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About the Definition of Port Variables for Contact Hamiltonian Systems

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Abstract. Extending the formulation of reversible thermodynamical transformations to the formulation of irreversible transformations of open thermodynamical systems different classes of nonlinear control systems has been defined in terms of control Hamiltonian systems defined on a contact manifold. In this paper we discuss the relation between the definition of variational control contact systems and the input-output contact systems. We have first given an expression of the variational control contact systems in terms of a nonlinear control systems. Secondly we have shown that the conservative input-output contact systems are a subclass of the contact variational systems with integrable output dynamics.

Keywords: Open irreversible thermodynamic systems · Nonlinear control systems · Hamiltonian systems on contact manifolds

1 Introduction

Extending the formulation of reversible thermodynamical transformations suggested in [11] to the formulation of irreversible transformations of open thermodynamical systems, a class of nonlinear control systems has been defined in terms of control Hamiltonian systems defined on a contact manifold [4,5,7,14]. Their dynamic properties as well as their feedback invariance and stabilization properties have been studied in [2,6,13,15]. An alternative definition, based on a variational formulation has been suggested in [10]. In this paper we shall discuss and compare this definition with the system-theoretic definition suggested in [13].

2 Control Hamiltonian Systems Defined on Contact Manifolds

Since Gibbs' work, it has been established that the Thermodynamic Phase Space is intrinsically defined as a *contact manifold*, that is a differentiable manifold

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$\mathcal{M} \ni \tilde{x}$ equipped with a contact form θ. In the sequel we shall denote by $\left(x_0, x, p^\top\right) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ a set of canonical coordinates[1].

It has also been established that the dynamics of thermodynamic systems subject to reversible and irreversible processes may be formulated in terms of contact Hamiltonian vector fields [4,5,7,11,14].

For *open* thermodynamic systems, a class of nonlinear control systems [12] has been defined, where the drift vector field and the input vector fields are both contact Hamiltonian vector fields [4,5,14]. Deriving from condition on structure preserving state feedback control, the natural output functions have then be defined as the contact Hamiltonian functions defining the input vector field [13]. An alternative definition of control contact systems, derived from a variational formulation, has been suggested in [10].

In this section we shall recall these two different definitions of control Hamiltonian systems and formulate the variational contact systems in terms of nonlinear control systems [12].

2.1 Input - Output Contact Systems [13]

Let us first recall the definition of input-output contact systems.

Definition 1 [13]**.** *An* input - output contact system *on the contact manifold* $(\mathcal{M}, \theta)$*, with input variable belonging the trivial vector bundle* $F = \mathcal{M} \times \mathbb{R}^m \ni (\tilde{x}, u)$ *over* $\mathcal{M}$ *and output variables being the dual vector bundle* $E = F^* \sim \mathcal{M} \times \mathbb{R}^m \ni (\tilde{x}, y)$*, is defined by the two functions* $K_0 \in C^\infty(\mathcal{M})$*, called the* internal contact Hamiltonian*,* $K_c \in C^\infty(\mathcal{M})$ *called the* interaction (or control) contact Hamiltonian*, and the state and output equations*

$$\frac{d\tilde{x}}{dt} = X_{K_0} + \sum_{i=1}^{m} X_{K_i} u_i \tag{1}$$

$$y_i = K_i(\tilde{x}) \quad i = 1, \ldots, m \tag{2}$$

where X_{K_0} *and* X_{K_i} *are the contact vector fields*[2] *of* $(\mathcal{M}, \theta)$ *generated by the contact Hamiltonians* K_0 *and* K_i *respectively.*

Note that input - output contact system are the analogue of input-output Hamiltonian systems defined on symplectic manifolds for driven mechanical systems [3,16,17] but extended to contact manifolds.

The models of physical systems such as heat diffusion or the Continuous Stirred Tank Reactor belong to a subclass of contact systems [4,5,14], called *conservative input-output contact systems.*

[1] The reader is referred to the classical textbooks [8, chap. V.] [1, app. 4.].

[2] Recall that a contact vector field X_K generated by the Hamiltonian function $K(\tilde{x})$ is the unique vector field satisfying

$$\begin{aligned} i_X\theta &= K \\ i_X d\theta &= -dK\left(\mathcal{H}\left(X\right)\right). \end{aligned} \tag{3}$$

Definition 2 [4]**.** *A* conservative *input-output contact system with respect to the Legendre submanifold* $\mathcal{L}$ *is an input-output contact system with the internal, respectively control, contact Hamiltonians* K_0*, respectively* K_i*, satisfying the two conditions:*

(i) they are invariants of the Reeb vector field, satisfying

$$i_E dK_0 = i_E dK_i = 0 \tag{4}$$

(ii) they satisfy the invariance condition

$$K_0\big|_{\mathcal{L}} = 0, \quad K_i\big|_{\mathcal{L}} = 0 \tag{5}$$

2.2 Control Contact System Arizing from a Variational Principle [10]

Arizing from the variational principle defined in [10] a more general class of contact systems has been defined which we briefly recall now.

Definition 3. *A* variational control contact system *[10] on the contact manifold* $(\mathcal{M}, \theta)$*, is defined by*

(i) the set of output variables *is defined by the vector bundle* $E \ni y$ *over*$\mathcal{M}$ *endowed with a (flat) covariant derivative* ∇
(ii) a bundle map $A : T^*\mathcal{M} \to E$ *with* $A(\theta) = 0$
(ii) the set of conjugated input variables *is the dual bundle* $E^* \ni u$ *over*$\mathcal{M}$
(iii) the input map *defined by the adjoint bundle map* $A^* : E^* \to T\mathcal{M}$
(iv) a smooth real function $K_0(\tilde{x})$*, called* internal contact Hamiltonian function

and the dynamical system $\frac{d\tilde{x}}{dt} = X(\tilde{x}, u, y)$ *associated with the unique vector field* $X(\tilde{x}, u, y)$ *satisfying*

$$\begin{aligned} i_{(X - A^* u)} d\theta + dK_0 = 0 \\ \theta(X) = i_X \theta = K_0 + \langle u, y \rangle \end{aligned} \tag{6}$$

Let us write the system explicitely in the form of a nonlinear control system. Firstly, notice that the condition $A(\theta) = 0$ is equivalent to

$$\operatorname{im} A^* \subset \ker \theta \tag{7}$$

that is, the image of A^* is contained in the field of contact elements $\ker\theta = \mathcal{C}$ (or *horizontal* with respect to θ).[3] Denoting by X_{K_0} the contact vector field

[3] The tangent bundle $T\mathcal{M}$ may be decomposed into

$$T\mathcal{M} = \ker d\theta \oplus \ker \theta \tag{8}$$

where $\ker d\theta$, called *vertical bundle* , is of rank 1 and is generated by the Reeb vector field and $\ker\theta$, called *horizontal bundle*, is of rank $2n$. Every vector field X on $\mathcal{M}$ may be decomposed in a unique way into

$$X = (i_X\theta)\, E + (X - (i_X\theta)\, E) \tag{9}$$

where $(i_X\theta)\, E \in \ker d\theta$ is vertical and $(X - (i_X\theta)\, E) = \mathcal{H}(X) \in \ker\theta = \mathcal{C}$ is horizontal with respect to the contact form θ.

generated by the internal contact Hamiltonian K_0 and using the decomposition of the tangent manifold (8), the vector field X defined by (6) becomes

$$\begin{aligned} X(\tilde{x}, u, y) &= \underbrace{(i_X\theta)\,E}_{\in \ker d\theta} + \underbrace{(X - (i_X\theta)\,E)}_{=\mathcal{H}(X)\in \ker\theta = \mathcal{C}} \\ &= (K_0 + \langle u, y\rangle)\,E + \underbrace{\mathcal{H}(X_{K_0}) + A^* u}_{\in \ker\theta = \mathcal{C}} \\ &= \underbrace{X_{K_0}}_{\text{drift contact vect. field}} + \underbrace{\underbrace{\langle u, y\rangle\,E}_{\in \ker d\theta} + \underbrace{A^* u}_{\in\ker\theta=\mathcal{C}}}_{\text{control vector field}} \end{aligned} \tag{10}$$

The second line of (10) shows the decomposition of the control vector field in terms of the vertical component which may be interpreted as the *power balance term* $K_0 + \langle u, y\rangle$ and the horizontal component which, using the tensor $\theta^\sharp$ mapping the semi-basic forms on the contact elements[4] , may be interpreted as a *Hamiltonian control system* defined on the contact elements

$$\theta^\sharp\,(dK_0 - (i_E dK_0)\,\theta) + A^* u \tag{12}$$

Note that these properties are due to the assumption (7).

The third line of (10) shows the decomposition of the control vector field into an *drift contact vector field* X_{K_0} defined by the internal Hamiltonian function K_0 and a *control vector field* decomposed into its vertical and horizontal parts.

The output variable y satisfies a dynamical equation on the output according to [10, p. 786–787]

$$\frac{d}{dt} y = A \circ d\theta\,(X(\tilde{x}, u, y)) \tag{13}$$

Using the expression (10), one obtains

$$\begin{aligned} d\theta\,(X(\tilde{x}, u, y)) &= i_{X(\tilde{x}, u, y)} d\theta \\ &= i_{X_{K_0}} d\theta + \langle u, y\rangle \underbrace{i_E d\theta}_{=0} + d\theta\,(A^*\,u) \\ &= [dK_0 - (i_E dK_0)\,\theta] + d\theta\,(A^*\,u) \end{aligned}$$

Using that $A(\theta) = 0$ hence the dynamics of the output (13) becomes

$$\tfrac{d}{dt} y = A\,([dK_0]) + (A \circ d\theta \circ A^*)\,u$$

[4] Any contact vector fields may be decomposed into

$$X_K = K\,E + \theta^\sharp\,(dK - (i_E dK)\,\theta) \tag{11}$$

where $K\,E$ is the vertical and $\theta^\sharp\,(dK - (i_E dK)\,\theta)$ is the horizontal components of the contact vector field where $\theta^\sharp$ denotes the inverse of the isomorphism $\theta^\flat\big|_{\mathcal{C}}$ from the vector space $\mathcal{C}$ of horizontal vector fields onto the space $\mathcal{F}$ of semi-basic 1-forms induced by the map $\theta^\flat(X) = -i_X d\theta$. [8, p. 293].

The Eq. (6) actually define the dynamical equations summarized in the following proposition.

Proposition 1. *The Eq. (6) defining the dynamics of a variational control contact system of definition 3, are equivalent to the dynamical system*

$$\frac{d\tilde{x}}{dt} = X_{K_0} + \langle u,\, y\rangle\, E + A^* u \tag{14}$$

$$\frac{dy}{dt} = A\left([dK_0]\right) + \left(A \circ d\theta \circ A^*\right) u \tag{15}$$

3 Relation Between Variational and Conservative Input-Output Contact Systems

In this section, we shall analyse the relations between conservative input-output contact systems of the definition 2 and the variational control contact systems of the definition 3. We shall give a direct proof that in this case the output dynamics (15) is *integrable* , that is when the output variable y may be expressed as a function of the state variable $\widetilde{x}$, as has been stated in [10, Sect. 4.1].

Proposition 2. *The conservative contact input-output system of definition 2 with internal contact Hamiltonian $K_0\left(\tilde{x}\right)$ and control contact Hamiltonians $-K_i\left(\tilde{x}\right)$ is a variational control contact system defined in definition 3 with internal contact Hamiltonian $K_0\left(\tilde{x}\right)$ and bundle map $A : T^*\mathcal{M} \to \mathbb{R}^n \times \mathcal{M}$ defined by*

$$A\left(\lambda\right) = \left(\langle \lambda,\, \mathcal{H}\left(K_i\right)\rangle\right)_{i=1,\,\dots,m} \tag{16}$$

Proof. Firstly, let us identify the dynamics Eqs. (14) and (1) by decomposing the input contact vector field into its vertical and horizontal part

$$\begin{aligned}\frac{d\tilde{x}}{dt} &= X_{K_0} - \sum_{i=1}^{m} X_{K_i} u_i \\ &= X_{K_0} - \sum_{i=1}^{m} K_i\left(\tilde{x}\right) u_i - \sum_{i=1}^{m} \mathcal{H}\left(X_{K_i}\right) u_i\end{aligned}$$

Comparing this expression with third line of (10), leads to the natural identification of the dual output bundle map $A^*\left(u\right) = \sum_{i=1}^{m} \mathcal{H}\left(X_{K_i}\right) u_i$ and the outputs $y_i = K_i\left(\tilde{x}\right)$. The map A^* obviously satisfies the condition (7) and its dual is by definition (16). Let us now check that the defined output indeed satisfies the dynamic Eq. (15). Using that that the functions K_i are invariants of the Reeb vector field : $i_E dK_i = 0$, let us compute the j-th component of $A\left(dK_i\right)$[5]

$$A\left(dK_i\right)_j = -\left\langle dK_i,\, \mathcal{H}\left(X_{K_i}\right)\right\rangle = \left[K_j,\, K_i\right]_\theta \quad i = 0, \dots, m\,,\ j = 1, \dots, m$$

[5] The *Jacobi bracket* $[f,\, g]_\theta$ of two differentiable functions f and g , defined by $[f,\, g]_\theta = i_E\left([X_f,\, X_g]\right)$ where $[\ ,]$denotes the Lie bracket on vector fields. We shall use the following identities $[f,\, g]_\theta = i_{X_f} dg - g\, i_E df = -i_{X_g} df + f\, i_E dg$.

Compute now the control term of the output Eq. (2), using again that that the functions K_i are invariants of the Reeb vector field

$$\begin{aligned}(A \circ d\theta \circ A^*)\, u &= A\left(d\theta\left(A^*\left(u\right)\right)\right) \\ &= -A\left(d\theta\left(\textstyle\sum_{i=1}^m u_i\, \mathcal{H}\left(X_{K_i}\right)\right)\right) \\ &= -\textstyle\sum_{i=1}^m u_i\, d\theta^\sharp\left(X_i, X_j\right) \\ &= -\textstyle\sum_{i=1}^m u_i \left(\left[K_j,\, K_i\right]_\theta\right)_{j=1,\ldots,\, m}\end{aligned}$$

Hence the second member of the dynamics (15) of the j-th component of output becomes

$$A\left(\left[dK_0\right]\right) + \left(A \circ d\theta \circ A^*\right) u = \left[K_j,\, K_0\right]_\theta - \textstyle\sum_{i=1}^m u_i \left[K_j,\, K_i\right]_\theta \tag{17}$$

Using that, for functions K_i are invariants of the Reeb vector field $\left[K_j,\, K_i\right]_\theta = L_{X_j} K_i$, one obtains

$$A\left(\left[dK_0\right]\right) + \left(A \circ d\theta \circ A^*\right) u = -\frac{dK_j}{dt} \tag{18}$$

Let us firstly notice that the output dynamics has a feedthrough term (depends explicitely on the input variables) which is linear in the Jacobi brackets of the control Hamiltonian functions. This resembles very much the situation for input-output Hamiltonian systems defined in symplectic or Poisson manifolds [9].

Let us discuss the example of integrable system given in [10, Sect. 4.1], for which the control contact Hamiltonians satisfy the conditions that they are in involution with respect to the Jacobi bracket. Indeed a contact manifold may be identified with the 1-jet of some manifold Q, (called *configuration manifold* in [10] and *manifold of independent extensive variables* in the context of Thermodynamics [4]). This 1-jet manifold may be identified with $\mathbb{R} \times T^*Q$ and equiped with the canonical contact structure. As the control Hamiltonian functions are chosen to be function of the configuration manifold only, they are in involution. If $\left[K_j,\, K_i\right]_\theta = 0\,, i,\, j = 1, \ldots,\, m$, then the output dynamics (17) does not depend on the control variables. It may be noticed that this condition is not fullfilled for the models of physical systems given in [4,5,14], except for the single input case of course.

4 Conclusion

In this paper we have discussed the relation between the definition of variational control contact systems suggested in [10] and the input-output contact systems defined in [13]. We have first given an expression of the variational control contact systems of [10] in terms of a nonlinear control systems. Secondly we have shown that the conservative input-output contact systems are a subclass of the contact variational systems defined in [10] with integrable output dynamics.

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