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Dynamic Feedback Synchronization of Lur'e Networks via Incremental Sector Boundedness

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***Abstract*—In this note, we generalize our results on synchronization of homogeneous Lur'e networks by static, relative state information based protocols from Zhang et al to the case that for each agent only relative measurements are available. We establish sufficient conditions under which a linear dynamic synchronization protocol exists for such networks. These conditions involve feasibility of two LMI's together with a coupling inequality, reminiscent of the well-known LMI conditions in H_∞ control by measurement feedback. We show that, regardless of the number of agents in the network, only the three inequalities are involved. In the computation of the protocol matrices, the eigenvalues of the Laplacian matrix of the interconnection graph occur. In particular, the matrices representing the protocol depend on the smallest nonzero eigenvalue and the largest eigenvalue of the Laplacian matrix. We validate our results by means of a numerical simulation example.**

***Index Terms*—Incremental sector boundedness, linear matrix inequalities, Lur'e system, synchronization.**

I. Introduction

This technical note deals with dynamic measurement feedback synchronization of homogeneous nonlinear multi-agent networks. We assume that each agent therein is represented by a Lur'e system, that is, a nonlinear system consisting of the negative feedback interconnection of a nominal linear system with a static nonlinearity [8]. The static nonlinearity is assumed to be *unknown*, but satisfies a condition of incremental sector boundedness within an *a priori* given sector. The problem we deal with is to find a single distributed protocol that synchronizes the network *for all* static nonlinearities that satisfy this incremental sector boundedness condition. This problem has been studied before in [23] for the special case that each agent has access to the *relative states* with respect to its neighbors. For that special case, in [23] a method to design a distributed *static* protocol was given. In the present technical note we consider the general case that the agents can only measure part of the relative states, i.e., only have access to the relative *outputs* with respect to their neighbors. As usual in control, in this case we need a *dynamic* protocol to synchronize the network.

In order to avoid confusion, we stress that in this technical note we do *not* assume that the agents themselves are passive or incrementally passive input-output systems. The incremental sector boundedness condition only serves to model the unknown feedback nonlinearity in the agent dynamics. In fact, this is the basic principle of the concept of Lur'e systems. There is no assumption on the nominal linear part of the Lur'e system.

Problems related to ours were considered in [16] and [25]. There, each agent was modeled as the feedback interconnection of a double integrator with an unknown static nonlinearity satisfying a Lipschitz condition with an *a priori* given bound. In the present technical note, the linear part of each agent is a general state space model. Also, the protocols to be designed in our technical note are linear state space systems themselves, while in [16] and [25] nonlinear functions of the state variables are proposed. Finally, although our incremental sector boundedness condition implies a Lipschitz-like condition, further analysis shows that they are not equivalent, especially not in the multivariable case considered in our technical note.

Synchronization of complex dynamical networks has been studied extensively before, see e.g. [1], [10], [18]–[20]. Largely, this is due to the fact that these networks have potential applications in many areas, such as spatiotemporal planning, cooperative multitasking and formation control [12]. For the case of linear multi-agent networks, synchronization has been studied in, for example, [11], [13]. Synchronization problems for nonlinear multi-agent networks, possibly with model uncertainties, time delays, data dropouts and quantized interconnections have also been addressed, for example in [2], [3], [18] and [21]. In [4], a passivity-based group coordination framework was proposed, especially applicable to nonlinear multi-agent networks, possibly in the presence of communication latencies, see [3]. A similar idea was applied in [15] to deal with networks of static output coupled incrementally passive oscillators. In [5], output synchronization and formation control were investigated in the context of passive and incrementally passive agents,

Many physical systems can indeed be represented as a Lur'e system. Examples are Chua's circuits, flexible robot arms, and aerospace applications. The static feedback loop can model different kinds of unknown nonlinearities such as saturation and deadzone phenomena.

Interconnections of Lur'e systems have been studied before in the literature. For example, in [17] dynamic feedback based master-slave synchronization of Lur'e systems was studied. In that context, dynamic feedback was used to recover a message signal in the presence of measurement noise. In [9], synchronization of static output coupled Lur'e systems with time delays was studied. In [6], the case was considered that the feedback nonlinearities are slope-restricted, but also exactly known. In fact, exact knowledge of the feedback nonlinearities is often employed in observer-based dynamic feedback stabilization of Lur'e systems. Clearly, the latter context is different from ours, as we allow all nonlinearities that satisfy a given incremental sector boundedness condition.

The remainder of this technical note is organized as follows. Section II introduces some preliminaries and formulates the dynamic feedback synchronization problem this technical note deals with. Our main results are presented in Section III. Sufficient synchronization conditions are established and it is discussed how to compute a suitable dynamic protocol. Some concluding remarks together with suggestions for future work close the technical note.

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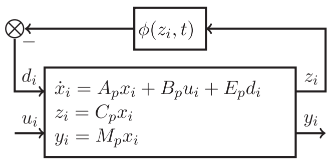

Fig. 1. Lur'e system.

II. Preliminaries and Problem Formulation

In this technical note, we use the standard notation $\mathbb{R}$ and $\mathbb{C}$ for the fields of real and complex numbers, respectively. We denote $\mathbb{R}^+ := [0,\infty)$. $\mathbb{R}^{m\times n}$ ($\mathbb{C}^{m\times n}$) denotes the space of $m\times n$ real (complex) matrices.

The interconnection topology of a network of bidirectionally interconnected dynamical systems will be represented by a simple undirected graph $\mathcal{G}$ with N nodes, where we always assume that $N \geq 2$. The relevant algebraic graph theory and particularly the concepts of adjacency matrix and Laplacian matrix can be found in [11]. Throughout this technical note it will be assumed that the graph $\mathcal{G}$ is connected. We consider a network of N identical Lur'e systems described by

$$\begin{cases} \dot{x}_i = A_p x_i + B_p u_i + E_p d_i \\ z_i = C_p x_i \\ y_i = M_p x_i \\ d_i = -\phi(z_i,t) \end{cases}, \quad i = 1,2,\ldots,N \tag{1}$$

where $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^m$, $z_i(t) \in \mathbb{R}^p$, and $y_i(t) \in \mathbb{R}^q$ are the state to be synchronized, the diffusive coupling input, the system output, and the measurement output of agent i, respectively. The equation $d_i = -\phi(z_i,t)$ represents a time-varying, memoryless, nonlinear negative feedback loop, see Fig. 1. The function $\phi(\cdot,t)$ from $\mathbb{R}^p \times \mathbb{R}^+$ to $\mathbb{R}^p$ is an unknown nonlinearity. It is, however, assumed to be incrementally sector bounded with *known* sector bounds:

Definition 1: Let S_1, $S_2 \in \mathbb{R}^{p\times p}$ be real symmetric matrices with $\mathbf{0} \leq S_1 < S_2$. Then $\phi(\cdot,t)$ is called *incrementally sector bounded* within sector $[S_1, S_2]$ if it satisfies

$$[d_1 - d_2 - S_1(z_1 - z_2)]^T \, [d_1 - d_2 - S_2(z_1 - z_2)] \leq 0 \tag{2}$$

for all z_1, $z_2 \in \mathbb{R}^p$ and $t \in \mathbb{R}^+$, where $d_1 = \phi(z_1,t)$, $d_2 = \phi(z_2,t)$.

The linear part of (1) is represented by the matrices A_p, B_p, C_p, E_p and M_p. They are assumed to be known constant matrices of compatible dimensions. We do not impose any assumptions of passivity or sector boundedness on the linear part of the Lur'e system.

Let $\mathcal{A} = [a_{ij}]$ be the adjacency matrix of the graph $\mathcal{G}$. We allow protocols that use relative information with respect to neighboring agents weighted by the weights on the edges, and hence we consider distributed dynamic protocols of the form

$$\begin{cases} \dot{w}_i = A_c w_i + B_c \sum\limits_{j=1}^{N} a_{ij}(y_i - y_j) + D_c \sum\limits_{j=1}^{N} a_{ij}(w_i - w_j) \\ u_i = C_c w_i, \quad i = 1,2,\ldots,N \end{cases} \tag{3}$$

where $w_i(t) \in \mathbb{R}^{n_c}$ is the protocol state for agent i, and A_c, B_c, C_c and D_c are the matrices representing the protocol. The protocol state space dimension n_c and the matrices A_c, B_c, C_c and D_c need to be determined. In order to be able to write down the equations of the overall network obtained by interconnecting the agents (1) using the protocol (3) we now first introduce some notation.

As usual, M^T denotes the transpose of a real matrix M, and M^* denotes the conjugate transpose of a complex matrix M. The Kronecker product of the matrices M_1 and M_2 is denoted by $M_1 \otimes M_2$. An important property of the Kronecker product is that $(M_1 \otimes M_2)(M_3 \otimes M_4) = (M_1 M_3) \otimes (M_2 M_4)$. We denote by $\mathbf{0}$ and I the zero and identity matrix, respectively, of compatible dimensions. By I_N we denote the identity matrix of dimension N.

By interconnecting the N Lur'e systems (1) using the dynamic protocol (3) we obtain the Lur'e dynamical network

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} I_N \otimes A_p & I_N \otimes B_p C_c \\ \mathcal{L} \otimes B_c M_p & I_N \otimes A_c + \mathcal{L} \otimes D_c \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \\ \qquad - \begin{bmatrix} I_N \otimes E_p \\ \mathbf{0} \end{bmatrix} \Phi(z,t) \\ z = (I_N \otimes C_p) x \end{cases} \tag{4}$$

where $x = (x_1^T, x_2^T, \ldots, x_N^T)^T$, $w = (w_1^T, w_2^T, \ldots, w_N^T)^T z = (z_1^T, z_2^T, \ldots, z_N^T)^T$, and

$$\Phi(z,t) = \left(\phi(z_1,t)^T, \phi(z_2,t)^T, \ldots, \phi(z_N,t)^T\right)^T.$$

In the above, $\mathcal{L}$ is the Laplacian matrix of the graph $\mathcal{G}$.

We are now ready to state the problem that we will consider in this technical note.

Problem: Given the sector bounds $\mathbf{0} \leq S_1 < S_2$, the problem is to design a protocol of the form (3) so that the overall interconnected network (4) is synchronized for all incrementally sector bounded functions, that is, $x_i(t) - x_j(t) \to \mathbf{0}$ and $w_i(t) - w_j(t) \to \mathbf{0}$ as $t \to \infty$, $\forall i,j = 1,2,\ldots,N$, for all initial conditions and all incrementally sector bounded functions $\phi(\cdot,t)$ within $[S_1, S_2]$.

Remark 1: The classical theory of Lur'e systems in general involves an ordinary sector boundedness condition. For given $\mathbf{0} \leq S_1 < S_2$ this condition requires $(\phi(z,t) - S_1 z)^T(\phi(z,t) - S_2 z) \leq 0$, $\forall z \in \mathbb{R}^p$, and $t \in \mathbb{R}^+$. Obviously, incremental sector bounedness implies sector boundedness. Since in the synchronization context one deals with "stabilizing" *relative* states, the *incremental* version is a natural condition in the context of this technical note. In the SISO case incremental sector boundedness is equivalent to slope-restrictedness, see [22]. If a SISO nonlinear function has finite slopes, as for example in saturation and deadzone nonlinearities, we can easily determine the corresponding sector bounds.

Before embarking on this problem, in the remainder of this section we will introduce some more notation, review some definitions and give some preliminary results. In the following, by $\mathbf{1}_N$ and $\mathbf{0}_N$ we denote the column vectors of dimension N with all elements equal to one and zero, respectively. It is well known that the graph $\mathcal{G}$ is connected if and only if its Laplacian eigenvalue 0 has geometric multiplicity one. In this case, the eigenvalues of the Laplacian matrix $\mathcal{L}$ associated with the graph $\mathcal{G}$ can be ordered as $\lambda_1 = 0 < \lambda_2 \leq \cdots \leq \lambda_N$. Furthermore, there exists an orthogonal matrix $\mathcal{U} = [(1/\sqrt{N})\mathbf{1}_N \mathcal{U}_2] \in \mathbb{R}^{N\times N}$, where $\mathcal{U}_2 \in \mathbb{R}^{N\times(N-1)}$, such that $\mathcal{U}^T \mathcal{L} \mathcal{U} = \operatorname{diag}(0, \lambda_2, \ldots, \lambda_N)$. It is obvious that $\mathcal{U}_2 \mathcal{U}_2^T = I_N - (1/N)\mathbf{1}_N \mathbf{1}_N^T$. Denote $\Lambda := \operatorname{diag}(0, \lambda_2, \ldots, \lambda_N)$, which can be partitioned as $\Lambda = \begin{bmatrix} 0 & \mathbf{0}_{N-1}^T \\ \mathbf{0}_{N-1} & \bar{\Lambda} \end{bmatrix}$, where $\bar{\Lambda} := \operatorname{diag}(\lambda_2, \ldots, \lambda_N)$.

The following lemma plays a crucial role in this technical note. For a proof we refer to [23].

Lemma 1: For any two vectors $a = [a_1^T, a_2^T, \ldots, a_N^T]^T$ and $b = [b_1^T, b_2^T, \ldots, b_N^T]^T$ with $a_i, b_i \in \mathbb{R}^n$, $i = 1,2,\ldots,N$, we have

$$a^T \left(\mathcal{U}_2 \mathcal{U}_2^T \otimes I_n\right) b = \frac{1}{N} \sum_{1 \leq I < j \leq N} (a_i - a_j)^T (b_i - b_j)$$

where $\mathcal{U}_2$ has been defined above.

We now review the definition of minimal left annihilator of a matrix. For more details we refer to [7]:

Definition 2: For a matrix $B \in \mathbb{C}^{m\times n}$ with rank $r < \min\{m, n\}$, we denote by $B^{\perp}$ any matrix in $\mathbb{C}^{(m-r)\times m}$ of full row rank such that $B^{\perp}B = \mathbf{0}$. Any such matrix $B^{\perp}$ is called *a minimal left annihilator* of B.

Note that a minimal left annihilator is only defined for matrices with linearly dependent rows. The set of all such matrices is given by $B^{\perp} = TU_2^*$, where T is any nonsingular matrix and U_2 is obtained from the singular value decomposition $B = [U_1 U_2]\begin{bmatrix}\Sigma & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}V_1^*\\ V_2^*\end{bmatrix}$. Thus, for a given B, $B^{\perp}$ is not unique. Throughout this technical note, $B^{\perp}$ will denote any choice from this set of matrices.

Without loss of generality, we assume that the dimensions m and q of the diffusive coupling inputs and the measurement outputs in the Lur'e systems (1) are strictly smaller than the state space dimension n. In this case the rows of B_p are linearly dependent and thus $B_p^{\perp}$ exists. Similarly, $\left(M_p^T\right)^{\perp}$ exists as well.

III. Main Results

In this section, our main result is presented. We establish sufficient conditions for the existence of a dynamic protocol (3) that synchronizes the network (4) for all incrementally sector bounded nonlinearities within a given sector. Our conditions are in terms of solvability of two linear matrix inequalities, together with a coupling inequality between the solutions of the LMI's. We will also give a conceptual algorithm to compute a synchronizing protocol. However, before stating our main result, we will first formulate two lemmas. The first one states that the network (4) is synchronized by a given protocol (3) if a set of $N-1$ linear matrix inequalities involving the nonzero eigenvalues $\lambda_2, \ldots, \lambda_N$ of the Laplacian of the graph $\mathcal{G}$ have a common positive definite solution:

Lemma 2: Let n_c be a positive integer and consider the protocol (3) represented by the matrices $A_c \in \mathbb{R}^{n_c\times n_c}$, $B_c \in \mathbb{R}^{n_c\times q}$, $C_c \in \mathbb{R}^{m\times n_c}$ and $D_c \in \mathbb{R}^{n_c\times n_c}$. Define $A := \begin{bmatrix}A_p & \mathbf{0}\\ \mathbf{0} & \mathbf{0}_{n_c\times n_c}\end{bmatrix}$, $B := \begin{bmatrix}B_p & \mathbf{0}\\ \mathbf{0} & I_{n_c}\end{bmatrix}$, $C := [C_p \mathbf{0}_{p\times n_c}]$, $H_i := \begin{bmatrix}\mathbf{0} & \lambda_i C_c\\ B_c & A_c + \lambda_i D_c\end{bmatrix}$, $M := \begin{bmatrix}M_p & \mathbf{0}\\ \mathbf{0} & I_{n_c}\end{bmatrix}$, $E := \begin{bmatrix}E_p\\ \mathbf{0}_{n_c\times p}\end{bmatrix}$. If there exists a positive definite matrix $\bar{P} \in \mathbb{R}^{(n+n_c)\times(n+n_c)}$ that solves all the $N-1$ LMI's

$$\left[\begin{array}{c|c}\begin{array}{c}\bar{P}\left(A - \frac{1}{2}E(S_1+S_2)C + BH_iM\right)\\ +\left(A - \frac{1}{2}E(S_1+S_2)C + BH_iM\right)^T\bar{P}\\ +\frac{1}{4}C^T(S_2-S_1)^2C\end{array} & -\bar{P}E\\ \hline -E^T\bar{P} & -I_p\end{array}\right] < \mathbf{0} \quad (5)$$

$i = 2, \ldots, N$, then the protocol (3) synchronizes the network (4) for all incrementally sector bounded $\phi(\cdot, t)$ within sector $[S_1, S_2]$.

Proof: The proof involves two steps. Firstly, analogously as in the proof of [23, Lemma 4], it can be shown that if the $N-1$ LMI's

$$\left[\begin{array}{c|c}\begin{array}{c}P(A+BH_iM)\\ +(A+BH_iM)^TP\\ -\tau C^T(S_1S_2+S_2S_1)C\end{array} & \begin{array}{c}-PE\\ +\tau C^T(S_1+S_2)\end{array}\\ \hline -E^TP+\tau(S_1+S_2)C & -2\tau I_p\end{array}\right] < \mathbf{0} \quad (6)$$

$i = 2, \ldots, N$, have a common solution pair (P, τ) with $P \in \mathbb{R}^{(n+n_c)\times(n+n_c)}$ positive definite and τ a positive real number, then the protocol (3) synchronizes the network (4) for all incrementally sector bounded $\phi(\cdot, t)$ within the sector $[S_1, S_2]$.

The second step is to note that the unknown τ can be removed from (6): by defining $\bar{P} := (1/2\tau)P$ and dividing (6) by 2τ, we see that the inequalities (6) have a common solution (P, τ) if and only if the inequalities (5) have a common solution $\bar{P}$ using Schur complement. □

Our second lemma states that the existence of matrices A_c, B_c, C_c and D_c such that the set of $N-1$ LMI's (5) has a common positive definite solution is equivalent to solvability of two LMI's together with a coupling condition:

Lemma 3: There exists a positive integer n_c and $A_c \in \mathbb{R}^{n_c\times n_c}$, $B_c \in \mathbb{R}^{n_c\times q}$, $C_c \in \mathbb{R}^{m\times n_c}$, $D_c \in \mathbb{R}^{n_c\times n_c}$ and $\bar{P} > \mathbf{0}$ such that (5) holds for all $i = 2, \ldots, N$ if and only if there exist matrices $X_p > \mathbf{0}$, $Y_p > \mathbf{0}$ of size $n\times n$ such that

$$\begin{bmatrix}B_p\\ \mathbf{0}\end{bmatrix}^{\perp}\begin{bmatrix}\bar{A}_pX_p + X_p\bar{A}_p^T + \frac{1}{4}E_pE_p^T & X_p\bar{C}_p^T\\ \bar{C}_pX_p & -I\end{bmatrix}\begin{bmatrix}B_p\\ \mathbf{0}\end{bmatrix}^{\perp T} < \mathbf{0} \quad (7)$$

$$\begin{bmatrix}M_p^T\\ \mathbf{0}\end{bmatrix}^{\perp}\begin{bmatrix}Y_p\bar{A}_p + \bar{A}_p^TY_p + \frac{1}{4}\bar{C}_p^T\bar{C}_p & Y_pE_p\\ E_p^TY_p & -I\end{bmatrix}\begin{bmatrix}M_p^T\\ \mathbf{0}\end{bmatrix}^{\perp T} < \mathbf{0} \quad (8)$$

$$Y_p - \frac{1}{4}X_p^{-1} > \mathbf{0} \quad (9)$$

where $\bar{A}_p := A_p - (1/2)E_p(S_1+S_2)C_p$, $\bar{C}_p := (S_2 - S_1)C_p$.

In this case, we can take $n_c = n$ and suitable A_c, B_c, C_c, D_c, and $\bar{P}$ are obtained as follows:

- Choose $r_2 > 0$ such that

$$\begin{bmatrix}\bar{A}_pX_p + X_p\bar{A}^T + \frac{1}{4}E_pE_p^T - 2r_2\lambda_2B_pB_p^T & X_p\bar{C}_p^T\\ \bar{C}_pX_p & -I\end{bmatrix} < \mathbf{0}. \quad (10)$$

- Choose $r_1 > 0$ such that

$$\begin{bmatrix}Y_p\bar{A}_p + \bar{A}_p^TY_p + \frac{1}{4}\bar{C}_p^T\bar{C}_p - 2r_1M_p^TM_p & Y_pE_p\\ E_p^TY_p & -I\end{bmatrix} < \mathbf{0}. \quad (11)$$

- Define $F := -r_2B_p^TX_p^{-1}$ and $G := -r_1Y_p^{-1}M_p^T$.
- Define

$$R_F^i := (\bar{A}_p + \lambda_iB_pF)^TX_p^{-1} + X_p^{-1}(\bar{A}_p + \lambda_iB_pF) + \frac{1}{4}X_p^{-1}E_pE_p^TX_p^{-1} + \bar{C}_p^T\bar{C}_p \quad (12)$$

$i = 2, \ldots, N$, and

$$R_G := (\bar{A}_p + GM_p)Y_p^{-1} + Y_p^{-1}(\bar{A}_p + GM_p)^T + \frac{1}{4}Y_p^{-1}\bar{C}_p^T\bar{C}_pY_p^{-1} + E_pE_p^T.$$

- Choose a real number $k \in (0, 1)$ such that

$$Y_pR_GY_p < \frac{1}{4}(1-k)R_F^N. \quad (13)$$

- Define $Z_p := Y_p - (1/4)X_p^{-1}$, $\bar{G} := Z_p^{-1}Y_pG$

$$\Delta_1 := \frac{1}{4}kZ_p^{-1}\left(\bar{A}_p^TX_p^{-1} + X_p^{-1}\bar{A}_p + \frac{1}{4}X_p^{-1}E_pE_p^TX_p^{-1} + \bar{C}_p^T\bar{C}_p\right)$$

$$\Delta_2 := -\frac{1}{4}Z_p^{-1}\left((1-k)F^TB_p^TX_p^{-1} - kX_p^{-1}B_pF\right).$$

- Choose

$$A_c := \bar{A}_p + \frac{1}{4}E_pE_p^TX_p^{-1} + \bar{G}M_p + \Delta_1 \quad (14)$$

$$B_c := -\bar{G} \quad (15)$$

$$C_c := F \quad (16)$$

$$D_c := B_pF + \Delta_2 \quad (17)$$

$$\bar{P} := \begin{bmatrix} Y_p & -Z_p \\ -Z_p & Z_p \end{bmatrix}.$$

Proof: (only if) Define $X := (1/4)\bar{P}^{-1}$. We get

$$\left[\begin{array}{c|c} (\bar{A}+BH_iM)X+X(\bar{A}+BH_iM)^T+\frac{1}{4}EE^T & X\bar{C}^T \\ \hline \bar{C}X & -I \end{array}\right] < \mathbf{0}$$

$i = 2,\ldots,N$, where $\bar{A} := A - (1/2)E(S_1+S_2)C$, $\bar{C} := (S_2 - S_1)C$, and thus

$$\begin{bmatrix} B \\ \mathbf{0} \end{bmatrix}^{\perp} \left[\begin{array}{c|c} \bar{A}X + X\bar{A}^T + \frac{1}{4}EE^T & X\bar{C}^T \\ \hline \bar{C}X & -I \end{array}\right] \begin{bmatrix} B \\ \mathbf{0} \end{bmatrix}^{\perp T} < \mathbf{0}.$$

Similarly, we have

$$\begin{bmatrix} M^T \\ \mathbf{0} \end{bmatrix}^{\perp} \left[\begin{array}{c|c} Y\bar{A} + \bar{A}^TY + \frac{1}{4}\bar{C}^T\bar{C} & YE \\ \hline E^TY & -I \end{array}\right] \begin{bmatrix} M^T \\ \mathbf{0} \end{bmatrix}^{\perp T} < \mathbf{0}$$

where $Y = \bar{P}$. Partition $X = \begin{bmatrix} X_p & X_{pc} \\ X_{pc}^T & X_c \end{bmatrix}$ and $Y = \begin{bmatrix} Y_p & Y_{pc} \\ Y_{pc}^T & Y_c \end{bmatrix}$ appropriately. Note that $\begin{bmatrix} B \\ \mathbf{0} \end{bmatrix}^{\perp} = \begin{bmatrix} B^{\perp} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix}$, $\begin{bmatrix} M^T \\ \mathbf{0} \end{bmatrix}^{\perp} = \begin{bmatrix} M^{T\perp} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix}$, $B^{\perp} = [B_p^{\perp}\ \mathbf{0}]$, $M^{T\perp} = [M_p^{T\perp}\mathbf{0}]$. Thus we obtain (7) and (8). Furthermore, $XY = (1/4)I$ implies that $X_pY_p + X_{pc}Y_{pc}^T = (1/4)I$ and $X_pY_{pc} + X_{pc}Y_c = \mathbf{0}$. Thus $Y_p - (1/4)X_p^{-1} = Y_{pc}Y_c^{-1}Y_{pc}^T \geq \mathbf{0}$. Since (7) and (8) are strict, we can always perturb X_p and Y_p slightly so that (7) and (8) still hold, but also the strict inequality (9) holds.

(if) By Finsler's lemma [7], (7) and (8) imply that there exist $r_2 > 0$ and $r_1 > 0$ such that (10) and (11) hold, respectively. By taking the Schur complement, (10) is equivalent to

$$\bar{A}_pX_p + X_p\bar{A}_p^T + \frac{1}{4}E_pE_p^T - 2r_2\lambda_2B_pB_p^T + X_p\bar{C}_p^T\bar{C}_pX_p < \mathbf{0}.$$

Thus, we have

$$R_F^i \leq (\bar{A}_p + \lambda_2B_pF)^TX_p^{-1} + X_p^{-1}(\bar{A}_p + \lambda_2B_pF) + \frac{1}{4}X_p^{-1}E_pE_p^TX_p^{-1} + \bar{C}_p^T\bar{C}_p < \mathbf{0}$$

for all $i = 2,\ldots,N$, and similarly, $R_G < \mathbf{0}$. Since we choose $k \in (0,1)$ such that $Y_pR_GY_p < (1/4)(1-k)R_F^N$, together with $R_F^N \leq R_F^{N-1} \leq \cdots \leq R_F^2$, we get $Y_pR_GY_p < (1/4)(1-k)R_F^i$, $\forall i = 2,\ldots,N$. Note that such k always exists.

Denote $A_i := \bar{A} + BH_iM$, $i = 2,\ldots,N$. Then (5) holds if and only if

$$\tilde{P}\tilde{A}_i + \tilde{A}_i^T\tilde{P} + \frac{1}{4}\tilde{C}^T\tilde{C} + \tilde{P}\tilde{E}\tilde{E}^T\tilde{P} < \mathbf{0}, \quad i = 2,\ldots,N \quad (18)$$

where $\tilde{P} = S^T\bar{P}S = \begin{bmatrix} (1/4)X_p^{-1} & \mathbf{0} \\ \mathbf{0} & Z_p \end{bmatrix}$, $\tilde{A}_i = S^{-1}A_iS$, $\tilde{E} = S^{-1}E$, $\tilde{C} = \bar{C}S$ and $S = \begin{bmatrix} I & \mathbf{0} \\ I & -I \end{bmatrix}$. We first compute

$$\tilde{A}_i = \left[\begin{array}{c|c} \bar{A}_p + \lambda_iB_pC_c & -\lambda_iB_pC_c \\ \hline \begin{array}{c} \bar{A}_p - B_cM_p + \lambda_iB_pC_c \\ -A_c - \lambda_iD_c \end{array} & -\lambda_iB_pC_c + A_c + \lambda_iD_c \end{array}\right]$$

$\tilde{C} = [\bar{C}_p\ \ \mathbf{0}]$, $\tilde{E} = \begin{bmatrix} E_p \\ E_p \end{bmatrix}$. By straightforward computation, the (1, 1) block of the left-hand side of (18) turns out to be $(1/4)R_F^i$. The (2, 1) and (2,2) blocks can be computed to be equal to $-(1/4)kR_F^i$ and $Y_pR_GY_p - (1/4)R_F^i + (1/2)kR_F^i$, respectively. Thus the left-hand side of (18) equals

$$\begin{bmatrix} \frac{1}{4}R_F^i & -\frac{1}{4}kR_F^i \\ -\frac{1}{4}kR_F^i & Y_pR_GY_p - \frac{1}{4}R_F^i + \frac{1}{2}kR_F^i \end{bmatrix}$$

for all $i = 2,\ldots,N$. The latter equals

$$\begin{bmatrix} \frac{1}{4}(1-k)R_F^i & \mathbf{0} \\ \mathbf{0} & Y_pR_GY_p - \frac{1}{4}(1-k)R_F^i \end{bmatrix} + \frac{1}{4}k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes R_F^i$$

for all $i = 2,\ldots,N$. Obviously, the first term above is negative define and the second term is negative semi-define since $k > 0$, $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \geq \mathbf{0}$ and $R_F^i < \mathbf{0}$. Therefore, (18) and also (5) hold for all $i = 2,\ldots,N$. This completes the proof. □

By combining the previous two lemmas we now obtain the main result of this technical note:

Theorem 1: There exists a protocol (3) such that the network (4) is synchronized for all incrementally sector bounded $\phi(\cdot,t)$ within sector $[S_1, S_2]$ if there exist positive definite solutions X_p and Y_p of the LMI's (7), (8) such that the coupling inequality (9) is satisfied. In that case, A_c, B_c, C_c, D_c representing a suitable protocol (3) of dynamic order $n_c = n$ are given by (14)–(17), respectively.

Remark 2: The protocol designed in Lemma 3 is inspired by the measurement feedback $\mathcal{H}_\infty$ controller construction for general linear systems in [14]. It is important to note that, whereas (5) involves a common solution to $N-1$ linear matrix inequalities, the conditions obtained in Lemma 3 only involve *two* LMI's together with a coupling inequality, regardless of the number of agents N. These conditions are *independent of the underlying graph*, and only depend on the linear part of the agent dynamics and on the sector bounds. In the algorithm leading to a suitable protocol, data on the network graph do enter. In particular, our protocol depends on the smallest nonzero and largest Laplacian eigenvalues via inequality (10), definition (12) and inequality (13). Therefore, in order to be able to compute the protocol, exact *a priori* knowledge of the network graph is required. In our previous work [23] we were able to circumvent this problem for the special case treated there.

Remark 3: We note that Lemma 2 can be given an interpretation in terms of synthesis of a set of $N-1$ feedback controllers (depending on the nonzero Laplacian eigenvalues) that *simultaneously* render a single system dissipative. Indeed, by the bounded real lemma, for given A_c, B_c, C_c, D_c the existence of $\bar{P}$ such that (5) holds for all $i = 2,\ldots,N$ is equivalent to the condition that the $N-1$ dynamic feedback controllers

$$\begin{cases} \dot{w}_i = A_cw_i + \lambda_iB_cy + \lambda_iD_cw_i \\ u_i = C_cw_i \end{cases}, \quad i = 2,\ldots,N$$

render the system

$$\begin{cases} \dot{x} = \left(A_p - \frac{1}{2}E_p(S_1+S_2)C_p\right)x + B_p u + E_p d \\ z = \frac{\sqrt{2\tau}}{2}(S_2 - S_1)C_p x \\ y = M_p x \end{cases}$$

dissipative with respect to the supply rate $s(d,z) = \tau d^T d - z^T z$ with the *common* storage function $\binom{w_i}{x}^T P \binom{w_i}{x}$.

IV. Simulation Results

In this section, we give a numerical example to illustrate the results obtained in this technical note. As in [23], a Chua's circuit model is taken as the individual agent dynamics

$$\begin{cases} \dot{x} = A_p x + B_p u + E_p d \\ z = C_p x \\ y = M_p x \\ d = -\phi(z) \end{cases} \tag{19}$$

where $x = [x_1, x_2, x_3]^T \in \mathbb{R}^3$, $B_p = [1\ 1\ 0]^T$, $C_p = [1\ 0\ 0]$,

$$A_p = \begin{bmatrix} -3.2 & 10 & 0 \\ 1 & -1 & 1 \\ 0 & -14.87 & 0 \end{bmatrix}, \quad M_p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

$E_p = [-2.95\ 0\ 0]^T$, and $\phi(z) = |z+1| - |z-1|$. Clearly, $\phi(z)$ is incrementally sector bounded within [0, 2]. We consider a connected undirected network of 50 agents. The interconnection topology is randomly generated, and its smallest nonzero and largest Laplacian eigenvalues are computed to be $\lambda_2 = 0.1784$ and $\lambda_{50} = 7.3237$, respectively. All computations below are performed using the Matlab LMI Control Toolbox.

First, we compute a suitable positive definite solution X_p of (7) as

$$X_p = \begin{bmatrix} 0.9529 & 0.6375 & 0.1156 \\ 0.6375 & 0.7789 & 0.0440 \\ 0.1156 & 0.0440 & 1.9730 \end{bmatrix}.$$

Then (9) becomes a linear matrix inequality, and together with (8), a suitable Y_p is computed as

$$Y_p = \begin{bmatrix} 3.3329 & -0.2383 & -0.0022 \\ -0.2383 & 3.3960 & -0.7572 \\ -0.0022 & -0.7572 & 3.1047 \end{bmatrix}.$$

Following the rest of the steps in Lemma 3, and using (14)–(17), protocol parameter matrices are computed as

$$A_c = \begin{bmatrix} -233.4831 & 31.4925 & -0.3681 \\ 26.3295 & -269.3691 & 0.4859 \\ 8.0665 & -84.8507 & -0.1220 \end{bmatrix}$$

$$B_c = \begin{bmatrix} 239.1310 & -23.4507 \\ -23.4507 & 264.3036 \\ -7.7791 & 68.4690 \end{bmatrix}$$

$$C_c = \begin{bmatrix} -229.7929 & -499.9370 & 24.6057 \end{bmatrix}$$

$$D_c = \begin{bmatrix} -236.7923 & -515.1647 & 25.3552 \\ -249.9538 & -543.7989 & 26.7645 \\ -4.2914 & -9.3364 & 0.4595 \end{bmatrix}.$$

For $i = 1, 2, \ldots, 50$, let $x_i = [x_{i1}, x_{i2}, x_{i3}]^T$ and $w_i = [w_{i1}, w_{i2}, w_{i3}]^T$ be the agent state and the protocol state, respectively. Denote $X_j := [x_{1j}, x_{2j}, \ldots, x_{50,j}]^T$ and $W_j := [w_{1j}, w_{2j}, \ldots, w_{50,j}]^T$ for $j = 1, 2, 3$. Choose the initial states as $x_i(0) = [i, i, i]^T$, $w_i(0) = \mathbf{0}_3$, $i = 1, 2, \ldots, 50$. The first components of the trajectories of the network (4), i.e., X_1, are plotted in Fig. 2, where two plots are given, using different time scales to visualize the synchronization process more clearly. Clearly, the network is synchronized. The first components of the trajectories of the protocol dynamics, i.e. W_1, are plotted in Fig. 3. As in [18], it can be shown that the dynamic part of the protocol in fact acts as an observer for the weighted sum of the relative states $\sum_{j=1}^{N} a_{ij}(x_i - x_j)$ based on the weighted sum of the relative outputs $\sum_{j=1}^{N} a_{ij}(y_i - y_j)$. Thus, for each i, the protocol state w_i is an estimate of the weighted sum of the relative states and the protocol states converge to zero.

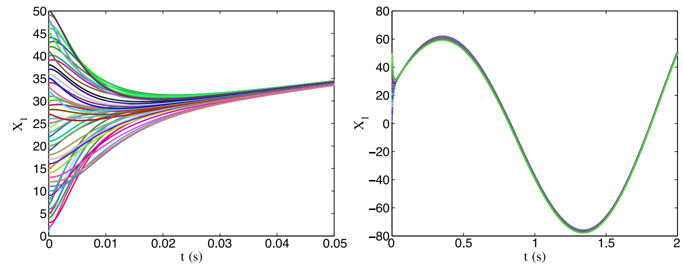

Fig. 2. Plots of X_1.

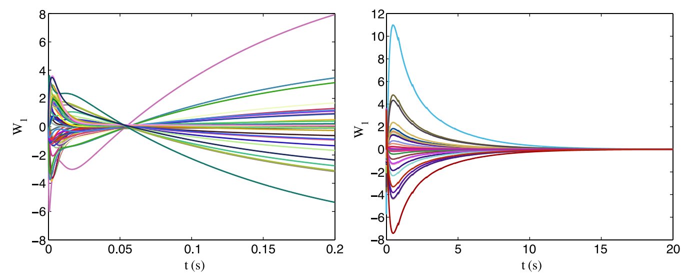

Fig. 3. Plots of W_1.

V. Conclusion

In this technical note, we have generalized our results in [23] on synchronization of homogeneous Lur'e networks by static, relative state based protocols to the case that only relative measurements are available. We have given sufficient conditions under which a linear dynamic protocol exists that synchronizes the network. These conditions involve feasibility of two LMI's together with a coupling inequality, reminiscent of the well-known LMI conditions in H_∞ control by measurement feedback. It is important to note that only three inequalities are involved, regardless of the number of agents in the network. Also, the three inequalities do not depend on the underlying graph, but only involve the linear part of the dynamics of the individual agents and the sector bounds. Only in the computation of the protocol the eigenvalues of the Laplacian matrix of the interconnection graph enter. In particular, the matrices defining the protocol depend on the smallest nonzero eigenvalue and the largest eigenvalue of the Laplacian matrix. We have validated our results by means of a numerical simulation example involving synchronization of a grid of Chua's circuits. As a possible venue for future research we mention the extension of our work to directed graphs. Our current methods are not applicable in that situation. Preliminary work on the directed graph case can be found in [24]. Another relevant future research direction is the extension of our work to time varying graphs.

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