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Published in:
IEEE Transactions on Automatic Control

DOI:
10.1109/TAC.2017.2655443

Nonlinear Eigenvalue Approach to Differential Riccati Equations for Contraction Analysis

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***Abstract*—In this paper, we extend the eigenvalue method of the algebraic Riccati equation to the differential Riccati equation (DRE) in contraction analysis. One of the main results is showing that solutions to the DRE can be expressed as functions of nonlinear eigenvectors of the differential Hamiltonian matrix. Moreover, under an assumption for the differential Hamiltonian matrix, real symmetry, regularity, and positive semidefiniteness of solutions are characterized by nonlinear eigenvalues and eigenvectors.**

***Index Terms*—Contraction analysis, differential Riccati equations (DREs), nonlinear eigenvalues, nonlinear systems.**

I. Introduction

In this paper, we present a novel eigenvalue method for the differential Riccati equation (DRE) in contraction analysis. Contraction analysis has been studied intensively in recent decades, which deals with trajectories of nonlinear systems with respect to one another [1]–[4]. One of the interesting ideas of contraction theory is considering an infinitesimal metric instead of a feasible distance function by lifting a function and vector field on a manifold to a function on its tangent and cotangent bundles. In such theoretical frameworks, for instance, stability analysis [1], [5], optimal control [2], [3], and balanced truncation [4] have been studied. In [2]–[4], a Riccati equation that we call a DRE plays an important role. The DRE is a nonlinear partial differential equation (nPDE) for an unknown matrix valued function of the state and time. The DRE can be viewed as an extension of algebraic Riccati equations (AREs) and DREs for linear time-invariant and variant systems rather than as the Hamilton-Jacobi equation (HJE).

One of the most important analysis methods for the ARE is the eigenvalue method [6]–[8]. This method shows that solutions to the ARE, a nonlinear algebraic equation, can be described as functions of eigenvectors of the Hamiltonian matrix, and in terms of eigenvalues and eigenvectors, real symmetry, regularity, and positive semidefiniteness of solutions have been studied. This method has been extended to the DRE for linear periodic systems [9]–[11], which is different from the equation considered in this paper.

Our main concern in this paper is extending the eigenvalue method to the DRE in contraction analysis in terms of recently introduced nonlinear eigenvalues and eigenvectors [12]–[14]. First, we demonstrate that solutions to the DRE can be expressed as functions of nonlinear right eigenvectors of the corresponding differential Hamiltonian matrix as in the linear case. Next, we investigate its solution structures when nonlinear right eigenvectors of the Hamiltonian matrix span the entire space. In this case, a nonlinear right eigenvalue is also a left eigenvalue and vice versa, and if λ is an (right or left) eigenvalue of the Hamiltonian matrix, then $-\lambda$, the complex conjugate of λ denoted by λ^*, and $-\lambda^*$ are also eigenvalues similarly to the linear case. Moreover, we study real symmetry, regularity, and positive semidefiniteness of solutions to the DRE in terms of nonlinear eigenvalues.

The nonlinear left and right eigenvalues and eigenvectors of the Jacobian matrix of a vector field correspond to a one-dimensional (1-D) invariant distributions and codistributions, respectively, in the time-invariant case. A similar concept can be found in the Koopman operator theory [15]. The Koopman eigenfunction coincides with an invariant subset under the Lie derivative of a function. The concepts of nonlinear eigenvalues are originally introduced in noncommutative algebra in relation to the pseudolinear transformation (PLT) [16], [17]. The PLT can be interpreted as a generalized notion of linear transformation to differential one-forms. Noncommutative algebra and the PLT are used for analysis of linear time-varying and nonlinear control systems [18], [19]. In contrast to nonlinear systems, there is no application of such eigenvalues to linear time-varying systems. The DRE of the linear periodic systems [9]–[11] is not analyzed in terms of such eigenvalues.

Notations: Let $\mathbf{R}$ and $\mathbf{C}$ be the fields of real and complex numbers, respectively. Let $\mathcal{K}_{\mathbf{R}}$ be the field of the real meromporphic functions in variables $x_1, x_2, \dots, x_n, t$. Let $\mathcal{K}$ be the set of functions $\{a + bj : a, b \in \mathcal{K}_{\mathbf{R}}\}$, where j is the imaginary unit, and the domain of definition of both a and b is $\mathbf{R}^n \times \mathbf{R}$. Note that $\mathcal{K}_{\mathbf{R}} \subset \mathcal{K}$, and $\mathcal{K}$ is a field. Then, $\mathcal{K}^{2n}$ is a vector space over $\mathcal{K}$. The reason why we consider (not commonly used) field $\mathcal{K}$ is that we exploit a concept of nonlinear eigenvalue of matrix $A \in \mathcal{K}^{n\times n}$. As will be shown in Example 2.3, for constant matrix $M \in \mathbf{C}^{n\times n}$, the set of nonlinear eigenvalues contains the set of eigenvalues in linear algebra. Since a linear eigenvalue can be a complex number even if M is in $\mathbf{R}^{n\times n}$, nonlinear eigenvalue can be an element in $\mathcal{K}$ even if matrix A is an element in $\mathcal{K}_{\mathbf{R}}^{n\times n}$. Therefore, we consider field $\mathcal{K}$ in this paper.

For a scalar-valued function $V(x,t) \in \mathcal{K}$, we denote a row vector consisting of the partial derivatives of V with respect to x_i ($i = 1, 2, \dots, n$) as $\partial V/\partial x$, and we denote $\partial^{\mathrm{T}} V/\partial x := (\partial V/\partial x)^{\mathrm{T}}$. For matrix $A(x,t) \in \mathcal{K}^{r_1 \times r_2}$, $\mathrm{rank}_{\mathcal{K}} A(x,t) = r$ means that the rank of $A(x,t)$ over field $\mathcal{K}$ is r. In particular, if $r_1 = r_2 = r$, A is said to be regular.

Next, we introduce an operator $\delta_f : \mathcal{K} \to \mathcal{K}$. By using real analytic vector-valued function $f(x,t) : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$, operator $\delta_f : \mathcal{K} \to \mathcal{K}$ is defined as

$$\delta_f(a(x,t)) = \frac{\partial a(x,t)}{\partial t} + \frac{\partial a(x,t)}{\partial x} f(x,t),\ a(x,t) \in \mathcal{K}. \quad (1)$$

Field $\mathcal{K}$ is a differential field with respect to δ_f. For matrix $X(x,t) = (X_{ij}(x,t)) \in \mathcal{K}^{n\times n}$, $\delta_f(X(x,t))$ denotes the matrix whose (i,j)th element is $\delta_f(X_{ij}(x,t))$. Operator δ_f coincides with the time derivative of a function along a solution to $\dot{x}(t) = f(x(t),t)$ because the

Manuscript received July 31, 2014; revised December 2, 2016; accepted January 12, 2017. Date of publication January 18, 2017; date of current version December 1, 2017. This work was supported in part by JSPS KAKENHI under Grant 15K18087 and Grant 15H02257 and in part by JST CREST. Recommended by Associate Editor L. Menini.

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Digital Object Identifier 10.1109/TAC.2017.2655443

time derivative of $a(x(t), t)$ is $da(x(t), t)/dt = \partial a(x(t), t)/\partial t + (\partial a(x(t), t)/\partial x) f(x(t), t)$. In systems and control, in general, we study a real-valued vector field. Thus, we assume that f is real valued. Throughout this paper, we leave out arguments of functions when these are clear from the context.

II. Eigenvalue Approach

A. Differential Riccati Equation

Let each element of $A(x,t) \in \mathcal{K}_{\mathbf{R}}^{n\times n}$, $R(x,t) = R^{\mathrm{T}}(x,t) \in \mathcal{K}_{\mathbf{R}}^{n\times n}$, and $Q(x,t) = Q^{\mathrm{T}}(x,t) \in \mathcal{K}_{\mathbf{R}}^{n\times n}$ be real analytic. In this paper, we study the following nPDE for unknown matrix $X(x,t) \in \mathcal{K}^{n\times n}$:

$$\begin{aligned}\delta_f(X(x,t)) \ &+ X(x,t)A(x,t) + A^{\mathrm{T}}(x,t)X(x,t) \\ &- X(x,t)R(x,t)X(x,t) = -Q(x,t). \qquad (2)\end{aligned}$$

Equation (2) is a generalization of the ARE, and thus we call (2) a (generalized) DRE. A real symmetric and positive definite solution $X \in \mathcal{K}_{\mathbf{R}}^{n\times n}$ plays an important role in systems and control theory such as that in contraction analysis [1], [2].

Example 2.1: A stabilizing controller is designed by using a solution to DRE. Consider a time-invariant real analytic system

$$\dot{x}(t) = f(x(t)) + Bu(t)$$

where $x \in \mathbf{R}^n$ and $u \in \mathbf{R}^m$. For $A = \partial f/\partial x$, $R = BB^{\mathrm{T}}$, and symmetric and positive definite $Q(x)$ at each $x \in \mathbf{R}^n$, suppose that DRE

$$\delta_f(X) + X\frac{\partial f}{\partial x} + \frac{\partial^{\mathrm{T}} f}{\partial x}X - XBB^{\mathrm{T}}X = -Q \qquad (3)$$

has a symmetric and positive definite solution $X \in \mathcal{K}_{\mathbf{R}}^{n\times n}$ at each $x \in \mathbf{R}^n$. Here, we show that if X satisfies $(\partial X_{ij}/\partial x)B = 0$, and if there exists a vector-valued function $k(x) \in \mathcal{K}^m$ such that $\partial k/\partial x = B^{\mathrm{T}}X$, then $u = -k(x)$ is a stabilizing controller. Under these assumptions, (3) can be rearranged as

$$\begin{aligned}&\delta_{f-Bk}(X) + X\frac{\partial(f-Bk)}{\partial x} + \frac{\partial^{\mathrm{T}}(f-Bk)}{\partial x}X \\ &= -Q - XBB^{\mathrm{T}}X.\end{aligned}$$

We notice that $V(x,\delta x) := \delta x^{\mathrm{T}} X \delta x$ is a contraction Riemannian metric for the closed-loop system and its variational system

$$\begin{aligned}\dot{x} &= f(x) - Bk(x), \qquad (4)\\ \frac{d}{dt}\delta x(t) &= \frac{\partial(f(x) - Bk(x))}{\partial x}\delta x(t).\end{aligned}$$

According to Forni and Sepulchre [1], the closed-loop system is incrementally globally asymptotically stable. Roughly speaking, any pair of trajectories of the closed-loop system converges to each other. If the system has an unique equilibrium point, the system is globally asymptotically stable. In summary, by solving DRE (3), we can construct a stabilizing controller $u = -k = -\int B^{\mathrm{T}}X dx$. In [2], a similar result is obtained for time-varying systems, and the integrability condition of $B^{\mathrm{T}}X$ is dropped by using a line integral.

Other applications of the DRE are, for instance, incremental optimal control [2] and balanced truncation [4]. In linear systems and control theory, the optimal controller is designed by solving an ARE. This result is extended in the contraction framework by using a DRE [2]. Moreover, the so-called differential balanced realization [4] is defined by using Lyapunov types of equations, which are specific DREs for $R \equiv 0$. In optimal control and balanced truncation, symmetric and positive definite solutions to DREs are used. Since the DRE is an nPDE for an unknown matrix, the structures of solutions have not been adequately studied. That is, it is unclear when a symmetric and positive definite solution exists. Here, our concern is investigating the solution structures by using nonlinear eigenvalues and eigenvectors [12], [14]. That is, we extend the eigenvalue method of the ARE [6]–[8].

B. Differential Hamiltonian Matrix

Solutions to the ARE are characterized by the eigenvalues and eigenvectors of the Hamiltonian matrix. The counterpart of the Hamiltonian matrix to the DRE is

$$\mathcal{H}(x,t) := \begin{bmatrix} A(x,t) & -R(x,t) \\ -Q(x,t) & -A^{\mathrm{T}}(x,t)\end{bmatrix}. \qquad (5)$$

We call this $\mathcal{H}(x,t) \in \mathcal{K}_{\mathbf{R}}^{2n\times 2n}$ a (generalized) differential Hamiltonian matrix. Since the elements of A, $R = R^{\mathrm{T}}$, and $Q = Q^{\mathrm{T}}$ are real analytic, the elements of $\mathcal{H}$ are also.

Next, we show the definition of the nonlinear eigenvalues and eigenvectors [12], [14], [17].

Definition 2.2: Consider δ_f defined in (1). Let $M \in \mathcal{K}^{n\times n}$.

1) $v \in \mathcal{K}^n \setminus \{0\}$ is a left eigenvector for M associated with left eigenvalue $\alpha \in \mathcal{K}$ if $v^{\mathrm{T}}M + \delta_f(v)^{\mathrm{T}} = v^{\mathrm{T}}\alpha$.
2) $w \in \mathcal{K}^n \setminus \{0\}$ is a right eigenvector for M associated with right eigenvalue $\beta \in \mathcal{K}$ if $Mw - \delta_f(w) = \beta w$.

Moreover, the sets of left and right eigenvalues of M are denoted by $\mathrm{lspec}_f(M)$ and $\mathrm{rspec}_f(M)$, respectively.

Nonlinear eigenvalues relate to invariant (co-) distributions when $M = \partial f/\partial x$ [3], [12]. The definitions of left and right eigenvalues are, respectively, rearranged as $\mathcal{L}_f(v^{\mathrm{T}}dx) = \alpha(v^{\mathrm{T}}dx)$ with the Lie derivative of one-forms along f and $[w, f] = \beta w$ with the Lie bracket of vector fields. Thus, $v^{\mathrm{T}}dx$ and w are, respectively, 1-D invariant codistribution and distribution.

Example 2.3: In the linear case when M is in $\mathbf{R}^{n\times n}$, the first equation in Definition 2.2 holds for linear eigenvalue $\alpha \in \mathbf{C}$ and left eigenvector $v \in \mathbf{C}^n$ (or eigenvalue $\beta \in \mathbf{C}$ and right eigenvector $v \in \mathbf{C}^n$). Thus, the linear eigenvalue and eigenvector are a nonlinear eigenvalue and eigenvector.

Nonlinear eigenvalues have similar properties to those in linear algebra. These are invariant under δ_f-conjugacy defined below, which relates to a change of basis over a differential field. Let $\{v_1,\dots,v_n\}$ and $\{w_1,\dots,w_n\}$ be bases for $\mathcal{K}^n$. Then, there exist matrices $M, N \in \mathcal{K}^{n\times n}$ such that $[\delta_f(v_1) \ \dots \ \delta_f(v_n)] = M[v_1 \ \dots \ v_n]$ and $[\delta_f(w_1) \ \dots \ \delta_f(w_n)] = N[w_1 \ \dots \ w_n]$. For two bases, there exists a regular matrix $T \in \mathcal{K}^{n\times n}$ such that $[v_1,\dots,v_n] = T[w_1,\dots,w_n]$. By applying δ_f from left, we have

$$M[v_1,\dots,v_n] = (TN + \delta_f(T))T^{-1}[v_1,\dots,v_n].$$

Since $\{v_1,\dots,v_n\}$ is a basis, $M = (TN + \delta_f(T))T^{-1}$. This pair of matrices (M, N) is said to be δ_f-conjugate.

Definition 2.4 ([16], [17]): A pair of matrices $(M, N) \in \mathcal{K}^{n\times n} \times \mathcal{K}^{n\times n}$ is δ_f-conjugate (with respect to T), if there exists a regular matrix $T \in \mathcal{K}^{n\times n}$ such that $M = (TN + \delta_f(T))T^{-1}$ holds.

Example 2.5: When $n = 1$, we have the definition given in [16] and [17] of δ_f-conjugation for elements in $a, b \in \mathcal{K}$. A pair (a, b) is δ_f-conjugate, if $b = a + \delta_f(c)/c$ for nonzero $c \in \mathcal{K}$.

Proposition 2.6 ([16], [17]): Let M be in $\mathcal{K}^{n\times n}$.

1) Let $(a, b) \in \mathcal{K} \times \mathcal{K}$ be δ_f-conjugate. If $a \in \mathrm{lspec}_f(M)$ (or $\mathrm{rspec}_f(M)$), then $b \in \mathrm{lspec}_f(M)$ (or $\mathrm{rspec}_f(M)$).
2) If (M, N) is δ_f-conjugate, $\mathrm{rspec}_f(M) = \mathrm{rspec}_f(N)$ and $\mathrm{lspec}_f(M) = \mathrm{lspec}_f(N)$.

Example 2.7: If (M, N) is δ_f-conjugate, then we have $N = (T^{-1}M + \delta_f(T^{-1}))T$, i.e. (N, M) is also δ_f-conjugate.

Example 2.8: If both (L, M) and (M, N) are δ_f-conjugate with respect to regular $T, S \in \mathcal{K}^{n\times n}$, respectively. Then, (L, N) is also δ_f-conjugate with respect to TS.

Example 2.9: Consider a system $\dot{x} = f(x)$ and its variational system $\delta\dot{x} = (\partial f/\partial x)\delta x$. After an analytic diffeomorphic coordinate transformation $z = \varphi(x)$, we have $\delta\dot{z} = (T(\partial f/\partial x) + \delta_f(T))T^{-1}\delta z$, where $T := \partial\varphi/\partial x$. Proposition 2.6 2) implies that $\partial f/\partial x$ and $(T(\partial f/\partial x) + \delta_f(T))T^{-1}$ have the same nonlinear left and right eigenvalues.

Proposition 2.6 1) comes from a scalar multiplication of eigenvectors. For some nonzero $a \in \mathcal{K}$, left eigenvalue α, and eigenvector v, we have

$$\begin{aligned} av^{\mathrm{T}}M + \delta_f(av^{\mathrm{T}}) &= av^{\mathrm{T}}M + a\delta_f(v^{\mathrm{T}}) + \delta_f(a)v^{\mathrm{T}} \\ &= (\alpha + \delta_f(a)/a)av^{\mathrm{T}}. \end{aligned}$$

Then, $\alpha + \delta_f(a)/a$ and av are also left eigenvalue and eigenvector, respectively. Note that $(\alpha, \alpha + \delta_f(a)/a)$ is δ_f-conjugate. For a similar relationship for right eigenvectors, see (19).

C. Main Theorem

Here, we show that solutions to the DRE can be expressed as functions of nonlinear eigenvectors of the corresponding differential Hamiltonian matrix $\mathcal{H}$.

Definition 2.10: A linear subspace $W \subset \mathcal{K}^{2n}$ is said to be right $\mathcal{H}$ invariant if $\mathcal{H}w - \delta_f(w) \in W$ for all $w \in W$. We denote the set of right eigenvalues of $\mathcal{H}$ in W by $\mathrm{rspec}_f(\mathcal{H}|_W)$, i.e.,

$$\begin{aligned} &\mathrm{rspec}_f(\mathcal{H}|_W) \\ &:= \{\beta \in \mathcal{K} : \mathcal{H}w - \delta_f(w) = \beta w,\ w \in W \setminus \{0\}\}. \end{aligned}$$

In [3], van der Schaft extends the connection between the Hamilton–Jacobi equation and Lagrangian submanifold to the DRE and a so-called Lagrangian subbundle. The $\mathcal{H}$-invariant subspace in this paper corresponds to the Lagrangian subbundle. In this paper, we study a more detailed algebraic structure of the $\mathcal{H}$-invariant subspace based on the following main theorem.

Theorem 2.11: Assume that there exists an n-dimensional $\mathcal{H}$ invariant subspace $W \subset \mathcal{K}^{2n}$. Consider matrices $U, V \in \mathcal{K}^{n\times n}$ such that

$$W = \mathrm{Im}\begin{bmatrix} U \\ V \end{bmatrix}. \tag{6}$$

If U is regular, $X := VU^{-1} \in \mathcal{K}^{n\times n}$ is a solution to (2) and satisfies

$$\mathrm{rspec}_f(A - RX) = \mathrm{rspec}_f(\mathcal{H}|_W). \tag{7}$$

Conversely, if $X \in \mathcal{K}^{n\times n}$ is a solution to (2), there exist $U, V \in \mathcal{K}^{n\times n}$ such that U is regular, and $X = VU^{-1}$. Moreover, for these U and V, subspace $W \subset \mathcal{K}^{2n}$ in (6) is an n-dimensional $\mathcal{H}$ invariant subspace and satisfies (7).

Proof: We prove the first part. Since W is $\mathcal{H}$ invariant, there exists some matrix $\Lambda \in \mathcal{K}^{n\times n}$ such that

$$\begin{bmatrix} A & -R \\ -Q & -A^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} U \\ V \end{bmatrix} - \begin{bmatrix} \delta_f(U) \\ \delta_f(V) \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix}\Lambda. \tag{8}$$

By multiplying U^{-1} from the right, we have

$$\begin{aligned} &\begin{bmatrix} A & -R \\ -Q & -A^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} I_n \\ VU^{-1} \end{bmatrix} - \begin{bmatrix} \delta_f(U)U^{-1} \\ \delta_f(V)U^{-1} \end{bmatrix} \\ &= \begin{bmatrix} I_n \\ VU^{-1} \end{bmatrix} U\Lambda U^{-1}. \end{aligned} \tag{9}$$

Next, by multiplying $[VU^{-1}\ \ -I_n]$ from the left, we obtain

$$\begin{aligned} &\delta_f(V)U^{-1} - VU^{-1}\delta_f(U)U^{-1} \\ &+ VU^{-1}A + A^{\mathrm{T}}VU^{-1} - VU^{-1}RVU^{-1} + Q = 0. \end{aligned}$$

It can be shown that $\delta_f(VU^{-1}) = \delta_f(V)U^{-1} - VU^{-1}\delta_f(U)U^{-1}$. Thus, $X := VU^{-1}$ is a solution to (2).

Next, from the upper half of (9)

$$A - RX = (U\Lambda + \delta_f(U))U^{-1}. \tag{10}$$

From Proposition 2.6 2)

$$\begin{aligned} \mathrm{rspec}_f(\Lambda) &= \mathrm{rspec}_f((U\Lambda + \delta_f(U))U^{-1}) \\ &= \mathrm{rspec}_f(A - RX). \end{aligned} \tag{11}$$

Let r be the maximum number of linearly independent right eigenvectors $w_1, \ldots, w_r \in W$ of $\mathcal{H}$ associated with right eigenvalues β_i $(i = 1, \ldots, r)$. Since W is an n-dimensional subspace, there exist $w_{r+1}, \ldots, w_n \in W$ such that $\mathrm{span}_{\mathcal{K}}\{w_1, \ldots, w_n\} = W$ holds. Let

$$\begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} := \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix}.$$

Then, there exists $\hat{\Lambda} \in \mathcal{K}^{n\times n}$ such that

$$\begin{aligned} &\begin{bmatrix} A & -R \\ -Q & -A^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} - \begin{bmatrix} \delta_f(\hat{U}) \\ \delta_f(\hat{V}) \end{bmatrix} = \begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix}\hat{\Lambda}, \\ &\hat{\Lambda} := \begin{bmatrix} B_{11} & B_{21} \\ 0 & B_{22} \end{bmatrix} \end{aligned}$$

where $B_{11} = \mathrm{diag}\{\beta_1, \ldots, \beta_r\}$ and $B_{21} \in \mathcal{K}^{(n-r)\times r}$, $B_{22} \in \mathcal{K}^{(n-r)\times(n-r)}$ are suitable matrices. Thus,

$$\mathrm{rspec}_f(\mathcal{H}|_W) = \mathrm{rspec}_f(B_{11}) = \mathrm{rspec}_f(\hat{\Lambda}). \tag{12}$$

From (11) and (12), it remains to show $\mathrm{rspec}_f(\Lambda) = \mathrm{rspec}_f(\hat{\Lambda})$. Since both $[U^{\mathrm{T}}\ V^{\mathrm{T}}]^{\mathrm{T}}$ and $[\hat{U}^{\mathrm{T}}\ \hat{V}^{\mathrm{T}}]^{\mathrm{T}}$ consist of bases of W, there exists a regular matrix $T \in \mathcal{K}^{n\times n}$ such that

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} T.$$

By substituting this equality into (8)

$$\mathcal{H}\begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} - \begin{bmatrix} \delta_f(\hat{U}) \\ \delta_f(\hat{V}) \end{bmatrix} = \begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix}(T\Lambda + \delta_f(T))T^{-1}$$

which implies $\hat{\Lambda} = (T\Lambda + \delta_f(T))T^{-1}$. From Proposition 2.6 2), the set of right eigenvalues of $\hat{\Lambda}$ and Λ are equivalent.

We prove the second part. Let $\Lambda := A - RX$. By premultiplying X, we have, from (2)

$$X\Lambda = XA - XRX = -Q - \delta_f(X) - A^{\mathrm{T}}X.$$

The above two equations yield

$$\begin{bmatrix} A & -R \\ -Q & -A^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} I_n \\ X \end{bmatrix} - \begin{bmatrix} \delta_f(I_n) \\ \delta_f(X) \end{bmatrix} = \begin{bmatrix} I_n \\ X \end{bmatrix}\Lambda. \tag{13}$$

Denote $U := I_n$ and $V := X$. Then, U is regular, and $X = VU^{-1}$ holds. Since U is regular, $w_i \in \mathcal{K}^{2n}$ $(i = 1, \ldots, n)$ defined by $[w_1, \ldots, w_n] := [U^{\mathrm{T}}\ V^{\mathrm{T}}]^{\mathrm{T}}$ spans $\mathcal{K}^n$, and thus W in (6) is an n-dimensional subspace. From (13), W is $\mathcal{H}$ invariant. Finally, it can be shown that (7) holds similarly to the proof of the first part. ∎

Remark 2.12: Solution $X \in \mathcal{K}^{n\times n}_{\mathbf{R}}$ does not depend on the choice of basis of W. Every basis of W can be represented with regular matrix $T \in \mathcal{K}^{n\times n}$ as

$$\begin{bmatrix} U \\ V \end{bmatrix} T = \begin{bmatrix} UT \\ VT \end{bmatrix}. \tag{14}$$

Since $(VT)(UT)^{-1} = VU^{-1} = X$ holds, X does not depend on the choice of basis of W.

As demonstrated in Example 2.1, a symmetric and positive definite solution $X \in \mathcal{K}^{n\times n}_{\mathbf{R}}$ to a DRE plays an important role in the contraction analysis. However, it is not guaranteed that a solution $X \in \mathcal{K}^{n\times n}$ to (2) has such a property for any n-dimensional $\mathcal{H}$ invariant subspace W in (6). In general, X is a complex-valued function because nonlinear eigenvalues and eigenvectors of $\mathcal{H}$, i.e., $U \in \mathcal{K}^{n\times n}$ and $V \in \mathcal{K}^{n\times n}$ can be complex-valued functions as in Example 2.13 below. In Section III, we give a characterization of W defining a symmetric and positive definite solution $X \in \mathcal{K}^{n\times n}_{\mathbf{R}}$ to a DRE under an assumption for differential Hamiltonian matrix $\mathcal{H}$.

Theorem 2.11 is an extension of the eigenvalue method for the ARE because Theorem 2.11 demonstrates that solutions to DRE (2) can be obtained by using the right eigenvectors of the corresponding differential Hamiltonian matrix $\mathcal{H}$.

Example 2.13: Based on Example 2.1, consider a stabilization problem of an RL-circuit with a nonlinear inductor

$$\begin{bmatrix} 1+x_1^2 & 0 \\ 0 & 1 \end{bmatrix} \dot{x} = -\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

Then, we have

$$f = \begin{bmatrix} \dfrac{-x_1 + x_2}{1+x_1^2} \\ x_1 - x_2 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; R = BB^{\mathrm{T}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} -\dfrac{1+2x_1x_2 - x_1^2}{(1+x_1^2)^2} & \dfrac{1}{1+x_1^2} \\ 1 & -1 \end{bmatrix}.$$

For positive definite $Q := \mathrm{diag}\{3 + 4x_1^2 + x_1^4, 1\}$ for all $x \in \mathbf{R}^2$, the differential Hamiltonian matrix is

$$\mathcal{H} = \begin{bmatrix} -\dfrac{1+2x_1x_2 - x_1^2}{(1+x_1^2)^2} & \dfrac{1}{1+x_1^2} & 0 & 0 \\ 1 & -1 & 0 & -1 \\ -(3+4x_1^2+x_1^4) & 0 & \dfrac{1+2x_1x_2 - x_1^2}{(1+x_1^2)^2} & -1 \\ 0 & -1 & -\dfrac{1}{1+x_1^2} & 1 \end{bmatrix}.$$

The right eigenvalues and eigenvectors of $\mathcal{H}$ are

$$\beta^1 := -\frac{2+x_1^2}{1+x_1^2}, \; w^1 = \begin{bmatrix} \dfrac{1}{1+x_1^2} & -1 & 1+x_1^2 & 0 \end{bmatrix}^{\mathrm{T}},$$

$$\beta^2 := -2 - 2x_1x_2 - x_1^2 - x_1^4 - \frac{(x_1 - x_2)c(x_1,x_1)}{1+x_1^2},$$

$$w^2 = \begin{bmatrix} 1 \\ -1 - x_1^2 - (x_1+x_2)c(x_1,x_1) \\ (1+x_1^2)(1+x_1^2 - (x_1+x_2)c(x_1,x_1)) \\ -(x_1+x_2)c(x_1,x_1) \end{bmatrix}$$

where

$$c(x_1,x_2) = \sum_{\{a : a^3 + 6a - x_1^3 - 3x_1 - 3x_2 = 0\}} \frac{(1+x_1^2)(a-x_1)^{\frac{a^2+3}{a^2+2}}}{\int_0^{x_1} (1+b^2)(a-b)^{\frac{a^2+3}{a^2+2}} db + 1}.$$

Since two of solutions to $a^3 + 6a - x_1^3 - 3x_1 - 3x_2 = 0$ are complex-valued functions, $U, V \in \mathcal{K}^{n\times n}$ are complex-valued functions. On the basis of Theorem 2.11, we define

$$U := \begin{bmatrix} \dfrac{1}{1+x_1^2} & 1 \\ -1 & -1 - x_1^2 - (x_1+x_2)c(x_1,x_2) \end{bmatrix},$$

$$V := \begin{bmatrix} 1+x_1^2 & (1+x_1^2)(1+x_1^2 - (x_1+x_2)c(x_1,x_2)) \\ 0 & -(x_1+x_2)c(x_1,x_2) \end{bmatrix}.$$

Then, a solution to DRE (3) is

$$X := VU^{-1} = \begin{bmatrix} 2(1+x_1^2)^2 & 1+x_1^2 \\ 1+x_1^2 & 1 \end{bmatrix}$$

and $(\partial X_{ij}/\partial x)B = 0$. Moreover, $X \in \mathcal{K}^{n\times n}_{\mathbf{R}}$ is positive definite for all $x \in \mathbf{R}^2$ while U and V are complex-valued functions. According to Example 2.1, the feedback controller

$$u = -\int_0^x B^{\mathrm{T}} X dx = -\left(x_1 + x_1^3/3 + x_2\right)$$

makes the closed-loop system globally incrementally asymptotically stable.

III. Detailed Properties in Simple Case

According to Example 2.1 and Manchester and Slotine [2], a stabilizing solution to the DRE is real symmetric and positive (semi)definite. In the linear case, real symmetry, regularity, and positive (semi)definiteness depend on a choice of n-eigenvectors of the Hamiltonian matrix, i.e., an n-dimensional $\mathcal{H}$-invariant subspace. Here, we study relationships between properties of solutions to the DRE and nonlinear eigenvalues and eigenvectors of the differential Hamiltonian matrix. As a first step, in this paper, we assume that the differential Hamiltonian matrix is simple.

Definition 3.1: A matrix $M \in \mathcal{K}^{n\times n}$ is said to be left (or right) simple if there exist n left (or right) eigenvectors $v_1, \ldots, v_n \in \mathcal{K}^n$ such that $\mathrm{span}_{\mathcal{K}}\{v_1, \ldots, v_n\} = \mathcal{K}^n$.

Note that, for any right eigenvector $w \in \mathcal{K}^{2n}$ of $\mathcal{H}$, $\{w\} \subset \mathcal{K}^{2n}$ is a 1-D $\mathcal{H}$-invariant subspace. Therefore, simplicity of $\mathcal{H}$ implies the existence of the $2n$-dimensional $\mathcal{H}$-invariant subspace.

It can readily be shown that a matrix M is left (or right) simple if and only if M is δ_f-conjugate to a diagonal matrix, which yields the following proposition.

Proposition 3.2: A matrix $M \in \mathcal{K}^{n\times n}$ is right simple if and only if it is left simple.

Since left and right simplicity are equivalent properties, a left or right simple matrix is called simple. Also, its left or right eigenvalue is called an eigenvalue.

If the differential Hamiltonian matrix $\mathcal{H}$ in (5) is simple, it is possible to show the following.

Theorem 3.3: Let $\mathcal{H}$ be simple. Let $W \subset \mathcal{K}^{2n}$ be an n-dimensional $\mathcal{H}$ invariant subspace.

1) There exist $U, V \in \mathcal{K}^{n\times n}$ in (6) and λ_i $(i = 1, \ldots, n)$ such that $\Lambda := \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$ holds in (8).
2) Denote λ^{δ_f} as the set of δ_f-conjugate elements of $\lambda \in \mathcal{K}$. Also, define for λ_i $(i = 1, \ldots, n)$ in 1)

$$\{\lambda_1, \ldots, \lambda_n\}^{\delta_f} := \{\lambda \in \lambda_i^{\delta_f} : i = 1, \ldots, n\}. \tag{15}$$

Then, $\{\lambda_1, \ldots, \lambda_n\}^{\delta_f} = \mathrm{rspec}_f(\mathcal{H}|_W)$.
3) $-\lambda_i, \lambda_i^*, -\lambda_i^* \in \mathrm{rspec}_f(\mathcal{H}|_W)$ for λ_i $(i = 1, \ldots, n)$ in 1), where λ_i^* is the complex conjugate of λ_i.

4) If $\mathcal{H}$ has no eigenvalue on the imaginary axis, then there is at least one W such that U^*V is Hermitian, and $U^{\mathrm{T}}V$ is symmetric for any choice of $U, V \in \mathcal{K}^{n\times n}$ in (6).
5) Suppose that U^*V is Hermitian or $U^{\mathrm{T}}V$ is symmetric for $U, V \in \mathcal{K}^{n\times n}$ in (6). Then, U is regular if and only if there is no λ_i $(i = 1, \ldots, n)$ in 1) satisfying

$$A^{\mathrm{T}}v + \delta_f(v) = -\lambda_i v,\ Rv = 0 \tag{16}$$

for nonzero $v \in \mathcal{K}^n$ such that $[0^{\mathrm{T}}, v^{\mathrm{T}}]^{\mathrm{T}} \in W$. Moreover, V is regular if and only if there is no λ_i $(i = 1, \ldots, n)$ in 1) satisfying

$$Au - \delta_f(u) = \lambda_i u,\ Qu = 0 \tag{17}$$

for nonzero $u \in \mathcal{K}^n$ such that $[u^{\mathrm{T}}\ 0^{\mathrm{T}}]^{\mathrm{T}} \in W$.
6) Suppose that U, V, Λ are chosen as in 1). Denote $\mathrm{Re}(\lambda_i)$ by the real part of $\lambda_i(x,t)$. Suppose that U, U^{-1}, V are defined on $\mathbf{R}^n \times \mathbf{R}$. Also, suppose that there is symmetric and positive semidefinite $\bar{Q} \in \mathbf{R}^{n\times n}$ such that $Q \geq \bar{Q}$ for all $(x,t) \in \mathbf{R}^n \times \mathbf{R}$. If for some $c < 0$, $\mathrm{Re}(\lambda_i) \leq c$ $(i = 1, \ldots, n)$ for all $(x,t) \in \mathbf{R}^n \times \mathbf{R}$, then $X := VU^{-1}$ is symmetric and positive semidefinite for all $(x,t) \in \mathbf{R}^n \times \mathbf{R}$.

Remark 3.4: In Theorem 3.3 4), we mention that the choice of a pair $U, V \in \mathcal{K}^{n\times n}$ in (6) is arbitrary. According to Remark 2.12, every pair U, V satisfying (6) is connected by (14). It is clear that if U^*V is Hermitian and $U^{\mathrm{T}}V$ is symmetric, then T^*U^*VT is Hermitian and $T^{\mathrm{T}}U^{\mathrm{T}}VT$ is symmetric.

Theorem 3.3 4) and 5) give characterizations of symmetry and regularity of a solution to the DRE. Denote $\Omega := U^*V$ and $\tilde{\Omega} := U^{\mathrm{T}}V$. If U is regular, we have

$$X = VU^{-1} = (U^{-1})^*\Omega U^{-1} = (U^{-1})^{\mathrm{T}}\tilde{\Omega}U^{-1}. \tag{18}$$

Thus, $X \in \mathcal{K}_{\mathbf{R}}^{n\times n}$ is real symmetric if both $\Omega = \Omega^*$ and $\tilde{\Omega} = \tilde{\Omega}^{\mathrm{T}}$ hold, and if U is regular. Regularity of U is characterized by (16).

Conditions (16) and (17) can be viewed as generalizations of Popov–Belevitch–Hautus accessibility and observability tests to nonlinear systems, respectively. In fact, there is no λ such that (16) and (17) hold if $\dot{x} = f(x) + Bu$ is locally strongly accessible [20] and if $\dot{x} = f(x)$, $y = h(x)$ is locally observable [20] when $A = \partial f/\partial x$, $R = BB^{\mathrm{T}}$, and $Q = \partial h/\partial x$ [13]. Thus, if these two systems are accessible and observable as in Example 2.13, and if $\mathcal{H}$ has no eigenvalue on the imaginary axis, then the DRE has at least one real symmetric and regular solution. Moreover, if the condition in Theorem 3.3 6) hold, one of real symmetric solutions is positive definite.

The remainder is dedicated to the proof of Theorem 3.3.

A. Proofs of Theorem 3.3 1) and 2)

Although the number of linearly independent right eigenvectors of differential Hamiltonian matrix $\mathcal{H}$ is at most $2n$, the number of eigenvalues can be infinite, which is different from the eigenvalues in linear algebra. Consider the right eigenvalue $\lambda \in \mathcal{K}$ and its associated right eigenvector $w \in \mathcal{K}^{2n} \setminus \{0\}$ of $\mathcal{H}$. For $a \in \mathcal{K} \setminus \{0\}$, from the definition of the right eigenvalue and eigenvector, we have

$$\begin{aligned}\mathcal{H}aw - \delta_f(aw) &= \mathcal{H}aw - a\delta_f(w) - \delta_f(a)w \\ &= (\lambda - \delta_f(a)a^{-1})aw.\end{aligned} \tag{19}$$

Thus, $\lambda - \delta_f(a)a^{-1}$ and aw are also right eigenvalue and eigenvector, respectively. These λ and $\lambda - \delta_f(a)a^{-1}$ are δ_f-conjugate.

Consider differential Hamiltonian matrix $\mathcal{H}$. An n-dimensional $\mathcal{H}$ invariant subspace $W \subset \mathcal{K}^{2n}$ can always be generated by linearly independent n right eigenvectors, which is demonstrated here. Let W be generated by $w_1, \ldots, w_n$, define $\hat{W} := [w_1, \ldots, w_n] \in \mathcal{K}^{n\times n}$, and let the column elements of $\hat{W}_2 := [\hat{w}_{n+1}, \ldots, \hat{w}_{2n}] \in \mathcal{K}^{n\times n}$ be n right eigenvectors associated with eigenvalues λ_i $(i = n+1, \ldots, 2n)$ such that

$$\mathrm{span}_{\mathcal{K}}\{w_1, \ldots, w_n, \hat{w}_{n+1}, \ldots, \hat{w}_{2n}\} = \mathcal{K}^{2n}.$$

Such $\hat{W}_2$ always exists because of the simplicity of $\mathcal{H}$. From the definitions of the n-dimensional $\mathcal{H}$ invariant subspace and the right eigenvalue and eigenvector, we have

$$\begin{aligned}&\mathcal{H}\left[\,\hat{W}\ \hat{W}_2\,\right] - \left[\,\delta_f(\hat{W})\ \delta_f(\hat{W}_2)\,\right] \\ &= \left[\,\hat{W}\ \hat{W}_2\,\right]\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22}\end{bmatrix}\end{aligned} \tag{20}$$

where $A_{11} \in \mathcal{K}^{n\times n}$ is a suitable matrix, and $A_{22} = \mathrm{diag}\{\lambda_{n+1}, \ldots, \lambda_{2n}\}$; consequently

$$\begin{aligned}&\left[\,\hat{W}\ \hat{W}_2\,\right]^{-1}\left(\mathcal{H}\left[\,\hat{W}\ \hat{W}_2\,\right] - \left[\,\delta_f(\hat{W})\ \delta_f(\hat{W}_2)\,\right]\right) \\ &= \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22}\end{bmatrix}.\end{aligned}$$

Since $\mathcal{H}$ is simple, A_{11} is also simple. Let column elements of $\hat{W}_1 := [\hat{w}_1, \ldots, \hat{w}_n]$ be linearly independent n right eigenvectors of A_{11} associated with eigenvalues λ_i $(i = 1, \ldots, n)$. Also, denote $\Lambda := \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$. From the definition of right eigenvalues and eigenvectors, we obtain $A_{11}\hat{W}_1 - \delta_f(\hat{W}_1) = \hat{W}_1\Lambda$. From this equality and (20)

$$\mathcal{H}\hat{W}\hat{W}_1 - \delta_f(\hat{W})\hat{W}_1 = \hat{W}A_{11}\hat{W}_1 = \hat{W}(\hat{W}_1\Lambda + \delta_f(\hat{W}_1))$$

and thus $\mathcal{H}\hat{W}\hat{W}_1 - \delta_f(\hat{W}\hat{W}_1) = \hat{W}\hat{W}_1\Lambda$. Because of $\Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$, all column elements of regular matrix $\hat{W}\hat{W}_1$ are right eigenvectors of $\mathcal{H}$. In summary, an n-dimensional $\mathcal{H}$ invariant subspace can always be generated by linearly independent n right eigenvectors if $\mathcal{H}$ is simple, which implies that $\{\lambda_1, \ldots, \lambda_n\}_f^{\delta} = \mathrm{rspec}_f(\mathcal{H}|_W)$ holds. Therefore, for simple $\mathcal{H}$, the set $\mathrm{rspec}_f(\mathcal{H}|_W)$ is obtained by finding n linearly independent right eigenvectors in W while the number of elements in $\mathrm{rspec}_f(\mathcal{H}|_W)$ can be infinite. Note that a solution to the DRE is uniquely determined irrespective of the choice of eigenvalues in $\mathrm{rspec}_f(\mathcal{H}|_W)$. Let $\hat{\lambda}_i$ be δ_f-conjugate to λ_i $(i = 1, \ldots, n)$. Then, there exists a_i such that $\hat{\lambda}_i = \lambda_i - \delta_f(a_i)/a_i$. Define $\hat{\Lambda} := \mathrm{diag}\{\hat{\lambda}_1, \ldots, \hat{\lambda}_n\}$ and $\hat{A} = \mathrm{diag}\{a_1, \ldots, a_n\}$. In a similar manner to the discussion in (19), $\mathcal{H}\hat{W}\hat{W}_1\hat{A} - \delta_f(\hat{W}\hat{W}_1\hat{A}) = \hat{W}\hat{W}_1\hat{A}\hat{\Lambda}$. Owing to Remark 2.12, $\hat{W}\hat{W}_1$ and $\hat{W}\hat{W}_1\hat{A}$ give the same solution X.

B. Proof of Theorem 3.3 3)

Owing to the specific structure of $\mathcal{H}$, we have the following relationship between the left and right eigenvalues of $\mathcal{H}$, where $\mathcal{H}$ does not need to be simple.

Proposition 3.5: $\beta \in \mathcal{K}$ is a right eigenvalue of $\mathcal{H}$ if and only if $-\beta$ is its left eigenvalue, or equivalently, if and only if $-\beta^*$ is its left eigenvalue, or equivalently, if and only if β^* is its right eigenvalue.

Proof: First, we show that if $\beta \in \mathcal{K}$ is a right eigenvalue, $-\beta$ is a left eigenvalue. Let $w \in \mathcal{K}^{2n}$ be a right eigenvector associated with right eigenvalue β, i.e., w and β satisfy

$$\mathcal{H}w - \delta_f(w) = \beta w. \tag{21}$$

For matrix $J \in \mathcal{K}^{2n\times 2n}$

$$J := \begin{bmatrix} 0 & I_n \\ -I_n & 0\end{bmatrix} \tag{22}$$

we have $J^{-1}\mathcal{H} = -\mathcal{H}^{\mathrm{T}} J^{-1}$. By premultiplying J^{-1} with both sides of (21), we have

$$-\mathcal{H}^{\mathrm{T}} J^{-1} w - \delta_f(J^{-1}w) = \beta J^{-1} w.$$

Therefore, $-\beta$ is a left eigenvalue of $\mathcal{H}$ with left eigenvector $J^{-1}w$, and the converse can readily be shown.

Since $\mathcal{H}$ is real analytic, by taking the conjugate transpose instead of the transpose in the above equations, we can show that $\beta \in \mathcal{K}$ is a right eigenvalue if and only if $-\beta^*$ is a left eigenvalue. Finally, from the above proof, $\hat{\beta} := -\beta^*$ is a left eigenvalue if and only if $-\hat{\beta} := \beta^*$ is a right eigenvalue. ■

Now, we are ready to prove 3).

Proof of 3): Let $\mathcal{H} \in \mathcal{K}^{2n\times 2n}$ be simple, and let $\lambda \in \mathcal{K}$ be its right eigenvalue. From Proposition 3.5, $-\lambda$, $-\lambda^*$, and λ^* are also right eigenvalues. ■

C. Proof of Theorem 3.3 4)

Let $\omega_{i,j}$ and $\tilde{\omega}_{i,j}$ be the (i,j) elements of $\Omega := U^*V$ and $\tilde{\Omega} := U^{\mathrm{T}} V$, respectively, i.e.,

$$\omega_{i,j} := u_i^* v_j,\ \tilde{\omega}_{i,j} := u_i^{\mathrm{T}} v_j,\ i,j = 1,2,\dots,n. \tag{23}$$

Conditions $\Omega = \Omega^*$ and $\tilde{\Omega} = \tilde{\Omega}^{\mathrm{T}}$ can be rewritten as

$$\omega_{i,j} - \omega_{j,i}^* = u_i^* v_j - v_i^* u_j = 0,\ i,j = 1,2,\dots,n, \tag{24}$$

$$\tilde{\omega}_{i,j} - \tilde{\omega}_{j,i}^{\mathrm{T}} = u_i^{\mathrm{T}} v_j - v_i^{\mathrm{T}} u_j = 0,\ i,j = 1,2,\dots,n. \tag{25}$$

These conditions are characterized by eigenvalues of $\mathcal{H}$.

Proposition 3.6: Let $w_i = [u_i^{\mathrm{T}}\ v_i^{\mathrm{T}}]^{\mathrm{T}}$ and $w_j = [u_j^{\mathrm{T}}\ v_j^{\mathrm{T}}]^{\mathrm{T}} \in \mathcal{K}^{2n}$ be right eigenvectors associated with right eigenvalues λ_i and $\lambda_j \in \mathcal{K}$ of $\mathcal{H}$, respectively. If λ_i^* and $-\lambda_j$ $(i,j = 1,\dots,n)$ are not δ_f-conjugate, (24) holds. Also, if λ_i and $-\lambda_j$ $(i,j = 1,\dots,n)$ are not δ_f-conjugate, (25) holds.

Proof: We prove that non δ_f-conjugacy of λ_i^* and $-\lambda_j$ $(i,j = 1,\dots,n)$ implies (24) by contraposition. That is, we show that $\omega_{i,j} - \omega_{j,i}^* \neq 0$ implies that λ_i^* and $-\lambda_j$ are δ_f-conjugate. For J in (22), $\mathcal{H}^{\mathrm{T}} J + J\mathcal{H} = 0$ holds. Since elements of $\mathcal{H}$ are real analytic functions, the definition of the right eigenvalue and eigenvector

$$\mathcal{H} w_j = \delta_f(w_j) + \lambda_j w_j \tag{26}$$

yields

$$w_i^* \mathcal{H}^{\mathrm{T}} = \delta_f(w_i^*) + \lambda_i^* w_i^*. \tag{27}$$

By computing $w_i^*(\mathcal{H}^{\mathrm{T}} J + J\mathcal{H}) w_j$ with (26) and (27), we have

$$\begin{aligned} &w_i^*(\mathcal{H}^{\mathrm{T}} J + J\mathcal{H}) w_j \\ &= (\delta_f(w_i^*) + \lambda_i^* w_i^*) J w_j + w_i^* J(\delta_f(w_j) + \lambda_j w_j) = 0. \end{aligned}$$

From (22) and $w_i = [u_i^{\mathrm{T}}\ v_i^{\mathrm{T}}]^{\mathrm{T}}$, we have

$$\begin{aligned} &(\delta_f(w_i^*) + \lambda_i^* w_i^*) J w_j + w_i^* J(\delta_f(w_j) + \lambda_j w_j) \\ &= \delta_f(u_i^* v_j - v_i^* u_j) + (\lambda_i^* + \lambda_j)(u_i^* v_j - v_i^* u_j). \end{aligned}$$

From (23) and $u_i^* v_j - v_i^* u_j = \omega_{i,j} - \omega_{j,i}^* \neq 0$, the equality can be rewritten as

$$\begin{aligned} &\delta_f(\omega_{i,j} - \omega_{j,i}^*) + (\lambda_i^* + \lambda_j)(\omega_{i,j} - \omega_{j,i}^*) = 0, \\ &\lambda_i^* + \delta_f(\omega_{i,j} - \omega_{j,i}^*)(\omega_{i,j} - \omega_{j,i}^*)^{-1} = -\lambda_j. \end{aligned}$$

Thus, λ_i^* and $-\lambda_j$ are δ_f-conjugate. In a similar manner, it is possible to show that (25) holds if λ_i and $-\lambda_j$ $(i,j = 1,\dots,n)$ are not δ_f-conjugate. ■

To proceed further analysis, we investigate δ_f-conjugacy of pairs of $(\lambda, -\lambda)$ and $(\lambda, -\lambda^*)$.

Proposition 3.7: Differential Hamiltonian matrix $\mathcal{H}$ has no left (or right) nonlinear eigenvalue on the imaginary axis if and only if for any left (or right) nonlinear eigenvalue λ of $\mathcal{H}$, neither pair $(\lambda, -\lambda)$ nor $(\lambda, -\lambda^*)$ is δ_f-conjugate.

Proof: (Necessity) We prove by contraposition. First, suppose that $(\lambda, -\lambda)$ is δ-conjugate. Then, there exists nonzero $a \in \mathcal{K}$ such that $2\lambda = \delta(a)/a$, which implies $\lambda = \delta(a^{1/2})/a^{1/2}$. Thus, $(\lambda, 0)$ is δ_f-conjugate with respect to $a^{1/2}$. From Proposition 2.6 1), 0 is a left (or right) eigenvalue of $\mathcal{H}$. Next, suppose that $(\lambda, -\lambda^*)$ is δ_f-conjugate. Then, $2\mathrm{Re}(\lambda) = \lambda + \lambda^* = \delta(a)/a$ for some nonzero $a \in \mathcal{K}$. Compute $4\mathrm{Re}(\lambda) = 2\mathrm{Re}(\lambda) + 2\mathrm{Re}(\lambda)^* = \delta(a)/a + \delta(a^*)/a^* = \delta(aa^*)/(aa^*)$, where aa^* is real valued, and consequently $4\lambda - \delta(aa^*)/(aa^*) = 4j\mathrm{Im}(\lambda)$. Thus, $(\lambda, j\mathrm{Im}(\lambda))$ is δ_f-conjugate with respect to $(aa^*)^{1/4}$. Therefore, $\mathcal{H}$ has a left (or right) eigenvalue on the imaginary axis.

(Sufficiency) We prove by contraposition. Let λ be a left (or right) eigenvalue of $\mathcal{H}$ on the imaginary axis. Then, $\lambda = -\lambda^*$. That is, $(\lambda, -\lambda^*)$ is δ_f-conjugate. Moreover, if $\lambda = 0$, $(\lambda, -\lambda)$ is δ_f-conjugate. ■

Now, we are ready to prove 4).

Proof of 4): Let $\{w_1, \dots, w_{2n}\}$ be the set of linearly independent eigenvectors of $\mathcal{H}$ associated with eigenvalues $\lambda_1, \dots, \lambda_{2n}$. Here, we show that $w_1, \dots, w_n$ can be chosen such that neither $(\lambda_i, -\lambda_j^*)$ nor $(\lambda_i, -\lambda_j)$ is δ_f-conjugate for any $i,j = 1,\dots,n$. Then, Proposition 3.6 implies that U^*V is Hermitian, and $U^{\mathrm{T}} V$ is symmetric for $W = \mathrm{span}_{\mathcal{K}}\{w_1, \dots, w_n\}$.

Let $\{a_1, \dots, a_r\}$ be the set of eigenvalues, where (a_i, a_j) is not δ_f-conjugate for any $i \neq j$, such that each λ_i $(i = 1,\dots,2n)$ is δ_f-conjugate to one of its elements. First, we focus on a_1. According to Theorem 3.3 3), $-a_1$, a_1^*, and $-a_1^*$ are also eigenvalues of $\mathcal{H}$. From Proposition 3.7, $(a_1, -a_1)$ is not δ_f-conjugate. That is, one of $a_2, \dots, a_r$ can be chosen as $-a_1$. Here, we chose $a_2 = -a_1$ without loss of generality. Moreover, if (a_1, a_1^*) is not δ_f-conjugate, any pair (b,c) $(b \neq c; b,c \in \{a_1, -a_1, a_1^*, -a_1^*\})$ is not δ_f-conjugate. Then, we can chose $a_3 = a_1^*$ and $a_4 = -a_1^*$ without loss of generality.

We proceed similar procedure for $a_5, \dots, a_r$. Then, we notice that r is an even number, i.e., $r = 2\hat{r}$ for some $\hat{r}$. Consider $\{a_1, a_3, \dots, a_{2\hat{r}-1}\}$. Then, neither $(a_i, -a_j)$ nor $(a_i, -a_j^*)$ $(i,j = 1,3,\dots,2\hat{r}-1)$ is δ_f-conjugate. Also, for the set $\{a_2, a_4, \dots, a_{2\hat{r}}\}$, neither $(a_i, -a_j)$ nor $(a_i, -a_j^*)$ $(i,j = 2,4,\dots,2\hat{r})$ is δ_f-conjugate. Therefore, if we construct W by using the eigenvectors of $\mathcal{H}$ associated with the eigenvalues, which are δ_f-conjugate to one of $a_1, a_3, \dots, a_{2\hat{r}-1}$ or the eigenvectors associated with the eigenvalues, which are δ_f-conjugate to one of $a_2, a_4, \dots, a_{2\hat{r}}$, then λ_i $(i = 1,\dots,n)$ satisfy conditions in Proposition 3.6. ■

D. Proof of Theorem 3.3 5)

Proof: Here, we prove 5) only for regularity of V when U^*V is Hermitian. In a similar manner, we can prove the other cases.

(Sufficiency) We prove this by contraposition. Let V be not regular. There exists a nonzero v such that

$$Vv = 0. \tag{28}$$

The lower half of (8) is $-QU - A^{\mathrm{T}} V - \delta_f(V) = V\Lambda$. By multiplying v, we have, from (28)

$$-QUv - \delta_f(V)v = V\Lambda v. \tag{29}$$

Note that from (28), $\delta_f(Vv) = \delta_f(V)v + V\delta_f(v) = 0$ holds, which yields $-\delta_f(V)v = V\delta_f(v)$. By using this, (29) can be rewritten as

$$-QUv + V\delta_f(v) = V\Lambda v. \tag{30}$$

By premultiplying v^*U^*, from $U^*V = V^*U$, we obtain

$$-v^*U^*QUv + v^*V^*U\delta_f(v) = v^*V^*U\Lambda v.$$

Since $Vv = 0$, the above equation implies

$$QUv = 0 \tag{31}$$

and thus, from (30)

$$V(\Lambda v - \delta_f(v)) = 0. \tag{32}$$

Note that (31) and (32) hold for all v satisfying $Vv = 0$.

Next, we show the existence of λ and nonzero $\hat{v}$ satisfying $V\hat{v} = 0$ and

$$\Lambda\hat{v} - \delta_f(\hat{v}) = \lambda\hat{v}. \tag{33}$$

We assume that v_1, the first element of v, is nonzero. Then, from (28), we have $V(v/v_1) = 0$ and from (32)

$$(1/v_1)V(\Lambda v - \delta_f(v)) = V(\Lambda(v/v_1) - \delta_f(v)/v_1) = 0.$$

Also, by using $\delta_f(v/v_1) = \delta_f(v)/v_1 + \delta_f(1/v_1)v$ and (28), we obtain

$$\begin{aligned} &V(\Lambda(v/v_1) - \delta_f(v/v_1) + \delta_f(1/v_1)v) \\ &= V(\Lambda(v/v_1) - \delta_f(v/v_1)) \\ &= V(\Lambda(v/v_1) - \delta_f(v/v_1) - \lambda_1(v/v_1)) = 0. \end{aligned}$$

This equality can also be expressed as $V\bar{v} = 0$, where

$$\bar{v} := \begin{bmatrix} \lambda_1 \\ \lambda_2(v_2/v_1) \\ \vdots \\ \lambda_n(v_n/v_1) \end{bmatrix} - \begin{bmatrix} 0 \\ \delta_f(v_2/v_1) \\ \vdots \\ \delta_f(v_n/v_1) \end{bmatrix} - \begin{bmatrix} \lambda_1 \\ \lambda_1(v_2/v_1) \\ \vdots \\ \lambda_1(v_n/v_1) \end{bmatrix}.$$

If $\bar{v} = 0$, let $\hat{v} := v/v_1$ and $\lambda := \lambda_1$. Then, $\hat{v}$ and λ satisfy $V\hat{v} = 0$ and (33). Otherwise, let $v := \bar{v}$. Then v_1, the first element of v, is zero. This v satisfies $Vv = 0$ and thus (32). We assume that v_2, the second element of v, is nonzero and repeat the above procedure for v. Finally, there exists $i \leq n$ such that $v = [0 \ \cdots \ 0 \ v_i \ 0 \cdots \ 0]^{\mathrm{T}}$ $(v_i \neq 0)$. For $\hat{v} := v/v_i$ and $\lambda = \lambda_i$, $V\hat{v} = 0$ and (33) hold. In summary, there exist λ and nonzero $\hat{v}$ satisfying $V\hat{v} = 0$ and (33).

From the upper half of (8), $V\hat{v} = 0$ and (33), we have

$$\begin{aligned} &AU\hat{v} + RV\hat{v} - \delta_f(U)\hat{v} = U\Lambda\hat{v}, \\ &AU\hat{v} - \delta_f(U)\hat{v} - U\delta_f(\hat{v}) = \lambda U\hat{v}, \\ &AU\hat{v} - \delta_f(U\hat{v}) = \lambda U\hat{v} \end{aligned}$$

where $U\hat{v} \neq 0$. Otherwise, $[U^{\mathrm{T}} \ V^{\mathrm{T}}]^{\mathrm{T}}\hat{v} = 0$, i.e., the column vectors of $[U^{\mathrm{T}} \ V^{\mathrm{T}}]^{\mathrm{T}}$ are linearly dependent, which contradicts that $W \subset \mathcal{K}^{2n}$ in Theorem 2.11 is an n-dimensional subspace. Since $\hat{v}$ satisfies (31), i.e., $QU\hat{v} = 0$, (17) holds for $w := U\hat{v}$ and λ.

(Necessity) Here, we prove by contraposition. That is, we show that if there is some λ_i in Theorem 3.3 1) satisfying (17) for nonzero $u \in \mathcal{K}^n$ such that $[u^{\mathrm{T}} \ 0^{\mathrm{T}}]^{\mathrm{T}} \in W$, then V is not regular. Let $w_i \in W$ $(i = 1, \ldots, n)$ be a right eigenvector of $\mathcal{H}$ associated with an eigenvalue λ_i $(i = 1, \ldots, n)$. If we choose λ_i $(i = 1, \ldots, n)$ such that Theorem 3.3 1) holds, we have

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix}. \tag{34}$$

for U, V in (6). In fact, one of w_i can be chosen as $[u^{\mathrm{T}} \ 0^{\mathrm{T}}]^{\mathrm{T}}$ as follows. For λ_i and nonzero u satisfying (17), we have

$$\begin{bmatrix} A & R \\ -Q & -A^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} u \\ 0 \end{bmatrix} - \begin{bmatrix} \delta_f(u) \\ 0 \end{bmatrix} = \lambda_i \begin{bmatrix} u \\ 0 \end{bmatrix}$$

which implies that $[u^{\mathrm{T}} \ 0^{\mathrm{T}}]^{\mathrm{T}}$ is a right eigenvector of $\mathcal{H}$ associated with λ_i. Furthermore, since $[u^{\mathrm{T}} \ 0^{\mathrm{T}}]^{\mathrm{T}} \in W$, one of w_i can be chosen as $w_i = [u^{\mathrm{T}} \ 0^{\mathrm{T}}]^{\mathrm{T}}$. Then, V is not regular for a basis $\{w_1, \ldots, w_n\}$ of W. Note that from Remark 2.12, regularity of V does not depend on the choice of basis. ■

E. Proof of Theorem 3.3 6)

Proof: From (18), if U is regular, positive semidefiniteness of X and U^*V are equivalent. Here, we prove positive semidefiniteness of U^*V. By respectively multiplying the upper and lower parts of (8) by V^* and U^* from left

$$V^*AU - V^*RV - V^*\delta_f(U) = V^*U\Lambda, \tag{35}$$

$$-U^*QU - U^*A^{\mathrm{T}}V - U^*\delta_f(V) = U^*V\Lambda. \tag{36}$$

By adding the complex conjugate of (36) to (35)

$$\delta_f(V^*U) + V^*U\Lambda + \Lambda^*V^*U = -V^*RV - U^*QU. \tag{37}$$

From the assumption for Q, there exists a symmetric and positive semidefinite matrix $\bar{U} \in \mathbf{R}^{n\times n}$ such that $-V^*RV + U^*QU \leq -\bar{U}$ for all $(x,t) \in \mathbf{R}^n \times \mathbf{R}$. Consider linear time-varying system $d\delta z/dt = \Lambda(\phi(x_0,t),t)\delta z$ along trajectory $\phi(x_0,t)$ of $\dot{x} = f(x,t)$ with the initial condition $x(t_0) = x_0$. Then, we have, from (37)

$$\begin{aligned} &\frac{d}{dt}(\delta z^*(t)V^*(\phi(x_0,t),t)U(\phi(x_0,t),t)\delta z(t)) \\ &\leq -\delta z^*(t)\bar{U}\delta z(t). \end{aligned} \tag{38}$$

Since $\mathrm{Re}(\lambda_i) < c$ $(i = 1, \ldots, n)$, i.e., $\mathrm{Re}(\Lambda) < cI_n$ for all $(x,t) \in \mathbf{R}^n \times \mathbf{R}$, the linear time-varying system is uniformly asymptotically stable at the origin [21]. Therefore, a time integral of (38) is

$$\begin{aligned} &\delta z^*(t_0)V^*(x_0,t_0)U(x_0,t_0)\delta z(t_0) \\ &= \int_{t_0}^{\infty} \delta z^*(t)\bar{U}\delta z(t)dt \geq 0 \end{aligned}$$

for any $\delta z(t_0) \in \mathbf{R}^n$. Therefore, V^*U is symmetric and positive semidefinite at each $(x_0,t_0) \in \mathbf{R}^n \times \mathbf{R}$. ■

IV. Conclusion

In this paper, we presented a nonlinear eigenvalue method for the DRE for contraction analysis. First, we showed that all solutions to the DRE can be expressed as functions of nonlinear eigenvectors of the corresponding differential Hamiltonian matrix. Next, in the simple case, we studied solution structures, e.g., real symmetry and regularity. Future work includes relaxing the simplicity assumption and constructing methods for finding nonlinear eigenvectors of the Hamiltonian matrix. As a solution method to the HJE, the generating function method [22], [23] is known. For the ARE, this method is useful for finding other eigenvectors of a Hamiltonian matrix from its eigenvectors, and this method may be extended to the DRE.

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