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Analysis of a slow–fast system near a cusp singularity

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Abstract

This paper studies a slow–fast system whose principal characteristic is that the slow manifold is given by the critical set of the cusp catastrophe. Our analysis consists of two main parts: first, we recall a formal normal form suitable for systems as the one studied here; afterwards, taking advantage of this normal form, we investigate the transition near the cusp singularity by means of the blow up technique. Our contribution relies heavily in the usage of normal form theory, allowing us to refine previous results.

Contents

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1. Introduction

A *slow–fast system* (SFS) is a singularly perturbed ordinary differential equation of the form

$$\begin{aligned}\dot{x} &= f(x, z, \varepsilon)\\ \varepsilon\dot{z} &= g(x, z, \varepsilon),\end{aligned} \tag{1}$$

where $x \in \mathbb{R}^m$, $z \in \mathbb{R}^n$ are local coordinates and where $\varepsilon > 0$ is a small parameter. The over-dot denotes the derivative with respect to the time parameter t. Throughout this text, we assume that the functions f and g are of class $\mathcal{C}^\infty$. In applications (e.g. [25]), $z(t)$ represents states or measurable quantities of a process while $x(t)$ stands for control parameters. The parameter ε models the difference of the rates of change between the variables z and x. That is why systems like (1) are often used to model phenomena with two time scales. Observe that the smaller ε is, the faster z evolves with respect to x. Therefore we refer to x (resp. z) as the *slow* (resp. *fast*) variable. The time parameter t is known as the *slow time*. For $\varepsilon \neq 0$, we can define a new time parameter τ by the relation $t = \varepsilon\tau$. With this time reparametrization (1) can be written as

$$\begin{aligned}x' &= \varepsilon f(x, z, \varepsilon)\\ z' &= g(x, z, \varepsilon),\end{aligned} \tag{2}$$

where now the prime denotes the derivative with respect to the rescaled time parameter τ, which we call *the fast time*. Since we consider only autonomous systems, we often omit to indicate the time dependence of the variables. In the rest of this document, we prefer to work with slow–fast systems presented as (2).

Observe that as long as $\varepsilon \neq 0$ and f is not identically zero, systems (1) and (2) are equivalent. A first approach to understand the qualitative behavior of slow–fast systems is to study the limit $\varepsilon \to 0$. The slow equation (1) restricted to $\varepsilon = 0$ reads as

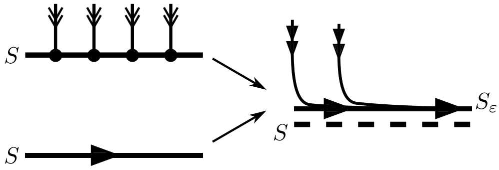

Fig. 1. A schematic representation of the persistence of a NHIM under the perturbation of the corresponding vector field. S denotes the slow manifold. Left–above: S is a set of hyperbolic equilibrium points of the layer equation. Left–below: S is the phase space of the constrained equation. Right: since S is a NHIM, it persists as an invariant manifold S_ε under small perturbations of the vector field.

$$\begin{aligned} \dot{x} &= f(x,z,0) \\ 0 &= g(x,z,0). \end{aligned} \tag{3}$$

A system of the form (3) is called *constrained differential equation* (CDE) [11,24]. On the other hand, in the limit $\varepsilon \to 0$, a system given by (2) becomes

$$\begin{aligned} x' &= 0 \\ z' &= g(x,z,0), \end{aligned} \tag{4}$$

which is called *the layer equation*. Associated to both systems, (3) and (4), the slow manifold S is defined by

$$S = \left\{(x,z) \in \mathbb{R}^m \times \mathbb{R}^n \,|\, g(x,z,0) = 0\right\}, \tag{5}$$

which serves as the phase space of the CDE (3) and as the set of equilibrium points of the layer equation (4). In the latter context, it is useful to recall the concept of Normally Hyperbolic Invariant Manifold (NHIM).

Definition 1.1 *(Normally Hyperbolic Invariant Manifold).* Consider a slow–fast system given by a vector field of the form

$$X_\varepsilon = \varepsilon f(x,z,\varepsilon)\frac{\partial}{\partial x} + g(x,z,\varepsilon)\frac{\partial}{\partial z}. \tag{6}$$

The associated slow (invariant) manifold $S = \{g(x,z,0) = 0\}$ is said to be normally hyperbolic if each point of S is a hyperbolic equilibrium point of X_0.

NHIMs are relevant in the context of the geometric study of slow–fast systems, see for example [8]. It is known that compact NHIMs persist under $\mathcal{C}^1$ small perturbation of the vector field [13,14]. In the particular context presented above, a normally hyperbolic compact subset of the slow manifold S persists as an invariant manifold of the slow–fast system X_ε. We show in Fig. 1 a schematic of the previous description.

After this introduction, we turn into the subject of this paper. Our goal is to understand the dynamics of a particular slow–fast system which has one fast and two slow variables given as

$$X_\varepsilon = \varepsilon(1+f_1)\frac{\partial}{\partial x_1} + \varepsilon f_2\frac{\partial}{\partial x_2} - \left(z^3 + x_2 z + x_1 + \varepsilon f_3\right)\frac{\partial}{\partial z}, \tag{7}$$

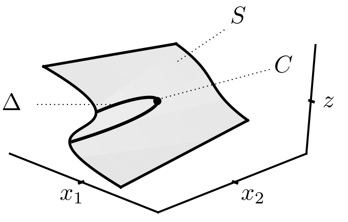

Fig. 2. The manifold S is two dimensional and can be defined as the critical set of the potential function $V(x_1, x_2, z) = \frac{1}{4}z^4 + \frac{1}{2}x_2z^2 + x_1z$. The curve Δ is defined by the set of degenerate critical points of V. Geometrically, Δ is the set of points of S where S is tangent to the fast direction, and C denotes the cusp point.

where the functions $f_i = f_i(x_1, x_2, z, \varepsilon)$, for $i = 1, 2, 3$, are smooth and vanish at the origin. The corresponding slow manifold is defined by

$$S = \left\{(x_1, x_2, z) \in \mathbb{R}^3 \,|\, z^3 + x_2z + x_1 = 0\right\}. \tag{8}$$

Remark 1.1. The slow manifold S can be regarded as the critical set of the cusp (or A_3) catastrophe, which is given as [1,5]

$$V(x_1, x_2, z) = \frac{1}{4}z^4 + \frac{1}{2}x_2z^2 + x_1z. \tag{9}$$

We denote by Δ the set of points in S at which S is tangent to the fast direction, that is

$$\Delta = \left\{(x_2, z) \in S \,|\, 3z^2 + x_2 = 0\right\}. \tag{10}$$

In other words, Δ is the set of degenerate critical points of (9). See Fig. 2 for a description of the slow manifold and the set Δ.

Our interest in studying (7) is due to the fact that the origin $(x_1, x_2, z) = (0, 0, 0)$ is a *non-hyperbolic equilibrium point* of X_0. This implies that a compact subset, around the origin, of the slow manifold S is not a NHIM of X_0, and therefore, the Geometric Singular Perturbation Theory [8,13,14] is not enough.

1.1. Motivation

There have been several studies, e.g. [16,17], dealing with a SFS of the form

$$X_\varepsilon = \varepsilon(1 + f_1)\frac{\partial}{\partial x_1} - \left(z^2 + x_1 + \varepsilon h\right)\frac{\partial}{\partial z}, \tag{11}$$

whose slow manifold is the critical set of the fold catastrophe. The next natural step is to consider the following case in the Thom list [22], i.e., a slow–fast system induced by the cusp catastrophe. That is

$$X_\varepsilon = \varepsilon(1 + f_1)\frac{\partial}{\partial x_1} + \varepsilon f_2\frac{\partial}{\partial x_2} - \left(z^3 + x_2z + x_1 + \varepsilon f_3\right)\frac{\partial}{\partial z}. \tag{12}$$

In [4], the system (12) is studied in a qualitative way. Here, however, we aim to refine the results by heavily using techniques from normal form theory. Moreover, we remark that the methods presented here are applicable to a larger class of slow–fast system given by

$$X_\varepsilon = \varepsilon(1+f_1)\frac{\partial}{\partial x_1} + \sum_{i=2}^{k-1}\varepsilon f_i \frac{\partial}{\partial x_i} - \left(z^k + \sum_{j=1}^{k-1} x_j z^{j-1} - \varepsilon f_k\right)\frac{\partial}{\partial z}, \tag{13}$$

which is called (regular) A_k-SFS, see [10].

1.2. Statement

We shall study the SFS

$$X_\varepsilon = \varepsilon(1+f_1)\frac{\partial}{\partial x_1} + \varepsilon f_2 \frac{\partial}{\partial x_2} - \left(z^3 + x_2 z + x_1 + \varepsilon f_3\right)\frac{\partial}{\partial z}, \tag{14}$$

where the functions $f_i = f_i(x_1, x_2, z, \varepsilon)$ are smooth. To avoid working with an ε-parameter family of vector fields as (14), it is customary to extend (14) by adding the trivial equation $\varepsilon' = 0$, and thus consider a smooth vector field in $\mathbb{R}^4$ which reads as

$$X = \varepsilon(1+f_1)\frac{\partial}{\partial x_1} + \varepsilon f_2 \frac{\partial}{\partial x_2} - \left(z^3 + x_2 z + x_1 + \varepsilon f_3\right)\frac{\partial}{\partial z} + 0\frac{\partial}{\partial \varepsilon}. \tag{15}$$

We regard (15) as a perturbation of “the principal part” F which is given as

$$F = \varepsilon\frac{\partial}{\partial x_1} + 0\frac{\partial}{\partial x_2} - \left(z^3 + x_2 z + x_1\right)\frac{\partial}{\partial z} + 0\frac{\partial}{\partial \varepsilon}. \tag{16}$$

Note that in a qualitative sense, F contains the essential elements of X. To state our main result, we first define the sections

$$\begin{aligned}\Sigma^- &= \left\{(x_1, x_2, z, \varepsilon) \in \mathbb{R}^4 \,|\, x_1 = -x_1^i\right\}\\ \Sigma^+ &= \left\{(x_1, x_2, z, \varepsilon) \in \mathbb{R}^4 \,|\, x_1 = x_1^f\right\},\end{aligned} \tag{17}$$

where $x_1^i > 0$ and $x_1^f > 0$ are arbitrarily large constants. For $\varepsilon > 0$ but sufficiently small, the sections Σ^- and Σ^+ are transversal to the flow of X_ε. Next, let $\Pi : \Sigma^- \to \Sigma^+$ be the Poincaré map induced by the flow of X_ε. We shall prove the following.

Transition along the cusp (See Theorem 3.1)**.** Consider a slow–fast system given by (15). Let Σ^-, Σ^+ and $\Pi : \Sigma^- \to \Sigma^+$ be defined as above. Then, we can choose coordinates in Σ^- and in Σ^+ such that the map Π reads as

$$\Pi(X_2, Z, \varepsilon) = (\tilde{X}_2, \tilde{Z}, \tilde{\varepsilon}), \tag{18}$$

where $\tilde{X}_2 = X_2 + H(X_2, \varepsilon)$ (with H flat at $(X_2, \varepsilon) = (0, 0)$), $\tilde{\varepsilon} = \varepsilon$ and where

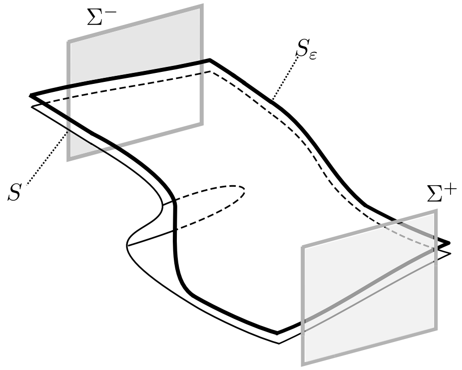

Fig. 3. Description of our main result. We may choose appropriate coordinates at the sections Σ^- and Σ^+ under which the invariant manifold S_ε is given by $Z = 0$. Moreover form (19) we have that all other trajectories starting at Σ^- are exponentially attracted to the invariant manifold S_ε. In this paper we provide quantitative information regarding this exponential contraction.

$$\tilde{Z} = \Phi(X_2, \varepsilon) + Z \exp\left(-\frac{1}{\varepsilon}(A(X_2, \varepsilon) + \varepsilon\Psi(X_2, Z, \varepsilon))\right), \tag{19}$$

where $A(X_2, 0) > 0$. Details of the functions Φ, A, and Ψ are given in Theorem 3.1. In an heuristic way, this result is described in Fig. 3.

1.3. *Idea of the proof*

Our proof consists of two main steps.

1. From [12], it is known that there exists a formal transformation bringing (15) into

$$F = \varepsilon\frac{\partial}{\partial x_1} + 0\frac{\partial}{\partial x_2} - \left(z^3 + x_2 z + x_1\right)\frac{\partial}{\partial z} + 0\frac{\partial}{\partial \varepsilon}. \tag{20}$$

Then, by Borel's lemma [5], the vector field F can be realized as a smooth normal form $X^N = F + R$ of (15) and where R is flat at $(x_1, x_2, z, \varepsilon) = (0, 0, 0, 0)$. See more details in Section 2.2.

Remark 1.2. Along this text, a function $f(x) : \mathbb{R}^n \to \mathbb{R}$ is flat at $x = 0$ means that it is infinitely flat at $x = 0$, i.e., $j^\infty(f)(0) = 0$.

2. Based the previous normalization, next we use the geometric desingularization or blow up method (as introduced in [7]) to study the flow of the normal form $X^N = F + R$. This is detailed in Section 3.

Remark 1.3. With this document we aim at two goals:

1. To refine the results of [4]. This is, we do not only provide a qualitative description of the transition Π, but details on the differentiability of such a map is also presented.

2. To prepare a framework for the geometric desingularization of A_k slow–fast systems. These are a generalization of (15) given as

$$X = \varepsilon(1+f_1)\frac{\partial}{\partial x_1} + \sum_{i=1}^{k-1}\varepsilon f_i \frac{\partial}{\partial x_i} - \left(z^k + \sum_{j=1}^{k-1} x_j z^{j-1} + \varepsilon f_k\right)\frac{\partial}{\partial z} + 0\frac{\partial}{\partial \varepsilon}. \tag{21}$$

The rest of this document is arranged as follows: in Section 2 we provide a brief recollection of preliminary results that will simplify our later studies. Next, in Section 3 we pose our result and prove it by means of the geometric desingularization method and the results of Section 2. For readability purposes, many technicalities have been put in the appendix.

2. Preliminaries of slow–fast systems

In this section, we provide a number preliminary results that will be used later in Section 3. First of all, we consider slow–fast systems along normally hyperbolic regions of the slow manifold. Afterwards, we recall a result from [12] dealing with the normal form of (15). We remark that we only consider SFS with one fast variable. Let us be more precise with the type of SFS that we shall study first.

Definition 2.1. A slow–fast system is said to be (locally) regular around a point p_0, if its corresponding slow manifold is normally hyperbolic in some neighborhood of p_0.

2.1. The slow vector field

Let us consider a slow–fast system given by

$$X_\varepsilon = \sum_{i=1}^{m} \varepsilon f_i(x,z,\varepsilon)\frac{\partial}{\partial x_i} + H(x,z,\varepsilon)\frac{\partial}{\partial z}, \tag{22}$$

where $x \in \mathbb{R}^m$, $z \in \mathbb{R}$, and as usual $0 < \varepsilon \ll 1$. Furthermore, assume that $f(0,0,0) \neq 0$, $H(0,0,0) = 0$ and $\frac{\partial H}{\partial z}(0,0,0) < 0$. Thus X_ε is regular around $0 \in \mathbb{R}^{m+2}$. The slow manifold associated to (22) is defined by

$$S = \left\{(x,z) \in \mathbb{R}^{m+1} \mid H(x,z,0) = 0\right\}. \tag{23}$$

From the defining assumptions of (22), we have that S is a NHIM in a neighborhood of the origin. By looking at the Jacobian of X_ε at 0, it follows that there exists an $m+1$ dimensional a center manifold. Since X_ε is smooth, we can choose a $\mathcal{C}^\ell$ center manifold $\mathcal{W}^C$ for any $\ell < \infty$. The manifold $\mathcal{W}^C$ is given as a graph $z = \phi(x,\varepsilon)$ where ϕ is a $\mathcal{C}^\ell$ function.

Remark 2.1. Along the rest of the document we frequently make use of a finite class of differentiability. As it is customary in the present context, when we say that a manifold (or a map) is $\mathcal{C}^\ell$, we mean that such a manifold (or map) is ℓ-differentiable for ℓ as large as necessary.

The slow manifold S is naturally given by the restriction $\mathcal{W}^C|_{\varepsilon=0} = S$. Next, let us consider the vector field $\frac{1}{\varepsilon}X_\varepsilon(x,\phi,\varepsilon)$. Since $\mathcal{W}^C$ is locally invariant, it follows that $\frac{1}{\varepsilon}X_\varepsilon$ is tangent to $\mathcal{W}^C$. Therefore the vector field

$$X^{slow} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}X_\varepsilon(x,\phi,\varepsilon), \tag{24}$$

is tangent to S at each point of S, and we call it *the slow vector field*. We remark that the slow vector field X^{slow} is only well defined whenever ϕ is invertible.

2.1.1. The slow divergence integral

Associated to a regular slow–fast system and the corresponding slow vector field, the *slow divergence integral* is defined here. For this, let Σ^- and Σ^+ be two sections which are transversal to the flow of X_ε given by (22). For $\varepsilon \neq 0$ but sufficiently small, these sections are also transversal to the slow manifold S. Let γ_ε be a solution curve of X_ε chosen along a center manifold $\mathcal{W}^C$, thus γ_ε is transversal to the sections Σ^- and Σ^+. In the limit $\varepsilon = 0$, the curve γ_0 is a curve along the slow manifold S. The idea now is to borrow the well-known divergence theorem [21] to get some sense on how the trajectories of X_ε are attracted to S (recall that we made the assumption $\frac{\partial H}{\partial z} < 0$). The divergence of X_ε (given by (22)) reads as

$$\operatorname{div} X_\varepsilon = \frac{\partial H(x,z,\varepsilon)}{\partial z} + O(\varepsilon). \tag{25}$$

We can now take the integral of $\operatorname{div} X_\varepsilon$ along the orbit γ_ε of X_ε parametrized by the fast time τ, we have

$$\int_{\gamma_\varepsilon} \operatorname{div} X_\varepsilon \, d\tau = \int_{\gamma_\varepsilon} \left(\frac{\partial H(x,z,\varepsilon)}{\partial z} + O(\varepsilon)\right) d\tau. \tag{26}$$

The *slow divergence integral* is defined by

$$I(t) = \int_{\gamma_0} \operatorname{div} X_0 \, dt, \tag{27}$$

where t is the slow time defined by the slow vector field X^{slow}. Our goal then is to relate the divergence integral (26) with I.

Proposition 2.1. *Under the assumptions made in this section, we have that*

$$\int_{\gamma_\varepsilon} \operatorname{div} X_\varepsilon \, d\tau = \frac{1}{\varepsilon}\left(I(t) + o(1)\right), \tag{28}$$

where $I(t)$ is the slow divergence integral.

Proof. Recall that the slow vector field reads as $X^{slow} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} X_\varepsilon(x, \phi, \varepsilon)$, where $\phi = \phi(x, \varepsilon)$ is a $\mathcal{C}^\ell$ function. By our assumptions, the curve γ_ε is transversal to the sections Σ^- and Σ^+ for ε small enough. Without loss of generality we can assume that γ_ε is parametrized by x_1. Then let x_1^- and x_1^+ be defined by $\gamma_\varepsilon(x_1^-) = \gamma_\varepsilon \cap \Sigma^-$ and $\gamma_\varepsilon(x_1^+) = \gamma_\varepsilon \cap \Sigma^+$. Next, the integral of the divergence of X_ε along γ_ε from Σ^- to Σ^+ reads as

$$\begin{aligned}\int_{\gamma_\varepsilon} \operatorname{div} X_\varepsilon \, d\tau &= \frac{1}{\varepsilon} \int_{x_1^-}^{x_1^+} \left(\frac{\partial H(x, z, 0)}{\partial z} + O(\varepsilon) \right) \frac{dx_1}{f_1(x, z, 0) + o(1)} \\ &= \frac{1}{\varepsilon} \left(\int_{x_1^-}^{x_1^+} \frac{\partial H(x, z, 0)}{\partial z} \frac{dx_1}{f_1(x, z, 0)} + o(1) \right) \\ &= \frac{1}{\varepsilon} \left(\int_{\gamma_0} \operatorname{div} X_0 \, dt + o(1) \right), \end{aligned} \tag{29}$$

where t is the slow time induced by X^{slow}, which in coordinates means that $\frac{dx_1}{dt} = f_1$. □

Observe that the slow divergence integral is a first order approximation of the divergence along orbits of X_ε. This will be useful when presenting our main result in Section 3.

2.1.2. Normal form and transition of a regular slow–fast system

Now we consider the problem of finding a suitable normal form of a regular SFS. The following is a well-known result but we recall it here for completeness.

Proposition 2.2. *Consider a regular slow–fast system on $\mathbb{R}^{m+3}$ given by*

$$X_\varepsilon = \varepsilon(1 + f_1)\frac{\partial}{\partial u} + \sum_{j=1}^{m} \varepsilon g_j \frac{\partial}{\partial v_j} + H\frac{\partial}{\partial z}, \tag{30}$$

where $(u, v_1, \ldots, v_m, z, \varepsilon) \in \mathbb{R}^{m+3}$; where the functions $f_1 = f_1(u, v, z, \varepsilon)$ and $g_j = g_j(u, v, z, \varepsilon)$, for $2 \geq j \geq k-1$, are smooth and vanish at the origin $(u, v, z, \varepsilon) = (0, 0, 0, 0)$. Furthermore, the function $H = H(u, v, z, \varepsilon)$ is smooth with $H(0, 0, 0, 0) = 0$ and $\frac{\partial H}{\partial z}(0, 0, 0, 0) < 0$. Then, the vector field X is $\mathcal{C}^\ell$-equivalent to a normal form given by

$$X_\varepsilon^N = \varepsilon\frac{\partial}{\partial U} + \sum_{j=1}^{m} 0\frac{\partial}{\partial V_j} - Z\frac{\partial}{\partial Z}, \tag{31}$$

where $\{Z = 0\}$ corresponds to a choice of the center manifold $\mathcal{W}^C$ of X_ε.

Proof. The first step is to divide the vector field X by $1 + f_1$. In a sufficiently small neighborhood of the origin this is a smooth equivalence relation. That is $Y = \frac{1}{1+f_1} X$ reads as

$$Y = \varepsilon \frac{\partial}{\partial u} + \sum_{j=1}^{m} \varepsilon \tilde{g}_j \frac{\partial}{\partial v_j} + \tilde{H} \frac{\partial}{\partial z}, \tag{32}$$

where $\tilde{g}_j$, for $2 \geq j \geq k-1$, and $\tilde{H}$ are smooth with $\tilde{H}(0) = 0$ and $\frac{\partial \tilde{H}}{\partial z}(0) < 0$. Now we note that the origin of $\mathbb{R}^{m+3}$ is a semihyperbolic equilibrium point with (u, v, ε) being center coordinates and z being the hyperbolic coordinate. We can now use Takens–Bonckaert results on normal forms of partially hyperbolic vector fields [2,3,23]. Thus, there exists a $\mathcal{C}^\ell$ change of coordinates (maybe respecting some constraints if required) under which Y is conjugated to

$$\bar{Y} = \varepsilon \frac{\partial}{\partial U} + \sum_{j=1}^{m} \varepsilon \bar{G}_j \frac{\partial}{\partial V_j} + \bar{H} Z \frac{\partial}{\partial Z}, \tag{33}$$

where $\bar{G}_j = \bar{G}_j(U, V, \varepsilon)$, for $2 \geq j \geq k-1$, and $\bar{H} = \bar{H}(U, V, \varepsilon)$ are $\mathcal{C}^\ell$ functions, and where $\{Z = 0\}$ corresponds to a choice center manifold which we denote by $\mathcal{W}^C$. We remark that in the vector field $\bar{Y}$, the functions $\bar{G}_j$ and $\bar{H}$ are independent of Z. Furthermore we have

$$\bar{H}(0,0,0) = \frac{\partial \tilde{H}}{\partial z}(0,0,0,0) < 0. \tag{34}$$

This means that in a small neighborhood of the origin $\bar{Y}$ can be divided by $|\bar{H}|$. In other words, $\bar{Y}$ is $\mathcal{C}^\ell$-equivalent to

$$\mathcal{Y} = \varepsilon \mathcal{G} \frac{\partial}{\partial U} + \sum_{j=1}^{m} \varepsilon \bar{K}_j \frac{\partial}{\partial V_j} - Z \frac{\partial}{\partial Z}, \tag{35}$$

where $\mathcal{G}(0,0,0) \neq 0$ and $\bar{K}_j = \bar{K}_j(U, V, \varepsilon)$, for $2 \geq j \geq k-1$, are $\mathcal{C}^\ell$. Next, since $\mathcal{W}^C = \{Z = 0\}$ is invariant under the flow of $\mathcal{Y}$, we can study the restriction $\mathcal{Y}|_{Z=0}$. This is

$$\mathcal{Y}|_{Z=0} = \varepsilon \mathcal{G} \frac{\partial}{\partial U} + \sum_{j=1}^{m} \varepsilon \bar{K}_j \frac{\partial}{\partial V_j}. \tag{36}$$

For $\varepsilon \neq 0$, the vector field $\mathcal{Y}|_{Z=0}$ is regular because $\mathcal{G}(0,0,0) \neq 0$. Thus, by the flow-box theorem, there exists a change of coordinates, depending in a $\mathcal{C}^\ell$ way on ε, under which $\mathcal{Y}|_{Z=0}$ can be written as

$$\varepsilon \frac{\partial}{\partial U} + \sum_{j=1}^{m} 0 \frac{\partial}{\partial V_j}. \tag{37}$$

We must note that the flow-box change of coordinates is $\mathcal{C}^\ell$ in ε even when $\varepsilon \to 0$. This is the case because $\mathcal{Y}$ is divisible by ε defining the slow vector field $\mathcal{Y}^{slow}$ (compare with Section 2.1) and Y^{slow} is regular. This in turn means that the limit of the flow-box, when $\varepsilon \to 0$, is a neighborhood tangent to the slow vector field.

Now, it follows that $\mathcal{Y}$ is $\mathcal{C}^\ell$-equivalent to

$$X_{reg}^N = \varepsilon\frac{\partial}{\partial U} + \sum_{j=1}^{m} 0\frac{\partial}{\partial V_j} - Z\frac{\partial}{\partial Z}, \tag{38}$$

as stated in the proposition. □

Motivated by Proposition 2.2 let us now discuss the dynamics of the vector field

$$X_{reg}^N = \varepsilon\frac{\partial}{\partial U} + \sum_{j=1}^{m} 0\frac{\partial}{\partial V_j} - Z\frac{\partial}{\partial Z}. \tag{39}$$

The slow manifold S, corresponding to the normal form (39), is given by

$$S = \{\varepsilon = 0,\ Z = 0\}. \tag{40}$$

Furthermore, we can parametrize the solution of (39) by U. Let us define the sections

$$\begin{aligned}\Sigma^- &= \left\{(U, V, Z, \varepsilon) \in \mathbb{R}\times\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}\,|\,U = U^-\right\}\\ \Sigma^+ &= \left\{(U, V, Z, \varepsilon) \in \mathbb{R}\times\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}\,|\,U = U^+\right\},\end{aligned} \tag{41}$$

where $U^- < U^+$. The sections Σ^- and Σ^+ are transversal to the manifold S and therefore, for $\varepsilon \neq 0$, are also transversal to the flow of (39). Associated to these sections, we define the transition

$$\begin{aligned}\Pi : \Sigma^- &\to \Sigma^+\\ (V, Z, \varepsilon) &\mapsto (\tilde{V}, \tilde{Z}, \tilde{\varepsilon}).\end{aligned} \tag{42}$$

To compute the component $\tilde{Z}$ we only need to integrate $\frac{dZ}{dU} = -\frac{1}{\varepsilon}Z$. Then it follows that $\tilde{Z} = Z(T)$, where T is the time to go from Σ^- to Σ^+, which is $T = U_f - U_i$. Then it follows that

$$\begin{aligned}\tilde{V} &= V\\ \tilde{Z} &= Z\exp\left(-\frac{1}{\varepsilon}(U_f - U_i)\right)\\ \tilde{\varepsilon} &= \varepsilon.\end{aligned} \tag{43}$$

Observe the particular format of the transition Π. The Z component is an *exponential* contraction towards the center manifold $\{Z = 0\}$. Maps with this characteristic appear frequently in our text and also in several other cases where slow–fast systems are studied. Therefore, in Appendix A we discuss in a rather general way, the properties of such maps.

2.2. Formal normal form of A_k slow–fast systems

In this section we recall a normal form of the so-called A_k slow–fast systems. A proof can be found in [12]. This normalization is important since it eliminates many unwanted terms from the system being studied here.

Definition 2.2. Let $k \in \mathbb{N}$ with $k \geq 2$. An A_k slow–fast system (A_k-SFS) is an ODE of the form

$$\begin{aligned} x_1' &= \varepsilon(1 + f_1) \\ x_j' &= \varepsilon f_j \\ z' &= -\left(z^k + \sum_{i=1}^{k-1} x_i z^{i-1}\right) + \varepsilon f_k \\ \varepsilon' &= 0, \end{aligned} \tag{44}$$

where $j = 2, \ldots, k-1$, and where the functions $f_i = f_i(x_1, \ldots, x_{k-1}, z, \varepsilon)$, for $1 \leq i \leq k$, are smooth and vanish at $(x_1, \ldots, x_{k-1}, z, \varepsilon) = (0, \ldots, 0, 0)$.

Remark 2.2.

- The system investigated in this work is an A_3-SFS.
- The slow manifold associated to an A_k-SFS is defined by

$$S = \left\{ (x, z) \in \mathbb{R}^k \mid z^k + \sum_{i=1}^{k-1} x_i z^{i-1} = 0 \right\}. \tag{45}$$

  The manifold S can equivalently be defined as the critical set of an A_k catastrophe [1]. Hence the name A_k-SFS.

Locally, we can regard (44) as $X = F + P$ where F and P are smooth vector fields of the form

$$F = \varepsilon \frac{\partial}{\partial x_1} + \sum_{j=2}^{k-1} 0 \frac{\partial}{\partial x_j} + g \frac{\partial}{\partial z} + 0 \frac{\partial}{\partial \varepsilon} \tag{46}$$

and

$$P = \sum_{i=1}^{k-1} \varepsilon f_i \frac{\partial}{\partial x_i} + \varepsilon f_k \frac{\partial}{\partial z} + 0 \frac{\partial}{\partial \varepsilon}, \tag{47}$$

respectively and where $g = -\left(z^k + \sum_{i=1}^{k-1} x_i z^{i-1}\right)$. We refer to F as the "principal part" and to P as the "perturbation". Briefly speaking we want to eliminate, via a change of coordinates, the perturbation. The procedure of normalizing the vector field X is motivated by [18], where normal forms of analytic perturbations of quasihomogeneous vector fields are investigated. The relevant result is the following

Theorem 2.1 *(Formal normal form [12]). Let $k \geq 2$ and let $X = F + P$ be a smooth vector field where*

$$F = \varepsilon\frac{\partial}{\partial x_1} + \sum_{i=2}^{k-1} 0\frac{\partial}{\partial x_i} - \left(z^k + \sum_{j=1}^{k-1} x_j z^{j-1}\right)\frac{\partial}{\partial z} + 0\frac{\partial}{\partial \varepsilon}, \tag{48}$$

and where

$$P = \sum_{i=1}^{k-1} P_i \frac{\partial}{\partial x_i} + P_k\frac{\partial}{\partial z} + 0\frac{\partial}{\partial \varepsilon}, \tag{49}$$

where each $P_i = P_i(x_1, \dots, x_{k-1}, z, \varepsilon)$ is a smooth function. Assume that the following conditions are satisfied

1. $P_i(x_1, \dots, x_{k-1}, z, 0) = 0$,
2. $\rho(\hat{P}_i) \geq 2k - i + 1$,

where $\hat{P}_i$ denotes the Taylor expansion of P_i and $\rho(\hat{P}_i)$ is the quasihomogeneous order of the polynomial $\hat{P}_i$. Then, there exists a formal diffeomorphism $\hat{\Phi}$ such that $\hat{\Phi}_* \hat{X} = F$.

Remark 2.3. The condition $\rho(\hat{P}_i) \geq 2k - i + 1$ is not restrictive for A_k-SFSs. In other words, any perturbation $P_i = \varepsilon f_i$, where $f_i(0) = 0$ (as in Definition 2.2) satisfies such condition. To see this note first that F is quasihomogeneous of degree $k-1$ and type $(k, k-1, \dots, 1, 2k-1)$ [12]. Next, observe that $\min(\rho(\hat{P}_i)) = \min(\rho(\varepsilon \hat{f}_i)) = 2k - 1 + \min(\rho(\hat{f}_i))$, because the quasihomogeneous weight of ε is $2k - 1$. From $f_i(0) = 0$ it follows that $\min(\rho(\hat{f}_i)) = 1$ and thus $\min(\rho(\hat{P}_i)) = 2k \geq 2k - i + 1$. Note however that if the perturbation is not of this form, the condition $\rho(\hat{P}_i) \geq 2k - i + 1$ does not necessarily hold.

In words, Theorem 2.1 shows that $\hat{X}$ and F are conjugated via $\hat{\Phi}$. It follows that, by Borel's lemma [5], the formal vector field $\hat{X}^N = F$ can be realized as a smooth vector field $X^N = F + \tilde{P}$ where $\tilde{P}$ is *flat* at $(x, z, \varepsilon) = (0, 0, 0)$. This has important consequences in the geometric desingularization of an A_3-SFS, presented in the following section.

3. Geometric desingularization of a slow–fast system near a cusp singularity

In this section we study an A_3 slow–fast system based on: a) the techniques introduced in Section 2 and in Appendix A, and b) the blow up method. To simplify the notation, let us now write the A_3-SFS as

$$X = \varepsilon(1 + f_1)\frac{\partial}{\partial a} + \varepsilon f_2 \frac{\partial}{\partial b} - (z^3 + bz + a + \varepsilon f_3)\frac{\partial}{\partial z} + 0\frac{\partial}{\partial \varepsilon}, \tag{50}$$

where thanks to Theorem 2.1 (see also Remark 2.3), the smooth functions $f_i = f_i(a, b, z, \varepsilon)$ are flat at the origin of $\mathbb{R}^4$. We investigate the transition associated to (50) between the sections

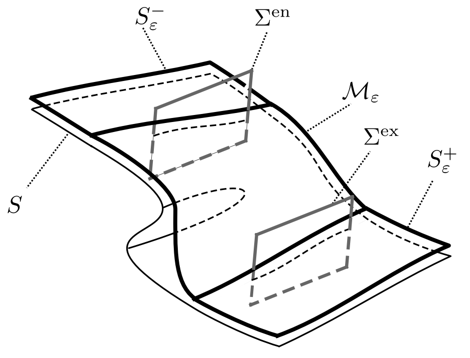

Fig. 4. Qualitative representation of the investigation performed in this section. The sections Σ^{en} and Σ^{ex} are arbitrarily close to the cusp point. On the other hand the sections Σ^- and Σ^+ (not shown) are parallel to Σ^{en} and Σ^{ex} but far away from the cusp point. In a qualitative sense, we will construct an invariant manifold $\mathcal{M}_\varepsilon$ and then extend it all the way up to the sections Σ^- and Σ^+. Our analysis aims for simplicity and thus depends extensively on the usage of normal forms. This, of course, makes our results coordinate-dependant.

$$\begin{aligned}\Sigma^- &= \left\{(a,b,z,\varepsilon)\in\mathbb{R}^4 \,|\, a=-a^-,\ z>0\right\}\\ \Sigma^+ &= \left\{(a,b,z,\varepsilon)\in\mathbb{R}^4 \,|\, a=a^+,\ z<0\right\},\end{aligned} \tag{51}$$

where $a^- > 0$ and $a^+ > 0$ are arbitrarily large constants. However, since the trajectories of X spend a long time along regular parts of S, it will be useful to define the "entry" and "exit" sections

$$\begin{aligned}\Sigma^{\text{en}} &= \left\{(a,b,z,\varepsilon)\in\mathbb{R}^4 \,|\, a=-a_0,\ z>0\right\}\\ \Sigma^{\text{ex}} &= \left\{(a,b,z,\varepsilon)\in\mathbb{R}^4 \,|\, a=a_0,\ z<0\right\},\end{aligned} \tag{52}$$

where a_0 is a positive but sufficiently small constant, for reference see Fig. 4.

It will be clear from our analysis in the blow up space (3.2) that the section Σ^- needs to be partitioned as follows.

Definition 3.1 *(The inner layer and the lateral regions).* Let $0 < L < M < \infty$ be constants. The inner layer $\Sigma^{\text{inner}} \subset \Sigma^-$ is defined as

$$\Sigma^- \supset \Sigma^{\text{inner}} = \left\{(b,z,\varepsilon)\in\Sigma^- \,|\, |b| < M\varepsilon^{2/5}\right\}. \tag{53}$$

On the other hand, the lateral regions are defined as

$$\begin{aligned}\Sigma^- \supset \Sigma^{+b} &= \left\{(b,z,\varepsilon)\in\Sigma^- \,|\, b > L\varepsilon^{2/5}\right\}\\ \Sigma^- \supset \Sigma^{-b} &= \left\{(b,z,\varepsilon)\in\Sigma^- \,|\, -b > L\varepsilon^{2/5}\right\}.\end{aligned} \tag{54}$$

Note that the set $\left\{\Sigma^{\text{inner}}, \Sigma^{+b}, \Sigma^{-b}\right\}$ is an open cover of Σ^-, see Fig. 5.

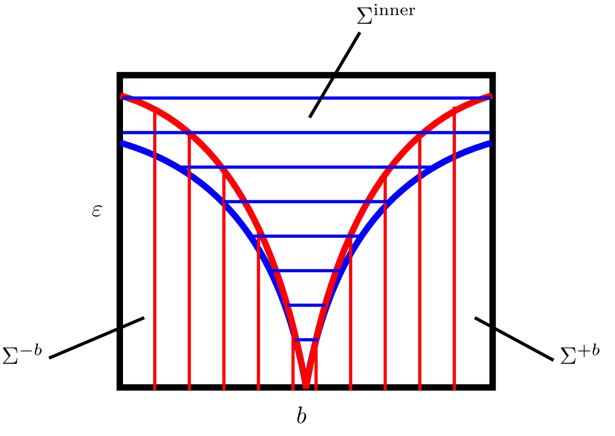

Fig. 5. The section Σ^- needs to be partitioned into three subsections: the inner layer Σ^{inner} and the lateral regions Σ^{+b}, Σ^{-b}. From a qualitative point of view, these three layers correspond to three different types of trajectories: 1. Trajectories starting at Σ^{inner} pass close to the cusp point. Observe that $\lim_{\varepsilon\to 0}(\Sigma^{\text{inner}}) = \{b = 0\}$ and then corresponds to a solution of the associated *CDE* passing exactly through the cusp point. 2. Trajectories starting at Σ^{+b} pass sufficiently away from the cusp point along the regular side of the manifold S. 3. Trajectories starting at Σ^{-b} pass sufficiently away from the cusp point along the folded side of the manifold S.

We are now in position to present our main result. In the following theorem, we characterize the transition $\Pi : \Sigma^- \to \Sigma^+$ under a suitable choice of coordinates at the section Σ^- and Σ^+. Furthermore, we give details on the differentiability of this map according to the cover of Σ^-, see Definition 3.1.

Theorem 3.1 *(Transition map of an A_3-SFS). Let X be an A_3 slow–fast system. This is, X is a vector field defined by*

$$X = \varepsilon(1 + f_1)\frac{\partial}{\partial a} + \varepsilon f_2 \frac{\partial}{\partial b} - \left(z^3 + bz + a + \varepsilon f_3\right)\frac{\partial}{\partial z} + 0\frac{\partial}{\partial \varepsilon}, \tag{55}$$

where each $f_i = f_i(a, b, z, \varepsilon)$, $i = 1, 2, 3$, is smooth. Let the sections Σ^-, Σ^+ be defined as above. Then we can choose suitable $\mathcal{C}^\ell$-coordinates (B, Z, ε) in Σ^- and $\mathcal{C}^\ell$-coordinates $(\tilde{B}, \tilde{Z}, \tilde{\varepsilon})$ in Σ^+ such that the transition $\Pi : (B, Z, \varepsilon) \mapsto (\tilde{B}, \tilde{Z}, \tilde{\varepsilon})$ is an exponential type map of the form

$$\Pi(B, Z, \varepsilon) = \left(B + h,\ \phi(B, \varepsilon) + Z \exp\left(-\frac{A(B, \varepsilon) + \Psi(B, Z, \varepsilon)}{\varepsilon}\right), \varepsilon\right), \tag{56}$$

where h is flat at the origin, $A > 0$ is $\mathcal{C}^\ell$, ϕ is $\mathcal{C}^\ell$-admissible with $\phi(B, 0) = 0$, and Ψ is $\mathcal{C}^\ell$-admissible with $\Psi(B, Z, 0) = 0$, see Appendix A for the definition of $\mathcal{C}^\ell$-admissible. Moreover, we have the following properties of the function A, ϕ and Ψ.

1. *$-A(B, 0) = I(B)$ where I is the slow divergence integral associated to (55).*
2. *Restricted to $(B, Z, \varepsilon) \in \Sigma^{\text{inner}}$, there are functions $\tilde{\phi}$ and $\tilde{\Psi}$ such that*

$$\phi(B, \varepsilon) = \tilde{\phi}\left(\mu, \varepsilon^{1/5}\right)$$

$$\Psi(B,Z,\varepsilon)=\tilde{\Psi}\left(|B|^{1/2},\varepsilon^{1/5},\varepsilon\ln\varepsilon,\mu,Z\right), \tag{57}$$

where $\tilde{\phi}$ and $\tilde{\Psi}$ are $\mathcal{C}^{\ell}$-admissible functions (see Definition A.1) with $\mu = B\varepsilon^{-2/5}$. Note that in this domain, μ is well defined in the sense that μ is bounded by a constant as $\varepsilon \to 0$.

3. *Restricted to $(B,Z,\varepsilon)\in\Sigma^{+b}$, there is a function $\tilde{\Psi}$ such that*

$$\begin{aligned}\phi(B,\varepsilon)&=0\\ \Psi(B,Z,\varepsilon)&=\tilde{\Psi}\left(|B|^{1/2},\varepsilon^{1/5},\varepsilon\ln(|B|),\sigma,Z\right),\end{aligned} \tag{58}$$

where $\tilde{\Psi}$ is a $\mathcal{C}^{\ell}$-admissible function (see Definition A.1) with $\sigma=\varepsilon|B|^{-5/2}$. Note that in this domain, σ is well defined since $|B|>0$.

4. *Restricted to $(B,Z,\varepsilon)\in\Sigma^{-b}$, there are functions $\tilde{\phi}$ and $\tilde{\Psi}$ such that*

$$\begin{aligned}\phi(B,\varepsilon)&=\tilde{\phi}\left(|B|^{1/2},\sigma\right)\\ \Psi(B,Z,\varepsilon)&=\tilde{\Psi}\left(|B|^{1/2},\varepsilon^{1/5},\varepsilon\ln(|B|),\sigma\right),\end{aligned} \tag{59}$$

where $\tilde{\phi}$ and $\tilde{\Psi}$ are $\mathcal{C}^{\ell}$-admissible functions (see Definition A.1) with $\sigma=\varepsilon|B|^{-5/2}$. Note that in this domain, σ is well defined since $|B|>0$.

Sketch of the proof. The first step is to recall Theorem 2.1, which shows that X is formally conjugate to

$$F=\varepsilon\frac{\partial}{\partial a}+0\frac{\partial}{\partial b}-\left(z^3+bz+a\right)\frac{\partial}{\partial z}+0\frac{\partial}{\partial\varepsilon}. \tag{60}$$

Next, by means of Borel's lemma [5], the vector field F can be realized as a smooth vector field $X^N=F+\varepsilon H$ where H is flat at $(a,b,z,\varepsilon)=(0,0,0,0)$. Thus, from now on, we only treat an A_3-SFS given as

$$X=\varepsilon(1+\varepsilon\tilde{f}_1)\frac{\partial}{\partial a}+\varepsilon^2\tilde{f}_2\frac{\partial}{\partial b}-\left(z^3+bz+a+\varepsilon\tilde{f}_3\right)\frac{\partial}{\partial z}+0\frac{\partial}{\partial\varepsilon}, \tag{61}$$

where each $\tilde{f}_i=\tilde{f}_i(a,b,z,\varepsilon)$ is flat at $(a,b,z,\varepsilon)=(0,0,0,0)$.

Another important ingredient of the proof is the blow up technique, which is described in Section 3.1. This method provides several local vector fields whose corresponding transitions are of exponential type, refer to Appendix A. Later all these local transitions are composed to produce an exponential type transition between the sections Σ^- and Σ^+. Along the analysis of the local vector fields (in the blow up space) we will take advantage of the flatness of the higher order terms of X. The complete proof follows Sections 3.1 to 3.5 and is given in Section 3.6.

Now, assuming that the transition Π is of the form (56), we can show that $A(B,0)$ is given by the slow divergence integral of X. For this, let us recall the Poincaré–Leontovich–Sotomayor formula [19], which in general is given as follows.

Proposition 3.1. *Let X be a vector field on a manifold M^n with a volume form Ω. Let Σ^- and Σ^+ be two open sections of M and transverse to the flow of X. Let γ_ε be an orbit of X along a center manifold $\mathcal{W}^C$ of X, starting at $p = \gamma_\varepsilon \cap \Sigma^-$ and reaching $q = \gamma_\varepsilon \cap \Sigma^+$ in finite time. Let $\Pi : \Sigma^- \to \Sigma^+$ be the transition map defined in a neighborhood of p. If $\psi^- : U \to \Sigma^-$ and $\psi^+ : V \to \Sigma^+$, with $U \subset \mathbb{R}^{n-1}$ and $V \subset \mathbb{R}^{n-1}$, are coordinates in Σ^- and in Σ^+ respectively, then*

$$\det\left(D\left((\psi^+)^{-1} \circ \Pi \circ \psi^-\right)\right)(s^-) = \frac{\langle \Omega(p), D\psi^-(s^-) \times X(p)\rangle}{\langle \Omega(q), D\psi^+(s^+) \times X(p)\rangle} \exp\left(\int_{\gamma_\varepsilon} \operatorname{div}_\Omega X\, d\tau\right), \quad (62)$$

where $s^- = (\psi^-)^{-1}(p)$ and $s^+ = (\psi^+)^{-1}(q)$. The integral is taken along the orbit γ_ε from p to q parametrized by the fast time τ.

So we have the following.

Proposition 3.2. *Consider an A_3-SFS and assume that the transition $\Pi : \Sigma^- \to \Sigma^+$ is given by (56). Then $-A(B, 0) = I(B)$, where $I(B)$ is the slow divergence integral associated to the A_3-SFS.*

Proof. The only relevant component is Z, so denote by Π_Z the Z-component of Π. The factor multiplying the exponential in (62) can be taken as a constant $C > 0$. Then we have that (62), for the vector field of Theorem 3.1, reads as

$$\frac{\partial \Pi_Z}{\partial Z} = C \exp\left(\int_{\gamma_\varepsilon} \operatorname{div}_\Omega X\, d\tau\right). \quad (63)$$

Here the volume form Ω can be taken as the standard one. In fact, the divergence $\operatorname{div}_\Omega X$ is independent of the chosen volume form [19,20]. Using the properties of the slow divergence integral described in (2.1.1), and since $C > 0$, we have

$$\begin{aligned}\frac{\partial \Pi_Z}{\partial Z} &= C \exp\left(\int_{\gamma_\varepsilon} \operatorname{div}_\Omega X\, d\tau\right)\\ &= \exp\left(\frac{1}{\varepsilon}\left(\int_{\gamma_0} \operatorname{div} X_0\, dt + \varepsilon \ln C + o(1)\right)\right)\\ &= \exp\left(\frac{1}{\varepsilon}(I + O(\varepsilon))\right), \end{aligned} \quad (64)$$

where I is the slow divergence integral of X along a curve in the slow manifold S from Σ^- to Σ^+. In principle, the limit $\varepsilon \to 0$ of (64) is not well defined. However, according to our Theorem 3.1, we have by differentiating (56) w.r.t. Z

$$\frac{\partial \Pi_Z}{\partial Z} = \exp\left(-\frac{A(B,\varepsilon)+\varepsilon\Psi(B,Z,\varepsilon)}{\varepsilon}\right). \tag{65}$$

Identifying (64) with (65) and taking the limit $\varepsilon \to 0$ we have indeed that

$$\lim_{\varepsilon\to 0}(I + O(\varepsilon)) = \lim_{\varepsilon\to 0}(-A(B,\varepsilon)+\varepsilon\Psi(B,Z,\varepsilon)), \tag{66}$$

which shows the claim. Note that the slow divergence integral in the coordinates (a, b, z) reads as

$$I(b) = \tilde{I}(b,\zeta^+) - \tilde{I}(b,\zeta^-), \tag{67}$$

where straightforward computations show that

$$\tilde{I}(b,\zeta) = \frac{9}{5}\zeta^5 + 2\zeta^3 b + b^2\zeta, \tag{68}$$

and where $\zeta^{\pm}$ is a constant defined by $(a^{\pm}, b, \zeta^{\pm}) \in \Sigma^{\pm} \cap S$.

On the other hand, in normal coordinates and along regular parts of the slow manifold, the A_3-SFS can be written as (see Section 2.1.2)

$$X(A,B,Z,\varepsilon) = \varepsilon\frac{\partial}{\partial A} + 0\frac{\partial}{\partial B} - Z\frac{\partial}{\partial Z} + 0\frac{\partial}{\partial \varepsilon}. \tag{69}$$

In these coordinates the slow divergence integral reads as

$$I = A^+ - A^-, \tag{70}$$

where A^+ and A^- are the corresponding parameterizations of Σ^+ and Σ^- (respectively) in the coordinates (A, B, Z, ε). □

3.1. Blow-up and charts

Let us briefly recall the blow up technique, for more details see e.g. [6,7,15]. The vector field X (50) is quasihomogeneous [1,12]. Therefore, it is convenient to use the *quasihomogeneous blow up*. This technique consists on performing a coordinate transformation defined by

$$a = r^3\bar{a},\ b = r^2\bar{b},\ z = r\bar{z},\ \varepsilon = r^5\bar{\varepsilon}, \tag{71}$$

which is called the blow up map, and where $\bar{a}^2 + \bar{b}^2 + \bar{z}^2 + \bar{\varepsilon}^2 = 1$ and $r \in [0, +\infty)$. That is $(\bar{a}, \bar{b}, \bar{z}, \bar{\varepsilon}, r) \in S^3 \times \mathbb{R}^+$. Since $\varepsilon \geq 0$, we can restrict the coordinates to $\bar{\varepsilon} \geq 0$. Note that $S^3 \times \{0\}$ is mapped, via the blow up map (71), to the origin of $\mathbb{R}^4$. The powers or weights of the blow up map (71) are obtained from the type of quasihomogeneity of X.

Let us denote by $\Phi(\bar{a}, \bar{b}, \bar{z}, \bar{\varepsilon})$ the blow up map (71). This map induces a smooth vector field $\tilde{X}$ on $S^3 \times \mathbb{R}^+$ defined by $\Phi_*\tilde{X} = X$. It is often the case in which the vector field $\tilde{X}$ is degenerate along $S^3 \times \{0\}$. Then one defines another vector field $\bar{X}$ by $\bar{X} = \frac{1}{r^m}\tilde{X}$ for a well chosen positive integer m so that $\bar{X}$ is non-degenerate along $S^3 \times \{0\}$. Since $r \in \mathbb{R}^+$, the phase portraits of $\tilde{X}$ and

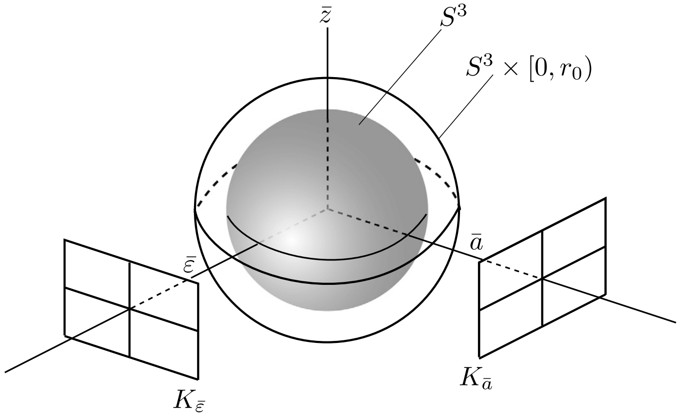

Fig. 6. The blow up space and the charts. Each chart K_ℓ parametrizes a region of the ball $S^3 \times [0, r_0)$. A local analysis in the charts provides a full picture of the dynamics of the vector field $\bar{X}$.

$\bar{X}$ are equivalent outside $S^3 \times \{0\}$, and therefore it is equally useful to study $\bar{X}$ instead of $\tilde{X}$. One obtains a complete description of the local flow of X near the cusp point by studying the flow of $\bar{X}$ for $(\bar{a}, \bar{b}, \bar{z}, \bar{\varepsilon}, r) \in S^3 \times [0, r_0)$ with $r_0 > 0$ sufficiently small.

For problems of dimension greater than 2, performing computations in spherical coordinates becomes tedious. Therefore, it is more convenient to consider charts which parametrize hemispheres of the ball $S^3 \times [0, r_0)$. In the present context, the useful charts are

$$K_{\mathrm{en}} = \{\bar{a} = -1\}, \ K_{\mathrm{ex}} = \{\bar{a} = 1\}, \ K_{\bar{\varepsilon}} = \{\bar{\varepsilon} = 1\}, \ K_{\pm} = \left\{\bar{b} = \pm 1\right\} \tag{72}$$

and we always keep $r \in [0, r_0)$. The previous setting is also known as directional blow up. A qualitative picture of the charts is given in Fig. 6.

Briefly speaking, our analysis goes as follows: first, we perform a local analysis on each chart given in (72). Next, we compose ("glue") the local results to provide a full description of the flow of X (50) in a small neighborhood of the cusp point. In this way, we construct an invariant manifold from Σ^{en} to Σ^{ex}. Later we "push away" this invariant manifold all the way up to the sections Σ^- and Σ^+ along regular parts of the slow manifold S.

To avoid confusion of the coordinates we adopt the following notation. Any object O defined in the chart K_{en} is denoted by O_1. Similarly any object defined in the chart K_{ex} is denoted by O_3. Finally, an object O defined in either of the charts $K_{\bar{\varepsilon}}$ or $K_\pm$ is denoted by O_2.

3.2. Analysis in the chart K_{en}

Taking into account our notation convention, the blow up map in this chart is given by

$$a = -r_1^3, \ b = r_1^2 b_1, \ z = r_1^3 z_1, \ \varepsilon = r_1^5 \varepsilon_1. \tag{73}$$

The corresponding vector field in this chart (after multiplication by 3) has the form

$$X_{\mathrm{en}}:\begin{cases} r_1' = -\varepsilon_1 r_1\left(1+\tilde{f}_1\right) \\ b_1' = 2\varepsilon_1 b_1\left(1+\tilde{f}_1\right) + r_1^6\varepsilon_1^2\tilde{f}_2 \\ z_1' = -3\left(z_1^3 + b_1 z_1 - 1 - \frac{1}{3}\varepsilon_1 z_1\right) + r_1^2\varepsilon_1\tilde{f}_3 \\ \varepsilon_1' = 5\varepsilon_1^2\left(1+\tilde{f}_1\right) \end{cases} \tag{74}$$

where the functions $\tilde{f}_i = \tilde{f}_i(r_1, b_1, z_1, \varepsilon_1)$ are flat along $r_1 = 0$, recall that $S^3 \times \{r = 0\} \mapsto 0 \in \mathbb{R}^4$ via the blow up map. We study a transition $\Pi_1 : \Delta_1^{\mathrm{en}} \to \Delta_1^{\mathrm{ex}}$ where

$$\begin{aligned} \Delta_1^{\mathrm{en}} &= \left\{(r_1, b_1, z_1, \varepsilon_1) \in \mathbb{R}^4 \,|\, r_1 = r_0, \varepsilon_1 < \delta,\ z_1 > 0\right\} \\ \Delta_1^{\mathrm{ex}} &= \left\{(r_1, b_1, z_1, \varepsilon_1) \in \mathbb{R}^4 \,|\, \varepsilon_1 = \delta, r_1 < r_0\right\}, \end{aligned} \tag{75}$$

where r_0 and δ are sufficiently small positive constants.

Remark 3.1. The section Δ_1^{en} corresponds to Σ^{en} in the blow up space, that is $\Sigma^{\mathrm{en}} = \Phi(\Delta_1^{\mathrm{en}})$, where Φ is the blow up map (73). This implies that trajectories of X crossing Σ^{en} correspond to trajectories of X_{en} crossing Δ_1^{en}.

Before going any further, let us provide a qualitative description of X_{en} as in [4]. This process can be repeated, following similar arguments, in all the local charts; however, for brevity we only detail it for the current one.

Qualitative description of the flow of X_{en}. The subspaces $\{r_1 = 0\}$, $\{\varepsilon_1 = 0\}$ and $\{r_1 = 0\} \cap \{\varepsilon_1 = 0\}$ are invariant. Therefore, it is useful to study the flow of X_{en} restricted to the aforementioned subspaces.

Restriction to $\{r_1 = 0\} \cap \{\varepsilon_1 = 0\}$. In this space X_{en} is reduced to

$$\begin{aligned} b_1' &= 0 \\ z_1' &= -3\left(z_1^3 + b_1 z_1 - 1\right). \end{aligned} \tag{76}$$

The set

$$\gamma_1 = \left\{(b_1, z_1) \,|\, z_1^3 + b_1 z_1 - 1 = 0\right\} \tag{77}$$

is a curve of equilibrium points. The phase portrait of (76) is shown in Fig. 7.

Remark 3.2. All the trajectories of (76) restricted to an initial condition $z_0 > 0$ are attracted to the curve $\gamma_1|_{z_1>0}$. Furthermore, due to our definition of Δ_1^{en}, we are interested *only* in trajectories satisfying this initial condition. Thus, from now on, we restrict our analysis to the subspace $\{z_1 > 0\}$.

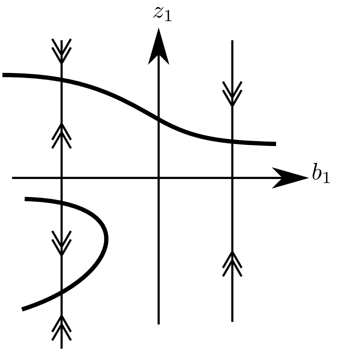

Fig. 7. The phase portrait of X_{en} restricted to the invariant space $\{r_1 = 0\} \cap \{\varepsilon_1 = 0\}$. The shown curve is γ_1 and it comprises a set of equilibrium points. Note that locally, all trajectories with initial condition $z_1(0) > 0$ are attracted to $\gamma_1|_{\{z_1>0\}}$.

Restriction to $\{\varepsilon_1 = 0\}$. In this space X_{en} is reduced to

$$\begin{aligned} r_1' &= 0 \\ b_1' &= 0 \\ z_1' &= -3\left(z_1^3 + b_1 z_1 - 1\right). \end{aligned} \tag{78}$$

The set $\Gamma_1 = \left\{(r_1, b_1, z_1) \,|\, z_1^3 + b_1 z_1 - 1 = 0\right\}$ is a surface of equilibrium points given by $\Gamma_1 = (r_1, \gamma_1)$. Since $r_1' = 0$, the phase space of (78) is foliated by two dimensional leaves in which the flow looks like Fig. 7.

Restriction to $\{r_1 = 0\}$. In this space X_{en} is reduced to

$$\begin{aligned} b_1' &= 2\varepsilon_1 b_1 \\ z_1' &= -3\left(z_1^3 + b_1 z_1 - 1 - \frac{1}{3}\varepsilon_1 z_1\right) \\ \varepsilon_1' &= 5\varepsilon_1^2. \end{aligned} \tag{79}$$

Once again, the set $\gamma_1 = \left\{(b_1, z_1, \varepsilon_1) \,|\, \varepsilon_1 = 0,\ z > 0,\ z_1^3 + b_1 z_1 - 1 = 0\right\}$ is a curve of equilibrium points. The Jacobian of (79) evaluated along γ_1 shows that, for small enough ε_1, there exists an invariant center manifold that passes through γ_1. Furthermore, the non-zero eigenvalue corresponding to the z-direction is negative along γ_1. The phase portrait of (79) is shown in Fig. 8.

Observe that the b_1 and the ε_1 directions are expanding. It is important to know the relation between such two expanding variables. We have

$$\frac{db_1}{d\varepsilon_1} = \frac{2}{5}\frac{b_1}{\varepsilon_1}, \tag{80}$$

which has the solution

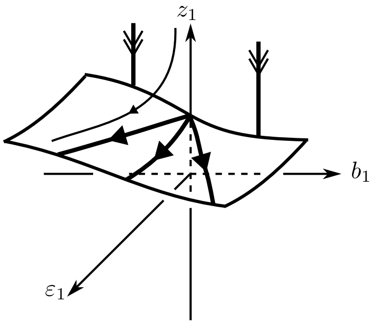

Fig. 8. Phase portrait of (79) restricted to $z_1 > 0$. The shown surface is an invariant center manifold, which is attracting in the z_1-direction.

$$b_1 = b_1^* \left(\frac{\varepsilon_1}{\varepsilon_1^*}\right)^{2/5}, \tag{81}$$

where $b_1^* \le b_1$ and $\varepsilon_1^* \le \varepsilon_1$ are the initial conditions, that is $(b_1^*, \varepsilon_1^*) = (b_1, \varepsilon_1)|_{\Delta_1^{\text{en}}}$. It is important to look at the ratio of initial conditions $\frac{b_1^*}{(\varepsilon_1^*)^{2/5}}$. This ratio tells us that b_1 is bounded as $\varepsilon_1 \to 0$ (and therefore as $\varepsilon_1^* \to 0$) if and only if $b_1^* \in O\left(\left(\varepsilon_1^*\right)^{2/5}\right)$. In other words, if the initial condition b_1^* is not of order $O((\varepsilon_1^*)^{2/5})$ then the value of b_1 at Δ_1^{ex} blows up as $\varepsilon_1^* \to 0$. This leads us to partition the section Δ_1^{en} into three open regions as follows.

$$\begin{aligned}
\Delta_1^{\text{en,inner}} &= \Delta_1^{\text{en}}|_{|b_1|<M\varepsilon_1^{2/5}} \\
\Delta_1^{\text{en},b_1} &= \Delta_1^{\text{en}}|_{b_1>K\varepsilon_1^{2/5}} \\
\Delta_1^{\text{en},-b_1} &= \Delta_1^{\text{en}}|_{-b_1>K\varepsilon_1^{2/5}},
\end{aligned} \tag{82}$$

where $0 < K < M < \infty$. Observe that the open sets $\Delta_1^{\text{en,inner}}$, Δ_1^{en,b_1} and $\Delta_1^{\text{en},-b_1}$ form an open cover of Δ_1^{en}. Accordingly, these sets induce an open cover of the entry section Σ^{en} via the blow up map (73). See Fig. 9 for a representation of the aforementioned partition.

Based on the partition of the entry section Δ_1^{en}, we define three transitions as follows

$$\begin{aligned}
\Pi_1^{\text{inner}} &: \Delta_1^{\text{en,inner}} \to \Delta_1^{\text{ex}} \\
\Pi_1^{+b_1} &: \Delta_1^{\text{en},+b_1} \to \Delta_1^{\text{ex},+b_1} \\
\Pi_1^{-b_1} &: \Delta_1^{\text{en},-b_1} \to \Delta_1^{\text{ex},-b_1},
\end{aligned} \tag{83}$$

where

$$\begin{aligned}
\Delta_1^{\text{ex}} &= \left\{(r_1, b_1, z_1, \varepsilon_1) \in \mathbb{R}^4 \,|\, \varepsilon_1 = \delta, r_1 < r_0\right\}, \\
\Delta_1^{\text{ex},\pm b_1} &= \left\{(r_1, b_1, z_1, \varepsilon_1) \in \mathbb{R}^4 \,|\, b_1 = \pm\eta, r_1 < r_0\right\}.
\end{aligned} \tag{84}$$

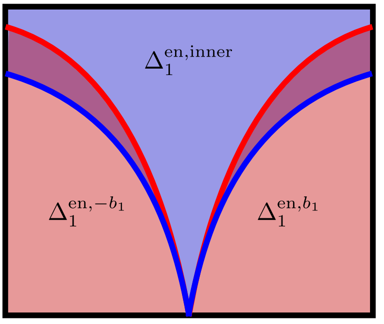

Fig. 9. Partition of Δ_1^{en}. Trajectories crossing through $\Delta_1^{\mathrm{en},\varepsilon_1}$ corresponding to the inner wedge area, have a continuation on the chart $K_{\bar{\varepsilon}}$. On the other hand, outside $\Delta_1^{\mathrm{en},\varepsilon_1}$ we must consider the lateral regions $\Delta_1^{\mathrm{en},b_1}$ and $\Delta_1^{\mathrm{en},-b_1}$.

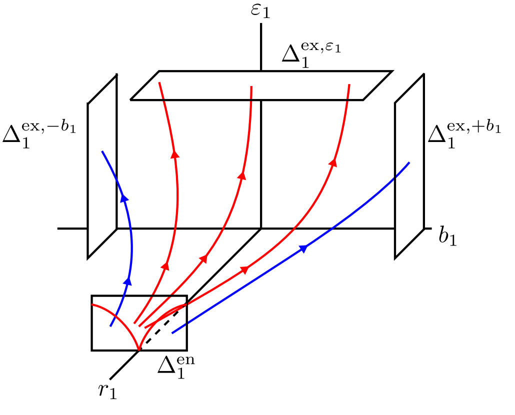

Fig. 10. Phase portrait of the trajectories of X_{en} depending on their initial condition. If the trajectories satisfy the estimate $y \in O(\varepsilon^{2/5})$, then they arrive to $\Delta_1^{\mathrm{ex},\varepsilon_1}$ in finite time. If the estimate $y \in O(\varepsilon^{2/5})$ is not satisfied, then we must choose one of the outgoing sections $\Delta_1^{\mathrm{ex},\pm b}$ in order to have a well defined transition map.

To finish with the qualitative description, note that there exists a (non-unique) 3-dimensional center manifold $\mathcal{W}_1^C$, which is shown to exist by evaluating the Jacobian of X_{en} all along the surface

$$\Gamma_1 = \left\{ (r_1, b_1, z_1, \varepsilon_1) \,|\, \varepsilon_1 = 0,\ z_1 > 0 z_1^3 + b_1 z_1 - 1 = 0 \right\}. \tag{85}$$

Moreover, by the analysis provided above, the center manifold $\mathcal{W}_1^C|_{z_1>0}$ is attracting for ε_1 small enough. Note that

$$\mathcal{W}_1^C|_{\varepsilon_1=0} = \Gamma_1.$$

This means that $\mathcal{W}_1^C$ can be interpreted as a perturbation of the slow manifold S, written in the coordinates of the current chart. See Fig. 10 for a representation of the previous exposition.

Let us recall that the vector field X_{en} is of the form

$$X_{\mathrm{en}}:\begin{cases} r_1' = -\varepsilon_1 r_1\left(1+\tilde{f}_1\right)\\ b_1' = 2\varepsilon_1 b_1\left(1+\tilde{f}_1\right)+r_1^6\varepsilon_1^2\tilde{f}_2\\ z_1' = -3\left(z_1^3+b_1z_1-1-\frac{1}{3}\varepsilon_1 z_1\right)+r_1^2\varepsilon_1\tilde{f}_3\\ \varepsilon_1' = 5\varepsilon_1^2\left(1+\tilde{f}_1\right)\end{cases} \tag{86}$$

We now proceed to describe the transitions Π_1 given by (83). For this, first we write (86) in a suitable normal form. Next, based on this normal form, we compute the corresponding transition.

First of all, let us move the origin to the point $(r_1, b, z_1, \varepsilon_1) = (0, 0, 1, 0)$. This is done by defining a new variable ζ_1 by $\zeta_1 = z_1 - 1$. With this variable we have a new local vector field Y_{en} which is defined by

$$Y_{\mathrm{en}}:\begin{cases} r_1' = -\varepsilon_1 r_1\left(1+\tilde{f}_1\right)\\ b_1' = 2\varepsilon_1 b_1\left(1+\tilde{f}_1\right)+r_1^6\varepsilon_1^2\tilde{f}_2\\ \varepsilon_1' = 5\varepsilon_1^2\left(1+\tilde{f}_1\right)\\ \zeta_1' = -3G(b_1,\varepsilon_1,\zeta_1)+\varepsilon_1\tilde{h},\end{cases} \tag{87}$$

where $G(0, 0, 0) = 0$ and $\frac{\partial G}{\partial \zeta_1}(0, 0, 0) = 3$. Now, we want to write Y_{en} in a suitable normal form. From Proposition C.1, we know that Y_{en} is $\mathcal{C}^\ell$ equivalent to

$$X_{\mathrm{en}}^N:\begin{cases} r_1' = -\varepsilon_1 r_1\\ B_1' = 2\varepsilon_1 B_1\\ \varepsilon_1' = 5\varepsilon_1^2\\ Z_1' = -9(1+H_1(r_1,B_1,\varepsilon_1))Z_1,\end{cases} \tag{88}$$

where H_1 is a C^ℓ-function vanishing at the origin. This normal form X_{en}^N is convenient since the chosen center manifold $\mathcal{W}_1^C$ is now simply given by $\mathcal{W}_1^C = \{Z_1 = 0\}$. Furthermore, from the format of X_{en}^N, it is evident the "hyperbolic nature" of the flow restricted to the center manifold: the restriction of X_{en}^N to the center manifold $\mathcal{W}_1^C$ has a simple structure, namely

$$X_{\mathrm{en}}^N|_{\mathcal{W}_1^C}:\begin{cases} r_1' = -\varepsilon_1 r_1\\ B_1' = 2\varepsilon_1 B_1\\ \varepsilon_1' = 5\varepsilon_1^2.\end{cases} \tag{89}$$

Note that for $\varepsilon_1 \neq 0$, the vector field $\frac{1}{\varepsilon_1}X_{\mathrm{en}}^N|_{\mathcal{W}_1^C}$ is hyperbolic.

The vector field X_{en}^N is of the form studied in Proposition C.4, therefore we have that the transition

$$\Pi_1^{\mathrm{inner}}:(B_1,\varepsilon_1,z_1)\mapsto(\tilde{r}_1,\tilde{B}_1,\tilde{Z}_1) \tag{90}$$

is of the form

$$\begin{aligned}\tilde{r}_1 &= r_0\left(\frac{\varepsilon_1}{\delta}\right)^{1/5}\\ \tilde{B}_1 &= B_1\left(\frac{\delta}{\varepsilon_1}\right)^{2/5}\\ \tilde{Z}_1 &= Z_1\exp\left(-\frac{9}{5\varepsilon_1}(1+\alpha_1\varepsilon_1\ln\varepsilon_1+\varepsilon_1 G_1)\right),\end{aligned} \tag{91}$$

where $\alpha_1 = \alpha_1(r_0|B_1|^{1/2}, r_0\varepsilon_1^{1/5})$ and $G_1 = G_1(r_0|B_1|^{1/2}, r_0\varepsilon_1^{1/5}, \mu)$ where $\mu = B_1\varepsilon_1^{-2/5}$. Recall that for this transition we have the condition $B_1 \in O(\varepsilon_1^{2/5})$ so μ is well defined.

On the other hand, the transition

$$\Pi_1^{\pm B_1} : (B_1, \varepsilon_1, Z_1) \mapsto (\tilde{r}_1, \tilde{\varepsilon}_1, \tilde{Z}_1) \tag{92}$$

is (see Proposition C.4) of the form

$$\begin{aligned}\tilde{r}_1 &= r_0\left(\frac{B_1}{\eta}\right)^{1/2}\\ \tilde{\varepsilon}_1 &= \varepsilon_1\left(\frac{\eta}{B_1}\right)^{5/2}\\ \tilde{Z}_1 &= Z_1\exp\left(-\frac{9}{5\varepsilon_1}(1+\beta_1\varepsilon_1\ln(|B_1|)+\varepsilon_1 H_1)\right),\end{aligned} \tag{93}$$

where $\beta_1 = \beta_1(r_0|B_1|^{1/2}, r_0\varepsilon_1^{1/5})$ and $H_1 = H_1(r_0|B_1|^{1/2}, r_0\varepsilon_1^{1/5}, \sigma)$, where $\sigma = \varepsilon_1|B_1|^{-5/2}$. Note that since $B_1 \notin O(\varepsilon_1^{2/5})$, σ is well defined. We observe that the transitions $\Pi_1^{\varepsilon_1}$ and $\Pi_1^{\pm B_1}$ are exponential type maps.

3.3. Analysis in the chart $K_{\bar{\varepsilon}}$

Taking into account our notation convention, the blow up map in this chart is given by

$$a = r_2^3 a_2,\; b = r_2^2 b_2,\; z = r_2^3 z_2,\; \varepsilon = r_2^5. \tag{94}$$

Then, the blown up vector field reads as

$$X_{\bar{\varepsilon}} : \begin{cases} r_2' = 0\\ a_2' = 1 + \tilde{g}_1\\ b_2' = r^6\tilde{g}_2\\ z_2' = -\left(z_2^3 + b_2 z_2 + a_2\right) + \tilde{g}_3,\end{cases} \tag{95}$$

where the function $\tilde{g}_i = \tilde{g}_i(r_2, a_2, b_2, z_2)$ are flat along $r_2 = 0$. Note that in this chart r_2 acts as a parameter and that the flow is regular. Furthermore, note that $X_{\bar{\varepsilon}}$ is not a slow–fast system, but a regular vector field. From the equation $a_2' = 1 + \tilde{g}_1$, we define the following "entry" and "exit" sections.

$$\Delta_2^{\mathrm{en},\bar{\varepsilon}} = \{(r_2, a_2, b_2, z_2) \,|\, a_2 = -A_0,\ z_2 \geq 0\},$$
$$\Delta_2^{\mathrm{ex},\bar{\varepsilon}} = \{(r_2, a_2, b_2, z_2) \,|\, a_2 = A_0,\ z_2 \leq 0\}. \tag{96}$$

Therefore, we define a transition $\Pi_2^{\bar{\varepsilon}}$ as

$$\Pi_2^{\bar{\varepsilon}} : \Delta_2^{\mathrm{en},\bar{\varepsilon}} \to \Delta_2^{\mathrm{en},\bar{\varepsilon}}$$
$$(r_2, b_2, z_2) \mapsto (\tilde{r}_2, \tilde{b}_2, \tilde{z}_2). \tag{97}$$

Since (95) is regular, by the flow-box theorem all trajectories starting at $\Delta_2^{\mathrm{en},\bar{\varepsilon}}$ arrive at $\Delta_2^{\mathrm{ex},\bar{\varepsilon}}$ in finite time. Moreover, the transition $\Pi_2^{\bar{\varepsilon}}$ is a diffeomorphism and then, from (95) we have that $\Pi_2^{\bar{\varepsilon}}$ reads as

$$\begin{aligned}\Pi_2\bar{\varepsilon}(r_2, b_2, z_2) &= (\tilde{r}_2, \tilde{b}_2, \tilde{z}_2)\\ &= (r_2, b_2 + h_{b_2}, \phi_1(r_2, b_2) + \phi_2(r_2, b_2)(1 + \phi_3(r_2, b_2, z_2))z_2),\end{aligned} \tag{98}$$

where the ϕ_i's are smooth functions. Observe that in this chart, the transition is not an exponential type map.

3.4. Analysis in the chart K_{ex}

Taking into account our notation convention, the blow up map in this chart is given by

$$a = r_3^3,\ b = r_3^2 b_3,\ z = r_3^3 z_3,\ \varepsilon = r_3^5 \varepsilon_3. \tag{99}$$

Then, the blown up vector field reads as

$$X_{\mathrm{ex}} : \begin{cases} r_3' = \varepsilon_3 r_3 \left(1 + \tilde{f}_1\right) \\ b_3' = -2\varepsilon_3 b_3 \left(1 + \tilde{f}_1\right) + r_3^6 \varepsilon_3^2 \tilde{f}_2 \\ z_3' = -3\left(z_3^3 + b_3 z_3 + 1 + \frac{1}{3}\varepsilon_3 z_3\right) + r_3^2 \varepsilon_3 \tilde{f}_3 \\ \varepsilon_3' = -5\varepsilon_3^2 \left(1 + \tilde{f}_1\right) \end{cases} \tag{100}$$

where the function $\tilde{f}_i = \tilde{f}_i(r_3, b_3, \varepsilon_3, z_3)$ are flat along $r_3 = 0$. Observe that the vector field X_{ex} resembles the vector field X_{en}. Therefore, we have a similar behavior of the trajectories, the main difference is that in the case of X_{ex}, there is one expanding (r_3) and three contracting (b_3, ε_3 and z_3) directions. The flow of X_{ex} is obtained following similar arguments as for the flow of X_{en}.

From the fact that X_{ex} has three contracting and one expanding direction, we define the entry sections

$$\begin{aligned}\Delta_3^{\mathrm{en},\bar{\varepsilon}} &= \{(r_3, b_3, \varepsilon_3, z_3) \,:\, \varepsilon_3 = \delta,\ z_3 < 0,\ r_3 < r_0\}\\ \Delta_3^{\mathrm{en},+b_3} &= \{(r_3, b_3, \varepsilon_3, z_3) \,:\, b_3 = \eta,\ z_3 < 0,\ r_3 < r_0\}\\ \Delta_3^{\mathrm{en},-b_3} &= \{(r_3, b_3, \varepsilon_3, z_3) \,:\, b_3 = -\eta,\ z_3 < 0,\ r_3 < r_0\},\end{aligned} \tag{101}$$

where all the constants are positive and sufficiently small, and the exit section

$$\Delta_3^{\mathrm{ex}} = \{(r_3, b_3, \varepsilon_3, z_3) \,:\, r_3 = r_0,\ z_3 < 0,\ \varepsilon_3 < \delta,\ |b_3| < \eta\}. \tag{102}$$

Then, accordingly, we define three transition maps as follows

$$\begin{aligned}\Pi_3^{\varepsilon_3} &: \Delta_3^{\mathrm{en},\bar{\varepsilon}} \to \Delta_3^{\mathrm{ex}}\\ &: (r_3, b_3, z_3) \mapsto (\tilde{b}_3, \tilde{\varepsilon}_3, \tilde{z}_3)\\ \Pi_3^{+b_3} &: \Delta_3^{\mathrm{en},+b_3} \to \Delta_3^{\mathrm{ex}}\\ &: (r_3, \varepsilon_3, z_3) \mapsto (\tilde{b}_3, \tilde{\varepsilon}_3, \tilde{z}_3)\\ \Pi_3^{-b_3} &: \Delta_3^{\mathrm{en},-b_3} \to \Delta_3^{\mathrm{ex}}\\ &: (r_3, \varepsilon_3, z_3) \mapsto (\tilde{b}_3, \tilde{\varepsilon}_3, \tilde{z}_3).\end{aligned} \tag{103}$$

Now we proceed to write X_{ex} in a normal form just as we did with X_{en} in (3.2). Following Proposition C.1 we have that X_{ex} is $\mathcal{C}^\ell$ equivalent to

$$X_{\mathrm{ex}}^N : \begin{cases} r_3' = \varepsilon_3 r_3\\ B_3' = -2\varepsilon_3 B_3\\ \varepsilon_3' = -5\varepsilon_3^2\\ Z_3' = -9(1 + H_3)Z_3, \end{cases} \tag{104}$$

where $H_3 = H_3(r_3, B_3, \varepsilon_3)$ is a C^ℓ function vanishing at the origin. Just as in the chart K_{en}, there exists a three dimensional center manifold $\mathcal{W}_3^C$ associated to X_{ex}^N and which has been chosen such that $\mathcal{W}_3^C = \{Z_3 = 0\}$. Since r_3 is the only expanding direction, we take as transition time $T_3 = \ln\left(\frac{r_0}{r_3}\right)$. This transition time is computed from the dynamics restricted to $\mathcal{W}_3^C$, that is, from the equation $r_3' = r_3$. In contrast to what happened in the chart K_{en}, the time T_3 is well defined for all the three transitions $\Pi_3^{\varepsilon_3}$, $\Pi_3^{+B_3}$ and $\Pi_3^{-B_3}$. Following Proposition C.4 we have

$$\begin{aligned}\tilde{B}_3 &= B_3 \left(\frac{r_3}{r_0}\right)^2\\ \tilde{\varepsilon}_3 &= \varepsilon_3 \left(\frac{r_3}{r_0}\right)^5\\ \tilde{Z}_3 &= Z_3 \exp\left(-\frac{9}{5\varepsilon_3}\left(\left(\frac{r_0}{r_3}\right)^5 - 1 + \alpha_3\varepsilon_3 \ln r_3 + \varepsilon_3 H_3\right)\right),\end{aligned} \tag{105}$$

where $\alpha_3 = \alpha_3(r_3|B_3|^{1/2}, r_3\varepsilon_3^{1/5})$ and $H_3 = H_3(r_3|B_3|^{1/2}, r_3\varepsilon_3^{1/5}, r_3)$. Therefore, by taking the definitions of the entry sections we have

$$\begin{aligned}
&\Pi_3^{\varepsilon_3}(r_3, B_3, Z_3)\\
&\quad = \left(B_3\left(\frac{r_3}{r_0}\right)^2, \delta\left(\frac{r_3}{r_0}\right)^5, Z_3\exp\left(-\frac{9}{5\delta}\left(\left(\frac{r_0}{r_3}\right)^5 - 1 + \alpha_3\delta\ln r_3 + \delta H_3\right)\right)\right)\\
&\Pi_3^{\pm b_3}(r_3, \varepsilon_3, Z_3)\\
&\quad = \left(\pm\eta\left(\frac{r_3}{r_0}\right)^2, \varepsilon_3\left(\frac{r_3}{r_0}\right)^5, Z_3\exp\left(-\frac{9}{5\varepsilon_3}\left(\left(\frac{r_0}{r_3}\right)^5 - 1 + \alpha_3\varepsilon_3\ln r_3 + \varepsilon_3 H_3\right)\right)\right).
\end{aligned} \tag{106}$$

Observe that these transitions are of exponential type.

3.5. *Analysis in the charts* $K_{\pm\bar{b}}$

In this section we study the local flow at the charts $K_{+\bar{b}}$ and $K_{-\bar{b}}$. In a qualitative sense, these charts come into play when the initial condition $b_0 = b|_{\Sigma^{\text{en}}}$ does not satisfy the estimate $b_0 \in O(\varepsilon^{2/5})$. This implies that the corresponding trajectory passes away from the cusp point. The chart $K_{+\bar{b}}$ "sees" trajectories with initial condition $b|_{\Sigma^{\text{en}}} > 0$ while $K_{-\bar{b}}$ "sees" trajectories with initial condition $b|_{\Sigma^{\text{en}}} < 0$.

Analysis in the chart $K_{+\bar{b}}$

In this chart the blow up maps reads

$$a = r_2^3 a_2,\ b = r_2^2,\ z = r_2 z_2,\ \varepsilon = r_2^5\varepsilon_2. \tag{107}$$

Then we have that the blow up vector field is given by

$$X_{+\bar{b}}: \begin{cases} r_2' = \varepsilon_2 \bar{f}_r\\ a_2' = \varepsilon_2(1+\bar{f}_{a_2}) + \varepsilon_2\bar{g}_{a_2}\\ \varepsilon_2' = -\varepsilon_2\bar{f}_{\varepsilon_2}\\ z_2' = -(z_2^3 + z_2 + a_2) + \varepsilon_2\bar{f}_{z_2} \end{cases} \tag{108}$$

where all the functions $\bar{f}_\ell$ are flat along $\{r_2 = 0\}$. Observe that the set

$$\Gamma_2 = \left\{(r_2, a_2, \varepsilon_2, z_2)\,|\,\varepsilon_2 = 0,\ z_2^3 + z_2 + a_2 = 0\right\} \tag{109}$$

is a NHIM of $X_{+\bar{b}}$. However, $X_{+\bar{b}}$ is not exactly a slow–fast system since $\varepsilon_2' \neq 0$, but the restriction of $X_{+\bar{b}}$ to $\{r_2 = 0\}$ is indeed a slow–fast system. This restriction reads as

$$X_{+\bar{b}}|_{\{r_2=0\}}: \begin{cases} a_2' = \varepsilon_2\\ \varepsilon_2' = 0\\ z_2' = -(z_2^3 + z_2 + a_2). \end{cases} \tag{110}$$

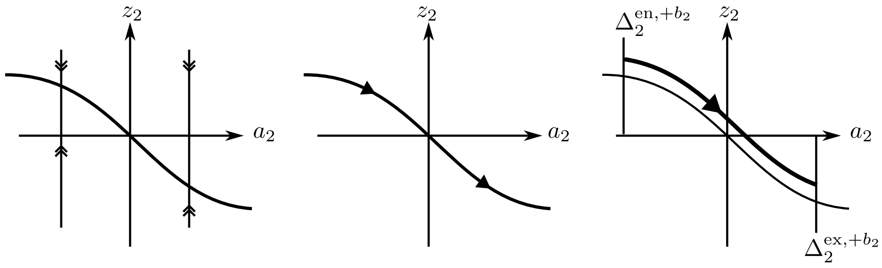

Fig. 11. Left: phase portrait of the corresponding layer equation of $X_{+\bar{b}}|_{\{r_2=0\}}$. Center: phase portrait of the corresponding CDE of $X_{+\bar{b}}|_{\{r_2=0\}}$. Right: Since the slow manifold is regular, by Fenichel theory we know that the manifold Γ_2 is perturbed to an invariant manifold Γ_{2,ε_2} which is at distance of order $O(\varepsilon_2)$ from Γ_2.

Remark 3.3. The subspace $\{r_2 = 0\}$ is invariant. Moreover, since $X_{+\bar{b}}$ is a flat perturbation of $X_{+\bar{b}}|_{\{r_2=0\}}$, it is equally useful to study the restriction $X_{+\bar{b}}|_{\{r_2=0\}}$. After all, by regular perturbation theory, their flows are equivalent.

The slow manifold of $X_{+\bar{b}}|_{\{r_2=0\}}$ is defined by $\Gamma_2|_{r_2=0}$ and is normally hyperbolic. Let us define the sections

$$\Delta_2^{\mathrm{en},+b_2} = \left\{ (r_2, a_2, \varepsilon_2, z_2) \in \mathbb{R}^4 \,|\, a_2 = -A_0 \right\}$$
$$\Delta_2^{\mathrm{ex},+b_2} = \left\{ (r_2, a_2, \varepsilon_2, z_2) \in \mathbb{R}^4 \,|\, a_2 = A_0 \right\}. \tag{111}$$

Accordingly, we study the transition

$$\Pi_2^{+b_2} : \Delta_2^{\mathrm{en},+b_2} \to \Delta_2^{\mathrm{ex},+b_2}$$
$$(r_2, \varepsilon_2, z_2) \mapsto (\tilde{r}_2, \tilde{\varepsilon}_2, \tilde{z}_2). \tag{112}$$

For a qualitative description of $X_{+\bar{b}}|_{\{r_2=0\}}$ and the objects defined above see Fig. 11.

We know from Section 2.1.2 that for sufficiently small ε_2, there exists a C^ℓ change of coordinates that transforms $X_{+\bar{b}}|_{\{r_2=0\}}$ into the vector field

$$Y^N : \begin{cases} a_2' = \varepsilon_2 \\ \varepsilon_2' = 0 \\ Z_2' = -Z_2, \end{cases} \tag{113}$$

From the definition of the entry and exit sections (111), the time of integration is $T = 2A_0$. To obtain the component Z_2 of the transition $\Pi_2^{+b_2}|_{\{r_2=0\}}$ we need to integrate

$$Z_2' = -\frac{1}{\varepsilon_2} Z_2, \tag{114}$$

and then $\tilde{Z}_2 = Z_2(T)$. Therefore we have that after choosing a center manifold $\mathcal{W}_2^C$, the transition $\Pi_2^{+b_2}$ reads as

$$\Pi_2^{+b_2}(0, \varepsilon_2, Z_2) = \left(0, \varepsilon_2, Z_2 \exp\left(-\frac{2A_0}{\varepsilon_2}\right)\right). \tag{115}$$

Note that $\Pi_2^{+b_2}$ is an exponential type map.

Analysis in the chart $K_{-\bar{b}}$

In this chart the blow up maps reads

$$a = r_2^3 a_2,\ b = -r_2^2,\ z = r_2 z_2,\ \varepsilon = r^5 \varepsilon_2. \tag{116}$$

Then we have that the blow up vector field is given by

$$X_{-\bar{b}} : \begin{cases} r_2' = -\varepsilon_2 \bar{f}_r \\ a_2' = \varepsilon_2(1 + \bar{f}_{a_2}) + \varepsilon_2 \bar{g}_{a_2} \\ \varepsilon_2' = \varepsilon_2 \bar{f}_{\varepsilon_2} \\ z_2' = -(z_2^3 - z_2 + a_2) + \varepsilon_2 \bar{f}_{z_2} \end{cases} \tag{117}$$

where all the functions $\bar{f}_\ell$ and $\bar{g}_{a_2}$ are flat along $\{r_2 = 0\}$. Observe that, as in the previous section, the subspace $\{r_2 = 0\}$ is invariant. The restriction of $X_{-\bar{b}}$ to this subspace reads as

$$X_{-\bar{b}}|_{\{r_2=0\}} : \begin{cases} a_2' = \varepsilon_2 \\ \varepsilon_2' = 0 \\ z_2' = -(z_2^3 - z_2 + a_2). \end{cases} \tag{118}$$

The flow of $X_{-\bar{b}}$ is a flat perturbation of the flow of $X_{-\bar{b}}|_{\{r_2=0\}}$. Therefore, let us continue our analysis restricted to the invariant space $\{r_2 = 0\}$.

The manifold Γ_2, which is defined by

$$\Gamma_2 = \left\{(r_2, a_2, \varepsilon_2, z_2) \,|\, r_2 = 0,\ \varepsilon_2 = 0,\ z_2^3 - z_2 + a_2 = 0\right\} \tag{119}$$

is normally hyperbolic except at the two points $p_\pm = \pm\left(\frac{2}{3\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. Let us define the sections

$$\begin{aligned} \Delta_2^{\mathrm{en},-b_2} &= \left\{(r_2, a_2, \varepsilon_2, z_2) \in \mathbb{R}^4 \,|\, a_2 = -A_0\right\} \\ \Delta_2^{\mathrm{ex},-b_2} &= \left\{(r_2, a_2, \varepsilon_2, z_2) \in \mathbb{R}^4 \,|\, a_2 = A_0\right\}, \end{aligned} \tag{120}$$

where $A_0 > 0$ is a sufficiently large constant. We are interested in the transition

$$\begin{aligned} \Pi_2^{-b_2} : \Delta_2^{\mathrm{en},-b_2} &\to \Delta_2^{\mathrm{ex},-b_2} \\ (r_2, \varepsilon_2, z_2) &\mapsto (\tilde{r}_2, \tilde{\varepsilon}_2, \tilde{z}_2). \end{aligned} \tag{121}$$

For a qualitative description of $X_{-\bar{b}}|_{\{r_2=0\}}$ and the objects defined above see Fig. 12.

Away from the fold points $p_\pm$, the manifold Γ_2 is regular and thus, Fenichel's theory applies. However, we need to take care of the transition near the fold point p_+. The local transition of a

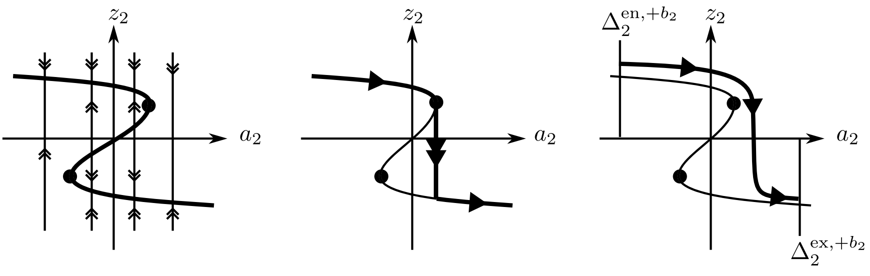

Fig. 12. Left: phase portrait of the corresponding layer equation of $X_{-\bar{b}}|_{\{r_2=0\}}$. Center: phase portrait of the corresponding CDE of $X_{-\bar{b}}|_{\{r_2=0\}}$. Right: The expected perturbed invariant manifold obtained from the flow of the corresponding CDE and layer equation.

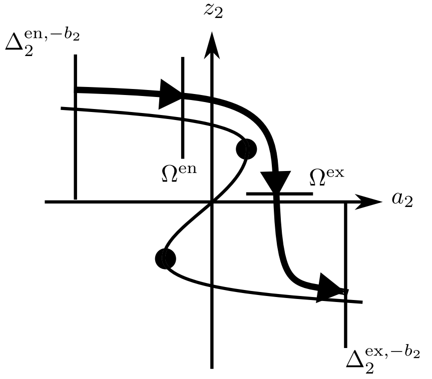

Fig. 13. The three different transitions in which $\Pi_2^{-b_2}$ is decomposed. The central transitions is locally an A_2 problem. The other two transitions at the sides are regular.

slow–fast system near a fold point is investigated in e.g. [16]. However, in our current problem this transition is not essential. By this we mean that the passage through the fold point is seen as a flat perturbation of the trajectory along the stable branch of Γ_2. In a qualitative sense, this is due to the fact that the transition $\Pi_2^{-b_2}$ goes along a large NHIM, which fails to be normally hyperbolic only at one point.

Proposition 3.3. *We can choose appropriate coordinates (Z_2, ε_2) in $\Delta_2^{\mathrm{en},-b_2}$ such that the transition $\Pi_2^{-b_2} : \Delta_2^{\mathrm{en},-b_2} \to \Delta_2^{\mathrm{ex},-b_2}$, restricted to $r_2 = 0$, is an exponential type map of the form*

$$\Pi_2^{-b_2}(0, \varepsilon_2, Z_2) = \left(0, \varepsilon_2, \phi_2(\varepsilon_2) + Z_2 \exp\left(-\frac{1}{\varepsilon_2}(A_0 + \varepsilon_2 \psi_2(Z_2, \varepsilon_2))\right)\right), \tag{122}$$

where ϕ_2 are flat at $\varepsilon_2 = 0$, ψ_2 is $\mathcal{C}^\ell$-admissible, and where A_0 is given by the slow divergence integral of $X_{-\bar{b}}|_{\{r_2=0\}}$.

Proof. To prove that A_0 is given by the slow divergence integral we proceed along the same reasoning as in Proposition 3.2, so we do not repeat it here. In Fig. 13 we show the three transitions that must considered.

The three transitions are defined as

$$\begin{aligned}\Pi_2^{reg_1} &: \Delta_2^{\mathrm{en},-b_2} \to \Omega^{\mathrm{en}}\\ \Pi_2^{fold} &: \Omega^{\mathrm{en}} \to \Omega^{\mathrm{ex}}\\ \Pi_2^{reg_2} &: \Omega^{\mathrm{ex}} \to \Delta_2^{\mathrm{ex},-b_2},\end{aligned} \tag{123}$$

where we define Ω^{en} and Ω^{en} as

$$\begin{aligned}\Omega^{\mathrm{en}} &= \left\{(a_2, \varepsilon_2, Z_2) \in \mathbb{R}^3 \mid a_2 = -a_{2,\mathrm{en}}\right\}\\ \Omega^{\mathrm{ex}} &= \left\{(a_2, \varepsilon_2, Z_2) \in \mathbb{R}^3 \mid Z_2 = -Z_{2,\mathrm{ex}}\right\},\end{aligned} \tag{124}$$

where $a_{2,\mathrm{en}}$ and $Z_{2,\mathrm{ex}}$ are sufficiently small positive constants. The total transition $\Pi_2^{+b_2}$ is given by $\Pi_2^{b_2} = \Pi_2^{reg_2} \circ \Pi_2^{fold} \circ \Pi_2^{reg_1}$. Recall from Appendix A that if we want to write the transition $\Pi_2^{+b_2}$ as an exponential type map, we require that $\Pi_2^{reg_1}$ is expressed as an exponential type map with no shift. The transition Π_2^{fold} is studied in e.g. [10,16]. In [10] is proved that there are local coordinates $(\bar{Z}_2, \varepsilon)$ in Ω^{en}, and $(\tilde{a}_2, \tilde{\varepsilon})$ in Ω^{ex}, such that the transition Π_2^{fold} is given by

$$\begin{aligned}\Pi_2^{fold}(\bar{Z}_2, \varepsilon_2) &= (\tilde{a}_2, \tilde{\varepsilon}_2)\\ &= \left(\varepsilon_2^{2/3} + O(\varepsilon_2), \varepsilon_2\right).\end{aligned} \tag{125}$$

Assume now that we have characterized an invariant manifold $\mathcal{M}_{\varepsilon_2}^{fold}$ from Ω^{en} to Ω^{ex} via the map Π_2^{fold}. Now we want to "extend" $\mathcal{M}_{\varepsilon_2}^{fold}$ all the way up to the sections $\Delta_2^{\mathrm{en},-b_2}$ and $\Delta_2^{\mathrm{ex},-b_2}$ via transitions along normally hyperbolic regions of Γ_2. For this, it is more convenient to regard $\mathcal{M}_{\varepsilon_2}^{fold}$ as a graph $\zeta_2 = \phi_{\varepsilon_2}(A_2)$ where (ζ_2, A_2) are local coordinates around the fold point p_+ and where ϕ_{ε_2} is a diffeomorphism for $\varepsilon_2 > 0$. In this way we can equivalently express the map Π_2^{fold} as

$$\begin{aligned}\Pi_2^{fold}(\zeta, \varepsilon_2) &= (\tilde{\zeta}_2, \tilde{\varepsilon}_2)\\ &= (\psi_{\varepsilon_2}(\zeta), \varepsilon_2)\end{aligned} \tag{126}$$

where ψ_{ε_2} is a diffeomorphism for $\varepsilon_2 > 0$ and only a homeomorphism for $\varepsilon_2 = 0$. Next, following Section 2.1.2 we can find coordinates (Z_2, ε_2) in $\Delta_2^{\mathrm{en},-b_2}$, and coordinates $(\tilde{Z}_2, \varepsilon_2)$ in $\Delta_2^{\mathrm{ex},-b_2}$ in such a way that the transitions $\Pi_2^{reg_1}$ and $\Pi_2^{reg_2}$ are given as

$$\begin{aligned}\Pi_2^{reg_1}(Z_2, \varepsilon_2) &= \left(Z_2 \exp\left(-\frac{1}{\varepsilon_2}(A_0 - a_{2,\mathrm{en}})\right)\right) = (\bar{Z}_2, \varepsilon_2)\\ \Pi_2^{reg_2}(-Z_{2,\mathrm{ex}}, \varepsilon_2) &= \left(-Z_{2,\mathrm{ex}} \exp\left(-\frac{1}{\varepsilon_2}(A_0 - \tilde{a}_2)\right)\right) = (\tilde{Z}_2, \varepsilon_2).\end{aligned} \tag{127}$$

Remark 3.4. Recall that along normally hyperbolic slow manifolds, it is possible to make a normal form transformation in such a way that this transformation respects certain constraint or structure of the vector field, [2,3]. In this particular case, we respect the choice of the invariant manifold $\mathcal{M}_{\varepsilon_2}^{fold}$.

Next, we can compute the composition $\Pi_2^{-b_2} = \Pi_2^{reg_2} \circ \Pi_2^{fold} \circ \Pi_2^{reg_1}$ by following Appendix A and it thus follows that

$$\Pi_2^{-b_2}(0, Z_2, \varepsilon_2) = \left(0, \bar{\psi}_{\varepsilon_2} + Z_2 \exp\left(-\frac{1}{\varepsilon_2}(A_1 + A_3 + \varepsilon_2\psi_2)\right), \varepsilon_2\right), \tag{128}$$

where $\bar{\psi}_{\varepsilon_2} = \psi_{\varepsilon_2}(0)\exp\left(-\frac{A_3}{\varepsilon_2}\right)$ and where $\psi_2 = \psi_2(Z, \varepsilon_2)$ is a $\mathcal{C}^\ell$-admissible function. Note that $\bar{\psi}_{\varepsilon_2}$ is flat at $\varepsilon_2 = 0$. □

3.6. *Proof of Theorem 3.1*

Let us first recall that, within the blow up space, we have three types of transitions according to the initial condition $b_1|_{\Delta_1^{en}}$, namely

- If $b_1|_{\Delta_1^{en}} \in O(\varepsilon_1^{2/5})$ then we construct a transition passing through the charts $K_{en} \to K_{\bar{\varepsilon}} \to K_{ex}$.
- If $b_1|_{\Delta_1^{en}} \notin O(\varepsilon_1^{2/5})$ and $b_1|_{\Delta_1^{en}} > 0$ then we construct a transition passing through the charts $K_{en} \to K_{+\bar{b}} \to K_{ex}$.
- If $b_1|_{\Delta_1^{en}} \notin O(\varepsilon_1^{2/5})$ and $b_1|_{\Delta_1^{en}} < 0$ then we construct a transition passing through the charts $K_{en} \to K_{-\bar{b}} \to K_{ex}$.

In Fig. 14 we give a qualitative diagram of the local transitions obtained and their relationship.

Let us only detail the transition through the inner layer Δ^{inner} corresponding to $b_1|_{\Delta_1^{en}} \in O(\varepsilon_1^{2/5})$, the other cases follow the same lines.

The transition $\Pi^{inner} : \Delta_1^{inner} \to \Delta_2^{ex}$ is given as

$$\Pi^{inner} = \Pi_3^{\varepsilon_3} \circ M_{\bar{\varepsilon}}^{ex} \circ \Pi_2^{\varepsilon_2} \circ M_{en}^{\bar{\varepsilon}} \circ \Pi_1^{inner} \tag{129}$$

where the matching maps are obtained from the blow up map. For example, to obtain the matching map from the chart K_{en} to the chart $K_{\bar{\varepsilon}}$ we relate the two directional blow up maps

$$a = -r_1^3,\ b = r_1^2 b_1,\ z = r_1 z_1,\ \varepsilon = r_1^5 \varepsilon_1 \tag{130}$$

and

$$a = r_2^3 a_2,\ b = r_2^2 b_2,\ z = r_2 z_2,\ \varepsilon = r_2^5. \tag{131}$$

Let us work out only with the z-component of the transitions as it is the only relevant one. Recall from Section 3.2 that Π_1^{inner} is an exponential type map with no shift. Next, the composition $\Pi^{central} = M_{\bar{\varepsilon}}^{ex} \circ \Pi_2^{\varepsilon_2} \circ M_{en}^{\bar{\varepsilon}}$ yields a diffeomorphism as $\Pi_2^{\varepsilon_2}$ is a diffeomorphism, and the

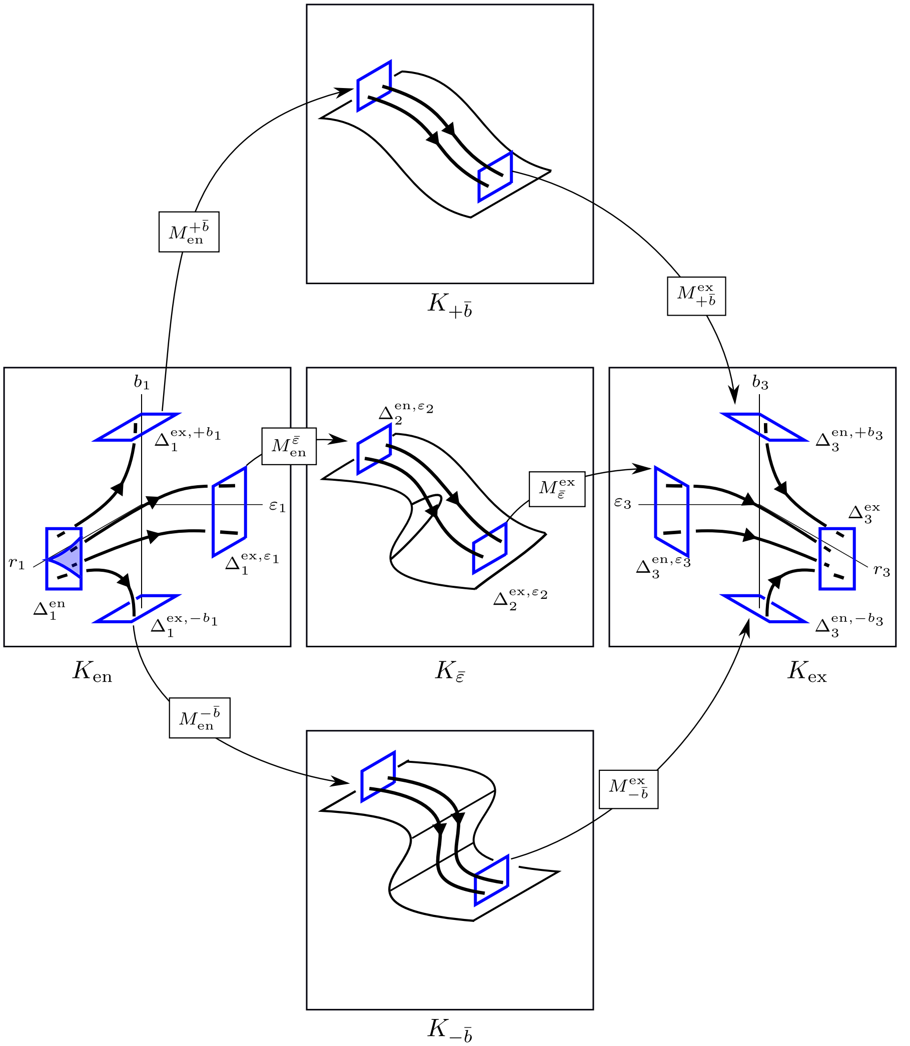

Fig. 14. All the transitions obtained in the charts. We have to compose all such transitions through the matching maps M_i^j. A matching map M_i^j relates the coordinates between the charts K_i and K_j.

matching maps are also diffeomorphisms on their domain of definition. Next, the last transition $\Pi_3^{\varepsilon_3}$ is an exponential type map with no shift, see Section 3.4. Therefore, following Appendix A we have that $\Pi_3^{\varepsilon_3} \circ \Pi^{central} \circ \Pi_1^{\text{inner}}$ is an exponential type map of the form

$$\Pi_{Z_1}^{\text{inner}} = \bar{\phi}(B_1, \varepsilon_1) + Z_1 \exp\left(-\frac{1}{\varepsilon_1}\left(\bar{\mathcal{A}}(B_1, \varepsilon_1) + \varepsilon_1 \bar{\Psi}(B_1, \varepsilon_1, Z_1)\right)\right), \tag{132}$$

where $\bar{\mathcal{A}} > 0$ and ϕ and Ψ are $\mathcal{C}^\ell$-admissible functions. The differentiability of ϕ and Ψ with respect to monomials is evident from the results of Section 3.2. By blowing down we obtain that the transition $\Pi^{\text{inner}} : \Sigma^{\text{en}} \to \Sigma^{\text{ex}}$ (in a small neighborhood of the cusp point and within the inner layer as domain) reads as

$$\Pi_Z^{\text{inner}} = \phi(B,\varepsilon) + Z \exp\left(-\frac{1}{\varepsilon}\left(\mathcal{A}(B,\varepsilon) + \varepsilon\bar{\Psi}(B,\varepsilon,Z)\right)\right). \tag{133}$$

To obtain the transition $\Pi : \Sigma^- \to \Sigma^+$ we now need to compose Π_Z^{inner} with exponential type maps on the left and on the right corresponding to

$$\begin{aligned}\Pi^- &: \Sigma^- \to \Sigma^{\text{en}}\\ \Pi^+ &: \Sigma^{\text{ex}} \to \Sigma^+.\end{aligned} \tag{134}$$

However, we must proceed with care. In order to express the transition Π as an exponential type map, we need to choose appropriate coordinates on Σ^- and on Σ^+ that respect the already chosen coordinates in Σ^{en} and in Σ^{ex}. Fortunately, this is possible with the extensions of Bonckaert [2,3] to the normalization results of Takens [23].

For sake of clarity, let $(B_{\text{en}}, Z_{\text{en}})$ be coordinates in Σ^{en} and $(B_{\text{ex}}, Z_{\text{ex}})$ be coordinates in Σ^{ex}. We have shown that these coordinates can be chosen in such a way that the "vertical" component of the transition map $\Pi^{\text{inner}} : \Sigma^{\text{en}} \to \Sigma^{\text{ex}}$ reads as

$$\begin{aligned}\Pi_{Z_{\text{en}}}(B_{\text{en}}, Z_{\text{en}}, \varepsilon) &= Z_{\text{ex}}\\ &= \phi(B_{\text{en}},\varepsilon) + Z_{\text{en}} \exp\left(-\frac{1}{\varepsilon}\left(\mathcal{A}(B_{\text{en}},\varepsilon) + \varepsilon\bar{\Psi}(B_{\text{en}},\varepsilon,Z_{\text{en}})\right)\right).\end{aligned} \tag{135}$$

In this case the invariant manifold, say $\mathcal{M}_\varepsilon$, is given by $Z_{\text{en}} = 0$. Using [2,3] we can find suitable coordinates (B_-, Z_-) in Σ^- in such a way that

$$\Pi_{Z_-}^-(B_-, Z_-, \varepsilon) = Z_- \exp\left(-\frac{1}{\varepsilon}(A_0)\right) = Z_{\text{en}}. \tag{136}$$

In other words, there is a change of coordinates respecting the invariant manifold $\mathcal{M}_\varepsilon$ under which the transition Π^- is an exponential type map with no shift and linear. Similar arguments hold for the choice of coordinates in Σ^+. Finally, following Appendix A, the composition $\Pi_{Z_+}^+ \circ \Pi_{Z_{\text{en}}} \circ \Pi_{Z_-}^-$ leads to the result.

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Appendix A. Exponential type functions

In this section, we discuss a particular type of function which will be found and used frequently throughout the main text. First, however, let us give a preliminary but useful definition. For this we need to extend the common concept of monomial. In our context, a monomial, e.g. in two variables, $\omega(u, v)$ is any expression of the form $u^\alpha v^\beta$ or of the form $u^\alpha(\ln v)^\beta$, with $\alpha, \beta \in \mathbb{R}$. In general, if we let $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$, we allow a monomial ω to be any expression of the type $u^p(\ln v)^q$, where $u^p = u_1^{p_1} \cdots u_m^{p_m}$ and $(\ln v)^q = (\ln v_1)^{q_1} \cdots (\ln v_n)^{q_n}$.

Definition A.1 *($\mathcal{C}^\ell$-admissible function).* Let $(U, V) \in \mathbb{R}^m \times \mathbb{R}^n$. We say that a function $f(U, V)$ is $\mathcal{C}^\ell$-admissible (with respect to a monomial ω), if f is $\mathcal{C}^\ell$ w.r.t. V in a neighborhood of $0 \in \mathbb{R}^n$, and if there is a quadrant $\mathcal{U} = [0, u_1) \times \cdots \times [0, u_n) \subset \mathbb{R}^m$ where the monomial ω is defined and such that the function $\tilde{f}(\omega, U, V) = f(U, V)$ is $\mathcal{C}^\ell$ with respect to ω in $\mathcal{U}$. Similarly, the function f is said to be $\mathcal{C}^\ell$-admissible (with respect to the monomials $\omega_1, \ldots, \omega_s$) if there is a quadrant $\mathcal{U}$ where the monomials are defined and such that the function $\tilde{f}(\omega_1, \ldots, \omega_s, U, V) = f(U, V)$ is $\mathcal{C}^\ell$ with respect to $\omega_1, \ldots, \omega_s$ in $\mathcal{U}$.

As an example of a $\mathcal{C}^\ell$-admissible function, consider $f(U) = U_1 \ln U_1 \phi(U)$ where $\phi(U)$ is smooth. This function is smooth away from $U = 0$ and C^0 at the origin. However, it is not differentiable w.r.t. U_1 at $U_1 = 0$ but it is differentiable with respect to $\omega = U_1 \ln U_1$ at $\omega = 0$.

Let $V \in \mathbb{R}^m$, $Z \in \mathbb{R}$, and as usual ε denotes a small parameter.

Definition A.2 *(Exponential type function).* A function $D(V, Z, \varepsilon)$ is called of exponential type if it has the following form

$$D(V, Z, \varepsilon) = \mathcal{B}(V, \varepsilon) + Z \exp\left(-\frac{\mathcal{A}(V, \varepsilon) + \Phi(V, \varepsilon, Z)}{\varepsilon}\right), \tag{A.1}$$

where $\mathcal{A}$ and $\mathcal{B}$, are $\mathcal{C}^\ell$-admissible functions with $\mathcal{A} > 0$, and $\mathcal{B}(V, 0) = 0$; and where Φ is $\mathcal{C}^\ell$-admissible with $\Phi(V, 0, Z) = 0$. We distinguish two particular cases

1. The exponential type function D is *without shift* if $\mathcal{B} \equiv 0$.
2. The exponential type function D is *linear* if $\Phi(V, Z, \varepsilon) \equiv \Phi(V, \varepsilon)$.

Remark A.1. Assume D is a given exponential type function, then the representation of D is unique in the sense that all the functions in r.h.s. of (A.1) are computable from D. In fact

$$\begin{aligned}
\mathcal{B} &= D(V, 0, \varepsilon) \\
\mathcal{A} &= \lim_{Z \to 0}\left(-\varepsilon \ln\left(\frac{D(V, Z, \varepsilon) - D(V, 0, \varepsilon)}{Z}\right)\right) \\
\Phi &= -\varepsilon \ln\left(\frac{D(V, Z, \varepsilon) - D(V, 0, \varepsilon)}{Z}\right) - \mathcal{A}.
\end{aligned} \tag{A.2}$$

We want to study the scenario where we have to compose D with some other functions and want to keep the exponential type structure. To be more precise, we consider D as a

(V,ε)-parameter family of functions (in Z) and compose it with a (V,ε)-parameter family of diffeomorphisms $\Psi_{(V,\varepsilon)}$ on $\mathbb{R}$.

Proposition A.1 *(Composition on the left). Let $\Psi_{(V,\varepsilon)}:\mathbb{R}\to\mathbb{R}$ be a family of diffeomorphisms, and let D be an exponential type function. Then, the composition $\Psi_{(V,\varepsilon)}\circ D$ is also of exponential function of the form*

$$\tilde{D}=\tilde{\mathcal{B}}(V,\varepsilon)+Z\exp\left(-\frac{\mathcal{A}(V,\varepsilon)+\tilde{\Phi}(V,Z,\varepsilon)}{\varepsilon}\right), \tag{A.3}$$

where $\tilde{\mathcal{B}}$ and $\tilde{\Phi}$ are admissible functions.

Proof. Let us simplify the notation by writing $\Psi=\Psi_{(V,\varepsilon)}$. Since Ψ is a diffeomorphism we can write $\Psi(a+b)=\Psi(a)+C(1+\psi(a,b))b$, near $b=0$, with ψ a $\mathcal{C}^{\ell}$ function such that $\psi(a,0)=0$ and with $C>0$. Then we have

$$\begin{aligned}\Psi\circ D(z)&=\Psi\left(\mathcal{B}+Z\exp\left(-\frac{\mathcal{A}+\Phi}{\varepsilon}\right)\right)\\&=\Psi(\mathcal{B})+C(1+\psi(V,Z,\varepsilon))Z\exp\left(-\frac{\mathcal{A}+\Phi}{\varepsilon}\right).\end{aligned} \tag{A.4}$$

Since $C>0$ we can take the logarithm of $C(1+\psi(V,Z,\varepsilon))$ and then we have

$$\begin{aligned}\Psi\circ D(z)&=\Psi(\mathcal{B})+\exp(\ln(C(1+\psi))Z\exp\left(-\frac{\mathcal{A}+\Phi}{\varepsilon}\right)\\&=\Psi(\mathcal{B})+Z\exp\left(-\frac{\mathcal{A}+\Phi+\varepsilon\ln(C(1+\psi)}{\varepsilon}\right).\end{aligned} \tag{A.5}$$

The result is obtained by setting $\tilde{\mathcal{B}}=\Psi(\mathcal{B})$ and $\tilde{\Phi}=\Phi+\varepsilon\ln(C(1+\psi)$. $\square$

Proposition A.2 *(Composition on the right). Let $\Psi_{(V,\varepsilon)}:\mathbb{R}\to\mathbb{R}$ be a family of diffeomorphisms with no shift, that is $\Psi_{(V,\varepsilon)}(0)=0$ for all (V,ε), and let D be an exponential type function. Then, the composition $D\circ\Psi_{(V,\varepsilon)}$ is also of exponential function of the form*

$$\tilde{D}=\tilde{\mathcal{B}}(V,\varepsilon)+Z\exp\left(-\frac{\mathcal{A}(V,\varepsilon)+\tilde{\Phi}(V,Z,\varepsilon)}{\varepsilon}\right), \tag{A.6}$$

where $\tilde{\mathcal{B}}$ and $\tilde{\Phi}$ are admissible functions.

Proof. Let us simplify the notation by writing $\Psi=\Psi_{(V,\varepsilon)}$. Since $\Psi(0)=0$ we can write $\Psi(z)=C(1+O(z))z$ with $C>0$. Then we have

$$\begin{aligned} D \circ \Psi(z) &= D(C(1+O(z))z) = \mathcal{B}(V,\varepsilon) + C(1+O(z))z \exp\left(-\frac{\mathcal{A}(V,\varepsilon)+\Phi(V,\varepsilon,\Psi)}{\varepsilon}\right) \\ &= \mathcal{B}(V,\varepsilon) + z\exp\left(-\frac{\mathcal{A}(V,\varepsilon)+\Phi(V,\varepsilon,\Psi)+\varepsilon\ln(C(1+O(z)))}{\varepsilon}\right). \end{aligned} \tag{A.7}$$

The result then is obtained by setting $\tilde{\Phi} = \Phi(V,\varepsilon,\Psi) + \varepsilon \ln(C(1+O(z)))$. □

Remark A.2. If we want the composition $\Pi \circ \Psi_{(V,\varepsilon)}$ to be of exponential type, the family $\Psi_{(V,\varepsilon)}$ cannot be arbitrary. In order to preserve the "exponential structure", $\Psi_{(V,\varepsilon)}$ should satisfy the hypothesis of Proposition A.2. In Corollary A.2 we show a particular case in which the diffeomorphism Ψ can have a shift and yet preserve the structure of the exponential type function.

Let us proceed by presenting a couple of useful corollaries.

Corollary A.1. *Let D_1 and D_2 be two exponential type functions of the form*

$$\begin{aligned} D_1(V,Z,\varepsilon) &= Z\exp\left(-\frac{\mathcal{A}_1(V,\varepsilon)+\Phi_1(V,Z,\varepsilon)}{\varepsilon}\right) \\ D_2(V,Z,\varepsilon) &= \mathcal{B}_2(V,\varepsilon) + Z\exp\left(-\frac{\mathcal{A}_2(V,\varepsilon)+\Phi_2(V,Z,\varepsilon)}{\varepsilon}\right), \end{aligned} \tag{A.8}$$

that is, D_1 is an exponential type function with no shift. Then $D_2 \circ D_1$ is an exponential type function.

Corollary A.2. *Let D_1 and D_2 be two exponential type functions with D_2 linear, this is*

$$\begin{aligned} D_1(V,Z,\varepsilon) &= \mathcal{B}_1(V,\varepsilon) + Z\exp\left(-\frac{\mathcal{A}_1(V,\varepsilon)+\Phi_1(V,Z,\varepsilon)}{\varepsilon}\right) \\ D_2(V,Z,\varepsilon) &= \mathcal{B}_2(V,\varepsilon) + Z\exp\left(-\frac{\mathcal{A}_2(V,\varepsilon)}{\varepsilon}\right). \end{aligned} \tag{A.9}$$

Then the composition $D_2 \circ D_1$ is of exponential type.

It is useful to consider the following: let $X(V,Z,\varepsilon)$ be a given vector field on $\mathbb{R}^{m+2}$, and let Σ_0 and Σ_1 be codimension 1 subsets of $\mathbb{R}^{m+2}$ which are transversal to the flow of X. For the moment it is sufficient to think of a section Σ_i given by $\{V_j = v_0\}$ or by $\{\varepsilon = \varepsilon_0\}$ with v_0 and ε_0 fixed constants. Induced from Definition A.2 we then have the following.

Definition A.3 *(Exponential type transition).* A transition $\Pi : \Sigma_0 \to \Sigma_1$ is called of exponential type if and only if its Z-component is an exponential type function. This is, an exponential type transition is of the form

$$\begin{aligned} \Pi(V,Z,\varepsilon) &= (G,D,H) \\ &= \left(G(V,\varepsilon),\, \mathcal{B}(V,\varepsilon) + Z\exp\left(-\frac{\mathcal{A}(V,\varepsilon)+\Phi(V,Z,\varepsilon)}{\varepsilon}\right),\, H(V,\varepsilon)\right), \end{aligned} \tag{A.10}$$

where $G:\mathbb{R}^{m+1}\to\mathbb{R}^m$ and $H:\mathbb{R}^{m+1}\to\mathbb{R}$ are $\mathcal{C}^\ell$ with $G(V,0)=V$ and $H(V,0)=0$; where A, B and Φ are $\mathcal{C}^\ell$-admissible functions. The names exponential type transition with no shift and linear are inherited as well from the type of D.

Suppose now that X is a given vector field on $\mathbb{R}^{m+2}$, as above, and let Σ_i with $i=0,1,2,3,4,5$, be disjoint sections which are all transversal to the flow of X. Assume that X induces exponential type transitions $\Pi_i:\Sigma_{i-1}\to\Sigma_i$ with $i=1,2,3,4,5$ of the following form

1. Π_1 is with no shift and linear
2. Π_2 is with no shift
3. Π_3 is a general diffeomorphism
4. Π_4 is with no shift
5. Π_5 is with no shift and linear.

We need to show that the composition of all these five maps is an exponential type transition.

Proposition A.3. *Let $\Pi_i:\Sigma_{i-1}\to\Sigma_i$ as described above. Then the composition $\Pi=\Pi_5\circ\Pi_4\circ\Pi_3\circ\Pi_2\circ\Pi_1$ is an exponential type map of the form*

$$\Pi=\left(\tilde{G}(V,\varepsilon),\ \tilde{\mathcal{B}}(V,\varepsilon)+Z\exp\left(-\frac{\tilde{\mathcal{A}}(V,\varepsilon)+\tilde{\Phi}(V,Z,\varepsilon)}{\varepsilon}\right),\ \tilde{H}(V,\varepsilon)\right), \tag{A.11}$$

where $\tilde{\mathcal{A}}=\mathcal{A}_1+\mathcal{A}_2+\mathcal{A}_4+\mathcal{A}_5$.

Proof. Let us write each of the transitions as follows.

1. $\Pi_1(V,Z,\varepsilon)=(G_1,D_1,H_1)=\left(G_1,Z\exp\left(-\frac{\mathcal{A}_1(V,\varepsilon)}{\varepsilon}\right),H_1\right)$
2. $\Pi_2(V,Z,\varepsilon)=(G_2,D_2,H_2)=\left(G_2,Z\exp\left(-\frac{\mathcal{A}_2(V,\varepsilon)+\Phi_2(V,Z,\varepsilon)}{\varepsilon}\right),H_2\right)$
3. $\Pi_3(V,Z,\varepsilon)=(G_3,D_3,H_3)$
4. $\Pi_4(V,Z,\varepsilon)=(G_4,D_4,H_4)=\left(G_4,Z\exp\left(-\frac{\mathcal{A}_4(V,\varepsilon)+\Phi_4(V,Z,\varepsilon)}{\varepsilon}\right),H_4\right)$
5. $\Pi_5(V,Z,\varepsilon)=(G_5,D_5,H_5)=\left(G_5,Z\exp\left(-\frac{\mathcal{A}_5(V,\varepsilon)}{\varepsilon}\right),H_5\right)$

For brevity let $\Pi_2\circ\Pi_1=(\tilde{G}_2,\tilde{D}_2,\tilde{H}_2)$. Then we have

$$\begin{aligned}&(\tilde{G}_2,\tilde{D}_2,\tilde{H}_2)=\\&\left(G_2(G_1,H_1),D_1\exp\left(-\frac{\mathcal{A}_2(G_1,H_1)+\Phi_2(G_1,D_1,H_1)}{H_1}\right),H_2(G_1,H_1)\right).\end{aligned} \tag{A.12}$$

Now, we take care only of the Z-component of the composition $\Pi_2\circ\Pi_1$. From the hypothesis on G_1 and H_1 we can write $G_1=V+O(\varepsilon)$ and $H_1=\alpha\varepsilon(1+O(\varepsilon))$ with $\alpha>0$, then

$$\tilde{D}_2=Z\exp\left(-\frac{\mathcal{A}_1(V,\varepsilon)+\mathcal{A}_2(V,\varepsilon)+\bar{\Phi}_2(V,Z,\varepsilon)}{\varepsilon}\right), \tag{A.13}$$

where we have gathered in $\bar{\Phi}_2$ the function Φ_1 and the terms resulting from taking $G_1 = V + O(\varepsilon)$ and $H_1 = \alpha\varepsilon(1 + O(\varepsilon))$. In a similar way, letting $\Pi_5 \circ \Pi_4 = (\tilde{G}_5, \tilde{D}_5, \tilde{H}_5)$ we get

$$\tilde{D}_5 = Z \exp\left(-\frac{\mathcal{A}_4(\varepsilon) + \mathcal{A}_5(\varepsilon) + \bar{\Phi}_5(V, Z, \varepsilon)}{\varepsilon}\right) \tag{A.14}$$

Next, and following similar arguments as above, we know from Proposition A.1 that the composition $\Pi_{321} = \Pi_3 \circ \Pi_2 \circ \Pi_1$ is of exponential type *with shift*. Finally since the transition $\Pi_{54} = \Pi_5 \circ \Pi_4$ is of exponential type with no shift, and using Proposition A.1, we have that $\Pi_{54} \circ \Pi_{321}$ is an exponential type transition as claimed in the proposition.

Remark A.3. In the case where Π_3 is an exponential type map, we get a similar result with $\tilde{\mathcal{A}} = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4 + \mathcal{A}_5$. □

Appendix B. First order differential equations (by R. Roussarie)

The contents of this section shall appear in greater detail in [20]. We reproduce some results here for completeness purposes and to use them in Appendix C.1.

Let $X(x)$ be a smooth vector field defined on $W \subset \mathbb{R}^n$, for arbitrary $n \in \mathbb{N}$ (here we include the possible parameters). Let $G(x, y) : W \times \mathbb{R} \to \mathbb{R}$ be a smooth function. We shall study the solutions of the first order differential equation

$$X \cdot K(x) = G(x, K(x)), \tag{B.1}$$

where $K(x)$ is the unknown function. We assume the following

1. There exists an open section $\Sigma \subset W$ which is transverse to X.
2. Let $\phi(t, x)$ denote the flow of X. We can choose an open domain W_Σ with the property that for any $x \in W_\Sigma$, there exists a unique smooth time $t(x)$ (possibly unbounded) such that $\phi(t(x), x) \in \Sigma$.
3. The vector field $Z(x, y) = X(x) + G(x, y)\partial_y$ has a complete flow.

The flow of Z takes the form $(\phi(t, x), \psi(t, x, y))$, where ϕ is the flow of X. It follows that *$K(x)$ is a solution of (B.1) if and only if the graph $\{y = K(x)\}$ is a surface tangent to the vector field Z*. Then we have the implicit formula

$$\psi(t(x), x, K(x)) = 0. \tag{B.2}$$

In our applications, the function G is affine in y, that is $G(x, y) = L(x)y + \Pi(x)$ where L and Π are smooth. If we write $\bar{L}(t, x) = L(\phi(t, x))$ and $\bar{\Pi}(t, x) = \Pi(\phi(t, x))$ (where ϕ is the flow of X), we have for ψ the following linear differential equation

$$\frac{d\psi}{dt}(t, x, y) = \bar{L}(t, x)\psi(t, x, y) + \bar{\Pi}(t, x). \tag{B.3}$$

Then we can integrate (B.3) with the initial condition $\psi(0,x,y)=y$ to obtain

$$\psi(t,x,y)=\exp\left(\int_0^t \bar{L}(\tau,x)d\tau\right)\left\{y+\int_0^t \bar{\Pi}(\tau,x)\left[\exp\left(-\int_0^\tau \bar{L}(\sigma,x)d\sigma\right)\right]d\tau\right\}. \quad \text{(B.4)}$$

Since $\exp\left(\int_0^t \bar{L}(\tau,x)d\tau\right)>0$ we can solve the implicit equation (B.2) obtaining

$$K(x)=-\int_0^{t(x)} \Pi(\phi(\tau,x))\left[\exp\left(-\int_0^\tau L(\phi(\sigma,x))d\sigma\right)\right]d\tau, \quad \text{(B.5)}$$

where we recall that ϕ is the flow of X and $t(x)$ is the time to go from x to the section Σ along this flow.

Let us now assume that the vector field X is partially hyperbolically attracting in the following sense: we assume coordinates $x=(a,b)\in\mathbb{R}^p\times\mathbb{R}^q$ and that the vector field X has a decomposition $X(x)=U(x)+V(x)$ where U is the component along $\mathbb{R}^p$ and V is the component along $\mathbb{R}^q$. Moreover, we assume that $V=0$ on $\mathbb{R}^p\times\{0\}$ (that is X is tangent to $\mathbb{R}^p\times\{0\}$). We also assume that at each point $x=(a,b)$ it is satisfied that $D_bV(a,0)$ has all its eigenvalues with strictly negative real part. We further suppose that X is given on $W=D\times\Delta$ where D is a domain diffeomorphic to a ball in $\mathbb{R}^p$ and Δ is a ball in $\mathbb{R}^q$. We choose $\Delta=\Delta_{\rho_0}$ for some $\rho_0>0$ where $\Delta_\rho=\{b\in\mathbb{R}^q \mid ||b||<\rho\}$. It then follows that under a linear change of coordinates $(a,b)\mapsto(a,A(a)b)$, the vector field X enters along $D\times\partial\Delta_\rho$ for $0<\rho\le\rho_0$ if we choose ρ_0 small enough. We now have the following

Proposition B.1. *Assume that $D_bV(a,0)$ has all its eigenvalues with a strictly negative real part and that ρ_0 is small enough as explained above. Let B be any domain diffeomorphic to a closed ball inside the interior of D and assume that the function $\Pi(x)$ is flat along $D\times\{0\}$. Then the equation*

$$X\cdot K(x)=L(x)K(x)+\Pi(x) \quad \text{(B.6)}$$

has a smooth solution $K(x)$ in $B\times\Delta$ which is flat along $B\times\{0\}$.

Proof. Let $f(a):\mathbb{R}^p\to[0,1]$ be a smooth function which is equal to 1 on B and equal to 0 on a neighborhood of ∂D. Define the vector field

$$T=V+fU. \quad \text{(B.7)}$$

This vector field T coincides with X on $B\times\Delta$. Moreover, T is tangent along $\partial D\times\Delta$ and enters the domain $D\times\Delta$ along $D\times\partial\Delta$. Let $\phi(t,x)=(\phi_a(t,x),\phi_b(t,x))\in\mathbb{R}^p\times\mathbb{R}^q$ denote the flow of T. It follows that $\phi(t,x)\in D\times\Delta$ for all $x\in D\times\Delta$ and all $t\ge 0$. From the assumption on V we have that there exists a positive constant $E>0$ such that

$$||\phi_b(t,x)||\le||b||\exp(-Et), \quad \text{(B.8)}$$

for any $x = (a, b) \in D \times \Delta$ and $t \in [0, +\infty)$. We now want to use this flow ϕ in (B.5) noting that if the integral converges, then $K(x)$ is a solution to the equation $T \cdot K = LK + \Pi$ on $D \times \Delta$ and then to the equation $X \cdot K = Lk + \Pi$ on $B \times \Delta$. In this setting (B.5) is written as

$$K(x) = -\int_0^\infty \Pi(\phi(\tau, x)) \left[\exp\left(-\int_0^\tau L(\phi(\sigma, x)) d\sigma \right) \right] d\tau. \tag{B.9}$$

Now, we need to prove that (B.9) defines a smooth function on $D \times \Delta$ which is flat along $D \times \{0\}$. In other words, we shall prove that K and all its partial derivatives are equal to 0 on $D \times \{0\}$. As L is bounded, there exists a constant $M_0 > 0$ such that

$$\exp\left(-\int_0^\tau L(\phi(\sigma, x)) d\sigma \right) \leq \exp(M_0 \tau). \tag{B.10}$$

Next, let $N \in \mathbb{N}$. Since Π is flat in v, there exists a constant $P_N > 0$ such that

$$|\Pi(a, b)| \leq P_N ||b||^N, \tag{B.11}$$

and then from (B.8) it follows that

$$|\Pi(\phi(\tau, x))| \leq P_N ||b||^N \exp(-NE\tau). \tag{B.12}$$

Using these estimates we have that

$$|K(x)| \leq P_N ||b||^N \int_0^{+\infty} \exp((M_0 - NE)\tau) d\tau. \tag{B.13}$$

The integral in (B.13) converges if N is large enough, strictly speaking if $N > \frac{M_0}{E}$. This proves that by choosing N sufficiently large, the right hand side of (B.9) defines a function which is continuous and equal to 0 on $D \times \{0\}$.

Let us now consider any partial derivation $\partial_\alpha K$ of K. Let us write

$$H(\tau, x) = \Pi(\phi(\tau, x)) \exp\left[-\int_0^\tau L(\phi(\sigma, x)) d\sigma \right], \tag{B.14}$$

the integrand in (B.9). Using chain rule on the derivative of (B.9), we have to prove that the integral

$$\int_0^{+\infty} \partial_\alpha H(\tau, x) d\tau \tag{B.15}$$

is convergent and that there is an estimate similar to (B.13) for N large enough. We do not want to give all the details here and refer the reader to [20]. The idea is that $\partial_\alpha H(\tau, x)$ is a finite sum of terms such that each of these terms is a product of factors which are partial derivatives in x and are of one of the following forms

1. $\partial_{\alpha_1}(\phi(\tau, x))$. Since Π is smooth and flat along $D \times \{0\}$, this is also the case for $\partial_{\alpha_1}(\phi(\tau, x))$. Therefore, for N sufficiently large, we can write an estimate of the form

$$|\partial_{\alpha_1}(\phi(\tau, x))| \leq P_{N_{\alpha_1}} ||b||^N \exp(-NE\tau), \tag{B.16}$$

   for constants $P_{N_{\alpha_1}} > 0$.
2. $\partial_{\alpha_2}\phi(\tau, x)$ (resp. $\partial_{\alpha_2}\phi(\sigma, x)$, note that $0 \leq \sigma \leq \tau$). By the usual variational method along trajectories, there exists constants $E_{\alpha_2} > 0$ such that $|\partial_{\alpha_2}\phi(\tau, x)| \leq \exp(E_{\alpha_2}\tau)$ (resp. $|\partial_{\alpha_2}\phi(\sigma, x)| \leq \exp(E_{\alpha_2}\sigma)$).
3. $\partial_{\alpha_3} L(\phi(\tau, x))$. As L is smooth in $D \times \Delta$, all these factors are bounded by a constant M_{α_3}.
4. $\exp\left(-\int_0^\tau L(\phi(\sigma, x))d\sigma\right)$. This factor is bounded by $\exp(M_0\tau)$.

Next, by remarking that a factor of the first type appears in each term of the expansion of $\partial_\alpha H$, and taking N large enough, it is possible to conclude that the integral (B.15) converges an is equal to 0 for $x \in D \times \{0\}$. Therefore, the partial derivative $\partial_\alpha K(x)$ exists, is continuous and is equal to 0 on $D \times \{0\}$. □

Appendix C. Normal form and transition of a semi-hyperbolic vector field

In this section, we present a rather general framework for the computation of a $\mathcal{C}^\ell$ normal form and the corresponding transition of a vector fields with a semi-hyperbolic singularity. The contents of this section are not only relevant for the object studied in this document, but for more general systems as well, cf. [10]. To make our computations simpler, we prove a lemma that allows us to “partition” a smooth function. As a simple example of this partition, let $f(u, v)$ be a smooth function on $\mathbb{R}^2$. We show that f can be written as $f(u, v) = f_1(uv, u) + f_2(uv, v)$, where f_1 and f_2 are smooth. This type of result becomes useful when computing the transition map that we present in Appendix C.3.

C.1. Normal form

Here we provide a $\mathcal{C}^\ell$ normal form of a semi-hyperbolic vector field which frequently appears in the analysis of slow–fast systems. The goal of obtaining such a normal form is that the computation of the corresponding transition becomes simpler.

Proposition C.1. *Let α, $\beta = (\beta_1, \ldots, \beta_m)$ and γ be non-zero constants, and consider the vector field X given by*

$$X : \begin{cases} u' = \alpha w u(1 + f) + wg \\ v_j' = \beta_j w v_j(1 + f) \\ w' = \gamma w^2(1 + f) \\ z' = -\Lambda + h, \end{cases} \tag{C.1}$$

where $j = 1, 2, \dots, m$; where the functions $f = f(u, v, w, z)$, $g = g(u, v, w, z)$ and $h = h(u, v, w, z)$ are smooth functions which are flat at the origin of $\mathbb{R}^{m+3}$, and where $\Lambda = \Lambda(u, v, w, z)$ is a smooth function such that $\Lambda(0) = 0$ and $\frac{\partial \Lambda}{\partial z}(0) > 0$. Then there exist a $\mathcal{C}^\ell$ coordinates $(U, V_1, \dots, V_m, W, Z)$ for which X is $\mathcal{C}^\ell$-equivalent to a normal form given by

$$X_{\text{sh}}^N : \begin{cases} U' = \alpha W U \\ V_j' = \beta_j W V_j \\ W' = \gamma W^2 \\ Z' = -GZ, \end{cases} \tag{C.2}$$

where $G = G(U, V, W)$ is a C^ℓ function such that $G(0) > 0$.

Proof of Proposition C.1. From the definition of the vector field X we note that the origin is a semi-hyperbolic singular point. The hyperbolic eigenspace is 1-dimensional while the center eigenspace is $(m + 2)$-dimensional. We now proceed in 4 steps as follows.

1. Define a new vector field Y by $Y = \frac{1}{1+f} X$, which reads as

$$Y : \begin{cases} u' = \alpha w u + w\bar{g} \\ v_j' = \beta_j w v_j \\ w' = \gamma w^2, \\ z' = -\Lambda + \bar{h}, \end{cases} \tag{C.3}$$

   where the functions $\bar{g}$ and $\bar{h}$ are flat at the origin of $\mathbb{R}^{m+3}$. Note that in a small neighborhood of $(u, v, w, z) = (0, 0, 0, 0)$ the vector fields X and Y are smoothly equivalent.
2. By looking at $DY(0)$, there exists an $(m + 2)$-dimensional center manifold $\mathcal{W}_2^C$ [9]. Let M_0 be the set of critical points of Y, that is

$$M_0 = \{(u, v, w, z) \mid \Lambda(u, v, 0, z) = 0\}. \tag{C.4}$$

   By definition, the manifold M_0 is invariant and normally hyperbolic. Now, assume $|w| \ll 1$. This condition appears naturally in our applications. By Fenichel's theory [8] the manifold M_0 persists as an invariant normally hyperbolic manifold M_w, for sufficiently small $w \neq 0$. We identify M_w with $\mathcal{W}_2^C$. In other words, there exists a C^ℓ function $m = m(u, v, w)$ such that the center manifold $\mathcal{W}_2^C$ is given as a graph

$$\mathcal{W}_2^C = \text{Graph}(u, v, w, m). \tag{C.5}$$

   Define $\zeta = z - m$, then $\zeta' = z' - m'$. But we know, due to invariance of $\mathcal{W}_2^C$ under the flow of Y, that $\zeta'|_{\zeta=0} = 0$. This is, there exists a C^ℓ function $H = H(u, v, w, \zeta)$ such that $\zeta' = -H\zeta$. With $H(0) = 0$ and $\dfrac{\partial H}{\partial \zeta}(0) > 0$.
   In conclusion of this step, there exists a C^ℓ transformation $\psi : (u, v, w, z) \mapsto (u, v, w, \zeta)$ that transforms the vector field Y into

$$\tilde{Y}:\begin{cases} u' = \alpha w u + w\bar{g} \\ v_j' = \beta_j w v_j \\ w' = \gamma w^2, \\ \zeta' = -H\zeta, \end{cases} \tag{C.6}$$

where $H = H(u, v, w, \zeta)$ is a C^ℓ function such that $H(0) = 0$ and where $\frac{\partial H}{\partial z}(0) = \frac{\partial \Lambda}{\partial z}(0) > 0$.

3. Observe that thanks to the previous step, the center manifold $\mathcal{W}_2^C$ has the simple expression $\mathcal{W}_2^C = \{\zeta = 0\}$. We now want to separate the variables on the center manifold (these are (u, v, w)) from those on the hyperbolic subspace (z). Additionally, we want to keep the simple format that $\tilde{Y}$ has in the center direction. This amounts to find a change of coordinates along ζ only. For this we use an extension of Takens's theorem on semi-hyperbolic vector fields [23] due to Bonckaert [2,3]. With this, it is possible to show there exists a C^ℓ transformation, fixing the center coordinates, that conjugates $\tilde{Y}$ to the vector field

$$\bar{Y}:\begin{cases} u' = \alpha w u + w\tilde{g} \\ v_j' = \beta_j w v_j \\ w' = \gamma w^2, \\ Z' = -\bar{H}Z, \end{cases} \tag{C.7}$$

where now the flat perturbation $\tilde{g}$ is independent of Z and $\bar{H} = \bar{H}(u, v, w)$ is a $\mathcal{C}^\ell$ function with $\bar{H}(0, 0, 0) > 0$.

4. In this last step we eliminate the flat perturbation from $\bar{Y}$, which appears only along u. Due to the previous step, the dynamics on the center manifold are independent of Z. The restriction of $\bar{Y}$ to $\mathcal{W}_2^C$ reads as

$$\bar{Y}|_{\mathcal{W}_2^C}:\begin{cases} u' = \alpha w u + w\tilde{g} \\ v_j' = \beta_j w v_j \\ w' = \gamma w^2. \end{cases} \tag{C.8}$$

Note that for $w \neq 0$, the vector field $\frac{1}{w}\bar{Y}|_{\mathcal{W}_2^C}$ is hyperbolic. Let $\mathcal{Y} = \frac{1}{w}\bar{Y}|_{\mathcal{W}_2^C}$, that is

$$\mathcal{Y}:\begin{cases} u' = \alpha u + \tilde{g} \\ v_j' = \beta_j v_j \\ w' = \gamma w. \end{cases} \tag{C.9}$$

Now we have a result that shows that there exists a change of coordinates, respecting the variables (v, w) that kills the term $\tilde{g}$. Keeping the coordinate w fixed is important because we want to prove an equivalence relation with $w\mathcal{Y}$ and not with $\mathcal{Y}$. The following proposition shall appear in a general context in [20].

Proposition C.2. *(See [20].) There exists a diffeomorphism* $(u, v, w) \mapsto (u + H(u, v, w), v, w)$ *with* H *flat at* $(u, v, w) = 0$ *which brings* $\mathcal{Y}$ *to*

$$\bar{\mathcal{Y}}: \begin{cases} u' = \alpha u \\ v_j' = \beta_j v_j \\ w' = \gamma w. \end{cases} \tag{C.10}$$

Proof. We shall use the path method to show that $\bar{\mathcal{Y}}$ is conjugate to $\mathcal{Y}$. Let s be a parameter and let us define the s-parameter family of vector fields

$$\mathcal{Y}^s = \mathcal{Y} + s\tilde{g}\frac{\partial}{\partial u}. \tag{C.11}$$

We call $\mathcal{Y}^s$ the path between $\mathcal{Y}$ and $\mathcal{Y} + \tilde{g}\frac{\partial}{\partial u}$. We now look for an s-parameter family of diffeomorphisms $\mathcal{H}^s$ with $\mathcal{H}^0 = \mathrm{Id}$ such that for each s we have the conjugacy

$$\mathcal{H}^s_* \mathcal{Y} = \mathcal{Y}^s. \tag{C.12}$$

In such a case, the vector fields $\mathcal{Y}$ and $\mathcal{Y} + \tilde{g}\frac{\partial}{\partial u}$ are conjugated by $\mathcal{H}^1$. By derivation of the family $\mathcal{H}^s$ along s, we obtain an s-parameter family of vector field ζ^s satisfying

$$\zeta^s(\mathcal{H}^s) = \frac{\partial H^s}{\partial s}. \tag{C.13}$$

This implies that by derivation of (C.11) with respect to s we obtain

$$[\mathcal{Y}^s, \zeta^s] = \frac{\partial \mathcal{Y}^s}{\partial s} = \tilde{g}\frac{\partial}{\partial u}. \tag{C.14}$$

Therefore, if are able to find a solution ζ^s of (C.14), the conjugacy $\mathcal{H}^s$ is obtained by integration of (C.13). In our particular case, we are looking for a solution along the u-direction, that is of the form $\zeta^s = P_s\frac{\partial}{\partial u}$. It follows that

$$\begin{aligned} [\mathcal{Y}^s, \zeta^s] &= \left[(\alpha u + s\tilde{g}) + \beta v\frac{\partial}{\partial v} + \gamma w\frac{\partial}{\partial w}, P_s\frac{\partial}{\partial u}\right] \\ &= \left(\mathcal{Y}^s(P_s) - \left(\alpha + s\frac{\partial \tilde{g}}{\partial u}\right) P_s\right)\frac{\partial}{\partial u}. \end{aligned} \tag{C.15}$$

Therefore we have reduced our conjugacy problem to solving the differential equation

$$\mathcal{Y}^s(P_s) - \left(\alpha + s\frac{\partial \tilde{g}}{\partial u}\right) P_s = \tilde{g}, \tag{C.16}$$

where we recall that $\tilde{g} = \tilde{g}(u, v, w)$ is flat at $(u, v, w) = (0, 0, 0)$. We now want to use Proposition B.1 to show that (C.16) has a solution $P_s = P_s(u, v, w)$ which is flat at $(u, v, w) = (0, 0, 0)$. For this, let $G_s = \alpha + s\frac{\partial \tilde{g}}{\partial u}$. Now, we only need a small adaptation: in the setting and notation of Proposition B.1 we may assume (under the suitable arrangement of coordinates) that $\mathcal{Y}^s$ (or X in Proposition B.1) is tangent to $\mathbb{R}^d \times \{0\}$ and $\{0\} \times \mathbb{R}^{n-d}$. Let $\mathcal{M}_s^\infty(a)$ and $\mathcal{M}_s^\infty(b)$ denote the space of germs of s-families of smooth functions that are flat at $\{a = 0\}$ and at $\{b = 0\}$ respectively. Using a blowing-up at $0 \in \mathbb{R}^n$ it can be shown that $\mathcal{M}_s^\infty(a, b) = \mathcal{M}_s^\infty(a) + \mathcal{M}_s^\infty(b)$ (see the arguments in Lemma C.1). From this formula, it follows that it is sufficient to solve (C.16) in the spaces $\mathcal{M}_s^\infty(a)$ and $\mathcal{M}_s^\infty(b)$ respectively. Naturally, these two cases are equivalent up to the change of $\mathcal{Y}^s$ by $-\mathcal{Y}^s$ and G_s by $-G_s$ in (B.6). In either case, the vector field $\mathcal{Y}^s$ (or $-\mathcal{Y}^s$) of (C.16) satisfies the hypothesis of Proposition B.1. Then for $\tilde{g}$ in $\mathcal{M}_s^\infty(a)$ (resp. in $\mathcal{M}_s^\infty(b)$) and applying Proposition B.1, we can solve (C.16) with P_s in $\mathcal{M}_s^\infty(a)$ (resp. in $\mathcal{M}_s^\infty(b)$). □

Thus, from Proposition C.2, we have that $\mathcal{Y} \sim \bar{\mathcal{Y}}$ respecting w, which implies $w\mathcal{Y} \sim w\bar{\mathcal{Y}}$. Therefore, we conclude that (C.7) can be written as stated in the proposition. □

C.2. Partition of a smooth function

In this section we investigate the problem of partitioning a smooth function. The result presented below is important since it is used to simplify the computation of transition maps. To be more specific, let us give a brief example. Consider the three dimensional differential equation

$$\begin{aligned} x' &= x \\ y' &= -y \\ z' &= g(x,y)z, \end{aligned} \tag{C.17}$$

where g is a smooth function. We want to take advantage from the fact that xy is a first integral. We show below that the function g can be partitioned as $g(x,y) = g_1(xy,x) + g_2(xy,y)$. This makes the integration of z' simpler.

Lemma C.1. *Let $u \in \mathbb{R}$ and $v \in \mathbb{R}^m$. Let $f = f(u,v)$ be a smooth function such that $f(0,0) = 0$. Then there exist smooth functions $f_0 = f_0(uv,u)$ and $f_1(uv,v)$ such that the function f can be written as*

$$f = f_0 + f_1, \tag{C.18}$$

where $f_0(0,0) = 0$ and $f_1(0,0) = 0$.

Proof of Lemma C.1. We proceed in two steps. The first consists in proving the formal version of the statement. The second step is to extend the formal result to the smooth case.

Formal step

Let $\hat{f}$ denote the formal expansion of the smooth function f. Let $p \in \mathbb{N}$ and $q \in \mathbb{N}^m$. We use the following notation:

- By $q \geq 0$ we mean $q_i \geq 0$ for all $i \in [1,m]$.
- For a vector $v \in \mathbb{R}^m$ we write $v^q = v_1^{q_1} \cdots v_m^{q_m}$.
- The L_1 norm of q is denote by $|q|$, and thus for $q > 0$ we have $|q| = \sum_{j=1}^m q_j$.
- We denote by $\tilde{q}_i$ the vector

$$\tilde{q}_i = (q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_m) \tag{C.19}$$

and therefore we have that $v^{\tilde{q}_i}$ reads as

$$v^{\tilde{q}_i} = \frac{v^q}{v_i^{q_i}} = v_1^{q_1} \cdots v_{i-1}^{q_{i-1}} v_{q_{i+1}}^{q_{i+1}} \cdots v_m^{q_m}. \tag{C.20}$$

Besides, we have that the L_1 norm of $\tilde{q}_i$ is given by $|\tilde{q}_i| = |q| - q_i = \sum_{j=1, j\neq i}^m q_j$.

The formal series expansion of f reads as

$$\hat{f} = \sum_{p\geq 0, q\geq 0} a_{pq} u^p v^q, \tag{C.21}$$

where $a_{00} = 0$. With the notation introduced above, we can partition $\hat{f}$ as follows

$$\hat{f} = \sum_{p\geq |q|} a'_{pq} (uv)^q u^{p-|q|} + \sum_{i=1}^m \sum_{q_i \geq p+|\tilde{q}_i|} a'_{pq} (uv_i)^p (v_i v)^{\tilde{q}_i} v_i^{q_i - p - |\tilde{q}_i|}, \tag{C.22}$$

where

$$\begin{aligned} (uv)^q &= (uv_1)^{q_1} \cdots (uv_m)^{q_m} \\ (v_i v)^{\tilde{q}_i} &= \frac{(v_i v)^q}{v_i^{2q_i}}, \end{aligned} \tag{C.23}$$

and where $a'_{pq} \in \mathbb{R}$ are suitable chosen coefficients. Let $r \in \mathbb{N}^m$, $s \in \mathbb{N}$. Define the following formal polynomials

$$\hat{h}(uv, u) = \sum_{r,s\geq 0} \alpha_{rs} (uv)^r u^s = \sum_{p\geq |q|} a'_{pq} (uv)^q u^{p-|q|}, \tag{C.24}$$

where $\alpha_{rs} \in \mathbb{R}$, and

$$\begin{aligned} \hat{g}_i(uv_i, v) &= \sum_{r,s,t\geq 0} \beta_{irs} (uv_i)^s v^r \\ &= \sum_{q_i \geq p+|\tilde{q}_i|} a'_{pq} (v_i v)^{\tilde{q}_i} (uv_i)^p v_i^{q_i - p - |\tilde{q}_i|}, \end{aligned} \tag{C.25}$$

where $\beta_{irs} \in \mathbb{R}$. The coefficients α_{rs} and β_{irs} are conveniently chosen to make the definitions hold. Let $uv = (uv_1, \ldots, uv_m)$. Define $\hat{g} = \hat{g}(uv, v)$ by $\hat{g}(uv, v) = \sum_{i=1}^m \hat{g}_i(uv_i, v)$, then we can write $\hat{f}$ as

$$\hat{f}(u, v) = \hat{h}(uv, u) + \hat{g}(uv, v). \tag{C.26}$$

This shows that the proposition holds for formal series.

Smooth step

By Borel's lemma [5], there exist smooth functions $h = h(uv, u)$ and $g = g(uv, v)$ (whose formal series expansions are $\hat{h}$ and $\hat{g}$ respectively) such that

$$f = h + g + R, \tag{C.27}$$

where R (reminder) is a flat function. We now show the following.

Proposition C.3. *Let $u \in \mathbb{R}$, $v \in \mathbb{R}^m$, and $R(u, v)$ be a smooth flat function at $(0, 0) \in \mathbb{R} \times \mathbb{R}^m$. There exist flat functions $r_0 = r_0(uv, u)$ and $r_1 = r_1(uv, v)$ such that*

$$R = r_0 + r_1. \tag{C.28}$$

Remark C.1. Proposition C.3 together with the formal step $\hat{f} = \hat{h} + \hat{g}$ imply our result.

Proof of Proposition C.3. For this proof we shall use the blow up technique. Let $\Phi : S^m \times \mathbb{R}^+ \to \mathbb{R}^{m+1}$ be a blow up map. The map Φ maps $S^m \times \{0\}$ to the origin in $\mathbb{R}^{m+1}$. Let $\tilde{R}$ be a function defined by $\tilde{R} = R \circ \Phi$. Since R is flat at the origin, the function $\tilde{R}$ is flat along the sphere S^m. We assume that the function $R = R(u, v)$ is defined on a small neighborhood $\mathcal{R}$ of the origin in $\mathbb{R} \times \mathbb{R}^m$; this neighborhood is defined as

$$\mathcal{R} = \{|u| \leq A,\ |v_i| \leq B_i\}, \tag{C.29}$$

for some A, B_i positive scalars. Let $0 < \delta < 1$. The sphere S^m can be partitioned into $m + 1$ regions as follows:

$$\begin{aligned} \mathcal{U} &= S^m \setminus \{|\bar{u}| \leq \delta\} \\ \mathcal{V}_i &= S^m \setminus \{|\bar{v}_i| \leq \delta\}, \end{aligned} \tag{C.30}$$

where $(\bar{u}, \bar{v}) = (\bar{u}, \bar{v}_1, \ldots, \bar{v}_m) \in S^m$. We can then take a partition of unity to split $\tilde{R}$ as

$$\tilde{R}(\bar{u}, \bar{v}) = \tilde{R}_0(\bar{u}, \bar{v}) + \sum_{i=1}^{m} \tilde{R}_i(\bar{u}, \bar{v}), \tag{C.31}$$

where $\mathrm{Supp}(\tilde{R}_0) \subset \mathcal{U}$ and $\mathrm{Supp}(\tilde{R}_i) \subset \mathcal{V}_i$ for $i \in [1, m]$. We define as R_0 and R_i the corresponding functions on $\mathbb{R}^{m+1}$ flat at the origin given by the blow up map Φ, that is $\tilde{R}_j = R_j \circ \Phi$, for $j = 0, 1, \ldots, m$. Note that $R \to \tilde{R}$ is an isomorphism between the space of functions on $(u, v) \in \mathbb{R}^{m+1}$ flat at the origin, and the space of functions on $((\bar{u}, \bar{v}), \rho) \in S^m \times \mathbb{R}^+$ flat at $S^m \times \{0\}$. Therefore, the splitting (C.31) induces the splitting

$$R(u, v) = R_0(u, v) + \sum_{i=1}^{m} R_i(u, v) \tag{C.32}$$

of functions on $\mathbb{R}^{m+1}$. We will now prove that there exist flat functions r_0 and r_i such that

$$\begin{aligned} R_0(u,v) &= r_0(uv,u) \\ R_i(u,v) &= r_i(uv_i,v). \end{aligned} \tag{C.33}$$

Let us detail only the case of R_0. The other functions are obtained in a similar way.

The function $\tilde{R}_0$ has support in $\mathcal{U}$. We can parametrize $\mathcal{U}$ by the directional blow up map Φ_u which reads as

$$(\bar{u}, \bar{v}_1, \dots, \bar{v}_m) \mapsto (\bar{u}, u\bar{v}_1, \dots, u\bar{v}_m) = (u, v_1, \dots, v_m). \tag{C.34}$$

Now, suppose that there exists a flat function $\tilde{P}_0$ defined by

$$\tilde{R}_0(u, \bar{v}) = \tilde{P}_0(u, u^2\bar{v}). \tag{C.35}$$

This implies that there is a function $\tilde{r}_0 = \tilde{P}_0 \circ \Phi_u^{-1}$ such that

$$R_0(u,v) = \tilde{r}_0(u, uv), \tag{C.36}$$

which is precisely what we want to prove. So, now we only need to show that indeed a function $\tilde{P}_0$ as above exists. For this let us define coordinates $(U, V_1, \dots, V_m)$ given by

$$U = u,\ V_1 = u^2\bar{v}_1,\ \dots,\ V_m = u^2\bar{v}_m, \tag{C.37}$$

and let $\tilde{P}_0(u, V)$ be a function defined as

$$\tilde{P}_0(u,V) = \tilde{R}_0\left(\frac{V}{u^2}, u\right). \tag{C.38}$$

Note that $\tilde{P}_0$ is flat at $(u, V) = 0$. This is seen as follows. Since $\tilde{R}_0$ is flat along $\{u = 0\}$, it follows that $\tilde{P}_0(0,0) = \tilde{R}_0|_{u=0} = 0$ and

$$\begin{aligned} \frac{\partial \tilde{P}_0}{\partial u}(0) &= \frac{\partial \tilde{R}_0}{\partial u}|_{u=0} = 0 \\ \frac{\partial \tilde{P}_0}{\partial V_i}(0) &= \frac{1}{u^2}\frac{\partial \tilde{R}_0}{\partial \bar{v}_i}|_{u=0} = 0, \end{aligned} \tag{C.39}$$

and so on for the higher order derivatives.

Finally, for convenience of notation we define $r_0(uv,u) = \tilde{r}_0(u,uv)$, thus we can write $R_0(u,v) = r_0(uv,u)$ Following similar arguments as above we find the functions $r_i = r_i(uv_i, v)$ such that $R_i(u,v) = r_i(uv_i, v)$ for $i \in [1,m]$. Then we define $r_1(uv,v) = \sum_{i=1}^m r_i(uv_i, v)$. It follows that

$$R(u,v) = r_0(uv,u) + r_1(uv,v). \qquad \square \tag{C.40}$$

With this last proposition we can now write the function f as

$$\begin{aligned} f &= h(uv, u) + g(uv, v) + R(u, v) \\ &= h(uv, u) + g(uv, v) + r_0(uv, u) + r_1(uv, v). \end{aligned} \tag{C.41}$$

Finally, to show the lemma we define the smooth functions f_1, f_2 of the statement by

$$\begin{aligned} f_1 &= h + r_0 \\ f_2 &= g + r_1. \qquad \square \end{aligned} \tag{C.42}$$

C.3. Transition

In this section we investigate the transitions for the vector field X_{sh}^N computed in Appendix C.1. Relabeling the coordinates we recall that X_{sh}^N reads as

$$X_{\mathrm{sh}}^N : \begin{cases} u' = \alpha w u \\ v_j' = \beta_j w v_j \\ w' = \gamma w^2 \\ Z' = -gZ, \end{cases} \tag{C.43}$$

where $j = 1, 2, \ldots, m$, and where $g = g(u, v, w)$ is a C^ℓ function such that $g(0) = \Lambda > 0$. We assume that $w \in \mathbb{R}^+$. For our applications, we are interested in only two particular situations.

1. The saddle 1 case where $\alpha = -1$, $\beta_j > 0$ for all $j \in [1, m]$, and $\gamma > 0$.
2. The saddle 2 case where $\alpha = 1$, $\beta_j < 0$ for all $j \in [1, m]$, and $\gamma < 0$.

Saddle 1

In this case we investigate the transitions of a vector field of the form

$$Y : \begin{cases} u' = -wu \\ v_j' = \beta_j w v_j \\ w' = \gamma w^2 \\ Z' = -gZ, \end{cases} \tag{C.44}$$

where the coefficients β_j, γ are positive. Observe that the flow in the direction of u and Z is a contraction while it expands in all the other directions. Roughly speaking, this implies that a transition can go out at any expanding direction v_j of w.

We investigate two types of transitions that are used in our applications. For this, let us define the following sections

$$\begin{aligned} \Sigma_{\mathrm{en}} &= \{(u, v, w, Z) \,|\, u = u_i\} \\ \Sigma_{\mathrm{ex}}^{w} &= \{(u, v, w, Z) \,|\, w = w_{\mathrm{out}}\} \\ \Sigma_{\mathrm{ex}}^{\pm v_j} &= \left\{(u, v, w, Z) \,|\, v_j = v_{j,\mathrm{out}}\right\}. \end{aligned} \tag{C.45}$$

In this section we compute the transitions

$$\begin{aligned} \Pi^w : \Sigma_{\text{en}} &\to \Sigma^w_{\text{ex}} \\ (v, w, Z) &\mapsto (\tilde{u}, \tilde{v}_i, \tilde{Z}), \end{aligned} \tag{C.46}$$

for all $i \in [i, m]$, and

$$\begin{aligned} \Pi^{\pm v_j} : \Sigma_i &\to \Sigma^{\pm v_j}_{\text{ex}} \\ (v, w, Z) &\mapsto (\tilde{u}, \tilde{v}_i, \tilde{w}, \tilde{Z}), \end{aligned} \tag{C.47}$$

for all $i \in [1, m]$ with $i \neq j$.

Proposition C.4. *Consider the vector field Y given by (C.44) and let Σ_{en}, Σ^w_{ex}, $\Sigma^{\pm v_j}_{\text{ex}}$ and Π^w, $\Pi^{\pm v_j}$ be as above. Then*

- *The transition Π^w is given by*

$$\begin{aligned} \tilde{u} &= u\left(\frac{w}{w_{\text{out}}}\right)^{1/\gamma}, \qquad \tilde{v}_i = v_i\left(\frac{w_{\text{out}}}{w}\right)^{\beta_i/\gamma} \\ \tilde{Z} &= Z\exp\left[-\frac{\Lambda}{\gamma w}\left(1 + \tilde{\alpha} w \ln(w) + w\tilde{G}\right)\right] \end{aligned} \tag{C.48}$$

  where $\tilde{\alpha} = \tilde{\alpha}(uv_i^{1/\beta_i}, uw^{1/\gamma})$ and $\tilde{G} = \tilde{G}(uv_i^{1/\beta_i}, uw^{1/\gamma}, \mu_i)$ are C^ℓ functions with $\mu_i = v_i^{1/\beta_i} w^{-1/\gamma}$.
- *The transition $\Pi^{\pm v_j}$ is given by*

$$\begin{aligned} \tilde{u} &= \left(\frac{v_j}{\eta_j}\right)^{1/\beta}, \qquad \tilde{v}_i = v_i\left(\frac{\eta_j}{v_j}\right)^{\beta_i/\beta_j}, \qquad \tilde{w} = w\left(\frac{\eta_j}{v_j}\right)^{\gamma/\beta_j} \\ \tilde{Z} &= Z\exp\left[-\frac{\Lambda}{\gamma w}\left(1 + \tilde{\alpha}' w \ln(v_j) + w\tilde{G}'\right)\right], \end{aligned} \tag{C.49}$$

  with $i \neq j$ and where

$$\begin{aligned} \tilde{\alpha}' &= \tilde{\alpha}'(uv_i^{1/\beta_i}, uw^{1/\gamma}) \\ \tilde{G}' &= \tilde{G}'(uv_i^{1/\beta_i}, uw^{1/\gamma}, \mu_w, \mu_i) \end{aligned} \tag{C.50}$$

  are C^ℓ functions with $\mu_w = w^{1/\gamma} v_j^{1/\beta_j}$ and $\mu_i = v_i^{1/\beta_i} v_j^{1/\beta_j}$.

Proof of Proposition C.4. We detail first the computations for the transition Π^w. The transition $\Pi^{\pm v_j}$ is computed in a similar way so we only highlight the key parts of the computation.

The transition Π^w

In this case, the time of integration is $T = \ln\left(\frac{w_{\text{out}}}{w}\right)^{1/\gamma}$, where $w_{\text{out}} = w(t)|_{\Sigma^w_{\text{ex}}}$ and $w = w(t)|_{\Sigma_{\text{en}}}$. This time of integration is obtained form the equation $w' = \gamma w$. We also make the assumption that $v_i \in O(w^{\beta_i/\gamma})$. This assumption appears our applications, but roughly speaking it ensures that $\tilde{v}_i$ is well defined when $w \to 0$. From the form of Y we evidently have

$$\begin{aligned} u(T) &= \tilde{u} = u\left(\frac{w}{w_{\text{out}}}\right)^{1/\gamma} \\ v_i(T) &= \tilde{v}_i = v_i\left(\frac{w_{\text{out}}}{w}\right)^{\beta_i/\gamma}. \end{aligned} \tag{C.51}$$

It only remains to compute the transition for the Z coordinate. Let us rewrite Y as follows

$$\begin{aligned} u' &= -u \\ v_i &= \beta_i v_i \\ w &= \gamma w \\ Z' &= -\frac{\Lambda + G(u, v, w)}{w} Z, \end{aligned} \tag{C.52}$$

where G is a C^ℓ function vanishing at the origin. Observe that we have the first integrals $u^{b_i} v_i$ and $u^\gamma w$. We shall take advantage of such a fact. We define new coordinates (U, V, W) given by

$$U = u,\ V_i^{\beta_i} = v_i,\ W^\gamma = w. \tag{C.53}$$

In these new coordinates we have the system

$$\begin{aligned} U' &= -U \\ V_i' &= V_i \\ W' &= W \\ Z' &= -\frac{\Lambda + G(U, V^{\beta_i}, W^\gamma)}{W^\gamma} Z. \end{aligned} \tag{C.54}$$

In the new coordinates, the time of integration is given as $T = \ln\left(\frac{W_o}{W}\right)$. To have an idea of the expression of $\tilde{Z}$, let us first study a simplified scenario.

The case $G = 0$

Let us suppose $G = 0$. Therefore we have $Z' = -\frac{\Lambda}{W^\gamma} Z$, which has the solution

$$Z(t) = Z(0)\exp\left(-\Lambda \int_0^t W(s)^{-\gamma} ds\right), \tag{C.55}$$

where $W(s) = W(0)\exp(s)$. Substituting the time of integration T we have

$$Z(T)=\tilde{Z}=Z\exp\left(-\frac{\Lambda}{W^{\gamma}}\int_{0}^{\ln\left(\frac{W_o}{W}\right)} e^{-\gamma s}ds\right)$$
$$=Z\exp\left(-\frac{\Lambda}{\gamma W^{\gamma}}\left(1-\left(\frac{W}{W_o}\right)^{\gamma}\right)\right). \tag{C.56}$$

Observe that $\tilde{Z}\to 0$ as $W\to 0$. Let us now study the general case. We expect that the general case $G\neq 0$ is a perturbation of (C.56).

The case $G\neq 0$

We now consider that $G\neq 0$, we have

$$Z(T)=\tilde{Z}=Z\exp\left(I_0+I_1\right), \tag{C.57}$$

where

$$I_0=-\Lambda\int_0^T \frac{1}{W(s)}ds$$
$$I_1=\int_0^T \frac{G(U(s),V(s)^{\beta_i},W(s)^{\gamma})}{W(s)^{\gamma}}ds. \tag{C.58}$$

The integral I_0 has already been computed above. Let us write $F(U,V,W)=\frac{G(U(s),V(s)^{\beta_i},W(s)^{\gamma})}{W(s)^{\gamma}}$. We can do this because $G(U,0,0)=0$ and $V^{\beta_i}\in O(W^{\gamma})$. Now we estimate the integral I_1. Using Lemma C.1, we can write

$$I_1=\int_0^T [F_1(s)+F_2(s)]ds, \tag{C.59}$$

where

$$F_1=F_1(UV_1,\dots,UV_m,UW,U)$$
$$F_2=F_2(UV_1,\dots,UV_m,UW,V_1,\dots,V_m,W). \tag{C.60}$$

Observe that UW and all the UV_j's' are first integrals. Let $J_1=\int F_1$ and $J_2=\int F_2$. Then we have

$$J_1=\int_0^T F_1(UV,UW,U(s))ds$$

$$= \int_{0}^{\ln\left(\frac{W_o}{W}\right)} F_1(UV, UW, Ue^{-s})ds. \tag{C.61}$$

Let us make the change of variables $y = e^{-s}$, we obtain

$$J_1 = -\int_{1}^{\frac{W}{W_o}} F_1(UV, UW, Uy)\frac{dy}{y}. \tag{C.62}$$

We expand the function F_1 in power of y that is

$$F_1(UV, UW, Uy) = F_1(UV, UW, 0) + O(y). \tag{C.63}$$

Then we have

$$J_1 = -\int_{1}^{\frac{W}{W_o}} \alpha_1 \frac{dy}{y} + \tilde{F}_1, \tag{C.64}$$

where $\alpha_1 = \alpha_1(UV, UW)$ and $\tilde{F}_1 = \tilde{F}_1(UV, UW, Uy(T))$ is some (unknown) C^ℓ function. Finally we get

$$J_1 = \alpha_1 \ln\left(\frac{W_0}{W}\right) + \tilde{F}_1\left(UV, UW, U\frac{W}{W_0}\right). \tag{C.65}$$

The function $\tilde{F}_1$ is C^ℓ but unknown, and W_0 is a fixed positive constant, then we can simplify the notation of $\tilde{F}_1$ as $\tilde{F}_1 = \tilde{F}_1(UV, UW)$.

Next we have

$$\begin{aligned} J_2 &= \int_{0}^{T} F_2(UV, UW, V(s), W(s))ds \\ &= \int_{0}^{\ln\left(\frac{W_o}{W}\right)} F_2(UV, UW, V_1e^{\beta_1 s}, \ldots, V_m e^{\beta_m s}, We^{\gamma s})ds. \end{aligned} \tag{C.66}$$

Let us make the change of variables $y = e^s$. Then we obtain

$$J_2 = \int_{1}^{\frac{W_o}{W}} F_2(UV, UW, V_1 y^{\beta_1}, \ldots, V_m y^{\beta_m}, Wy^{\gamma})\frac{dy}{y}. \tag{C.67}$$

As above, we expand in powers of y, that is

$$F_2 = \alpha_2 + O(y), \tag{C.68}$$

and then we have

$$J_2 = \alpha_2 \ln\left(\frac{W_0}{W}\right) + \tilde{F}_2, \tag{C.69}$$

where $\alpha_2 = \alpha_2(UV, UW)$, $F_2 = F_2(UV, UW, \mu_i)$ is a C^ℓ function with $\mu_i = V_i W^{-1}$ for all $i \in [1, m]$. Recall that since $v_i \in O(w^{\beta_i/\gamma})$ we also have that $V \in O(W)$, that is μ_i is well defined.

Now we can write the integral I_1 as

$$\begin{aligned} I_1 &= J_1 + J_2 \\ &= \alpha_1 \ln\left(\frac{W_0}{W}\right) + \tilde{F}_1 + \alpha_2 \ln\left(\frac{W_0}{W}\right) + \tilde{F}_2 \\ &= \alpha \ln\left(\frac{W_0}{W}\right) + \tilde{F}, \end{aligned} \tag{C.70}$$

where $\alpha = \alpha(UV, UW)$ and $\tilde{F} = \tilde{F}(UV, UW, \mu_i)$ are C^ℓ functions. Finally we write $\tilde{Z}$ in the original coordinates as follows

$$\begin{aligned} \tilde{Z} &= Z \exp(I_0 + I_1) \\ &= Z \exp\left[-\frac{\Lambda}{\gamma w}\left(1 - \frac{w}{w_{\text{out}}}\right) + \frac{1}{\gamma}\alpha \ln\left(\frac{w_{\text{out}}}{w}\right) + \tilde{F}\right] \\ &= Z \exp\left[-\frac{\Lambda}{\gamma w}\left(1 + \tilde{\alpha} w \ln(w) + w\tilde{G}\right)\right], \end{aligned} \tag{C.71}$$

where $\tilde{\alpha} = \tilde{\alpha}(uv_i^{1/\beta_i}, uw^{1/\gamma})$ and $\tilde{G} = \tilde{G}(uv_i^{1/\beta_i}, uw^{1/\gamma}, \mu_i)$ are C^ℓ functions with $\mu_i = v_i w^{-\beta_i/\gamma}$.

The transition $\Pi^{\pm v_j}$

In this case the time of integration is given by $T = \ln\left(\frac{\eta_j}{v_j}\right)^{1/\beta_j}$. Such a time of integration is obtained from the equation $v_j' = \beta_j v_j$. The we have

$$\begin{aligned} \tilde{u} &= u\left(\frac{v_j}{\eta_j}\right)^{1/\beta} \\ \tilde{v}_i &= v_i\left(\frac{\eta_j}{v_j}\right)^{\beta_i/\beta_j} \\ \tilde{w} &= w\left(\frac{\eta_j}{v_j}\right)^{\gamma/\beta_j}. \end{aligned} \tag{C.72}$$

It then only rests to compute $\tilde{Z}$. Following similar arguments as for the transition Π^w we get in this case

$$\tilde{Z} = Z \exp\left[-\frac{\Lambda}{\gamma w} \left(1 + \tilde{\alpha}' w \ln(v_j) + w \tilde{G}' \right) \right], \tag{C.73}$$

where now

$$\begin{aligned} \tilde{\alpha}' &= \tilde{\alpha}'(u v_i^{1/\beta_i}, u w^{1/\gamma}) \\ \tilde{G}' &= \tilde{G}'(u v_i^{1/\beta_i}, u w^{1/\gamma}, \mu_w, \mu_i) \end{aligned} \tag{C.74}$$

are C^ℓ functions with $\mu_w = w v_j^{-\gamma/\beta_j}$ and $\mu_i = v_i v_j^{-\beta_i/\beta_j}$. $\square$

Saddle 2

In this case we investigate the transitions of a vector field of the form

$$Y : \begin{cases} u' = wu \\ v_j' = -\beta_j w v_j \\ w' = -\gamma w^2 \\ Z' = -gZ, \end{cases} \tag{C.75}$$

where the coefficients β_j, γ are positive. We assume that $u \in \mathbb{R}^+$. Observe that now, in contrast with case 1, we only have one expanding direction, which is u. This makes the study of the transition easier. Due to the same reason, it is more convenient to study a transition

$$\Pi^u : \Sigma_{\text{en}} \to \Sigma_{\text{ex}}, \tag{C.76}$$

where to be general, we let Σ_{en} be any codimension 1 subset of $\mathbb{R}^{m+3}$ obtained by setting one of the coordinates (v, w) to a constant and with $u < u_{\text{out}}$; and where

$$\Sigma_{\text{ex}} = \left\{ (u, \tilde{v}, \tilde{w}, \tilde{Z}) \,|\, \tilde{u} = u_{\text{out}} \right\}. \tag{C.77}$$

Proposition C.5. *Consider the vector field Y given by (C.75) and let Σ_{en}, Σ_{ex} and Π^u be as above. Then*

$$\begin{aligned} \tilde{v}_i &= v_i \left(\frac{u}{u_{\text{out}}} \right)^{\beta_i} \\ \tilde{w} &= w \left(\frac{u}{u_{\text{out}}} \right)^{\gamma} \\ \tilde{Z} &= Z \exp\left[-\frac{\Lambda}{\gamma w} \left(\left(\frac{u_{\text{out}}}{u} \right)^{\gamma} - 1 + \alpha w \ln(u) + w \tilde{F} \right) \right] \end{aligned} \tag{C.78}$$

where $\alpha = \alpha(u^{\beta_i} v_i, u^\gamma w)$ and $\tilde{F} = \tilde{F}(u^{\beta_i} v_i, u^\gamma w, u)$ are C^ℓ functions.

Proof of Proposition C.5. We have that the time of integration is $T = \ln\left(\frac{u_{\text{out}}}{u}\right)$. It follows that

$$\tilde{v}_i = v_i \left(\frac{u}{u_{\text{out}}}\right)^{\beta_i}$$
$$\tilde{w} = w \left(\frac{u}{u_{\text{out}}}\right)^{\gamma}. \tag{C.79}$$

It only remains to compute $\tilde{Z}$. Following similar arguments as in case 1 we have

$$\tilde{Z} = Z \exp\left[-\frac{\Lambda}{\gamma w}\left(\left(\frac{u_{\text{out}}}{u}\right)^{\gamma} - 1 + \alpha w \ln(u) + w\tilde{F}\right)\right], \tag{C.80}$$

where $\alpha = \alpha(u^{\beta_i} v_i, u^{\gamma} w)$ and $\tilde{F} = \tilde{F}(u^{\beta_i} v_i, u^{\gamma} w, u)$ are C^{ℓ} functions. □

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