_{i \notin S_3, j \in S_4} b_{ij}(t) \leq M \sum_{i \in S_3, j \notin S_4} b_{ij}(t) \\ &= M \sum_{i \in S_1, j \notin S_2} c_{ij}(t). \end{aligned}$$

Remark 7: Note that if $B(t)$, $t \geq 0$, satisfy the cut-balance condition, $C(t) = [c_{ij}(t)]_{N \times N}$, $t \geq 0$, do not preserve the cut-balance property in general while $C'(t)$, $t \geq 0$, do. However, the evolution of $z_i(t)$ does not satisfy $z_i(t+1) = \sum_{j=1}^N c'_{ij}(t) z_j(t)$, $i \in \mathcal{V}$, in general. In this case, though $C'(t) = [c'_{ij}(t)]_{N \times N}$, $t \geq 0$, satisfy the cut-balance condition, the evolution of $z_i(t)$ cannot be directly expressed using $C'(t)$ ■

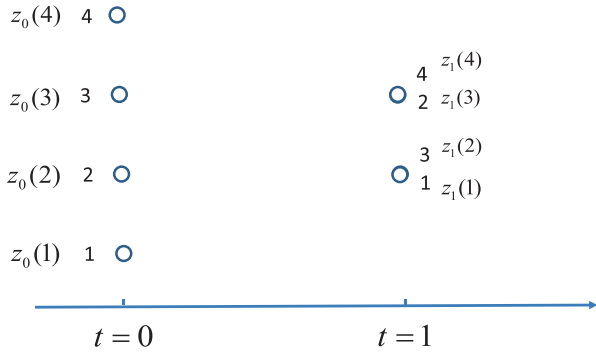


Fig. 1. System state evolution at the first step.

and is not easy to be analyzed making use of the property of $C'(t)$. In addition, the diagonal elements of the newly defined matrix $C(t)$, $t \geq 0$, are not necessarily positive any more and some existing results in the literature cannot be directly applied to the system (39).

Example 1: Consider the system (33) consisting of four agents. Let

$$B(t) = \begin{cases} \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, & \text{if } t \text{ is even} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, & \text{if } t \text{ is odd.} \end{cases}$$

Let the initial state of the system be $y(0) = [0, 1, 2, 3]^T$. We only consider the first step evolution of the system as shown in Fig. 1.

One can easily see that

$$y(1) = B(0)y(0) = [1, 2, 1, 2]^T.$$

Since $y(0)$ is already sorted, $\sigma_0(i) = i$, $i = 1, \dots, 4$ and $z(0) = y(0)$. By the definition of σ_t , we have $\sigma_1(1) = 1, \sigma_1(2) = 3, \sigma_1(3) = 2, \sigma_1(4) = 4$, and $z(1) = [1, 1, 2, 2]^T$. By the definitions of $c'_{ij}(t) \doteq b_{\sigma_t(i), \sigma_t(j)}(t)$ and $c_{ij}(t) \doteq b_{\sigma_{t+1}(i), \sigma_t(j)}(t)$, one has

$$C'(0) = B(0)$$

and

$$\begin{aligned} C(0) &= [b_{\sigma_1(i), \sigma_0(j)}(0)]_{N \times N} \\ &= \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}. \end{aligned}$$

Note that the matrix $B(0)$ has positive diagonal elements while $C(0)$ does not. It can be directly verified that $z(1) \neq y(1) = C'(0)z(0)$ and $z(1) = C(0)z(0)$. \square

Lemma 9: Assume that the matrix B satisfies that

$$\sum_{i \notin S_1, j \in S_2} b_{ij} \leq M \sum_{i \in S_1, j \notin S_2} b_{ij}(t) \quad (41)$$

for a constant $M \geq 1$ and any two nonempty proper subsets S_1, S_2 of \mathcal{V} with the same cardinality. Let σ and μ be permutations of \mathcal{V} and $c_{ij} = b_{\mu(i), \sigma(j)}$. Then, for any sorted vector $z \in \mathbb{R}^n$ and $1 \leq l \leq N-1$, one has

$$\begin{aligned} &\sum_{i=1}^l M^{-i} \left(\sum_{j=1}^N c_{ij}(z_j - z_i) \right) \\ &\geq (z_{l+1} - z_l) M^{-l} \sum_{i=l+1}^N \sum_{j=1}^l c_{ji} \geq 0. \end{aligned} \quad (42)$$

Remark 8: The proof of Lemma 9 is similar to that of [9, Lemma 2] and [12, Lemma 9] and, hence, is omitted here. Note that if the matrix B only satisfies the cut-balance condition (35), then the inequality (42) may not hold since the matrix $C = [c_{ij}]_{N \times N}$ defined in Lemma 9 does not satisfy the cut-balance condition any more in general.

Proof of Proposition 1: Note that $z_i(t)$ satisfies $z_i(t+1) = \sum_{j=1}^N c_{ij}(t)z_j(t)$, $i \in \mathcal{V}$. In addition, $z(t) = [z_1(t), \dots, z_N(t)]^T$ is a sorted state vector and $\Phi(t) = z_n(t) - z_1(t)$ for $t \geq 0$. With the key inequality (42) in Lemma 9 in hand, using similar ideas to the proofs of [12, Sec. 4.2, Lemmas 10, 11, and Proposition 2], one can derive (38). \square

Next consider system (33) with $B(t)$ satisfying the cut-balance condition (34) and $b_{ii}(t) \geq \gamma$ for all $i \in \mathcal{V}, t \geq 0$. We show that when the persistent graph \mathbb{G}'_p contains a directed spanning tree, then the matrix sequence $B(t)$, $t \geq 0$, has the absolute infinite flow property. First note that under the cut-balance condition, if the persistent graph contains a directed spanning tree then it is strongly connected. For every sequence $S(t)$, $t \geq 0$, of nonempty proper subsets of \mathcal{V} , if there are an infinite number of pairs of $S(t)$ and $S(t+1)$ such that $S(t) \neq S(t+1)$, then for each of this pair, one has

$$\sum_{\substack{i \notin S(t+1) \\ j \in S(t)}} b_{ij}(t) \geq \gamma$$

since $b_{ii}(t) \geq \gamma$ for all $i \in \mathcal{V}, t \geq 0$. It follows that:

$$\sum_{t=0}^{\infty} \sum_{\substack{i \notin S(t+1) \\ j \in S(t)}} b_{ij}(t) = \infty.$$

If there are only a finite number of pairs of $S(t)$ and $S(t+1)$ such that $S(t) \neq S(t+1)$, then there exists an integer T_0 such that for $t \geq T_0$, $S(t) = S$. It follows that:

$$\sum_{t=0}^{\infty} \sum_{\substack{i \notin S(t+1) \\ j \in S(t)}} b_{ij}(t) = \sum_{t=T_0}^{\infty} \sum_{\substack{i \notin S \\ j \in S}} b_{ij}(t).$$

Since the persistent graph \mathbb{G}'_p is strongly connected, there must exist an arc from S to \bar{S} and one concludes that the previous expression is equal to ∞ . One concludes that the matrix sequence $B(t)$, $t \geq 0$, has the absolute infinite flow property. Then, we can define a time sequence t_0, t_1, \dots based on (37) for the cut-balanced system in the same way as for the balanced asymmetric system. Note that when $B(t)$, $t \geq 0$, satisfy the cut-balance condition (34), they also satisfy the balanced asymmetry condition with $M = \max\{M_*, \frac{N-1}{\gamma}\}$. We immediately have the following proposition by applying Proposition 1.

Proposition 2: For system (33), assume that the matrices $B(t)$, $t \geq 0$, satisfy the cut-balance condition (34) and $b_{ii}(t) \geq \gamma$ for all $i \in \mathcal{V}, t \geq 0$. If the persistent graph \mathbb{G}'_p contains a directed spanning tree, then

$$\Phi(t_{p+1}) \leq \left(1 - M^{-\lfloor \frac{N}{2} \rfloor} / (8N^2)^{\lfloor \frac{N}{2} \rfloor}\right) \Phi(t_p) \quad (43)$$

where $M = \max\{M_*, \frac{N-1}{\gamma}\}$ and global consensus is reached.

Remark 9: Proposition 2 gives a convergence rate of the system (33) satisfying the two assumptions in Lemma 7. Note that the proof of the convergence result for the consensus system under noninstantaneous reciprocal interactions in [13] made use of the intermediate result Lemma 7. With the help of Proposition 2, one can relate the convergence rate of the system discussed in [13] to the amount of interactions having taken place as well.

Proof of Theorem 2 (ii): For system (1) and any given initial time $t_0 \geq 0$, let $k_0^0 = k_0 = 0$ and define a finite time sequence $k_p^0, k_p^1, \dots, k_p^{\lfloor \frac{N}{2} \rfloor}$, $p \geq 0$. k_p^{q+1} is defined by

$$k_p^{q+1} \doteq \inf \left\{ t \geq k_p^q + 1 : \min_{|S(k)|=\dots=|S(t-1)|} W \sum_{k=k_p^q}^{t-1} \sum_{\substack{i \notin S(k+1) \\ j \in S(k)}}^{L-1} a_{ij}(kL + u + t_0) \geq 1 \right\} \quad (44)$$

where $W = \frac{\eta^L}{(N-1)L}$ is a constant. Let $k_{p+1} = k_p^{\lfloor \frac{N}{2} \rfloor}$ and $k_{p+1}^0 = k_{p+1}$. We derive an infinite time sequence k_0, k_1, k_2, \dots . Under Assumptions 1 and 3, it can be shown that when the persistent graph \mathbb{G}_p contains a directed spanning tree, the time sequence k_0, k_1, k_2, \dots is well defined.

We first show that if the persistent graph \mathbb{G}_p contains a directed spanning tree, then

$$\Psi(k_{p+1}L + t_0) \leq \left(1 - K_*^{-\lfloor \frac{N}{2} \rfloor} / (8N^2)^{\lfloor \frac{N}{2} \rfloor}\right) \Psi(k_pL + t_0) \quad (45)$$

where $K_* = \max\{\frac{(N-1)K}{\eta^{L-1}}, \frac{N-1}{\eta^L}\}$.

Consider system (24) derived based on system (1). It has been shown in Lemma 6 that system (24) satisfies the assumptions of Proposition 2 with $M_* = (N-1)K\eta^{-L+1}$ and $\gamma = \eta^L$. Next, we verify that k_p^{q+1} defined in (44) satisfies that

$$\sum_{k=k_p^q}^{k_p^{q+1}-1} \sum_{\substack{i \notin S(k+1) \\ j \in S(k)}} b_{ij}(k) \geq 1.$$

Let S_1, S_2 be two nonempty proper subsets of \mathcal{V} with the same cardinality. If $S_1 = S_2$, then it follows from Lemma 4 that

$$\eta^{L-1} \sum_{\substack{i \notin S_1 \\ j \in S_1}} \sum_{u=0}^{L-1} a_{ij}(kL + u + t_0) \leq \sum_{\substack{i \notin S_1 \\ j \in S_1}} b_{ij}(k).$$

If $S_1 \neq S_2$, then $\sum_{\substack{i \notin S_1 \\ j \in S_2}} \sum_{u=0}^{L-1} a_{ij}(kL + u + t_0) \leq (N-1)L$ and $\sum_{\substack{i \notin S_1 \\ j \in S_2}} b_{ij}(k) \geq \eta^L$ since $b_{ii}(k) \geq \eta^L$ for all $i \in \mathcal{V}, k \geq 0$. This implies that

$$\begin{aligned} \sum_{\substack{i \notin S_1 \\ j \in S_2}} \sum_{u=0}^{L-1} a_{ij}(kL + u + t_0) &\leq (N-1)L \\ &\leq \frac{(N-1)L}{\eta^L} \sum_{\substack{i \notin S_1 \\ j \in S_2}} b_{ij}(k). \end{aligned}$$

Hence, for all S_1, S_2 , and $k \geq 0$, it always holds that

$$\begin{aligned} \frac{\eta^L}{(N-1)L} \sum_{\substack{i \notin S_1 \\ j \in S_2}} \sum_{u=0}^{L-1} a_{ij}(kL + u + t_0) \\ = W \sum_{\substack{i \notin S_1 \\ j \in S_2}} \sum_{u=0}^{L-1} a_{ij}(kL + u + t_0) &\leq \sum_{\substack{i \notin S_1 \\ j \in S_2}} b_{ij}(k). \end{aligned}$$

Combining with (44), one has that

$$\begin{aligned} \sum_{k=k_p^q}^{k_p^{q+1}-1} \sum_{\substack{i \notin S(k+1) \\ j \in S(k)}} b_{ij}(k) \\ \geq W \sum_{k=k_p^q}^{k_p^{q+1}-1} \sum_{\substack{i \notin S(k+1) \\ j \in S(k)}} \sum_{u=0}^{L-1} a_{ij}(kL + u + t_0) &\geq 1. \end{aligned}$$

Note that $\Phi(t) = \Psi(tL + t_0)$ and applying (43) in Proposition 2 immediately gives (45).

Next, we prove (8). Note that for any $k \geq 0$ and any sequence $S(k)$, $k \geq 0$, of nonempty proper subsets of \mathcal{V} with the same cardinality, it always holds that

$$W \sum_{\substack{i \notin S(k+1) \\ j \in S(k)}} \sum_{u=0}^{L-1} a_{ij}(kL + u + t_0) \leq WL(N-1) = \eta^L.$$

It follows from the definition of k_p^{q+1} in (44) that for any sequence $S(k)$, $k \geq 0$, of nonempty proper subsets of \mathcal{V} with the same cardinality, and any $p \geq 0, 0 \leq q \leq \lfloor \frac{N}{2} \rfloor - 1$

$$W \sum_{k=k_p^q}^{k_p^{q+1}-1} \sum_{\substack{i \notin S(k+1) \\ j \in S(k)}} \sum_{u=0}^{L-1} a_{ij}(kL + u + t_0) \leq \eta^L + 1.$$

Therefore,

$$W \sum_{k=0}^{k_{\omega_2}-1} \sum_{\substack{i \notin S(k+1) \\ j \in S(k)}} \sum_{u=0}^{L-1} a_{ij}(kL + u + t_0) \leq \omega_2 \left\lfloor \frac{N}{2} \right\rfloor (\eta^L + 1).$$

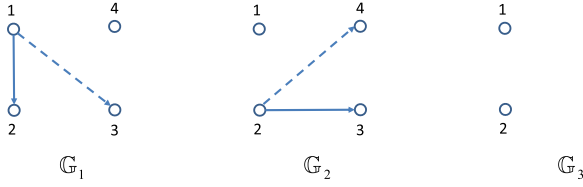


Fig. 2. Interaction graph switches periodically among \mathbb{G}_1 , \mathbb{G}_2 , and \mathbb{G}_3 .

By the definition of (9), $k^* \geq k_{\omega_2}$. Applying (45), one has that if $t \geq k^*L + t_0$, then

$$\begin{aligned} \Psi(t) &\leq \Psi(k^*L + t_0) \leq \Psi(k_{\omega_2}L + t_0) \\ &\leq \left(1 - \frac{K_* - \lfloor \frac{N}{2} \rfloor}{(8N^2)^{\lfloor \frac{N}{2} \rfloor}}\right)^{\omega_2} \Psi(t_0) \\ &\leq \epsilon \Psi(t_0). \end{aligned}$$

This proves the desired contraction rate.

V. EXAMPLES

We first provide an example to validate the results derived in Section II.

Example 2: Consider a four-agent system. Assume that the interaction graph switches periodically among three graphs \mathbb{G}_1 , \mathbb{G}_2 , and \mathbb{G}_3 given in Fig. 2. Let the initial time $t_0 = 1$.

- 1) For $t = 3k + 1$, $a_{21}(t) = \frac{1}{t} a_{31}(t) = \frac{1}{t^2}$.
- 2) For $t = 3k + 2$, $a_{32}(t) = \frac{1}{t} a_{42}(t) = \frac{1}{t^2}$.
- 3) For $t = 3k + 3$, $a_{43}(t) = \frac{1}{t}$.

$k \geq 0$, all the other values of $a_{ij}(t)$ that are not explicitly given are zero, and $a_{ii}(t)$, $i \in \mathcal{V}$ can be calculated such that the system matrix $A(t)$ is a stochastic matrix. It is easy to see that the persistent graph \mathbb{G}_p contains three persistent arcs $\{(1, 2), (2, 3), (3, 4)\}$ and, therefore, \mathbb{G}_p is a directed tree. The system matrix satisfies Assumption 2 with $L = 3$ and $K = 2$, but does not satisfy the cut-balance or arc-balance condition for $L = 1$. Though the graph sequence associated with the system matrix is repeatedly jointly rooted [6], the nonzero weights of those arcs decay to zero. The results in the literature do not apply here while Theorem 1 implies the convergence of the system to consensus as shown in Fig. 3. However, the convergence is not exponentially fast.

For the consensus system (1), the assumption that the nonzero elements of $A(t)$ are lower bounded by a positive constant η is often imposed [2], [6], [18]. Assumption 1 has relaxed this by discarding the requirement on the positive lower boundness for the off-diagonal elements of $A(t)$. However, the existence of η as a lower bound for $a_{ii}(t)$ is critical for the convergence to consensus of system (1). Next, we give an example to illustrate that if the diagonal elements are not lower bounded by η , then consensus may not be reached under the same conditions as in Theorem 2.

Example 3: Consider a three-agent system. Assume that the interaction graph switches periodically among three graphs \mathbb{G}_1 , \mathbb{G}_2 , and \mathbb{G}_3 given in Fig. 4. Let the initial time $t_0 = 1$.

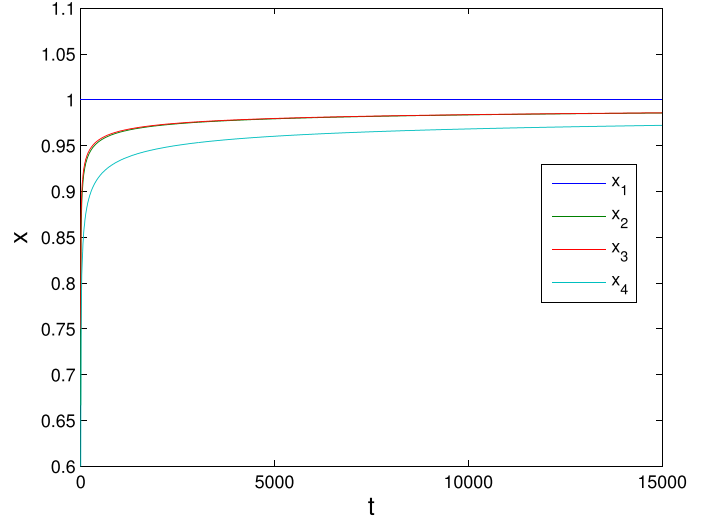


Fig. 3. System state reaches a consensus but takes a long time.

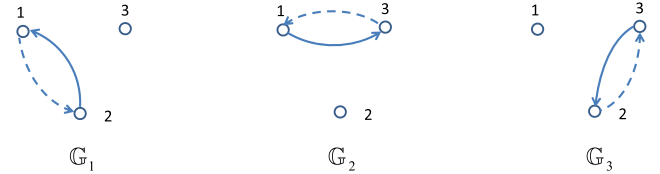


Fig. 4. Interaction graph switches periodically among \mathbb{G}_1 , \mathbb{G}_2 , and \mathbb{G}_3 .

The system matrix $A(t)$ is given by

$$\begin{aligned} A(3k+1) &= \begin{bmatrix} \frac{1}{3k+1} & 1 - \frac{1}{3k+1} & 0 \\ 1 - \frac{1}{(3k+1)^2} & \frac{1}{(3k+1)^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ A(3k+2) &= \begin{bmatrix} \frac{1}{(3k+2)^2} & 0 & 1 - \frac{1}{(3k+2)^2} \\ 0 & 1 & 0 \\ 1 - \frac{1}{3k+2} & 0 & \frac{1}{3k+2} \end{bmatrix} \\ A(3k+3) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3k+3} & 1 - \frac{1}{3k+3} \\ 0 & 1 - \frac{1}{(3k+3)^2} & \frac{1}{(3k+3)^2} \end{bmatrix} \end{aligned}$$

for $k \geq 0$. Note that though the matrix $A(t)$ has positive diagonals for all $t \geq 1$, there does not exist a positive constant $\eta > 0$ such that $a_{ii}(t) \geq \eta$ for all $t \geq 1$ since $A(3k+r)$ has some positive element converging to 0 for all $r = 1, 2, 3$ as $k \rightarrow \infty$.

One can verify that the matrix sequence $A(t)$, $t \geq 1$, satisfies Assumption 3 with $K = 2$ and $L = 1$ in (5) since $\frac{1 - \frac{1}{t^2}}{1 - \frac{1}{t}} = \frac{t+1}{t} \leq 2$ for all $t \geq 1$. However, it does not satisfy the balanced asymmetry condition in (35). To see this, consider the matrix $A(3k+1)$, $k \geq 0$, and let $S_1 = \{1\}$ and $S_2 = \{2\}$. It is easy to see that

$$\sum_{\substack{i \in S_2 \\ j \in S_1}} a_{ij}(3k+1) = \frac{1}{3k+1} = (3k+1) \sum_{\substack{i \in S_1 \\ j \in S_2}} a_{ij}(3k+1).$$

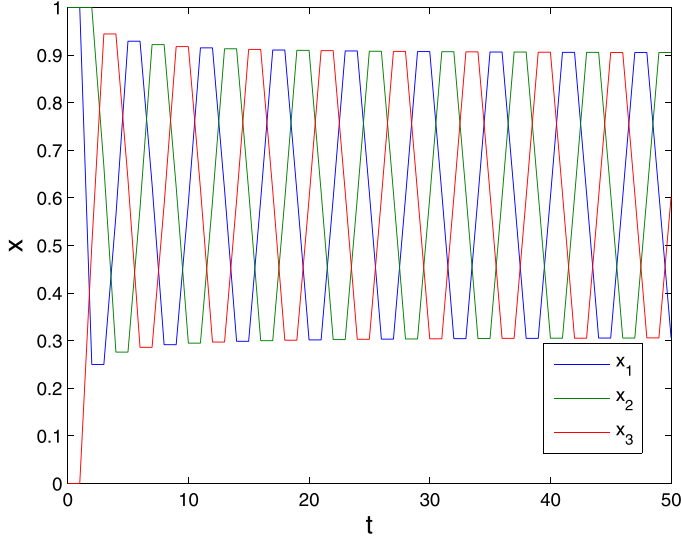


Fig. 5. System state does not reach a consensus.

One concludes that the sequence $A(t)$, $t \geq 1$ does not satisfy the balanced asymmetry condition since $3k + 1$ is not bounded as $k \rightarrow \infty$.

It is obvious that the persistent graph \mathbb{G}_p is strongly connected. In addition, the matrix sequence $A(t)$, $t \geq 1$, has the absolute infinite flow property. To verify this, one only has to consider the sequence $S(t)$, $t \geq 1$, of sets with cardinality equal to 1 since $\mathcal{V} \setminus S(t)$ also appears in the definition of absolute infinite flow property and there are three agents in total. Assume that each $S(t)$ has cardinality equal to 1. For any $t = 3k + 1$, $k \geq 0$ and the set $S(3k + 1) = \{1\}$, one can see that

$$\sum_{\substack{i \notin S(3k+2) \\ j \in S(3k+1)}} a_{ij}(3k+1) + \sum_{\substack{i \in S(3k+2) \\ j \notin S(3k+1)}} a_{ij}(3k+1) \geq \frac{1}{3k+1} \quad (46)$$

for any $S(3k + 2)$. For $S(3k + 1) = \{2\}$, the above inequality also holds for any $S(3k + 2)$. For $S(3k + 1) = \{3\}$, if $S(3k + 2)$ is $\{1\}$ or $\{2\}$, then the left-hand side of (46) is at least 2; if $S(3k + 2) = \{3\}$, then the left-hand side of (46) is 0, in which case it is clear that for any $S(3k + 3)$

$$\sum_{\substack{i \notin S(3k+3) \\ j \in S(3k+2)}} a_{ij}(3k+2) + \sum_{\substack{i \in S(3k+3) \\ j \notin S(3k+2)}} a_{ij}(3k+2) \geq \frac{1}{3k+2}.$$

To sum up, in all cases one has

$$\sum_{r=1}^3 \left(\sum_{\substack{i \notin S(3k+r+1) \\ j \in S(3k+r)}} a_{ij}(3k+r) + \sum_{\substack{i \in S(3k+r+1) \\ j \notin S(3k+r)}} a_{ij}(3k+r) \right) \geq \frac{1}{3k+2}$$

for all nonempty proper subset $S(3k + r)$ of \mathcal{V} satisfying $|S(3k + r)| = |S(3k + r + 1)|$, $r = 1, 2, 3$, and $k \geq 0$, which

implies that

$$\sum_{t=1}^{\infty} \left(\sum_{\substack{i \notin S(t+1) \\ j \in S(t)}} a_{ij}(t) + \sum_{\substack{i \in S(t+1) \\ j \notin S(t)}} a_{ij}(t) \right) \geq \sum_{k=0}^{\infty} \frac{1}{3k+2} = \infty$$

for all nonempty proper sequence $S(t)$, $t \geq 1$ of subsets of \mathcal{V} with the same cardinality. One concludes that the matrix sequence $A(t)$, $t \geq 1$ has the absolute infinite flow property.

We next show that global consensus cannot be reached. Consider the initial condition $x(1) = [1, 1, 0]^T$. It is obvious that $\Psi(1) = \Psi(2) = 1$. For $t \geq 2$, one can show that

$$\Psi(t+1) \geq \left(1 - \frac{1}{t^2}\right) \Psi(t).$$

It follows that:

$$\lim_{t \rightarrow \infty} \Psi(t) \geq \prod_{t=2}^{\infty} \left(1 - \frac{1}{t^2}\right) M(2) = \prod_{t=2}^{\infty} \left(1 - \frac{1}{t^2}\right) > 0$$

since $\sum_{t=2}^{\infty} \frac{1}{t^2} < \infty$. The evolution of the system state is depicted in Fig. 5, which illustrates the disagreement of the system states. ■

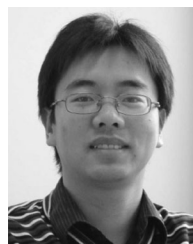
VI. CONCLUSION

In this paper, we have generalized the cut-balance and arc-balance conditions in the literature so as to allow for noninstantaneous reciprocal interactions between agents. The assumption on the existence of a lower bound on the nonzero weights a_{ij} of the arcs has been relaxed. Illustrative examples have been provided to show the necessity of imposing a positive lower bound on the self-weights of the agents. It has been shown that global consensus is reached if and only if the persistent graph contains a directed spanning tree. The estimate of the convergence rate of the discrete-time system has been given, which is not established for the cut-balance case in [13]. Future work may consider multiagent systems consisting of agents interacting with each other through attractive and repulsive couplings [1], [8], [21], [28].

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