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Truncated Long-Range Percolation on Oriented Graphs

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Abstract We consider different problems within the general theme of long-range percolation on oriented graphs. Our aim is to settle the so-called truncation question, described as follows. We are given probabilities that certain long-range oriented bonds are open; assuming that the sum of these probabilities is infinite, we ask if the probability of percolation is positive when we truncate the graph, disallowing bonds of range above a possibly large but finite threshold. We give some conditions in which the answer is affirmative. We also translate some of our results on oriented percolation to the context of a long-range contact process.

Keywords Contact processes · Oriented percolation · Long-range percolation · Truncation

Mathematics Subject Classification 60K35 · 82B43

1 Introduction

Let $G = (\mathbb{V}(G), \mathbb{E}(G))$ be the graph with set of vertices $\mathbb{V} = \mathbb{Z}^d$ and set of (unoriented) bonds $\mathbb{E} = \{\langle \vec{x}, \vec{x} + i \cdot \vec{e}_m \rangle : \vec{x} \in \mathbb{Z}^d,\ i \in \mathbb{Z},\ m \in \{1, \ldots, d\}\}$, where $\vec{e}_1, \ldots, \vec{e}_d$ denote the vectors in the canonical basis of $\mathbb{Z}^d$. Let $(p_i)_{i=1}^{\infty}$ be a sequence in the interval $[0, 1]$ and consider a Bernoulli bond percolation model where each bond $e \in \mathbb{E}$ is open with probability $p_{\|e\|}$, where $\|e\|$ denotes the l_∞ distance between the two endpoints of e. That is, take $(\Omega, \mathcal{A}, P)$, where $\Omega = \{0, 1\}^{\mathbb{E}}$, $\mathcal{A}$ is the canonical product σ-algebra, and $P = \prod_{e \in \mathbb{E}} \mu_e$, where $\mu_e(\omega_e = 1) = p_{\|e\|} = 1 - \mu_e(\omega_e = 0)$. An element $\omega \in \Omega$ is called a percolation configuration. As usual, the set $\{0 \leftrightarrow \infty\}$ denotes the set of configurations such that the origin is connected to infinitely many vertices by paths of open bonds (bonds where $\omega_e = 1$).

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We now consider a truncation of the sequence $(p_i)_i$ at some finite range k. More precisely, for each $k > 0$ consider the truncated sequence $(p_i^k)_{i=1}^{\infty}$, defined by

$$p_i^k = \begin{cases} p_i, & \text{if } i \leq k, \\ 0, & \text{if } i > k. \end{cases} \tag{1}$$

and the measure $P^k = \prod_{e \in \mathbb{E}} \mu_e^k$, where $\mu_e^k(\omega_e = 1) = p_{\|e\|}^k = 1 - \mu_e^k(\omega_e = 0)$. Then, the *truncation question* is: in case $P\{0 \leftrightarrow \infty\} > 0$, do we have $P^k\{0 \leftrightarrow \infty\} > 0$ for k large enough?

The problem seems to have been considered first in [14], who studied the question for exponentially decaying $(p_i)_i$ in two or more dimensions, where an affirmative answer to the truncation question was found. Afterwards, affirmative answers have been derived under different sets of assumptions on the dimension d and the sequence $(p_i)_i$ in the works [2,7,8, 13,15,17].

The principal assumption concerning the sequence $(p_i)_i$ for us will be

$$\sum_{i=1}^{\infty} p_i = \infty, \tag{2}$$

so that, by the Borel–Cantelli Lemma, we have $P\{0 \leftrightarrow \infty\} = 1$. [7] gives an affirmative answer to the truncation question for $d \geq 3$ and no assumption on (p_i) other than (2); moreover, this work shows how the analogous question for the long-range Potts model can be studied via a long range percolation model. We would like to mention that the general truncation question for $d = 2$ under the assumption (2) is still open and it is not difficult to see that for $d = 1$ the answer is negative.

In the nonsummable situation, the positive answer to the truncation question (in dimensions more than 1) appears to be more robust than in the summable case. Indeed, the presence of first-order transitions in the occupation density, or in a temperature-like parameter for summable infinite-range models, causes the truncation question to have a negative answer, as observed in [7]. Although continuity of the transition is known for Ising models, and their associated random-cluster models, in considerable generality (see for example the recent work [1]), this is not the case for independent percolation, where even in $d = 3$ it is a famous open question in the nearest-neighbor model, while for q-state Potts models first-order transitions are quite common for $q \geq 3$ (see the Refs. [4,6,9]).

In this paper, we consider the truncation question in an oriented graph. Let $\mathcal{G} = (\mathbb{V}(\mathcal{G}), \mathbb{E}(\mathcal{G}))$ be the oriented graph defined as follows. The vertex set is $\mathbb{V}(\mathcal{G}) = \mathbb{Z}^d \times \mathbb{Z}_+$, where $\mathbb{Z}_+ = \{0, 1, \ldots\}$; elements of $\mathbb{V}(\mathcal{G})$ will be denoted $(\vec{x}, n)$, where $\vec{x} \in \mathbb{Z}^d$ and $n \in \mathbb{Z}_+$. The set $\mathbb{E}(\mathcal{G})$ of oriented bonds is

$$\Big\{\big\langle(\vec{x}, n), (\vec{x} + i \cdot \vec{e}_m, n + 1)\big\rangle : \vec{x} \in \mathbb{Z}^d,\ n \in \mathbb{Z}_+,\ m \in \{1, \ldots, d\},\ i \in \mathbb{Z}\Big\}. \tag{3}$$

Again we are given a sequence $(p_i)_{i=1}^{\infty}$ satisfying (2) and we assume each bond $\langle(\vec{x}, n), (\vec{x} + i \cdot \vec{e}_m, n + 1)\rangle$ is open with probability p_i independently of each other. Again denoting by P the probability measure corresponding to this percolation configuration and by $\{(\vec{0}, 0) \leftrightarrow \infty\}$ the event that there exists an infinite open oriented path starting from $(\vec{0}, 0)$, Borel–Cantelli gives $P\{(\vec{0}, 0) \leftrightarrow \infty\} = 1$. For each $k > 0$, we then consider the truncated sequence given in (1) and the corresponding measure P^k and ask the truncation question, that is, whether $P^k\{(\vec{0}, 0) \leftrightarrow \infty\} > 0$. We prove:

Theorem 1 *For any $d \geq 2$, if the sequence $(p_i)_i$ satisfies (2), the truncation question has an affirmative answer for the graph $\mathcal{G}$. Moreover,*

$$\lim_{k\to\infty} P^k\{(\vec{0}, 0) \leftrightarrow \infty\} = 1.$$

Oriented percolation is an active field of study, and the model is both studied for its own sake, as well as a tool for analysis of and comparison with other models (see for example [3, 16]). The separate study of oriented versus unoriented versions of lattice models is justified by several considerations. First, in some cases one of the two versions turns out to be technically more accessible, or to lend itself to be studied by different approaches. For example, the oriented Toom model has been shown to have a phase transition (see [12]), which is still open for the related symmetric majority probabilistic cellular automaton and interacting particle system [5]. Another example is the present paper: we obtain some $(2+1)$-dimensional results whereas the equivalent result is still open in the unoriented 2-dimensional case (known results for the unoriented case do not imply our results in any dimension). Second, there are cases in which the unoriented and oriented models present distinct properties. For instance, in percolation the upper critical dimensions are 6 for unoriented and $4+1$ for oriented models (see [18]). Third, the two models typically serve distinct purposes: in the oriented case, one of the coordinates plays a separate role, as it often models time, as opposed to space which is modeled by the other coordinate(s).

We also notice that, given a particular unoriented percolation problem (such as the truncation question), there is often more than one way to consider analogous questions for oriented models, which is why in the present work we consider a number of different settings.

Theorem 1 is proved in the next section. In Sect. 3, we will treat a related question for the contact process and also for a different oriented graph.

2 Proof of Theorem 1

We obtain Theorem 1 as an immediate consequence of a stronger result, which we now describe. We fix $d = 2$ and consider $\mathcal{G}$ defined as above, with vertex set $\mathbb{Z}^2 \times \mathbb{Z}_+$ and set of oriented bonds given in (3). We take two sequences (p_i), (q_i) and now prescribe that bonds of the form $\langle(\vec{x}, n), (\vec{x} + i \cdot \vec{e}_1, n+1)\rangle$ are open with probability p_i and bonds of the form $\langle(\vec{x}, n), (\vec{x} + i \cdot \vec{e}_2, n+1)\rangle$ are open with probability q_i. The truncated measure P^k is obtained by truncating both sequences $(p_i)_i$ and $(q_i)_i$ at range k.

Proposition 1 *If $(p_i)_{i=1}^{\infty}$ satisfies (2) and $(q_i)_{i=1}^{\infty}$ is not identically zero, then* $\lim_{k\to\infty} P^k\{(\vec{0}, 0) \leftrightarrow \infty\} = 1$.

Proof By assumption, we can fix $\beta \in \mathbb{N}$ such that $q_\beta > 0$.

We will define certain *bifurcation events* which will imply that a point $(\vec{x}, n)$ is connected to two new points $(\vec{y}, n+2)$ and $(\vec{z}, n+2)$. For each $(\vec{x}, n) \in \mathcal{G}$, define the *bifurcation event*

$$E_{(\vec{x},n)} = \bigcup_{a,a' \in \mathbb{Z}} \left\{ \begin{array}{l} \omega_{\langle(\vec{x},n),(\vec{x}+a\vec{e}_1,n+1)\rangle} \\ = \omega_{\langle(\vec{x}+a\vec{e}_1,n+1),(\vec{x}+a\vec{e}_1+\beta\vec{e}_2,n+2)\rangle} \\ = \omega_{\langle(\vec{x}+a\vec{e}_1,n+1),(\vec{x}+a\vec{e}_1+a'\vec{e}_1,n+2)\rangle} = 1 \end{array} \right\}.$$

We have

$$P^k\left(E_{(\vec{x},n)}\right) = 1 - \prod_{a:|a|\leq k} \left(1 - p_{|a|} \cdot q_\beta \cdot \left(1 - \prod_{a':|a'|\leq k} (1 - p_{|a'|})\right)\right) = \gamma_k,$$

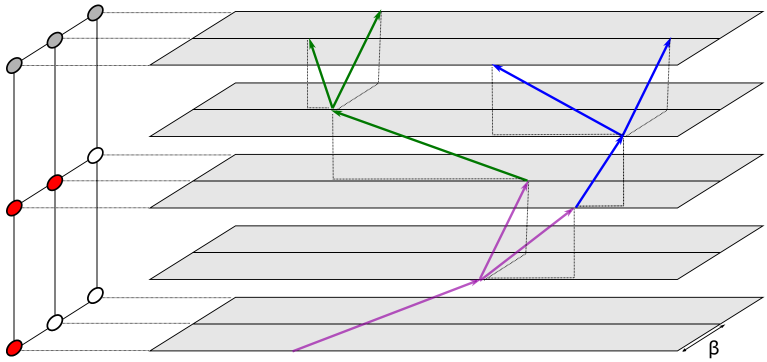

Fig. 1 The occurrence of each bifurcation event is represented by a triple of *arrows with the same color*. On the *left side* of the picture, we represent a certain projection which will be defined from these events: *red vertices* will appear at the (projected) starting points of bifurcations. With the information available in the picture, it is impossible to tell whether or not the three vertices on top are *red*

which can be made arbitrarily close to 1 by increasing k, by (2). Also note that

$$\{(\vec{0},0) \leftrightarrow (\vec{x},n)\} \cap E_{(\vec{x},n)} \subseteq \bigcup_{a,a' \in \mathbb{Z}_+} \left\{ \begin{array}{l} (\vec{0},0) \leftrightarrow (\vec{x}+a\vec{e}_1+a'\vec{e}_1, n+2), \\ (\vec{0},0) \leftrightarrow (\vec{x}+a\vec{e}_1+\beta\vec{e}_2, n+2) \end{array} \right\}. \tag{4}$$

Finally, under P and P^k, $E_{((a,b),m)}$ and $E_{((a',b'),n)}$ are independent and identically distributed as soon as either $b \neq b'$ or $|m-n| \geq 2$.

The next step is to prove that, if k is large enough, a certain projection of the k-truncated process dominates an oriented supercritical Bernoulli percolation on $\mathbb{Z}_+^2$. Define the following order in $\mathbb{Z}_+^2$: given $(m_1,n_1), (m_2,n_2) \in \mathbb{Z}_+^2$ we say that $(m_1,n_1) \prec (m_2,n_2)$ if and only if $n_1 < n_2$ or ($n_1 = n_2$ and $m_1 < m_2$). Given $X \subset \mathbb{Z}_+^2$, we define the exterior boundary of X as the set

$$\partial_e X = \{(m,n) \in \mathbb{Z}_+^2 \backslash X : (m,n-1) \in X \text{ or } (m-1,n-1) \in X\}.$$

We define the vertex $(m,n) \in \mathbb{Z}_+^2$ as *red* if and only if the following event occurs: $\bigcup_{a\in\mathbb{Z}} \left(\left\{(\vec{0},0) \leftrightarrow ((a,m\beta),2n)\right\} \cap E_{((a,m\beta),2n)}\right)$.

To avoid confusion, let us emphasize that, if a vertex in $\mathbb{Z}_+^2$ has coordinates (m,n), then this vertex is defined as red through an event in the original lattice $\mathbb{Z}^2 \times \mathbb{Z}_+$; this event involves a bifurcation with some starting point in the line $\{((a,m\beta),2n) : a \in \mathbb{Z}\}$. In particular, in Fig. 1, one horizontal unit and one vertical unit in the lattice depicted on the left correspond respectively to β units and 2 units in the lattice on the right.

We will construct a red cluster dynamically, defining inductively two sequences $(A_i)_i$ and $(B_i)_i$ of subsets of $\mathbb{Z}_+^2$. Set $A_0 = B_0 = \emptyset$ and $x_0 = (0,0)$. Assuming A_j, B_j and x_j have been defined for $j = 0,\ldots,i$, we let

$$A_{i+1} = \begin{cases} A_i \cup \{x_i\}, & \text{if } x_i \text{ is red,} \\ A_i, & \text{otherwise,} \end{cases} \qquad B_{i+1} = \begin{cases} B_i, & \text{if } x_i \text{ is red,} \\ B_i \cup \{x_i\}, & \text{otherwise.} \end{cases}$$

Now, if $(\partial_e A_{i+1})\backslash B_{i+1} = \emptyset$, we stop our recursive definition. Otherwise we let x_{i+1} be the minimal point of $(\partial_e A_{i+1})\backslash B_{i+1}$ with respect to the order $\prec$ defined above, and continue the recursion. Regardless of whether or not the recursion ever ends, we let $\mathcal{C}$ be the union of all sets A_i that have been defined. It follows from (4) that $\{|\mathcal{C}| = \infty\} \subseteq \{(\vec{0},0) \leftrightarrow \infty\}$.

Now, observe that

$$P^k(x_i \text{ is red} \mid (A_j, B_j) : 0 \leq j \leq i) \geq \gamma_k.$$

This implies that $\mathcal{C}$ stochastically dominates the cluster of the origin in Bernoulli oriented site percolation on $\mathbb{Z}_+^2$ with parameter γ_k (see Lemma 1 of [10]). As noted earlier, γ_k can be made arbitrarily close to 1; this proves that $\lim_{k\to\infty} P^k(|\mathcal{C}| = \infty) = 1$. □

3 Contact Process and Oriented Percolation on Other Graphs

3.1 The Contact Process

Here we will give a counterpart of Theorem 1 for the contact process obtained from truncating an infinite set of rates. Let us define precisely the model that we have in mind. We are given a sequence of non-negative real numbers, $(\lambda_i)_{i=1}^\infty$. We take a family of independent Poisson point processes on $[0, \infty)$:

- a process $D^{\vec{x}}$ of rate 1 for each $\vec{x} \in \mathbb{Z}^d$;
- a process $B^{(\vec{x},\vec{y})}$ of rate $\lambda_{|i|}$ for each ordered pair $(\vec{x}, \vec{y})$ with $\vec{x} \in \mathbb{Z}^d$ and $\vec{y} = \vec{x} + i \cdot \vec{e}_m$ with $i \in \mathbb{Z}$ and $m \in \{1, \ldots, d\}$.

We view each of these processes as a random discrete subset of $[0, \infty)$ and write, for $0 \leq a < b$, $D^{\vec{x}}_{[a,b]} = D^{\vec{x}} \cap [a, b]$ and $B^{(\vec{x},\vec{y})}_{[a,b]} = B^{(\vec{x},\vec{y})} \cap [a, b]$.

Fix $k \in \mathbb{N}$. Given $\vec{x}, \vec{y} \in \mathbb{Z}^d$ and $0 \leq s \leq t$, we say $(\vec{x}, s)$ and $(\vec{y}, t)$ are k-connected, and write $(\vec{x}, s) \overset{k}{\leftrightarrow} (\vec{y}, t)$, if there exists a function $\gamma : [s, t] \to \mathbb{Z}^d$ that is right-continuous, constant between jumps and satisfies:

$$\begin{aligned} \gamma(s) = \vec{x},\ \ \gamma(t) = \vec{y}\ \ \text{and, for all } r \in [s,t],\ & \gamma(r) \notin D^{\gamma(r)}, \\ & r \in B^{(\gamma(r-),\gamma(r))} \text{ if } \gamma(r) \neq \gamma(r-), \\ & |\gamma(r) - \gamma(r-)| \leq k. \end{aligned}$$

We then define

$$\xi_{t,k}(\vec{x}) = I\{(\vec{0}, 0) \overset{k}{\leftrightarrow} (\vec{x}, t)\}, \qquad \vec{x} \in \mathbb{Z}^d,\ t \geq 0.$$

$(\xi_{t,k})_{t\geq 0}$ is then a Markov process on the space $\{0, 1\}^{\mathbb{Z}^d}$ for which the configuration that is identically equal to 0 (denoted here by $\underline{0}$) is absorbing. In case $\lambda_i > 0$ only for $i = 1$, $(\xi_{t,1})$ is the contact process of Harris [11].

Theorem 2 *For all $d \geq 2$, if $\sum_{i=1}^\infty \lambda_i = \infty$, then*

$$\lim_{k\to\infty} P\left(\xi_{t,k} \neq \underline{0} \ \text{ for all } t\right) = 1.$$

Proof It is enough to prove the case $d = 2$. Fix $\delta > 0$ and $k \in \mathbb{Z}_+$. Let $t_n = n\delta$, for $n \in \{0, 1, \ldots\}$. Fix b such that $\lambda_b > 0$.

For $\vec{x} \in \mathbb{Z}^d$ and $n \in \mathbb{Z}_+$, let $F_{(\vec{x},n)}$ be the event

$$\left\{D^{\vec{x}}_{[t_n,t_{n+1}]} = \varnothing\right\} \cap \bigcup_{a\in\mathbb{Z}} \left\{ \begin{array}{l} D^{\vec{x}+a\vec{e}_1}_{[t_n,t_{n+1}]} = D^{\vec{x}+a\vec{e}_1+b\vec{e}_2}_{[t_n,t_{n+1}]} = \varnothing, \\ B^{(\vec{x},\vec{x}+a\vec{e}_1)}_{[t_n,t_n+\delta/2]} \neq \varnothing,\ \ B^{(\vec{x}+a\vec{e}_1,\vec{x}+a\vec{e}_1+b\vec{e}_2)}_{[t_n+\delta/2,t_{n+1}]} \neq \varnothing \end{array} \right\}.$$

Then,

$$P^k(F_{(\vec{x},n)}) = e^{-\delta}\left(1 - \prod_{a=-k}^{k}\left(1 - e^{-2\delta}\cdot\left(1 - e^{-\frac{\lambda_{|a|}\delta}{2}}\right)\cdot\left(1 - e^{-\frac{\lambda_{|b|}\delta}{2}}\right)\right)\right).$$

By first taking δ small and then taking k large, the probability of these events can be made arbitrarily close to 1. Moreover,

$$\{\xi_{t_n,k}(\vec{x}) = 1\} \cap F_{(\vec{x},n)} \subseteq \bigcup_{a\in\mathbb{Z}} \left\{ \begin{array}{l} \xi_{t_{n+1},k}(\vec{x} + a\vec{e}_1) \\ = \xi_{t_{n+1},k}(\vec{x} + a\vec{e}_1 + b\vec{e}_2) = 1 \end{array} \right\}.$$

The proof is then completed with a comparison with oriented percolation almost identical to the one that established Proposition 1. □

3.2 Other Oriented Graphs

In this section we consider a graph $\mathcal{G}^* = (\mathbb{V}(\mathcal{G}^*), \mathbb{E}(\mathcal{G}^*))$. Once more, the vertex set is $\mathbb{V}(\mathcal{G}^*) = \mathbb{Z}^d \times \mathbb{Z}_+,\ d \geq 1$. The set of bonds $\mathbb{E}(\mathcal{G}^*)$ consists of two disjoint subsets; one of them, denoted $\mathbb{E}_v$, only contains oriented bonds, and the other, $\mathbb{E}_h$, only unoriented bonds. These subsets are given by

$$\mathbb{E}_v = \left\{\langle(\vec{x}, n), (\vec{x}, n+1)\rangle : \vec{x} \in \mathbb{Z}^d, n \in \mathbb{Z}_+\right\},$$
$$\mathbb{E}_h = \{\langle(\vec{x}, n), (\vec{x} + i\cdot\vec{e}_m, n)\rangle : \vec{x} \in \mathbb{Z}^d,\ n \in \mathbb{Z}_+,\ i \in \mathbb{Z},\ m \in \{1, \ldots, d\}\}.$$

That is, we are considering the hypercubic lattice where there are nearest neighbour, oriented bonds along the vertical direction and long range, unoriented bonds parallel to all other coordinate axes.

We consider an anisotropic oriented Bernoulli percolation on this graph. Given $\epsilon \in (0, 1)$ and a sequence $(p_i)_{i=1}^{\infty}$ in the interval $[0, 1]$, each bond $e \in \mathbb{E}$ is open with probability ϵ or $p_{\|e\|}$, if $e \in \mathbb{E}_v$ or $e \in \mathbb{E}_h$, respectively.

Given two vertices $(\vec{x}, m)$ and $(\vec{y}, n)$ with $m < n$, we say that $(\vec{x}, n)$ and $(\vec{y}, m)$ are connected if there exists a path

$$\left\langle(\vec{x}, n) = (\vec{x}_0, n_0), (\vec{x}_1, n_1), \ldots, (\vec{x}_s, n_s) = (\vec{y}, m)\right\rangle$$

such that $\langle(\vec{x}_i, n_i), (\vec{x}_{i+1}, n_{i+1})\rangle \in \mathbb{E}_h$ or ($\vec{x}_i = \vec{x}_{i+1}$ and $n_{i+1} = n_i + 1$) for all $i = 0, \ldots, s-1$, and the bonds $\langle(\vec{x}_i, n_i), (\vec{x}_{i+1}, n_{i+1})\rangle$ are open for all $i = 0, \ldots, s-1$. That is, all allowed paths use vertical bonds only in the upward direction. We use the notation $\{(\vec{0}, 0) \overset{*}{\leftrightarrow} \infty\}$ to denote the set of configurations in which there is an infinite open path starting at $(\vec{0}, 0)$. We use also the notations P and P^k to denote the non-truncated and the truncated (in the range k) probability measures, respectively.

Theorem 3 *For any $d \geq 2$, any $\epsilon > 0$ and any sequence $(p_i)_{i=1}^{\infty}$ such that $\sum_{i\in\mathbb{N}} p_i = \infty$, we have* $\lim_{k\to\infty} P^k\{(\vec{0}, 0) \overset{*}{\leftrightarrow} \infty\} = 1$.

A weaker result was proven in [7] (see Theorem 6 therein) in the context of non-oriented and isotropic percolation. The proof of Theorem 3 is inspired by the proof thereof [7].

Proof It is sufficient to prove the theorem for $d = 2$.

Let $\gamma : \mathbb{Z} \to \mathbb{Z}^2$ be the function satisfying

$$\gamma(0) = \vec{0}, \qquad \gamma(m+1) - \gamma(m) = \begin{cases} \vec{e}_1 & \text{if } m \text{ is even,} \\ -\vec{e}_2 & \text{if } m \text{ is odd.} \end{cases}$$

Define the events

$$H_{m,n} = \left\{ \begin{array}{l} (\gamma(m), n) \text{ and } (\gamma(m+1), n) \text{ are connected} \\ \text{by a path of open bonds of } \mathbb{E}_h \text{ that is} \\ \text{entirely contained in the line that contains} \\ (\gamma(m), n) \text{ and } (\gamma(m+1), n) \end{array} \right\}, \; m \in \mathbb{Z}, \; n \in \mathbb{Z}_+.$$

Clearly, $P^k(H_{m,n}) = P^k(H_{0,0})$ for all m, n. Also note that, if $(m_1, n_1) \neq (m_2, n_2)$, then the line that contains $(\gamma(m_1), n_1)$ and $(\gamma(m_1+1), n_1)$ does not share any bonds of $\mathbb{E}_h$ with the line that contains $(\gamma(m_2), n_2)$ and $(\gamma(m_2+1), n_2)$. Hence, the events $H_{m,n}$ defined above are independent. Moreover, we have

$$\lim_{k\to\infty} P^k(H_{m,n}) = 1 \tag{5}$$

(a proof of this can be found in the first few lines of the proof of Theorem 6 in [7]).

Now, fix $\epsilon > 0$ and $\delta > 0$. Let N be an integer satisfying $(1-(1-\epsilon)^N)^2 > 1 - \delta/2$. Then, using (5), choose $k > 0$ such that $(P^k(H_{0,0}))^{2N} > 1 - \delta/2$. Then let

$$\Lambda_0 = \{(a, n) \in \mathbb{Z} \times \mathbb{Z}_+ : a + n \text{ is even}\}.$$

For each $(a, n) \in \Lambda_0$, let $\zeta(a, n)$ be the indicator function of the event

$$\left(\bigcap_{m=aN}^{aN+2N-1} H_{m,n} \right) \cap \left(\bigcup_{m=aN}^{aN+N-1} \{\langle(\gamma(m), n), (\gamma(m), n+1)\rangle \text{ is open}\} \right) \cap$$

$$\left(\bigcup_{m=aN+N}^{aN+2N-1} \{\langle(\gamma(m), n), (\gamma(m), n+1)\rangle \text{ is open}\} \right).$$

Then, the elements of the sequence of random variables $(\zeta(a, n))_{(a,n)\in\Lambda_0}$ are independent and, by the choice of N, each of them is equal to 1 with probability $1 - \delta$. Now note that an infinite sequence $(a_i)_{i=0}^{\infty}$ such that $a_0 = 0$, $|a_{i+1} - a_i| = 1$ and $\zeta(a_i, i) = 1$ for each i necessarily corresponds to an infinite open path in G. Moreover, the probability of existence of such a sequence can be taken arbitrarily close to 1 since δ is arbitrary. □

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