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A note on strict complementarity for the doubly non-negative cone

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ABSTRACT
In this paper, we consider a closed convex cone $\mathcal{K}$ given by the intersection of two cones $\mathcal{K}_1$ and $\mathcal{K}_2$. We study faces and complementary faces of $\mathcal{K}$ in terms of $\mathcal{K}_1$ and $\mathcal{K}_2$. Based on complementary faces, the tangent spaces of $\mathcal{K}$ can be characterized as well. Moreover, many numerical methods assume regularity conditions such as strict complementarity. We provide necessary and sufficient conditions for strict complementarity for the cone $\mathcal{K}$. All these results can be applied to the doubly non-negative cone. Finally, a numerically efficient procedure for checking strict complementarity of (X, Y) for the doubly non-negative cone is provided when X has exactly one zero eigenvalue.

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1. Introduction

Let us denote the space of real symmetric $m \times m$ matrices by $\mathcal{S}_m$ and the entries of a matrix $M \in \mathcal{S}_m$ by $M(i,j), i,j = 1, \ldots, m$. Conic programming includes linear programming, semi-definite programming, doubly non-negative cone programming by, respectively, taking (where * denotes the dual cone)

- the non-negative cone: $\mathcal{N}_m = \mathcal{N}_m^* = \{M \in \mathcal{S}_m \mid M(i,j) \geq 0 \text{ for all } i,j\}$,
- the positive semi-definite cone:
  $\mathcal{S}_m^+ = (\mathcal{S}_m^+)^* = \{M \in \mathcal{S}_m \mid z^{\mathrm{T}}Mz \geq 0 \text{ for all } z \in \mathbb{R}^m\}$,
- the doubly non-negative cone and its dual:
  $\mathcal{DNN}_m = \mathcal{S}_m^+ \cap \mathcal{N}_m \quad \text{and} \quad (\mathcal{DNN}_m)^* = \mathcal{S}_m^+ + \mathcal{N}_m.$

The feasible set of a linear conic program is given by the intersection of an affine space and a convex cone. Thus, the cone under consideration plays an important role as the objective function and all the other constraints are linear. The better we know the structure of a cone or a local description of a cone, the more efficiently we can solve these conic problems. The geometry of cones, in particular, that of the semi-definite and of the non-negative cone, was investigated in [1]. Properties such as non-degeneracy and strict complementarity were generalized from linear programming to conic programming and are defined in terms of minimal faces and complementary faces [1]. In this paper, we review and study faces, complementary faces and tangent spaces of cones which are given by the intersection of two closed convex cones, for example, the doubly non-negative cone.

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Numerical methods often assume regularity conditions such as Slater's condition or strict complementarity (see, e.g. [2]). Even though there is a general definition of strict complementarity in terms of complementary faces for cones in [1], there may be an even sharper equivalent description of strict complementarity depending on the cone under consideration. In linear programming, it is well known that complementarity requires $X(i,j)Y(i,j) = 0$ for all i,j and strict complementarity says that both components $X(i,j)$ and $Y(i,j)$ cannot be zero at the same time. In semi-definite programming, if $X, Y \in \mathcal{S}_m^+$ are complementary, i.e. $\langle X, Y\rangle = 0$, then the pair is strictly complementary if and only if $\text{rank}X + \text{rank}Y = m$ (see, e.g. [3]). It is well known that $\mathcal{N}_m$ and $\mathcal{S}_m^+$ are self-dual and facially exposed, while $(\mathcal{DNN}_m)^*$ is neither self-dual nor facially exposed (see, e.g. [4]). If the cones under consideration are not self-dual, then the strict complementarity definition in [1] can be stated from a primal or a dual cone perspective, which is not necessarily equivalent in general. Different definitions of strict complementarity for the doubly non-negative cone and their relations are studied in [2]. In Section 2, we describe the strict complementarity condition for a cone $\mathcal{K}$ in terms of two cones whose intersection is $\mathcal{K}$. In Section 3, we consider a pair (X, Y) of primal and dual solutions where X is a doubly non-negative matrix with exactly one zero eigenvalue, and we provide a numerically efficient procedure to check strict complementarity for this setting.

2. Strict complementarity and other properties for cones given by the intersection of two cones

In this section, we summarize the findings concerning cones given by the intersection of two closed convex cones and provide some other complementary results. The cone $\mathcal{K}$ in consideration is closed and convex. Let us first introduce some notations and definitions. We denote the closure, relative interior, boundary and linear span of a set S as cl$S$, ri$S$, bd$S$ and lin$S$, respectively. Let us denote a segment connecting points $X, Y \in \mathcal{S}_m$ as $[X, Y] := \{\lambda X + (1-\lambda)Y \mid \lambda \in [0, 1]\}$. Recall that $trace(M) := \sum_{i=1}^{m} M(i,i)$ denotes the trace of a matrix $M \in \mathcal{S}_m$. The standard inner product, sometimes referred to as Frobenius inner product, in the space $\mathcal{S}_m$ is given by $\langle X, Y\rangle := trace(XY)$ for $X, Y \in \mathcal{S}_m$. As usual, the dual cone $\mathcal{K}^*$ of $\mathcal{K}$ with respect to the standard inner product $\langle\cdot,\cdot\rangle$ in $\mathcal{S}_m$ is given as

$$\mathcal{K}^* := \{Y \in \mathcal{S}_m \mid \langle Y, X\rangle \geq 0 \text{ for all } X \in \mathcal{K}\}.$$

A pair $X \in \mathcal{K}, Y \in \mathcal{K}^*$ is called complementary if $\langle X, Y\rangle = 0$. A non-empty convex set $F \subseteq \mathcal{K}$ is called a face of $\mathcal{K}$ if the condition $\text{ri}[X, Y] \cap F \neq \emptyset$ implies that $[X, Y] \subset F$ for any segment $[X, Y] \subset \mathcal{K}$. If $F \neq \mathcal{K}$, then F is called a proper face. If a proper face F of $\mathcal{K}$ can be given as an intersection of a hyperplane and $\mathcal{K}$, then we say that F is an exposed face. Note that any non-empty intersection of $\mathcal{K}$ and a supporting hyperplane of $\mathcal{K}$ is an exposed face of $\mathcal{K}$. A cone is called facially exposed if all its proper faces are exposed.

Given $X \in \mathcal{K}$, we denote the minimal face of the cone $\mathcal{K}$ containing X by face$(X, \mathcal{K})$. By definition, we have $X \in \text{ri face}(X, \mathcal{K})$ for each $X \in \mathcal{K}$, see, e.g. the proof of [5, Theorem 1]. For a face F of $\mathcal{K}$, we define the complementary face as

$$F^{\triangle} := \{Q \in \mathcal{K}^* \mid \langle Q, S\rangle = 0 \text{ for all } S \in F\}.$$

Clearly, $F^{\triangle} \subseteq \mathcal{K}^*$ is a closed convex cone. Moreover, it is not difficult to see (cf., e.g. [1]) that if $X \in \text{ri } F$, then we have

$$F^{\triangle} = \{Q \in \mathcal{K}^* \mid \langle Q, X\rangle = 0\}.$$

For brevity, we write $\text{face}^{\triangle}(X, \mathcal{K})$ for $(\text{face}(X, \mathcal{K}))^{\triangle}$.

Definition 2.1: A pair $X \in \mathcal{K}$ and $Y \in \mathcal{K}^*$ is called strictly complementary for cone $\mathcal{K}$, if

$$Y \in \text{ri face}^{\triangle}(X, \mathcal{K}). \tag{1}$$

In [1], strict complementarity for $X \in \mathcal{K}, Y \in \mathcal{K}^*$ is defined by

$$\text{face}^{\triangle}(X, \mathcal{K}) = \text{face}(Y, \mathcal{K}^*) \tag{2}$$

and it is shown in [5, Theorem 2] that (1) and (2) are equivalent. From a dual perspective, the condition

$$\text{face}^{\triangle}(Y, \mathcal{K}^*) = \text{face}(X, \mathcal{K}) \tag{3}$$

is equivalent to $X \in \text{ri face}^{\triangle}(Y, \mathcal{K}^*)$. We will refer to (3) as the dual strict complementarity condition. For an illustrative example of these 'asymmetric' definitions of strict complementarity, we refer to [2, Example 1]. Neither of conditions (2) or (3) implies the other one unless $\mathcal{K}$ or $\mathcal{K}^*$ are facially exposed, as noted in [1, Remark 3.3.2]. In particular, when the primal cone is facially exposed, we state the following result for later use. In an analogous way, a similar proposition can be derived from the dual cone perspective.

Proposition 2.2: *Let $\mathcal{K}$ be facially exposed. Then the primal strict complementarity condition (2) implies the dual strict complementarity condition (3).*

Proof: Consider the complementary face of $\text{face}^{\triangle}(X, \mathcal{K})$. Using condition (2), we derive

$$\text{face}^{\triangle\triangle}(X, \mathcal{K}) = \text{face}^{\triangle}(Y, \mathcal{K}^*). \tag{4}$$

As $\mathcal{K}$ is facially exposed, we have $\text{face}\,(X, \mathcal{K}) = \text{face}^{\triangle\triangle}(X, \mathcal{K})$ (see [6]). Combining the latter with (4), we obtain the dual strict complementarity condition $\text{face}\,(X, \mathcal{K}) = \text{face}^{\triangle}(Y, \mathcal{K}^*)$. ■

In the remainder of this paper, we consider cones which are given by the intersection of two convex and closed cones $\mathcal{K}_1$ and $\mathcal{K}_2$. Let $\mathcal{K} = \mathcal{K}_1 \cap \mathcal{K}_2$, then it is well known that its dual cone is $\mathcal{K}^* = \text{cl}(\mathcal{K}_1^* + \mathcal{K}_2^*)$ (see, e.g. [4]). We assume that $\mathcal{K}^* = \mathcal{K}_1^* + \mathcal{K}_2^*$ holds throughout this paper, i.e. the set $\mathcal{K}_1^* + \mathcal{K}_2^*$ is closed. A well-known sufficient condition for the closedness is $\text{ri}\,\mathcal{K}_1 \cap \text{ri}\,\mathcal{K}_2 \neq \emptyset$, and there are other weaker conditions for $\mathcal{K}^* = \mathcal{K}_1^* + \mathcal{K}_2^*$ to hold. Especially, for $X \in \text{ri}\,(\mathcal{K}_1 \cap \mathcal{K}_2)$, the following condition is shown to be necessary and sufficient in [7, Theorem 5.1] if $\mathcal{K}_1$ and $\mathcal{K}_2$ are so-called nice cones:

$$\text{dir}(X, \mathcal{K}_1) \cap \text{dir}(X, \mathcal{K}_2) = \text{cl dir}(X, \mathcal{K}_1) \cap \text{cl dir}(X, \mathcal{K}_2),$$

where $\text{dir}(X, \mathcal{K}_i) := \{Z \mid X + tZ \in \mathcal{K}_i \text{ for some } t > 0\}$ is the set of feasible directions. We refer to [7] for the definition of a nice cone and further equivalent conditions.

Faces of $\mathcal{K}$ are fully described by the intersection of faces of $\mathcal{K}_1$ and $\mathcal{K}_2$ (see, e.g. [8]). Moreover, following the proof of [1, Theorem 3.3.1(2a)], the statement can be sharpened with respect to their corresponding affine spaces. For completeness, let us state these two results regarding the minimal faces:

Proposition 2.3: *For any $X \in \mathcal{K} = \mathcal{K}_1 \cap \mathcal{K}_2$, we have*

(a) $\text{face}(X, \mathcal{K}) = \text{face}(X, \mathcal{K}_1) \cap \text{face}\,(X, \mathcal{K}_2)$.
(b) $\text{lin face}(X, \mathcal{K}) = \text{lin face}(X, \mathcal{K}_1) \cap \text{lin face}(X, \mathcal{K}_2)$.

Some properties such as non-degeneracy in cone programming can be described by using complementary faces (see, e.g. [1]). We look at some relation between complementary faces of $\mathcal{K}_1, \mathcal{K}_2$ and $\mathcal{K}$.

Proposition 2.4: *Let $\mathcal{K} = \mathcal{K}_1 \cap \mathcal{K}_2$ and $\mathcal{K}^* = \mathcal{K}_1^* + \mathcal{K}_2^*$. The following holds for any $X \in \mathcal{K}$:*

$$\text{face}^{\triangle}(X, \mathcal{K}) = \text{face}^{\triangle}(X, \mathcal{K}_1) + \text{face}^{\triangle}(X, \mathcal{K}_2).$$

Proof: For $i = 1,2$, let $Y_i \in \text{face}^{\triangle}(X, \mathcal{K}_i)$. Since $Y_i \in \mathcal{K}_i^* \subset \mathcal{K}^*$ and $\langle X, Y_i \rangle = 0$, we have $Y_i \in \text{face}^{\triangle}(X, \mathcal{K})$. Using the fact that a complementary face is a convex cone, we derive that $Y_1 + Y_2 \in \text{face}^{\triangle}(X, \mathcal{K})$. Therefore, it is clear that $\text{face}^{\triangle}(X, \mathcal{K}_1) + \text{face}^{\triangle}(X, \mathcal{K}_2) \subseteq \text{face}^{\triangle}(X, \mathcal{K})$.

On the other hand, for any $Y \in \text{face}^{\triangle}(X, \mathcal{K}) \subset \mathcal{K}^* = \mathcal{K}_1^* + \mathcal{K}_2^*$, there exists a decomposition $Y = Z_1 + Z_2$ such that $Z_1 \in \mathcal{K}_1^*$ and $Z_2 \in \mathcal{K}_2^*$. As $X \in \mathcal{K}_1 \cap \mathcal{K}_2$ and $0 = \langle X, Y \rangle = \langle X, Z_1 \rangle + \langle X, Z_2 \rangle$, we obtain $\langle X, Z_i \rangle = 0$ by duality, and hence $Z_i \in \text{face}^{\triangle}(X, \mathcal{K}_i)$ for $i = 1, 2$. Therefore, $\text{face}^{\triangle}(X, \mathcal{K}) \subseteq \text{face}^{\triangle}(X, \mathcal{K}_1) + \text{face}^{\triangle}(X, \mathcal{K}_2)$ holds as desired. ■

Recall that $\mathcal{K} \subseteq \mathcal{S}_m$ is a closed convex cone and its tangent space at $X \in \mathcal{K}$ is defined as

$$\tan(X, \mathcal{K}) := \{Z \in \mathcal{S}_m \mid \text{dist}(X \pm tZ, \mathcal{K}) = o(t)\}.$$

where $dist(X \pm tZ, \mathcal{K})$ denotes the distance between point $X \pm tZ$ and cone $\mathcal{K}$.

Corollary 2.5: *Let* ri $\mathcal{K}_1 \cap$ ri $\mathcal{K}_2 \neq \emptyset$. *For any $X \in \mathcal{K} = \mathcal{K}_1 \cap \mathcal{K}_2$, we have*

$$\tan(X, \mathcal{K}) = \tan(X, \mathcal{K}_1) \cap \tan(X, \mathcal{K}_2) = [\text{face}^{\triangle}(X, \mathcal{K}_1) + \text{face}^{\triangle}(X, \mathcal{K}_2)]^{\perp}.$$

Proof: Under the assumption ri $\mathcal{K}_1 \cap$ ri $\mathcal{K}_2 \neq \emptyset$, we have $\mathcal{K}^* = \mathcal{K}_1^* + \mathcal{K}_2^*$ (see, e.g. [7]) and it is shown in [1, Proposition 3.2.3] that the tangent space at $X \in \mathcal{K}$ is $\tan(X, \mathcal{K}) = \tan(X, \mathcal{K}_1) \cap \tan(X, \mathcal{K}_2)$. Furthermore, it follows from [1, 9, Lemma 3.2.1] that the tangent space of a convex cone is characterized by its complementary face as $\tan(X, \mathcal{K}) = \text{face}^{\triangle}(X, \mathcal{K})^{\perp}$. By combining these two arguments together with Proposition 2.4, the statement follows directly. ■

Moreover, using Proposition 2.4, we can describe exposed minimal faces of the dual cone $\mathcal{K}^*$.

Corollary 2.6: *Let $\mathcal{K} = \mathcal{K}_1 \cap \mathcal{K}_2$ and $\mathcal{K}^* = \mathcal{K}_1^* + \mathcal{K}_2^*$. Assume that $Y \in \mathcal{K}^*$ and* $\text{face}(Y, \mathcal{K}^*)$ *is exposed. Then we have*

$$\text{face}(Y, \mathcal{K}^*) = \text{face}^{\triangle}(X, \mathcal{K}_1) + \text{face}^{\triangle}(X, \mathcal{K}_2)$$

for any $X \in$ ri $\text{face}^{\triangle}(Y, \mathcal{K}^*)$.

Proof: Condition $X \in$ ri $\text{face}^{\triangle}(Y, \mathcal{K}^*)$ is equivalent to the dual strict complementarity property (3) of (X, Y). By considering the complementary face of $\text{face}(X, \mathcal{K})$ and using (3), we obtain

$$\text{face}^{\triangle}(X, \mathcal{K}) = \text{face}^{\triangle\triangle}(Y, \mathcal{K}^*) = \text{face}(Y, \mathcal{K}^*)$$

where the last equality is due to the exposedness of $\text{face}(Y, \mathcal{K}^*)$, see [6, Theorem 6.7]. By Proposition 2.4, the statement follows directly. ■

Now let us look at a necessary and sufficient condition for strict complementarity.

Theorem 2.7: *Let $\mathcal{K} = \mathcal{K}_1 \cap \mathcal{K}_2$ and $\mathcal{K}^* = \mathcal{K}_1^* + \mathcal{K}_2^*$. The pair $X \in \mathcal{K}$ and $Y \in \mathcal{K}^*$ is strictly complementary for $\mathcal{K}$ if and only if there exists a decomposition $Y = Y_1 + Y_2$ with $Y_1 \in \mathcal{K}_1^*$ and $Y_2 \in \mathcal{K}_2^*$ such that $(X, Y_1) \in \mathcal{K}_1 \times \mathcal{K}_1^*$ is strictly complementary for cone $\mathcal{K}_1$ and $(X, Y_2) \in \mathcal{K}_2 \times \mathcal{K}_2^*$ is strictly complementary for cone $\mathcal{K}_2$.*

Proof: As the relative interior of a Minkowski sum is preserved, see, e.g. [10, Lemma 1.3.12] or [11], and by Proposition 2.4, we obtain

$$\text{ri face}^{\triangle}(X, \mathcal{K}) = \text{ri face}^{\triangle}(X, \mathcal{K}_1) + \text{ri face}^{\triangle}(X, \mathcal{K}_2)$$

and the statement follows directly. ■

3. Strict complementarity for the doubly non-negative cone

Optimization over the doubly non-negative cone provides significantly tight bounds for some combinatorial problems (see, e.g. [2, 12]). In this section, we specialize our findings to the doubly non-negative cone. Let us first present the non-negative cone and the semi-definite cone.

Non-negative cone: Let $X \in \mathcal{N}_m$. The corresponding faces and complementary faces are

$$\text{face}(X, \mathcal{N}_m) = \{Z \in \mathcal{N}_m \mid Z(i,j) = 0 \text{ for all } i,j \text{ such that } X(i,j) = 0\} \tag{5}$$

and

$$\text{face}^{\triangle}(X, \mathcal{N}_m) = \{Y \in \mathcal{N}_m \mid Y(i,j) = 0 \text{ for all } i,j \text{ such that } X(i,j) > 0\}. \tag{6}$$

Furthermore, it can easily be seen that a complementary pair $X \in \mathcal{N}_m$ and $Y \in \text{face}^{\triangle}(X, \mathcal{N}_m)$ is strictly complementary for $\mathcal{N}_m$ if and only if $X(i,j) + Y(i,j) > 0$ for all i,j.

Semi-definite cone: Let $X \in \mathcal{S}^+_m$. The faces and the complementary faces of the positive semi-definite cone are well known (see, e.g. [1]):

$$\text{face}(X, \mathcal{S}^+_m) = \{Z \in \mathcal{S}^+_m \mid \mathcal{R}(Z) \subseteq \mathcal{R}(X)\} \tag{7}$$

and

$$\text{face}^{\triangle}(X, \mathcal{S}^+_m) = \{Z \in \mathcal{S}^+_m \mid \mathcal{R}(Z) \subseteq \mathcal{R}(X)^{\perp}\}, \tag{8}$$

where $\mathcal{R}(X)$ is the linear subspace spanned by the columns of the matrix X. Understanding facial structures and faces leads to many results in semi-definite programming (see, e.g. [1]). Consider a pair (X, Y) of positive semi-definite $m \times m$ matrices such that $Y \in \text{face}^{\triangle}(X, \mathcal{S}^+_m)$. Let $\text{rank}\, X = r$ and $\text{rank}\, Y = s$. It is known that X and Y are strictly complementary for the positive semi-definite cone if and only if $r+s=m$ (see, e.g. [3]).

Let $X \in \mathcal{DNN}_m$. By Proposition 2.3 and combining (5), (7), its minimal face is given by

$$\begin{aligned}\text{face}(X, \mathcal{DNN}_m) = \{Z \in \mathcal{DNN}_m \mid \mathcal{R}(Z) \subseteq \mathcal{R}(X) \\ \text{and } Z(i,j) = 0 \text{ for all } i,j \text{ such that } X(i,j) = 0\}.\end{aligned}$$

For the complementary face, using Proposition 2.4 and combining (6) and (8), we have

$$\begin{aligned}\text{face}^{\triangle}(X, \mathcal{DNN}_m) = \{Z_1 + Z_2 \mid Z_1 \in \mathcal{S}^+_m, Z_2 \in \mathcal{N}_m \text{ such that} \\ \mathcal{R}(Z_1) \subseteq \mathcal{R}(X)^{\perp} \text{ and } Z_2(i,j) = 0 \text{ for all } i,j \text{ such that } X(i,j) > 0\}.\end{aligned}$$

Since the intersection of two facially exposed cones is facially exposed, the doubly non-negative cone is facially exposed while its dual is not [4]. In such cases, sometimes strict complementarity is defined by including both (2) and (3) in the literature (see, e.g. [2]). However, by Proposition 2.2, the primal strict complementarity definition (2) is sufficient for the both conditions for the doubly non-negative cone. Moreover, Theorem 2.7 can be applied to check strict complementarity.

Example 3.1: Consider $\mathcal{K} = \mathcal{DNN}_2 = \mathcal{S}^+_2 \cap \mathcal{N}_2$. Let

$$X = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

By construction, we have $Y \in \text{face}^{\triangle}(X, \mathcal{K})$. We search for a decomposition $Y = Y_1 + Y_2$ with $Y_1 \in \text{ri face}^{\triangle}(X, \mathcal{S}^+_2)$ and $Y_2 \in \text{ri face}^{\triangle}(X, \mathcal{N}_2)$. Using strict complementarity for the cones $\mathcal{S}^+_2$ and $\mathcal{N}_2$, it is easy to check that the strict complementary decomposition matrices must have the following form

$$Y_1 = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Y_2 = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_2 & 0 \end{pmatrix} \quad \text{with } \alpha, \beta_1, \beta_2 > 0.$$

It is clear that there does not exist any such decomposition with $Y = Y_1 + Y_2$ and thus $Y \notin \text{ri face}^{\triangle}(X, \mathcal{K})$. In other words, the pair (X, Y) is not strictly complementary for the doubly non-negative cone.

The following example illustrates that the strict complementarity decomposition $Y = Y_1 + Y_2$ is not unique and there are non-strict complementary decompositions as well.

Example 3.2: Consider $\mathcal{K} = \mathcal{DNN}_2$ and a pair $X = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $Y = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 4 \end{smallmatrix}\right)$. By choosing

$$Y_1 = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \quad \text{and} \quad Y_2 = \begin{pmatrix} 0.5 & 2 \\ 2 & 3.5 \end{pmatrix},$$

it is straightforward to check that $Y_1 \in \text{ri face}^{\triangle}(X, \mathcal{S}_2^+)$ and $Y_2 \in \text{ri face}^{\triangle}(X, \mathcal{N}_2)$. As $Y = Y_1 + Y_2$, we have $Y \in \text{ri face}^{\triangle}(X, \mathcal{DNN})$. However, if we consider a decomposition with $\tilde{Y}_1 = Y$ and $\tilde{Y}_2 = 0$, then $(X, \tilde{Y}_1)$ is not strictly complementary for cone $\mathcal{S}_2^+$ as $\tilde{Y}_1$ is a rank one matrix and we have $\tilde{Y}_1 \in \text{bd face}^{\triangle}(X, \mathcal{S}_2^+)$. Similarly, $(X, \tilde{Y}_2)$ is not strictly complementary for cone $\mathcal{N}_2$.

By Theorem 2.7, a complementary pair (X, Y) is strictly complementary for the doubly non-negative cone if and only if there exists a decomposition $Y = Y_1 + Y_2$ such that (X, Y_1) is strictly complementary for the positive semi-definite cone and (X, Y_2) is strictly complementary for the non-negative cone. Thus, the problem of checking strict complementarity of a given pair (X, Y) can be modelled as a rank-constrained semi-definite program. Let $\text{rank}X = r$.

$$\begin{aligned} \max \quad & \varepsilon \\ \text{s.t.} \quad & \text{rank}(Y - Y_2) = (m - r) \\ & (Y - Y_2) \in \mathcal{S}_m^+ \\ & Y_2(i,j) = 0 \quad \text{if } X(i,j) > 0 \\ & Y_2(i,j) \geq \varepsilon \quad \text{if } X(i,j) = 0. \end{aligned}$$

Let $\varepsilon > 0$ and Y_2 be a solution of this problem. Then $\langle X, Y_2 \rangle = 0$ by construction, and from complementarity condition, we have $\langle X, Y \rangle = 0$. Thus, we have $\langle X, Y - Y_2 \rangle = 0$ and $\text{rank}(Y - Y_2) + \text{rank}X = m$. Therefore, $(X, Y - Y_2)$ is strictly complementary for the positive semi-definite cone and (X, Y_2) is strictly complementary for the non-negative cone if and only if there exists a solution and $\varepsilon > 0$.

Now, when $\text{rank}X = (m - 1)$, we provide a numerically efficient way to check strict complementarity of a complementary pair (X, Y).

Theorem 3.3: *Consider $X \in \mathcal{DNN}_m$ with $\text{rank}X = (m - 1)$ and $Y \in (\mathcal{DNN}_m)^*$ such that $Y \in \text{face}^{\triangle}(X, \mathcal{DNN}_m)$. Let $X = Q\,\text{Diag}(\lambda_1, \ldots, \lambda_{m-1}, 0)Q^{\mathrm{T}}$ be an eigenvalue decomposition of $X \in \mathcal{S}_m^+$ where Q is an orthonormal matrix whose columns are the eigenvectors of X. Define the matrices*

$$\overline{Y}_1 = Q\,\text{Diag}(0, \ldots, 0, \omega_m)Q^{\mathrm{T}} \tag{9}$$

and

$$\overline{Y}_2 = \begin{cases} \overline{Y}_2(i,j) = 0 & \text{if } X(i,j) > 0 \\ \overline{Y}_2(i,j) = \alpha_{ij} & \text{if } X(i,j) = 0. \end{cases} \tag{10}$$

Then (X, Y) is strictly complementary for $\mathcal{DNN}_m$ if and only if $Y = \overline{Y}_1 + \overline{Y}_2$ has a solution with $\omega_m > 0$ and $\alpha_{ij} > 0$ for all i,j.

Proof: First let us consider strict complementarity of $(X, \overline{Y}_1)$ for the positive semi-definite cone. The complementarity condition $\langle X, \overline{Y}_1\rangle = 0$ for $X, \overline{Y}_1 \in \mathcal{S}_m^+$ imply (see, e.g. [3])

$$0 = X\overline{Y}_1 = Q^{\mathrm{T}}XQQ^{\mathrm{T}}\overline{Y}_1Q = \mathrm{Diag}(\lambda_1, \dots, \lambda_{m-1}, 0)Q^{\mathrm{T}}\overline{Y}_1Q = \begin{pmatrix} \lambda_1 v_1^{\mathrm{T}} \\ \vdots \\ \lambda_{m-1}v_{m-1}^{\mathrm{T}} \\ 0v_m^{\mathrm{T}} \end{pmatrix},$$

where v_j^{T} are the rows of $Q^{\mathrm{T}}\overline{Y}_1Q$. We thus find $v_j^{\mathrm{T}} = 0$ for $j = 1, \dots, m-1$ and $v_m^{\mathrm{T}} = (0, \dots, 0, \omega_m)$ with $\omega_m \geq 0$ because $Q^{\mathrm{T}}\overline{Y}_1Q \in \mathcal{S}_m^+$. The strict complementarity condition $\mathrm{rank}X + \mathrm{rank}\overline{Y}_1 = m$ further leads to $\omega_m > 0$. As $Q^{\mathrm{T}}\overline{Y}_1Q = \mathrm{Diag}(0, \dots, 0, \omega_m)$, the matrix $\overline{Y}_1$ must have the form (9) with $\omega_m > 0$.

Next, $(X, \overline{Y}_2)$ is strictly complementary for the non-negative cone if and only if $X(i,j) + \overline{Y}_2(i,j) > 0$ and $X(i,j)\overline{Y}_2(i,j) = 0$ for all i,j (see, e.g. [1]). Thus, $\overline{Y}_2$ has to have the form (10) and α_{ij} has to be positive. Combining the above arguments, the pair (X, Y) is strictly complementary for the doubly non-negative cone if and only if the linear system of equations $Y = \overline{Y}_1 + \overline{Y}_2$ has a positive solution. ■

Therefore, when (X, Y) is a complementary pair for the doubly non-negative cone and $\mathrm{rank}X = (m-1)$, we can check strict complementarity by finding the eigenvalue decomposition of X and solving a system of linear equations.

4. Conclusion

Consider a closed convex cone $\mathcal{K}$ given by the intersection of two cones $\mathcal{K}_1$ and $\mathcal{K}_2$. We summarize minimal and complementary faces of $\mathcal{K}$ and provide a necessary and sufficient condition for strict complementarity for cone $\mathcal{K}$ in Theorem 2.7. Moreover, the procedure to check strict complementarity of (X, Y) for the doubly non-negative cone is given in Theorem 3.3 when X has exactly one zero eigenvalue.

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