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Robust cooperative output regulation of heterogeneous Lur'e networks

Fan Zhang[1,*,†], Harry L. Trentelman[2] and Jacquelien M. A. Scherpen[3]

[1]*Department of Mechanical and Biomedical Engineering, City University of Hong Kong, Hong Kong*
[2]*Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, Groningen, The Netherlands*
[3]*Engineering and Technology Institute Groningen, University of Groningen, Groningen, The Netherlands*

SUMMARY

In this paper, we study robust cooperative output regulation problems for a directed network of Lur'e systems that consist of a nominal linear dynamics with an unknown static nonlinearity around it through negative feedback. We assume that the linear part of each agent is identical, but the nonlinearities are allowed to be different for distinct agents. In this sense, the network is heterogeneous. As is common in the context of Lur'e systems, the unknown nonlinearities are assumed to be sector bounded within one given sector. The interconnection graph among these agents is assumed to contain a directed spanning tree. Similar to classical output regulation problems, there is a virtual exosystem generating a reference signal in which all the agents are required to track cooperatively. Our designed distributed dynamic state/output feedback protocol makes a copy of the reference signal at each agent asymptotically, and then the robust cooperative output regulation problem becomes a robust tracking problem that can be handled by each agent via local information. It turns out that our cooperative protocols are fully distributed. Sufficient conditions on the existence of output synchronization protocols are given along with some discussions on these conditions. Finally, two simulation examples illustrate our design. Copyright © 2016 John Wiley & Sons, Ltd.

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KEY WORDS: Lur'e system; heterogeneous network; cooperative output regulation; robust synchronization

1. INTRODUCTION

Synchronization of multi-agent networks has many potential applications and thus has attracted many attention from multidisciplinary researchers over the past decade [1–3].

Multi-agent networks in the presence of model uncertainties, external disturbances, etc. are indeed heterogeneous, and their synchronization problems can be usually handled by means of robust control techniques [4]. However, in practice, multi-agent networks are often *intrinsically* heterogeneous because of non-identical agent dynamics. In this case, output regulation theory and particularly the internal model principle has been explored to tackle synchronization problems, see, for example, [5], where heterogeneous linear networks were studied. The main idea is that the models of the individual agents together with their 'local' controllers must embed an internal model of the (virtual) exosystem that generates the reference signal. In virtue of robust output regulation theory, output synchronization of uncertain heterogeneous linear networks was also performed, see, for example, [6]. In [7], the role of the internal model principle was discussed in the distributed coordination of heterogeneous nonlinear networks as well. Recently, output synchronization in diverse collective motion patterns has been studied for heterogeneous nonlinear networks [8].

*Correspondence to: Fan Zhang, The Lab of Cooperative Control for Complex Systems, Southeast University, China.
†E-mail: dr.fanzhang@outlook.com

In this paper, we will study uncertain heterogeneous nonlinear networks, in which the model of each agent is represented by a Lur'e system, that is, a nonlinear system consisting of a nominal linear dynamics with an *unknown* static nonlinearity around it through negative feedback. A single Lur'e system can represent many control systems such as flexible robotic arms and flight vehicles [9]. The study on its absolute stability greatly promoted the development of Lyapunov theory and established some well-known results, for example, Kalman–Yakubovich–Popov lemma. Synchronization of two coupled chaotic circuits in the form of Lur'e systems can be applied to secure communication [10]. It is promising that a large scale of chaotic circuits can improve the communication security. Also, biomedical oscillator networks are good applications.

In our setting, these Lur'e systems are assumed to be identical, in the sense that their linear parts are identical and their unknown nonlinear parts are sector bounded within the same sector. The problem formulation in this paper, however, allows the actual nonlinearities that occur to differ from agent to agent. In other words, the network is heterogeneous in the sense that the nonlinearities can be different for each agent. In our previous work [11–14], these nonlinearities were assumed to be identical for each agent and satisfy the assumption of *incremental* passivity or *incremental* sector boundedness. In our present paper, only the condition of sector boundedness is imposed. For the passivity case, the reader is referred to the conference version [15], where only dynamic *state* feedback protocols were designed. In the present paper, the dynamic *output* feedback protocol design will be discussed as well. The case that both the nominal linear parts as well as the unknown nonlinearities are non-identical can be considered in a similar, albeit technically more involved way and is omitted from this paper.

We stress that this paper deals with agent dynamics with *functional uncertainties*. This setting is more challenging than that of parametric uncertainties, which has been mostly considered in the literature. A convergence-based controller for output regulation of Lur'e systems was designed in [16]. The structure of the unknown nonlinearities in the present paper is more general. Besides, in [16], the authors did not give an explicit description of the uncertainty involved, and robustness only holds with respect to a certain neighborhood of the nominal system. Cooperative output regulation of a class of nonidentical nonlinear systems in the form of a Lur'e system was also discussed in [17]. There, besides the special structure of the linear parts, the nonlinear parts are assumed to be known precisely. Furthermore, in our opinion, it is not reasonable to assume that every agent is influenced by the same exosystem.

The remainder of this paper is organized as follows. In Section 2, the robust cooperative output regulation problems we deal with are formulated, and some preliminaries are provided. Our solutions to the problems of robust cooperative output regulation by dynamic state feedback and respectively dynamic output feedback are presented in Sections 3 and 4 along with some discussions in Section 5. Two numerical simulation examples are given in Section 6. Some concluding remarks and possible future work close the paper.

2. PRELIMINARIES AND PROBLEM FORMULATION

Let $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively. $\mathbb{R}^{n\times m}$ ($\mathbb{C}^{n\times m}$) denotes the space of $n \times m$ real (complex) matrices. Matrices, if not explicitly stated, are assumed to have compatible dimensions. The superscript $(\cdot)^T$ denotes the transpose of a real matrix, and the superscript $(\cdot)^*$ denotes the conjugate transpose of a complex matrix. The Kronecker product of matrices M_1 and M_2 is denoted by $M_1 \otimes M_2$. An important property of the Kronecker product is that $(M_1 \otimes M_2)(M_3 \otimes M_4) = (M_1M_3) \otimes (M_2M_4)$. We denote by $\mathbf{0}$ and I the zero and identity matrices, respectively, of compatible dimensions.

In this paper, the interconnection topology of a network of unidirectionally interconnected dynamical systems is described by a simple directed graph $\mathcal{G}$ that consists of a finite, nonempty node set $\mathcal{V} = \{1, 2, \cdots, N\}$, and an edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, where $N \geqslant 2$. The relevant algebraic graph theory and particularly the concepts of adjacency matrix and Laplacian matrix can be found in [3]. It is well known that the Laplacian matrix associated with a (directed) graph has a simple zero eigenvalue if and only if the graph contains a directed spanning tree. The remaining Laplacian eigenvalues have strictly positive real parts.

In the following, we give the definition of minimal left annihilator of a matrix.

Definition 1 ([18]*)*
For a matrix $B \in \mathbb{C}^{n\times m}$ with rank $r < \min\{n, m\}$, we denote by $B^{\perp}$ any matrix in $\mathbb{C}^{(n-r)\times n}$ of full row rank such that $B^{\perp}B = \mathbf{0}$. Any such matrix $B^{\perp}$ is called a *minimal left annihilator* of B.

Note that the minimal left annihilator is only defined for matrices with linearly dependent rows. The set of all such matrices is given by $B^{\perp} = UU_2^*$, where U is an arbitrary nonsingular matrix and U_2 is obtained from the singular value decomposition $B = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}^*$. Thus, for a given B, $B^{\perp}$ is not unique. Throughout this paper, $B^{\perp}$ will denote any choice from such set of matrices.

In this paper, we will consider a directed network of $N(\geqslant 2)$ non-identical Lur'e systems represented by

$$\begin{cases} \dot{x}_i = Ax_i + Bu_i + Ed_i \\ y_i = Cx_i \quad z_i = Hx_i \\ d_i = -\phi_i(y_i) \end{cases}, \qquad i = 1, 2, \cdots, N, \tag{1}$$

where $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^m$, $y_i(t) \in \mathbb{R}^p$ and $z_i(t) \in \mathbb{R}^q$ are the system state, the diffusive coupling input, the feedback loop input, and the system output to be regulated of agent i, respectively. A, B, C, E, and H are known constant matrices of compatible dimensions. Without loss of generality, we assume that the dimension m of the diffusive coupling inputs and the dimension q of the system outputs are strictly less than the state space dimension n. In this case, the rows of matrix B are linearly dependent, and thus, $B^{\perp}$ exists. Similarly, $\left(H^T\right)^{\perp}$ exists as well. The equation $d_i = -\phi_i(y_i)$ represents a memoryless, nonlinear negative feedback loop (Figure 1). The function $\phi_i(\cdot) : \mathbb{R}^p \to \mathbb{R}^p$ denotes an unknown static nonlinearity around the nominal linear dynamics. The $\phi_i(\cdot)$'s are assumed to be *sector bounded*, that is,

$$(\phi_i(y) - S_1 y)^T(\phi_i(y) - S_2 y) \leqslant 0, \ \forall\ y \in \mathbb{R}^p, \ i = 1, 2, \cdots, N,$$

where the matrices $S_1, S_2 \in \mathbb{R}^{p\times p}$ satisfy $\mathbf{0} \leqslant S_1 < S_2$. In the SISO case, y and $\phi_i(\cdot)$ are scalars, and hence, $S_1 = \alpha$, $S_2 = \beta$ with $0 \leqslant \alpha < \beta$ [19]. The interconnection topology among the agents (1) is assumed to be described by a graph $\mathcal{G}$ that contains a directed spanning tree.

Remark 1
As stated in Section 1, in this paper, we consider networks of identical Lur'e systems, in the sense that their nominal linear parts and the sector bounds are identical for all agents. In contrast to our previous work (e.g., [11]), however, the nonlinearities that actually occur are allowed to differ from agent to agent. For this reason, we use the notation $\phi_i(\cdot)$ to denote the nonlinearity in the dynamics of agent i.

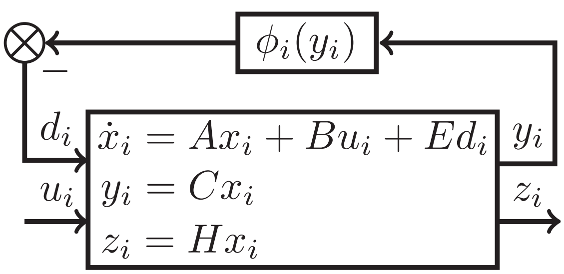

Figure 1. Lur'e system.

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Similar to classic output regulation problems [20], we assume that we have an exosystem given by

$$\dot{w} = Sw, \quad z = Rw, \tag{2}$$

where $w(t) \in \mathbb{R}^s$ and $z(t) \in \mathbb{R}^q$ are its state and output, respectively. We assume that the matrix S has all its eigenvalues on the imaginary axis. The pair (S, R) is assumed to be detectable. For any specified initial state of the exosystem, z is the reference signal that the outputs z_i's of the agents (1) are required to track asymptotically. In this sense, the exosystem can be viewed as a virtual leader.

In order to proceed, we define an augmented graph $\hat{\mathcal{G}}$ as follows. We introduce a new node, indexed by '0'. The dynamics associated with this new node is given by the exosystem (2). We assume that one of the root nodes in the original graph $\mathcal{G}$ has direct access to the reference signal z generated by the exosystem. Without loss of generality, let '1' be the index of this root node. Then the augmented graph is given by $\hat{\mathcal{G}} := \{\{0\} \cup \mathcal{V}, (0, 1) \cup \mathcal{E}\}$. The entry a_{10} of the associated adjacency matrix is defined to be equal to 1. Note that the node 0 is now the unique root node of the augmented graph $\hat{\mathcal{G}}$.

Remark 2
Concerning the leader-following architecture, assumption 1 on network topologies in [21] is often used, which sounds more general than that we use in this paper. However, it is easily checked that technically, there are no big differences. Note that our assumption was also used in, for example, [22]. On the other hand, our assumption is more suitable for the virtual leader case. The exosystem is usually called a *virtual* leader. Physically, it can be only embedded in an agent and constructed using embedded software together with the protocol for this agent. Assumption 1 in [21] in fact implies that the neighboring agents of the root agent associated with the exosystem might have access to the exosystem through this root agent, which can only happen over the original network topology $\mathcal{G}$. Clearly, the augmented graph in assumption 1 in [21] is in fact exactly the original one in the presence of a virtual leader. Hence, it makes sense that we assume the exosystem to be connected to one root agent.

In this paper, we first assume that the agents (1) can be interconnected by *dynamic state feedback* protocols of the form

$$\begin{cases} \dot{w}_1 = Sw_1 + T(z - Rw_1) \\ \dot{w}_i = Sw_i + \sum_{j=1}^{N} a_{ij}(w_j - w_i),\ i = 2, \cdots, N, \end{cases} \tag{3a}$$

$$u_i = Fx_i + Kw_i, \quad i = 1, 2, \cdots, N, \tag{3b}$$

where $w_i(t) \in \mathbb{R}^s$ is the state of the dynamic protocol for agent i, T is a matrix such that $S - TR$ is Hurwitz, $\mathcal{A} = [a_{ij}]$ is the adjacency matrix associated with the graph $\mathcal{G}$, F and K are feedback, and respectively, feedforward gain matrices to be determined.

Subsequently, we assume that the agents (1) can be interconnected by *dynamic output feedback* protocols of the form

$$\begin{cases} \dot{w}_1 = Sw_1 + T(z - Rw_1) \\ \dot{w}_i = Sw_i + \sum_{j=1}^{N} a_{ij}(w_j - w_i),\ i = 2, \cdots, N, \end{cases} \tag{4a}$$

$$\begin{cases} \dot{v}_i = A_c v_i + B_c(z_i - Rw_i) \\ u_i = C_c v_i + D_c z_i + Kw_i \end{cases}, \quad i = 1, 2, \cdots, N, \tag{4b}$$

where $w_i(t) \in \mathbb{R}^s$ and $v_i(t) \in \mathbb{R}^{n_c}$ are the states of the dynamic protocol for agent i, T is a matrix such that $S - TR$ is Hurwitz, $\mathcal{A} = [a_{ij}]$ is the adjacency matrix associated with the graph $\mathcal{G}$, and A_c, B_c, C_c, D_c, and K are gain matrices to be determined. The dimension n_c is also a crucial parameter to be chosen. For the later dynamic output feedback protocol design, H is assumed to have full row rank (Theorems 3 and 4).

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Remark 3
In the dynamic state feedback case, agent i uses the relative information $w_j - w_i$, the absolute information w_i, and its own state x_i; in the dynamic output feedback case, agent i uses the relative information $w_j - w_i$, the absolute information w_i, v_i and its own output z_i. Note that (3a) (or (4a)) presents a distributed observer for the state w of the exosystem (2). In fact, the interconnections among these agents only happen in the distributed observer network. Such framework is commonly used to deal with heterogeneous networks, see, for example, figure 2 in [7]. In our design, agent 1 does not use the relative information with respect to its neighbors in order to reduce consumption of communication and computation.

Remark 4
The internal states of the distributed observer (3a) (or (4a)) spread through the network, which can be performed only by wireless communications. However, signal broadcasting could cause, for example, quantized data and time delays. These will obtain the network performance down. Therefore, only relative measurement-based coordination is advocated, which can be performed by sensors [23]. It would be a possible future topic to design relative measurement-based synchronization protocols for heterogeneous Lur'e networks. Here, we want to stress that the agent dynamics we consider contains *functional uncertainties*, while parametric uncertainties are considered in [23]. Whereas output regulation with parametric uncertainties has been extensively studied, there are few efforts on robust output regulation against functional uncertainties. In the presence of parametric uncertainties, people usually assume that there exists a known nominal parameter and the real one is away from it within a predefined radius. Then robust control techniques are competent. In contrast, we do not assume there is a nominal linear or nonlinear function for unknown Lur'e-type nonlinearties. Without known nominal functions, it is not possible to solve regulation equations involving unknown Lur'e-type nonlinearities. Furthermore, for specific nonlinear agent dynamics, the 'local' regulator design for each agent must be addressed case by case, as we did in this paper. In particular, our studied nonlinearities are not allowed to be involved in the regulator design.

Under the earlier setting, we study the following robust cooperative output regulation problem:
Problem: The network of agents (1) with the protocol (3a)–(3b) and respectively (4a)–(4b) is robustly output regulated if $z_i(t) - z(t) \to \mathbf{0}$ as $t \to \infty$ for all initial conditions on the agents and the exosystem and all nonlinearities $\phi_i(\cdot)$, $i = 1, 2, \cdots, N$, satisfying the sector boundedness condition.

Before moving on, a basic preliminary result will be given in the following.

Lemma 1
For all solutions w and w_i, $i = 1, 2, \cdots, N$, of the interconnection of the exosystem (2) and the distributed observer (3a) (or (4a)), we have $w_i(t) - w(t) \to \mathbf{0}$ as $t \to \infty$ exponentially.

Proof
Let $\tilde{w}_i = w_i - w$, $i = 1, 2, \cdots, N$. We have

$$\dot{\tilde{w}}_1 = (S - TR)\tilde{w}_1.$$

Because $S - TR$ is Hurwitz, $\tilde{w}_1$ converges to $\mathbf{0}$ exponentially. On the other hand, we have

$$\dot{\tilde{w}}_i = S\tilde{w}_i + \sum_{j=1}^{N} a_{ij}\left(\tilde{w}_j - \tilde{w}_i\right), \quad i = 2, \cdots, N,$$

that is,

$$\dot{\tilde{w}} = \left(I_{N-1} \otimes S - \tilde{L} \otimes I_s\right)\tilde{w} - l_{21} \otimes \tilde{w}_1,$$

where $\tilde{w} = \left[\tilde{w}_2^T, \cdots, \tilde{w}_N^T\right]^T$, $\mathcal{L} = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & \tilde{L} \end{bmatrix}$. Here, $\mathcal{L}$ is the Laplacian matrix associated with the graph $\mathcal{G}$. Because agent 1 does not use the relative information with respect to its neighbors, the Laplacian matrix describing the interconnection relations *in our protocols* is given by

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$\tilde{\mathcal{L}} = \begin{bmatrix} 0 & \mathbf{0} \\ l_{21} & \tilde{L} \end{bmatrix}$. It is easily seen that $\tilde{\mathcal{L}}$ has a unique zero eigenvalue and its remaining eigenvalues have strictly positive real parts. Therefore, $-\tilde{L}$ is Hurwitz. Let $v(t) = \left(I_{N-1} \otimes e^{-St}\right) \tilde{w}(t)$. We obtain

$$\begin{aligned} \dot{v} &= -\left(I_{N-1} \otimes Se^{-St}\right) \tilde{w} + \left(I_{N-1} \otimes e^{-St}\right) \left[\left(I_{N-1} \otimes S - \tilde{L} \otimes I_s\right) \tilde{w} - l_{21} \otimes \tilde{w}_1\right] \\ &= -\left(\tilde{L} \otimes e^{-St}\right) \tilde{w} - l_{21} \otimes \left(e^{-St} \tilde{w}_1\right) = -\left(\tilde{L} \otimes I_s\right) v - l_{21} \otimes \left(e^{-St} \tilde{w}_1\right). \end{aligned}$$

Because S has all its eigenvalues on the imaginary axis and $\tilde{w}_1$ vanishes exponentially, $e^{-St}\tilde{w}_1$ goes to zero exponentially. In addition, $-\tilde{L} \otimes I_s$ is Hurwitz. Thus, $\tilde{w}$ as well as v tends to zero exponentially. This completes the proof. □

In the next two sections, we will discuss robust cooperative output regulation of the network of agents (1) with the protocols (3a)–(3b) and (4a)–(4b), respectively. In Section 3, we will consider the dynamic state feedback protocol design. Later on, in Section 4, the dynamic output feedback case will be studied.

3. DYNAMIC STATE FEEDBACK

In this section, we will discuss the design of a dynamic state feedback protocol in the form of (3a)–(3b).

Theorem 1
Let (Π, Γ) be a solution pair to the regulator equations

$$\begin{cases} \Pi S = A\Pi + B\Gamma \\ \mathbf{0} = C\Pi \\ \mathbf{0} = H\Pi - R \end{cases}. \tag{5}$$

If there exists a positive definite matrix P, a matrix F, and a positive real number τ such that

$$\left[\begin{array}{c|c} P(A+BF)+(A+BF)^T P - \tau C^T(S_1S_2+S_2S_1)C & -PE+\tau C^T(S_1+S_2) \\ \hline -E^T P + \tau(S_1+S_2)C & -2\tau I \end{array}\right] < \mathbf{0}, \tag{6}$$

then the network of agents (1) with protocol (3a)–(3b), where $K = \Gamma - F\Pi$, is robustly output regulated.

Proof
Let $\tilde{x}_i = x_i - \Pi w_i$, $i = 1, 2, \cdots, N$, where Π together with Γ satisfies (5). We obtain

$$\dot{\tilde{x}}_1 = (A+BF)\tilde{x}_1 - E\phi_1(C\tilde{x}_1) - \Pi T(z - Rw_1), \tag{7a}$$

$$\dot{\tilde{x}}_i = (A+BF)\tilde{x}_i - E\phi_i(C\tilde{x}_i) - \Pi \sum_{j=1}^{N} a_{ij}(w_j - w_i),\ i = 2, \cdots, N. \tag{7b}$$

Denote $\Sigma_1 := \Pi T(z - Rw_1)$ and $\Sigma_i := \Pi \sum_{j=1}^{N} a_{ij}(w_j - w_i)$, $i = 2, \cdots, N$. By Lemma 1, $\Sigma_i(t) \to \mathbf{0}$ as $t \to \infty$ exponentially, $i = 1, 2, \cdots, N$.

Choose a Lyapunov function candidate $V_1(\tilde{x}_i) = \tilde{x}_i^T P \tilde{x}_i$, $i = 1, 2, \cdots, N$, where $P > \mathbf{0}$ together with F and $\tau > 0$ satisfies (6). Obviously, $V_1(\tilde{x}_i)$ is positive definite and radially unbounded. Then the time derivative of $V_1(\tilde{x}_i)$ along the trajectories of $\tilde{x}_i$ governed by (7a)–(7b) is given by

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$$\dot{V}_1(\tilde{x}_i) = 2\tilde{x}_i^T P[(A+BF)\tilde{x}_i - E\phi_i(C\tilde{x}_i) - \Sigma_i]$$
$$= \begin{bmatrix} \tilde{x}_i \\ \phi_i(C\tilde{x}_i) \end{bmatrix}^T \left[\begin{array}{c|c} P(A+BF)+(A+BF)^T P & -PE \\ \hline -E^T P & \mathbf{0} \end{array}\right] \begin{bmatrix} \tilde{x}_i \\ \phi_i(C\tilde{x}_i) \end{bmatrix} - 2\tilde{x}_i^T P\Sigma_i.$$

On the other hand, we have

$$\begin{bmatrix} \tilde{x}_i \\ \phi_i(C\tilde{x}_i) \end{bmatrix}^T \begin{bmatrix} \frac{1}{2}C^T(S_1S_2+S_2S_1)C & -\frac{1}{2}C^T(S_1+S_2) \\ -\frac{1}{2}(S_1+S_2)C & I \end{bmatrix} \begin{bmatrix} \tilde{x}_i \\ \phi_i(C\tilde{x}_i) \end{bmatrix} \leqslant \mathbf{0}$$

because of the property of sector boundedness. Thus, using (6), there always exists a positive real number α such that

$$\begin{aligned} \dot{V}_1(\tilde{x}_i) &\leqslant -\alpha\tilde{x}_i^T\tilde{x}_i - \alpha\phi_i(C\tilde{x}_i)^T\phi_i(C\tilde{x}_i) + 2\|\tilde{x}_i\|\|P\Sigma_i\| \\ &\leqslant -\alpha\tilde{x}_i^T\tilde{x}_i - \alpha\phi_i(C\tilde{x}_i)^T\phi_i(C\tilde{x}_i) + \beta\|\tilde{x}_i\|^2 + \frac{1}{\beta}\|P\Sigma_i\|^2 \\ &= -(\alpha-\beta)\tilde{x}_i^T\tilde{x}_i - \alpha\phi_i(C\tilde{x}_i)^T\phi_i(C\tilde{x}_i) + \frac{1}{\beta}\|P\Sigma_i\|^2, \end{aligned}$$

where $0 < \beta < \alpha$. It follows that the systems (7a)–(7b) are globally input-to-state stable with Σ_i, $i = 1, 2, \cdots, N$, as the inputs, respectively. Because $\Sigma_i(t) \to \mathbf{0}$ as $t \to \infty$ exponentially, by taking Σ_i as the output of a stable linear time-invariant system with zero input, it is easily seen that $\tilde{x}_i(t)$ goes to zero as $t \to \infty$. Together with $w_i(t) - w(t) \to \mathbf{0}$ as $t \to \infty$ exponentially, the proof is completed. □

Remark 5
From the earlier analysis, it is clear how the case of non-identical nominal linear parts can be also tackled. In this case, a set of N non-identical regulator equations will be required to have a common solution pair, see, for example, (6) in [5]. In addition, similar to classic output regulation problems, disturbance rejection at each agent can be considered, which just introduces more regulator equations [7]. These problems are left to the reader.

Note that Theorem 1 does not tell us how to compute a suitable F for a given heterogeneous Lur'e network, and consequently, a suitable K is unknown either. Referring to our previous work, particularly lemma 3.3 in [11], the following result complements Theorem 1 by giving a suitable F and subsequently a suitable K.

Lemma 2
There exists a positive definite matrix P, a matrix F, and a positive real number τ such that (6) holds if and only if there exists a positive definite matrix Q and a positive real number ρ such that the following LMI holds:

$$\begin{bmatrix} B \\ \mathbf{0} \end{bmatrix}^{\perp} \left[\begin{array}{c|c} Q\left(A-\frac{1}{2}E(S_1+S_2)C\right)^T + \left(A-\frac{1}{2}E(S_1+S_2)C\right)Q + \frac{1}{2}\rho EE^T & QC^T \\ \hline CQ & -2\rho(S_2-S_1)^{-2} \end{array}\right] \left(\begin{bmatrix} B \\ \mathbf{0} \end{bmatrix}^{\perp}\right)^T < \mathbf{0}. \tag{8}$$

In this case, a suitable P is given by $P = Q^{-1}$, a suitable τ is given by $\tau = \frac{1}{\rho}$, and a suitable F is given by $F = \mu B^T Q^{-1}$, where the real number μ is chosen to satisfy

$$\left[\begin{array}{c|c} Q\left(A-\frac{1}{2}E(S_1+S_2)C\right)^T + \left(A-\frac{1}{2}E(S_1+S_2)C\right)Q + \frac{1}{2}\rho EE^T + 2\mu BB^T & QC^T \\ \hline CQ & -2\rho(S_2-S_1)^{-2} \end{array}\right] < \mathbf{0}. \tag{9}$$

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Because the earlier result is slightly different from theorem 2 in [11], its proof is given in the following.

Proof
For the 'only if' part, by taking the Schur complement, (6) is equivalent to

$$\left[\begin{array}{c:c} \begin{array}{c}(A-\frac{1}{2}E(S_1+S_2)C)^TP+P(A-\frac{1}{2}E(S_1+S_2)C)\\ +F^TB^TP+PBF+\frac{1}{2\tau}PEE^TP\end{array} & C^T \\ \hdashline C & -\frac{2}{\tau}(S_2-S_1)^{-2}\end{array}\right]<\mathbf{0}. \tag{10}$$

Let $Q=P^{-1}$ and $\rho=\frac{1}{\tau}$. Then we obtain

$$\left[\begin{array}{c:c} \begin{array}{c}Q(A-\frac{1}{2}E(S_1+S_2)C)^T+(A-\frac{1}{2}E(S_1+S_2)C)Q\\ +QF^TB^T+BFQ+\frac{1}{2}\rho EE^T\end{array} & QC^T \\ \hdashline CQ & -2\rho(S_2-S_1)^{-2}\end{array}\right]<\mathbf{0}.$$

It is easily verified that

$$\begin{bmatrix} B \\ \mathbf{0}\end{bmatrix}^{\perp}=\begin{bmatrix} B^{\perp} & \mathbf{0} \\ \mathbf{0} & I\end{bmatrix}. \tag{11}$$

By premultiplying with (11) and postmultiplying with the transpose of (11), (8) is obtained.
For the 'if' part, again by taking the Schur complement, (8) implies

$$B^{\perp}\left[Q\left(A-\frac{1}{2}E(S_1+S_2)C\right)^T+\left(A-\frac{1}{2}E(S_1+S_2)C\right)Q\right.$$
$$\left.+\frac{1}{2}\rho EE^T+\frac{1}{2\rho}QC^T(S_2-S_1)^2CQ\right]\left(B^{\perp}\right)^T<\mathbf{0}.$$

By Finsler's lemma [18], it follows that there exists a real number μ such that

$$Q\left(A-\frac{1}{2}E(S_1+S_2)C\right)^T+\left(A-\frac{1}{2}E(S_1+S_2)C\right)Q$$
$$+\frac{1}{2}\rho EE^T+\frac{1}{2\rho}QC^T(S_1-S_2)^2CQ+2\mu BB^T<\mathbf{0},$$

that is, (9). Let $P=Q^{-1}$, $\tau=\frac{1}{\rho}$ and $F:=\mu B^TP$. Then we obtain

$$\left(A-\frac{1}{2}E(S_1+S_2)C\right)^TP+P\left(A-\frac{1}{2}E(S_1+S_2)C\right)$$
$$+F^TB^TP+PBF+\frac{1}{2\tau}PEE^TP+\frac{1}{2}\tau C^T(S_2-S_1)^2C<\mathbf{0},$$

that is, (10). This completes the proof. □

Hence, the computation of a dynamic state feedback protocol (3a)–(3b) can be performed as follows:

- Compute a solution pair (Π,Γ) to (5);
- Compute a $Q>\mathbf{0}$ and a $\rho>0$ such that (8) holds;
- Compute a μ such that (9) holds;
- Compute $F:=\mu B^TQ^{-1}$;
- Compute $K:=\Gamma-F\Pi$.

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Remark 6
Note that in the earlier computation, the knowledge of the entire interconnection topology, which is a kind of global information, is not required. This is in contrast to our work in [11] where we had to employ an adaptive protocol to remove such requirement. In this sense, our designed protocol in the present paper is *naturally* fully distributed thanks to the result of Lemma 1.

4. DYNAMIC OUTPUT FEEDBACK

In this section, we will discuss the design of a dynamic output feedback protocol in the form of (4a)–(4b).

Theorem 2
Let (Π, Γ) be a solution pair to the regulator Eq. (5). If there exists a positive definite matrix P, a matrix F and a positive real number τ such that

$$\left[\begin{array}{c|c} P(A_f + B_f F H_f) + (A_f + B_f F H_f)^T P & -PE_f + \\ -\tau C_f^T (S_1 S_2 + S_2 S_1) C_f & \tau C_f^T (S_1 + S_2) \\ \hline -E_f^T P + \tau (S_1 + S_2) C_f & -2\tau I \end{array}\right] < \mathbf{0}, \tag{12}$$

where $A_f = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, $B_f = \begin{bmatrix} B & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix}$, $C_f = \begin{bmatrix} C & \mathbf{0} \end{bmatrix}$, $E_f = \begin{bmatrix} E \\ \mathbf{0} \end{bmatrix}$, $H_f = \begin{bmatrix} H & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix}$ and $F = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}$, then the network of agents (1) with protocol (4a)–(4b), where $K = \Gamma - D_c H \Pi$, is robustly output regulated.

Proof
Let $\tilde{x}_i = x_i - \Pi w_i, i = 1, 2, \cdots, N$, where Π together with Γ satisfies (5). We obtain

$$\begin{cases} \dot{\tilde{x}}_1 = (A + BD_c H)\tilde{x}_1 + BC_c v_1 - E\phi_1 (C\tilde{x}_1) - \Pi T(z - Rw_1) \\ \dot{v}_1 = B_c H \tilde{x}_1 + A_c v_1 \end{cases}, \tag{13a}$$

$$\begin{cases} \dot{\tilde{x}}_i = (A + BD_c H)\tilde{x}_i + BC_c v_i - E\phi_i (C\tilde{x}_i) - \Pi \sum_{j=1}^N a_{ij}(w_i - w_j) \\ \dot{v}_i = B_c H \tilde{x}_i + A_c v_i, \ i = 2, \cdots, N \end{cases}. \tag{13b}$$

Denote $\Sigma_1 := \Pi T(z - Rw_1)$ and $\Sigma_i := \Pi \sum_{j=1}^N a_{ij}(w_j - w_i)$, $i = 2, \cdots, N$. By Lemma 1, $\Sigma_i(t) \to \mathbf{0}$ as $t \to \infty$ exponentially, $i = 1, 2, \cdots, N$.

Partition $P > \mathbf{0}$ in (12) appropriately as $P = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}$. Choose a Lyapunov function candidate $V_2(\tilde{x}_i, v_i) = \begin{pmatrix} \tilde{x}_i \\ v_i \end{pmatrix}^T P \begin{pmatrix} \tilde{x}_i \\ v_i \end{pmatrix}$, $i = 1, 2, \cdots, N$. Obviously, $V_2(\tilde{x}_i, v_i)$ is positive definite and radially unbounded. Then the time derivative of $V_2(\tilde{x}_i, v_i)$ along the trajectories of $\tilde{x}_i$ and vi governed by (13a)–(13b) is given by

$$\begin{aligned} \dot{V}_2(\tilde{x}_i, v_i) &= \begin{pmatrix} \tilde{x}_i \\ v_i \end{pmatrix}^T \left[P(A_f + B_f F H_f) + (A_f + B_f F H_f)^T P \right] \begin{pmatrix} \tilde{x}_i \\ v_i \end{pmatrix} \\ &\quad - 2 \begin{pmatrix} \tilde{x}_i \\ v_i \end{pmatrix}^T \begin{bmatrix} P_1 E \\ P_2^T E \end{bmatrix} \phi_i (C\tilde{x}_i) - 2\tilde{x}_i^T P_1 \Sigma_i - 2v_i^T P_2^T \Sigma_i \\ &= \begin{pmatrix} \tilde{x}_i \\ v_i \\ \phi_i (C\tilde{x}_i) \end{pmatrix}^T \begin{bmatrix} P(A_f + B_f F H_f) + (A_f + B_f F H_f)^T P & -PE_f \\ -E_f^T P & \mathbf{0} \end{bmatrix} \begin{pmatrix} \tilde{x}_i \\ v_i \\ \phi_i (C\tilde{x}_i) \end{pmatrix} \\ &\quad - 2\tilde{x}_i^T P_1 \Sigma_i - 2v_i^T P_2^T \Sigma_i. \end{aligned}$$

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On the other hand, we have

$$\begin{pmatrix} \tilde{x}_i \\ v_i \\ \phi_i(C\tilde{x}_i) \end{pmatrix}^T \begin{bmatrix} \frac{1}{2}C_f^T(S_1S_2 + S_2S_1)C_f & -\frac{1}{2}C_f^T(S_1 + S_2) \\ -\frac{1}{2}(S_1 + S_2)C_f & I \end{bmatrix} \begin{pmatrix} \tilde{x}_i \\ v_i \\ \phi_i(C\tilde{x}_i) \end{pmatrix} \leqslant \mathbf{0}$$

because of the property of sector boundedness. Thus, using (12), there always exists a positive real number α such that

$$\dot{V}_2(\tilde{x}_i, v_i) \leqslant -\alpha\tilde{x}_i^T\tilde{x}_i - \alpha v_i^T v_i - \alpha\phi_i(C\tilde{x}_i)^T\phi_i(C\tilde{x}_i) - 2\tilde{x}_i^T P_1\Sigma_i - 2v_i^T P_2^T\Sigma_i.$$

Following a similar analysis as in the proof of Theorem 1, this proof can be completed. □

Next, we will now move to the existence and computation of suitable $P > \mathbf{0}$, F and $\tau > 0$ such that (12) holds.

Theorem 3

Assume that matrix H has full row rank. There exist $P > \mathbf{0}$, F and $\tau > 0$ such that (12) holds if and only if there exist matrices $X > \mathbf{0}$, $Y > \mathbf{0}$ and positive real numbers α, β such that $XY = I$ and $\alpha\beta = 1$,

$$\begin{bmatrix} B_f \\ \mathbf{0} \end{bmatrix}^{\perp} \left[\begin{array}{c|c} \left(A_f - \frac{1}{2}E_f(S_1+S_2)C_f\right)X + X\left(A_f - \frac{1}{2}E_f(S_1+S_2)C_f\right)^T + \frac{1}{2}\alpha E_f E_f^T & XC_f^T \\ \hline C_f X & -2\alpha(S_2-S_1)^{-2} \end{array}\right] \begin{bmatrix} B_f \\ \mathbf{0} \end{bmatrix}^{\perp T} < \mathbf{0}, \tag{14}$$

$$\begin{bmatrix} H_f^T \\ \mathbf{0} \end{bmatrix}^{\perp} \left[\begin{array}{c|c} Y\left(A_f - \frac{1}{2}E_f(S_1+S_2)C_f\right) + \left(A_f - \frac{1}{2}E_f(S_1+S_2)C_f\right)^T Y + \frac{1}{2}\beta C_f^T(S_2-S_1)^2C_f & YE_f \\ \hline E_f^T Y & -2\beta I \end{array}\right] \begin{bmatrix} H_f^T \\ \mathbf{0} \end{bmatrix}^{\perp T} < \mathbf{0}. \tag{15}$$

In this case, a suitable P is given by $P = X^{-1}$, a suitable τ is given by $\tau = \alpha^{-1}$, and a suitable F is given by $F = -rB_f^T\Theta_x^{-1}XH_f^T\left(H_fX\Theta_x^{-1}XH_f^T\right)^{-1}$, where r and Θ_x are determined as follows: choose a positive real number r such that

$$\Theta_x := rB_fB_f^T - Q_X - \frac{1}{2}\alpha E_fE_f^T - \frac{1}{2\alpha}XC_f^T(S_2-S_1)^2C_fX > \mathbf{0}, \tag{16}$$

where $Q_X := \left[A_f - \frac{1}{2}E_f(S_1+S_2)C_f\right]X + X\left[A_f - \frac{1}{2}E_f(S_1+S_2)C_f\right]^T$.

Proof

The existence of solutions F, $P > \mathbf{0}$ and $\tau > 0$ to (12) is equivalent to the existence of F, $X > \mathbf{0}$ and $\alpha > 0$ to

$$\begin{bmatrix} B_fFH_fX + (B_fFH_fX)^T + Q_X + \frac{1}{2}\alpha E_fE_f^T & XC_f^T \\ C_fX & -2\alpha(S_2-S_1)^{-2} \end{bmatrix} < \mathbf{0}. \tag{17}$$

This can be seen by taking $X = P^{-1}$, $\alpha = \tau^{-1}$ and considering appropriate Schur complements.

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(only if) Let X be a positive definite solution to (17). Define $Y = X^{-1}$. Then $X > \mathbf{0}$, $Y > \mathbf{0}$ and $XY = I$. Obviously, (14) holds. Equation (17) implies that

$$\begin{bmatrix} YB_fFH_f + (YB_fFH_f)^T + Q_Y + \frac{1}{2}\beta C_f^T(S_2 - S_1)^2 C_f & YE_f \\ E_f^T Y & -2\beta I \end{bmatrix} < \mathbf{0},$$

where $Q_Y := Y\left[A_f - \frac{1}{2}E_f(S_1 + S_2)C_f\right] + \left[A_f - \frac{1}{2}E_f(S_1 + S_2)C_f\right]^T Y$, $\beta = \alpha^{-1}$, which then implies (15).

(if) By Finsler's lemma [18], (14) implies that there exists a $r > 0$ such that

$$\begin{bmatrix} rB_fB_f^T - Q_X - \frac{1}{2}\alpha E_f E_f^T & -XC_f^T \\ -C_f X & 2\alpha(S_2 - S_1)^{-2} \end{bmatrix} > \mathbf{0},$$

equivalently, $\Theta_x > \mathbf{0}$. Similarly, (15) implies that there exists a matrix $S > \mathbf{0}$ such that

$$\begin{bmatrix} H_f^T S^{-1} H_f - Q_Y - \frac{1}{2}\beta C_f^T(S_2 - S_1)^2 C_f & -YE_f \\ -E_f^T Y & 2\beta I \end{bmatrix} > \mathbf{0},$$

equivalently, $\Theta_y := XH_f^T S^{-1} H_f X - Q_X - \frac{1}{2\alpha} XC_f^T(S_2 - S_1)^2 C_f X - \frac{1}{2}\alpha E_f E_f^T > \mathbf{0}$.

Define $\Xi := rI - r^2 B_f^T \Theta_x^{-1} B_f + r^2 B_f^T \Theta_x^{-1} X H_f^T \left(H_f X \Theta_x^{-1} X H_f^T\right)^{-1} H_f X \Theta_x^{-1} B_f$, where $H_f X \Theta_x^{-1} X H_f^T$ is positive definite because $\Theta_x > \mathbf{0}$, $X > \mathbf{0}$ and H_f has full row rank. Obviously, $\Xi > \mathbf{0}$ if and only if there exists a matrix $Z > \mathbf{0}$ such that

$$rI - r^2 B_f^T \Theta_x^{-1} B_f + r^2 B_f^T \Theta_x^{-1} X H_f^T \left(H_f X \Theta_x^{-1} X H_f^T + Z\right)^{-1} H_f X \Theta_x^{-1} B_f > \mathbf{0},$$

or equivalently, using the matrix inversion lemma [24], $rI - r^2 B_f^T \left(\Theta_x + XH_f^T Z^{-1} H_f X\right)^{-1} B_f > \mathbf{0}$, equivalently, using the Schur complement lemma,

$$\begin{bmatrix} rI & rB_f^T \\ rB_f & \Theta_x + XH_f^T Z^{-1} H_f X \end{bmatrix} > \mathbf{0},$$

and equivalently,

$$\begin{aligned} &\Theta_x + XH_f^T Z^{-1} H_f X - rB_f B_f^T \\ &= -Q_X - \frac{1}{2}\alpha E_f E_f^T - \frac{1}{2\alpha} XC_f^T (S_2 - S_1)^2 C_f X + XH_f^T Z^{-1} H_f X > \mathbf{0}. \end{aligned}$$

Earlier, it has been shown that this holds if we take $Z = S$, in this case, the earlier inequality is exactly $\Theta_y > \mathbf{0}$. This shows that $\Xi > \mathbf{0}$.

Now, clearly,

$$\begin{aligned} &\left[F + rB_f^T \Theta_x^{-1} X H_f^T \left(H_f X \Theta_x^{-1} X H_f^T\right)^{-1}\right] H_f X \Theta_x^{-1} X H_f^T \\ &\left[F + rB_f^T \Theta_x^{-1} X H_f^T \left(H_f X \Theta_x^{-1} X H_f^T\right)^{-1}\right]^T = \mathbf{0} < \Xi \end{aligned}$$

for the particular choice $F = -rB_f^T \Theta_x^{-1} X H_f^T \left(H_f X \Theta_x^{-1} X H_f^T\right)^{-1}$. The latter inequality holds if and only if $\left(rB_f^T + FH_f X\right) \Theta_x^{-1} \left(rB_f^T + FH_f X\right)^T < rI$, which in turn is equivalent to

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$$\begin{bmatrix} \Theta_x & \left(rB_f^T + FH_fX\right)^T \\ rB_f^T + FH_fX & rI \end{bmatrix} > \mathbf{0},$$

and to

$$\frac{1}{r}\left(rB_f^T + FH_fX\right)^T\left(rB_f^T + FH_fX\right) < \Theta_x = rB_fB_f^T - Q_X - \frac{1}{2}\alpha E_fE_f^T - \frac{1}{2\alpha}XC_f^T(S_2 - S_1)^2C_fX.$$

It follows that

$$B_fFH_fX + (B_fFH_fX)^T + \frac{1}{r}XH_f^TF^TFH_fX + Q_x + \frac{1}{2}\alpha E_fE_f^T + \frac{1}{2\alpha}XC_f^T(S_2 - S_1)^2C_fX < \mathbf{0},$$

which yields (17) because $\frac{1}{r}XH_f^TF^TFH_fX$ is always positive semi-definite. This completes the proof. □

Note that Theorem 3 is not yet entirely satisfactory because it does not enable us to compute a suitable protocol. The problem is that it does not tell us how to choose the dimension n_c of the protocol state space. The following result solves this problem.

Theorem 4

Assume that matrix H has full row rank. There exists a nonnegative integer n_c, matrices $X > \mathbf{0}$, $Y > \mathbf{0}$ of size $(n + n_c) \times (n + n_c)$ and $\alpha > 0$, $\beta > 0$ such that the conditions in Theorem 3 hold if and only if there exist matrices $X_p > \mathbf{0}$, $Y_p > \mathbf{0}$ of size $n \times n$ and $\alpha > 0$, $\beta > 0$ such that $\alpha\beta = 1$,

$$\begin{bmatrix} B \\ \mathbf{0} \end{bmatrix}^{\perp} \begin{bmatrix} Q_x + \frac{1}{2}\alpha EE^T & X_pC^T \\ CX_p & -2\alpha(S_2 - S_1)^{-2} \end{bmatrix} \begin{bmatrix} B \\ \mathbf{0} \end{bmatrix}^{\perp T} < \mathbf{0}, \tag{18}$$

$$\begin{bmatrix} H^T \\ \mathbf{0} \end{bmatrix}^{\perp} \begin{bmatrix} Q_y + \frac{1}{2}\beta C^T(S_2 - S_1)^2C & Y_pE \\ E^TY_p & -2\beta I \end{bmatrix} \begin{bmatrix} H^T \\ \mathbf{0} \end{bmatrix}^{\perp T} < \mathbf{0}, \tag{19}$$

$$\begin{bmatrix} X_p & I \\ I & Y_p \end{bmatrix} \geqslant \mathbf{0}, \tag{20}$$

$$\operatorname{rank}\begin{bmatrix} X_p & I \\ I & Y_p \end{bmatrix} \leqslant n + n_c, \tag{21}$$

where $Q_x := \left[A - \frac{1}{2}E(S_1 + S_2)C\right]X_p + X_p\left[A - \frac{1}{2}E(S_1 + S_2)C\right]^T$, $Q_y := Y_p[A - \frac{1}{2}E(S_1 + S_2)C] + \left[A - \frac{1}{2}E(S_1 + S_2)C\right]^T Y_p$.

Proof

(only if) Assume that there exists a nonnegative integer n_c, $X > \mathbf{0}$, $Y > \mathbf{0}$ of size $(n+n_c)\times(n+n_c)$ and $\alpha > 0$, $\beta > 0$ such that $XY = I$, $\alpha\beta = 1$, (14) and (15) hold. Partition $X = \begin{bmatrix} X_p & X_{pc} \\ X_{pc}^T & X_c \end{bmatrix}$ and $Y = \begin{bmatrix} Y_p & Y_{pc} \\ Y_{pc}^T & Y_c \end{bmatrix}$ appropriately. Note that $B_f^{\perp} = \begin{bmatrix} B^{\perp} & \mathbf{0} \end{bmatrix}$, $H_f^{T\perp} = \begin{bmatrix} H^{T\perp} & \mathbf{0} \end{bmatrix}$, $\begin{bmatrix} B_f \\ \mathbf{0} \end{bmatrix}^{\perp} = \begin{bmatrix} B_f^{\perp} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix}$, $\begin{bmatrix} H_f^T \\ \mathbf{0} \end{bmatrix}^{\perp} = \begin{bmatrix} H_f^{T\perp} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix}$. In this way, we obtain (18) and (19). $XY = I$ implies that $X_pY_p + X_{pc}Y_{pc}^T = I$ and $X_pY_{pc} + X_{pc}Y_c = \mathbf{0}$. Thus,

$$Y_p - X_p^{-1} = Y_{pc}Y_c^{-1}Y_{pc}^T \geqslant \mathbf{0}. \tag{22}$$

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Using the Schur complement, this is equivalent to (20). In addition,

$$\operatorname{rank}\begin{bmatrix} X_p & I \\ I & Y_p \end{bmatrix} = \operatorname{rank}(X_p) + \operatorname{rank}\left(Y_p - X_p^{-1}\right) = n + \operatorname{rank}\left(Y_{pc}Y_c^{-1}Y_{pc}^T\right) \leqslant n + n_c.$$

So (21) holds.

(if) Let Y_{pc} and $Y_c > \mathbf{0}$ be any matrices satisfying (22), while $X_p > \mathbf{0}$, $Y_p > \mathbf{0}$, $\alpha > 0$ and $\beta > 0$ satisfy (18), (19), (20), and $\alpha\beta = 1$, respectively, and n_c is chosen so that (21) is satisfied. It can be verified that a matrix pair (X, Y) such that $Y = \begin{bmatrix} Y_p & Y_{pc} \\ Y_{pc}^T & Y_c \end{bmatrix}$, $X = Y^{-1}$ together with the earlier α and β satisfies the conditions in Theorem 3. This completes the proof. □

Hence, the computation of a dynamic output feedback protocol (4a)–(4b) can be performed as follows:

- Compute a solution pair (Π, Γ) to (5);
- Compute $X_p > \mathbf{0}$, $Y_p > \mathbf{0}$, $\alpha > 0$ and $\beta > 0$ through (18), (19), (20), and $\alpha\beta = 1$;
- Choose n_c as $n_c = \operatorname{rank}\begin{bmatrix} X_p & I \\ I & Y_p \end{bmatrix} - n$, then define A_f, B_f, C_f, E_f, and H_f as introduced before in Theorem 2;
- Choose $Y_c > \mathbf{0}$ and Y_{pc} satisfying (22), consequently, $Y > \mathbf{0}$ and $X > \mathbf{0}$ are obtained;
- Compute $r > 0$ such that $\Theta_x > \mathbf{0}$;
- Definite F by $F = -rB_f^T\Theta_x^{-1}XH_f^T\left(H_fX\Theta_x^{-1}XH_f^T\right)^{-1}$;
- Partition F as $\begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}$;
- Compute $K = \Gamma - D_cH\Pi$.

Remark 7

We want to stress that the results in Theorems 2–4 can be used to attack *robust* output feedback stabilization of Lur'e systems in the sense that output feedback controllers can be designed without the precise knowledge of Lur'e-type nonlinearities. In the existing publications, for example, [25, 26], Lur'e-type nonlinearities were always assumed to be known precisely and used in the controller design. We also leave this problem to the reader.

5. DISCUSSIONS

In this section, we will elaborate on some of the conditions obtained in the previous two sections.

1. Note that in (5), besides the conventional linear regulator equations, the condition $\mathbf{0} = C\Pi$ is imposed, which is in fact a necessary requirement in our robust output regulation problem. This can be shown by contradiction. Suppose that $C\Pi \neq \mathbf{0}$ and take the closed-loop dynamics of agent 1 as the example. By defining $\tilde{x}_1 = x_1 - \Pi w_1$, where Π together with Γ only satisfies the first and third equations in (5), we obtain $\dot{\tilde{x}}_1 = (A + BF)\tilde{x}_1 - E\phi_1(C\tilde{x}_1 + C\Pi w_1) - \Pi T(z - Rw_1)$, which implies that $E\phi_1(C\Pi w_1) = \mathbf{0}$ as long as w_1 and $\tilde{x}_1$ reach w and zero, respectively. However, because $\phi_1(\cdot)$ is sector bounded and w_1 approaching w is a persistently exciting signal, $E\phi_1(C\Pi w_1) = \mathbf{0}$ cannot always hold. We have the same argument for the rest agents. In other words, the condition $\mathbf{0} = C\Pi$ guarantees that the unknown nonlinearities $\phi_i(\cdot)$'s, $i = 1, 2, \cdots, N$, vanish when the cooperative output regulation is achieved. Otherwise, we have to solve nonlinear regulator equations involving these unknown nonlinearities, which is impossible. This necessity as well as an example regarding output regulation of a specific Lur'e system has been discussed in [16].

   Again, there are few results on robust output regulation against functional uncertainties that involves stringent conditions. Hence, it deserves more attention than robust output regulation against parametric uncertainties that have been extensively studied.

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2. The feasibility of the regulator Eq. (5) is critical in our protocol design. By applying theorem 9.8 in [27], necessary and sufficient conditions for solvability of (5) are obtained subsequently. For the sake of completeness, theorem 9.8 in [27] is recalled:

   Lemma 3
   Consider the linear matrix equation

$$\sum_{i=1}^{k} A_i X q_i(B) = C, \tag{23}$$

   where A_i, B, and C are given matrices with B square, $q_i(s)$'s are real polynomials, and X is unknown. (23) has a solution X if and only if the polynomial matrices $\begin{bmatrix} A(s) & \mathbf{0} \\ \mathbf{0} & sI - B \end{bmatrix}$ and $\begin{bmatrix} A(s) & C \\ \mathbf{0} & sI - B \end{bmatrix}$ have the same Smith form. Here, the polynomial matrix $A(s)$ is defined by $A(s) = \sum_{i=1}^{k} q_i(s) A_i$.

   By applying Lemma 3 with $k = 2$, $B = S$, $q_1(s) = 1$, $q_2(s) = s$, $A_1 = \begin{bmatrix} -A & -B \\ C & \mathbf{0} \\ H & \mathbf{0} \end{bmatrix}$, $A_2 = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, $X = \begin{bmatrix} \Pi \\ \Gamma \end{bmatrix}$, $C = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ R \end{bmatrix}$, and thus, $A(s) = \begin{bmatrix} sI - A & -B \\ C & \mathbf{0} \\ H & \mathbf{0} \end{bmatrix}$, we obtain the following result:

   Corollary 1
   The regulator Eq. (5) have a solution pair (Π, Γ) if and only if

$$\begin{bmatrix} sI - A & -B & \mathbf{0} \\ C & \mathbf{0} & \mathbf{0} \\ H & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & sI - S \end{bmatrix} \text{ and } \begin{bmatrix} sI - A & -B & \mathbf{0} \\ C & \mathbf{0} & \mathbf{0} \\ H & \mathbf{0} & R \\ \mathbf{0} & \mathbf{0} & sI - S \end{bmatrix}$$

   have the same Smith form.

   Certainly, we could just leave this feasibility problem to some software solvers, for example, Matlab.
3. The feasibility of (8) has been discussed in [11]. In general, it can only be checked numerically whether it has solutions. This holds for other LMI's (e.g., (14) and (15)) and matrix inequalities as well.

6. SIMULATION EXAMPLES

In this section, we present two numerical simulations to illustrate the results obtained in this paper. We consider the dynamics of agents described by the following nonlinear ordinary differential equations:

$$\begin{cases} \dot{x} = Ax + Bu + Ed \\ y = Cx \quad z = Hx \\ d = -\phi(y) \end{cases}, \tag{24}$$

where $x = [x(1), x(2), x(3)]^T$, $A = [-3.2, 10, 0; 1, -1, 1; 0, -14.87, 0]$, $B = [1, 0; 1, 0; 0, 1]$, $C = [1, 0, 0]$, $E = [-2.95; 0; 0]$, and $H = [0, 0, 1]$. The nonlinearity $\phi_i(\cdot)$ for agent i is taken as $(0.4i - 0.1)\text{atan}(\cdot)$, $i = 1, 2, 3, 4$, where 'atan' denotes the arctangent function. It is easily checked that $(0.4i - 0.1)\text{atan}(\cdot)$, $i = 1, 2, 3, 4$ is sector bounded with $S_1 = 0$ and $S_2 = 2$ (Figure 2).

Consider a network of four such agents as shown in Figure 3, where the interconnection graph contains a directed spanning tree and agent 1 has direct access to the exosystem. The dotted edge $(3, 1)$ means that agent 1 can obtain the relative information with respect to agent 3 but will not use it in its protocol.

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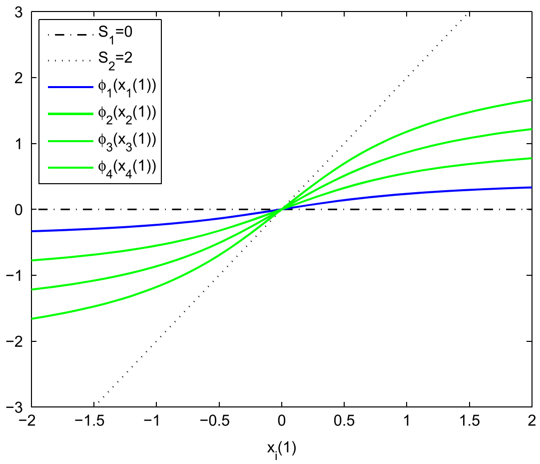

Figure 2. The nonlinearities. [Colour figure can be viewed at wileyonlinelibrary.com]

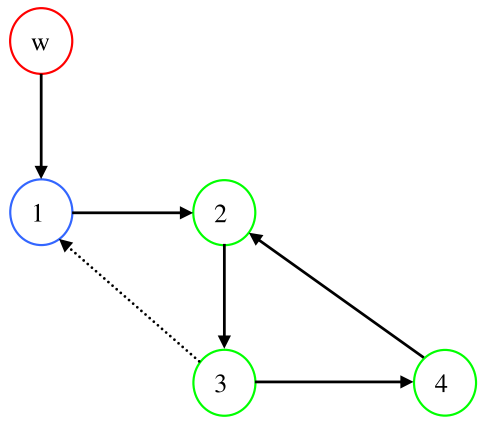

Figure 3. The interconnection graph. [Colour figure can be viewed at wileyonlinelibrary.com]

The exosystem is taken as $\dot{w} = Sw$, $z = Rw$, where $w = [w(1), w(2)]^T$, $S = [0, 1; -1, 0]$ and $R = [1, 0]$. A suitable T can be chosen as $[2; 1]$ such that $S - TR$ is Hurwitz.

Choose the initial conditions for w and w_i as $[1; 1]$ and $[i + 1; i + 1]$, $i = 1, 2, 3, 4$, respectively. Using Matlab, the trajectories of w and w_i's are plotted in Figure 4. Clearly, all the w_i's converge to the state w of the exosystem.

Example 1

We first consider the dynamic state feedback case. In this case, we just need to obtain the feedback and feedforward gain matrices. We refer to classic monographs on solving linear regulator equations, for example, [20]. The LMI's (8) and (9) are standard and can be solved in the Matlab LMI Control Toolbox successively. Using Matlab, we can easily find a solution pair to (5): $\Pi = \begin{bmatrix} 0 & 0 \\ 0.0902 & -0.0082 \\ 1 & 0 \end{bmatrix}$ and $\Gamma = \begin{bmatrix} -0.9016 & 0.082 \\ 1.3407 & 0.8781 \end{bmatrix}$ and obtain the design parameters in Lemma 2: $Q = \begin{bmatrix} 35.9811 & 19.0388 & -0.8627 \\ 19.0388 & 36.1012 & 0.8627 \\ -0.8627 & 0.8627 & 55.08 \end{bmatrix}$, $\rho = 69.0989$, $\mu = -1497.4$, $F = \begin{bmatrix} -27.2814 & -27.089 & -0.003 \\ -1.383 & 1.38 & -27.2283 \end{bmatrix}$ and $K = \begin{bmatrix} 1.5438 & -0.1401 \\ 28.4447 & 0.8894 \end{bmatrix}$. Choosing the initial conditions for agent i as $(i + 1) \cdot [1; 1; 1]$, $i = 1, 2, 3, 4$, their output trajectories are plotted in Figure 5. All the outputs can asymptotically track the reference signal generated by the exosystem in the presence of heterogeneous unknown nonlinearities.

Example 2

In this example, the dynamic output feedback case will be considered. By implementing the computation procedure given after Theorem 4, all the design parameters can be obtained successively. First, $\alpha > 0$ and $X_p > \mathbf{0}$ can be computed to be $\alpha = 69.0989$, and respectively, $X_p = \begin{bmatrix} 35.9811 & 19.0388 & -0.8627 \\ 19.0388 & 36.1012 & 0.8627 \\ -0.8627 & 0.8627 & 55.0800 \end{bmatrix}$. Then $\beta = 1/\alpha = 0.0145$. Here, we enforce the left-hand side of (20) being strictly positive definite. Thus, Y_p determined by LMI (19) together with the strict version of (20) can be computed in the Matlab LMI Control Toolbox as well. So Y_p is computed to be $Y_p = \begin{bmatrix} 0.0788 & -1.0095 & 1.2935 \\ -1.0095 & 41.6257 & 12.1358 \\ 1.2935 & 12.1358 & 291.9815 \end{bmatrix}$. Then the state space dimension n_c of a possible dynamic protocol can be $n_c = 3$. Without loss of generality, we choose Y_c as $Y_c = I_3$. It follows

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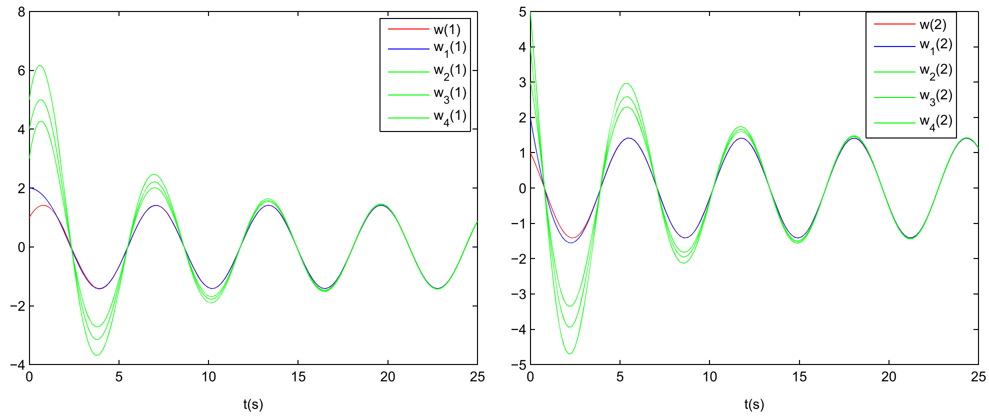

Figure 4. The plots of $w(t)$ and $w_i(t)$. [Colour figure can be viewed at wileyonlinelibrary.com]

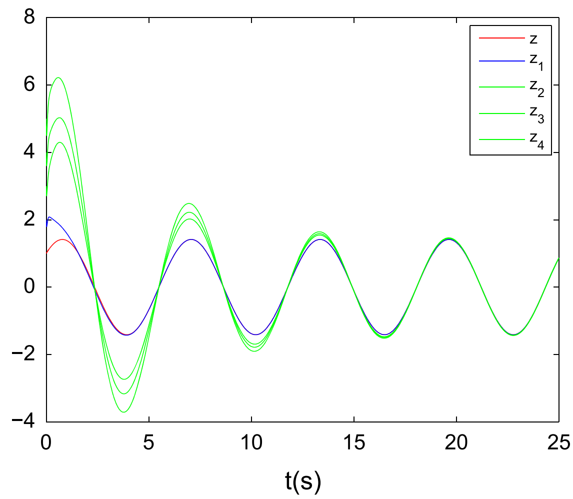

Figure 5. The plots of $z(t)$ and $z_i(t)$ in Example 1. [Colour figure can be viewed at wileyonlinelibrary.com]

that $Y_{pc} = \begin{bmatrix} 0.2006 & 0 & 0 \\ -4.9316 & 4.1553 & 0 \\ 6.4449 & 10.5695 & 11.7776 \end{bmatrix}$ from (22) by using Cholesky decomposition. Thus, a suitable Y is

$$Y = \begin{bmatrix} 0.0788 & -1.0095 & 1.2935 & 0.2006 & 0 & 0 \\ -1.0095 & 41.6257 & 12.1358 & -4.9316 & 4.1553 & 0 \\ 1.2935 & 12.1358 & 291.9815 & 6.4449 & 10.5695 & 11.7776 \\ 0.2006 & -4.9316 & 6.4449 & 1.0000 & 0 & 0 \\ 0 & 4.1553 & 10.5695 & 0 & 1.0000 & 0 \\ 0 & 0 & 11.7776 & 0 & 0 & 1.0000 \end{bmatrix},$$

and consequently, a suitable X is

$$X = 10^3 \begin{bmatrix} 0.0360 & 0.0190 & -0.0009 & 0.0922 & -0.0700 & 0.0102 \\ 0.0190 & 0.0361 & 0.0009 & 0.1687 & -0.1591 & -0.0102 \\ -0.0009 & 0.0009 & 0.0551 & -0.3506 & -0.5858 & -0.6487 \\ 0.0922 & 0.1687 & -0.3506 & 3.0736 & 3.0044 & 4.1287 \\ -0.0700 & -0.1591 & -0.5858 & 3.0044 & 6.8534 & 6.8988 \\ 0.0102 & -0.0102 & -0.6487 & 4.1287 & 6.8988 & 7.6412 \end{bmatrix}.$$

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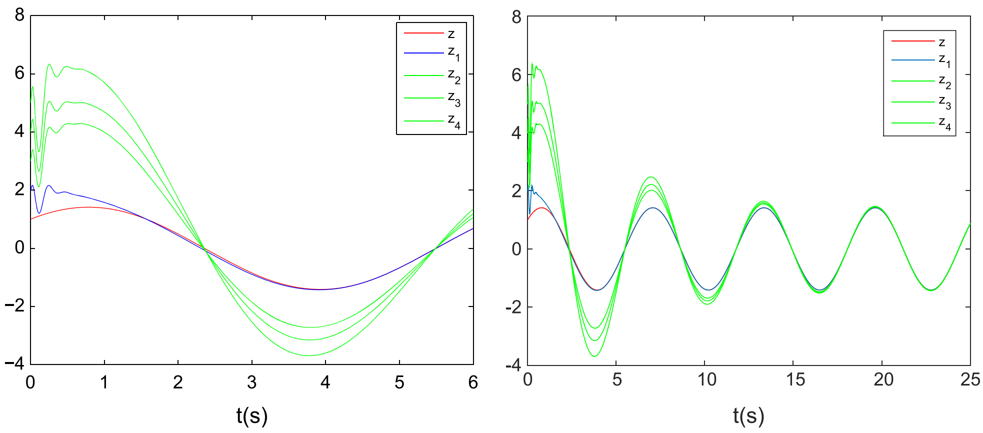

Figure 6. The plots of $z(t)$ and $z_i(t)$ in Example 2. [Colour figure can be viewed at wileyonlinelibrary.com]

Now, r is computed as $r = 158480$, and hence, Θ_x is also known. By computing F and partitioning it appropriately, we obtain, finally, $A_c = 10^5 \begin{bmatrix} 0.0189 & 0.0234 & 1.0265 \\ -0.0123 & -0.0151 & 1.1271 \\ 0.0000 & 0.0000 & -1.5848 \end{bmatrix}$, $B_c = 10^6 \begin{bmatrix} 1.2461 \\ 1.3038 \\ -1.8665 \end{bmatrix}$, $C_c = 10^4 \begin{bmatrix} 0.0368 & 0.0454 & 0.4685 \\ -0.0029 & -0.0040 & -1.2546 \end{bmatrix}$, and $D_c = 10^5 \begin{bmatrix} 0.6236 \\ -1.4840 \end{bmatrix}$. Then, $K = \Gamma - D_c H \Pi$ is computed as $K = 10^5 \begin{bmatrix} -0.6236 & 0.0000 \\ 1.4840 & 0.0000 \end{bmatrix}$.

By using the same initial conditions as in Example 1 and let the initial states of v_i's, $i = 1, 2, 3, 4$, be zeros, we obtain the output trajectories that are plotted in Figure 6. Here, we use two different time scales in order to make the synchronization behavior be clear at the beginning as well. Its synchronization progress is a little bit different from the dynamic state feedback case in Example 1, which makes sense.

7. CONCLUSIONS

In this paper, we have studied the robust cooperative output regulation problems for directed Lur'e networks. The networks are allowed to be heterogeneous in the sense that the Lur'e-type nonlinearities therein are allowed to differ for distinct agents. By designing a fully distributed observer, a copy of the reference signal generated by a virtual exosystem is made asymptotically at each agent, and thus, these agents can asymptotically track the reference signal via local information. Both the dynamic state feedback and the dynamic output feedback cases have been considered. In the near future, we will study the case that the linear part of a Lur'e system is uncertain as well as its nonlinearity. Meanwhile, measurement feedback based protocols will be employed.

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