Control of port-Hamiltonian systems

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Mechanical Systems: Velocity Estimation and Output Feedback Stabilization

"Strive for perfection in everything you do. Take the best that exists and make it better. When it does not exist, design it." - Sir Henry Royce.

In the first and second chapters, we had given a brief introduction to mechanical systems which are modeled by the Hamiltonian equations (2.9). As we discussed, in such systems the velocity measurements (which is the port-Hamiltonian output) can be subjected to noise and hence can be inaccurate while the position measurements may be much more accurate. Therefore, it is of interest to build observers for these systems, to estimate the velocity based on the position measurements and subsequently design a stabilizing control law for the system using the position measurements and the velocity estimates. In this chapter, we deal with both these problems for a special class of mechanical systems.

3.1 Introduction

We recall the Hamiltonian equations used for modeling a mechanical system being given by

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{pmatrix} \nabla_q H \\ \nabla_p H \end{pmatrix} + \begin{pmatrix} 0 \\ G(q) \end{pmatrix} u, \tag{3.1}$$

where $q, p \in \mathbb{R}^n$ are the generalized positions and momenta, respectively, $u \in \mathbb{R}^m$ is the control input, $m \leq n$ and $G : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ is a full rank matrix. The Hamiltonian function $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the total energy of the system and is given as

$$H(q,p) = \frac{1}{2} p^\top M^{-1}(q) p + U(q), \tag{3.2}$$

where $M : \mathbb{R}^n \to \mathbb{R}^{n\times n}_P$ is the mass matrix and $U : \mathbb{R}^n \to \mathbb{R}$ is the potential energy function, with $\mathbb{R}^{n\times n}_P$ being the set of $n \times n$ positive definite matrices. Sometimes, frictional effects are also included in the model of the mechanical system in which case, our proposed observer incorporates a term that requires the knowledge of the frictional forces. Since, in many applications, these forces are negligible or usually quite uncertain, we chose to omit their presence while designing the observer.

The observer design problem is formulated as follows. We assume that q is measurable, p is unmeasurable and the input signal $u(t)$ is such that the system (3.1) is forward complete, that is, trajectories exist for all $t \geq 0$. Our first objective is to design an asymptotically convergent observer for p. The second objective is to prove that the observer can be used in conjunction with the interconnection and damping assignment passivity–based controller (IDA–PBC) preserving asymptotic stability by assuming the existence of a full state feedback (IDA–PBC) that asymptotically stabilizes a desired equilibrium point $(q_\star, 0)$.

We focus on mechanical systems that can be rendered linear in the unmeasurable momenta via a change of coordinates of the form $(q, P) = (q, \Psi^\top(q)p)$, with $\Psi : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ being a full rank matrix. The class of systems that satisfy this property, which is fully determined by the inertia matrix M, will be called henceforth “Partially Linearizable via Coordinate Changes” (PLvCC). As illustrated in [11, 25, 44, 80, 92], achieving linearity in P simplifies the observer design as well as the control problem. However, the class of mechanical systems considered in the literature in the context of linearization is only a small subset of all PLvCC systems. Further, the linearization conditions considered in the literature imposes quite restrictive assumptions on the inertia matrix M. In contrast to this situation, we give a complete characterization of PLvCC systems, in terms of solvability of a set of partial differential equations (PDE)’s and show that the class contains many examples of practical interest.

We propose for the PLvCC systems, a globally (exponentially) convergent reduced order immersion and invariance (I&I) observer [6]. This design imposes an integrability condition, which is equivalent to solving a second set of PDEs. We propose a systematic procedure to solve these PDEs for a special subclass of PLvCC systems and illustrate it with several practical examples. Furthermore, we show that the integrability condition can be obviated using the full order I&I observer with dynamic scaling recently proposed in [44]. However, (as we shall show) the full-order observer design based on [44] increases the complexity of the computations and also involves the injection of high gain (which is not very desired). A final contribution of our work is the proof that the proposed observer solves the position feedback stabilization problem mentioned above.

Notation used in this chapter: For any matrix $A \in \mathbb{R}^{n\times n}$, $A_i \in \mathbb{R}^n$ denotes the i–th column, A^i the i–th row and A_{ij} the ij–th element. That is, with $e_i,\ i \in \bar{n} := \{1, \ldots, n\}$, the Euclidean basis vectors, $A_i := Ae_i$, $A^i := e_i^\top A$ and

$A_{ij} := e_i^\top A e_j$.

3.2 Characterization of the Class of PLvCC Systems

In this section, we identify the class of mechanical systems for which a change of coordinates of the form $(q, P) = (q, \Psi^\top(q)p)$, where Ψ has a full rank for all q, renders the system linear in P. As we shall see, this property is uniquely defined by the inertia matrix M.

We first state two important lemmas which will be used by us later for proving our main result.

Lemma 3.1. *Let $\Psi(q)$ be an $n \times n$ full rank matrix. Define the $n \times n$ matrix*

$$\mathcal{J} := \sum_{i=1}^{n} \{(p^\top \nabla_{q_i} \Psi)^\top (e_i^\top \Psi) - (\Psi^\top e_i)(p^\top \nabla_{q_i} \Psi)\} \tag{3.3}$$

Then, the $(jk)^{th}$ element of the matrix $\mathcal{J}$ is given by

$$\mathcal{J}_{jk} = -p^\top [\Psi_j, \Psi_k], \tag{3.4}$$

where $[\Psi_j, \Psi_k]$ is the standard Lie bracket of the column vectors Ψ_j and Ψ_k.

Proof. We prove the lemma by computing the (jk)–th element of $\mathcal{J}$ as

$$\begin{aligned} e_j^\top \mathcal{J} e_k &= \sum_{i=1}^{n} \{(p^\top (\nabla_{q_i} \Psi) e_j) \Psi_{ik} - (p^\top (\nabla_{q_i} \Psi) e_k) \Psi_{ij}\} \\ &= p^\top \sum_{i=1}^{n} \{(\nabla_{q_i} \Psi_j) \Psi_{ik} - (\nabla_{q_i} \Psi_k) \Psi_{ij}\} \\ &= p^\top \{(\nabla_q \Psi_j) \Psi_k - (\nabla_q \Psi_k) \Psi_j\} \\ &= -p^\top [\Psi_j, \Psi_k]. \end{aligned}$$

□

Lemma 3.2. *Define the $n \times n$ matrices*

$$\bar{J}_i := \sum_{j=1}^{n} [\Psi_i, \Psi_j] \Psi_j^\top (\Psi \Psi^\top)^{-1} M^{-1}, \quad i \in \bar{n}.$$

Then

$$\sum_{i=1}^{n} (\Psi^\top e_i)(p^\top \nabla_{q_i} \Psi) \Psi^{-1} M^{-1} p - \dot{\Psi}^\top p = \sum_{i=1}^{n} e_i (p^\top \bar{J}_i p).$$

Proof. We first note that

$$\begin{aligned}\dot{\Psi}^\top p &= \sum_{i=1}^{n}(p^\top \nabla_{q_i}\Psi)^\top (e_i^\top M^{-1}p)\\ &= \sum_{i=1}^{n}(p^\top \nabla_{q_i}\Psi)^\top (e_i^\top \Psi)\Psi^{-1}M^{-1}p.\end{aligned}$$

Replacing (3.3) of Lemma 3.1 we obtain

$$\dot{\Psi}^\top p - \sum_{i=1}^{n}(\Psi^\top e_i)(p^\top \nabla_{q_i}\Psi)\Psi^{-1}M^{-1}p = \mathcal{J}\Psi^{-1}M^{-1}p.$$

Computing the i–th element of the vector completes the proof

$$\begin{aligned}e_i^\top \mathcal{J}\Psi^{-1}M^{-1}p &= e_i^\top \mathcal{J}\Psi^\top(\Psi\Psi^\top)^{-1}M^{-1}p\\ &= \sum_{k=1}^{n}\mathcal{J}_{ik}e_k^\top \Psi^\top(\Psi\Psi^\top)^{-1}M^{-1}p\\ &= -\sum_{k=1}^{n}p^\top[\Psi_i,\Psi_k]\Psi_k^\top(\Psi\Psi^\top)^{-1}M^{-1}p\\ &= -p^\top \bar{J}_i p,\end{aligned}$$

where we have used (3.15) to obtain the third identity. □

We next state the following proposition.

Proposition 3.3. *Let Ψ be an $n\times n$ full rank matrix. The dynamics of (3.1) expressed in the coordinates (q,P), where $P=\Psi^\top(q)p$, is linear in P if and only if for $i\in\bar{n}$, $\bar{n}:=\{1,\dots,n\}$, we have*

$$\mathcal{B}_{(i)}(q)+\mathcal{B}_{(i)}^\top(q)=0, \tag{3.5}$$

where the matrices $\mathcal{B}_{(i)}:\mathbb{R}^n\to\mathbb{R}^{n\times n}$ are defined as

$$\mathcal{B}_{(i)}(q) := \sum_{j=1}^{n}\Big\{[\Psi_i,\Psi_j]\Psi_j^\top(M\Psi\Psi^\top)^{-1}+\frac{1}{2}\Psi_{ji}\Psi\nabla_{q_j}(\Psi^\top M\Psi)^{-1}\Psi^\top\Big\}, \tag{3.6}$$

with $[\Psi_i,\Psi_j]$ being the standard Lie bracket. Under the condition (3.5), the dynamics becomes

$$\dot{q}=(\Psi^\top M)^{-1}P,\ \dot{P}=-\Psi^\top\left(\nabla U - Gu\right). \tag{3.7}$$

Proof. The equation for $\dot{q}$ follows trivially from the definition of P. Now, $\dot{P}$ can be expressed as

$$\begin{aligned}\dot{P} &= \dot{\Psi}^\top p+\Psi^\top\dot{p}\\ &= -\mathbf{D}_\Psi(q,p)-\Psi^\top(\nabla U(q)-G(q)u),\end{aligned} \tag{3.8}$$

where the parameterized mapping $\mathbf{D}_\Psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$\mathbf{D}_\Psi(q,p) := \Psi^\top \nabla_q \{\frac{1}{2} p^\top M^{-1} p\} - \dot{\Psi}^\top p. \tag{3.9}$$

We will now show that each element of the vector $\mathbf{D}_\Psi$ is a quadratic form in p, that is,

$$\mathbf{D}_\Psi = \sum_{i=1}^{n} e_i p^\top \mathcal{B}_{(i)} p, \tag{3.10}$$

that becomes zero for all p if and only if the condition 3.5 is satisfied. To show this, we compute

$$\begin{aligned}
\nabla_q \{\frac{1}{2} p^\top M^{-1} p\} &= \nabla_q \Big\{\frac{1}{2} p^\top \Psi (\Psi^\top M \Psi)^{-1} \Psi^\top p\Big\} \\
&= \frac{1}{2} \sum_{i=1}^{n} e_i \Big\{ 2p^\top (\nabla_{q_i} \Psi)(\Psi^\top M \Psi)^{-1} \Psi^\top p \\
&\quad + p^\top \Psi \nabla_{q_i} ((\Psi^\top M \Psi)^{-1}) \Psi^\top p \Big\} \\
&= \sum_{i=1}^{n} e_i \Big\{ p^\top (\nabla_{q_i} \Psi) \Psi^{-1} M^{-1} p \\
&\quad + \frac{1}{2} p^\top \Psi \nabla_{q_i} ((\Psi^\top M \Psi)^{-1}) \Psi^\top p \Big\}.
\end{aligned} \tag{3.11}$$

Replacing (3.11) in (3.9) we obtain

$$\begin{aligned}
\mathbf{D}_\Psi &= -\dot{\Psi}^\top p + \sum_{i=1}^{n} \Big\{ (\Psi^\top e_i)(p^\top \nabla_{q_i} \Psi) \Psi^{-1} M^{-1} p \\
&\quad + \frac{1}{2} (\Psi^\top e_i) [p^\top \Psi \nabla_{q_i} ((\Psi^\top M \Psi)^{-1}) \Psi^\top p] \Big\}, \\
&= \sum_{i=1}^{n} \Big\{ e_i (p^\top \bar{J}_i p) + \frac{1}{2} (\Psi^\top e_i) [p^\top \Psi \nabla_{q_i} ((\Psi^\top M \Psi)^{-1}) \Psi^\top p] \Big\}, \qquad (3.12) \\
&= \sum_{i=1}^{n} e_i p^\top \mathcal{B}_{(i)} p, \qquad (3.13)
\end{aligned}$$

where we use Lemma 3.2 to obtain the first term in (3.12) and the definition of $\mathcal{B}_{(i)}$ given in (3.6) to obtain (3.13). Hence, the proof follows. □

If the inertia matrix M satisfies the condition (3.5) for some full rank matrix Ψ, then the mechanical system (3.1) becomes linear in P and (as described earlier) is said to be PLvCC. The class of M for which there exists such a Ψ is denoted by $\mathcal{S}_{\texttt{PLvCC}}$. That is, $M \in \mathcal{S}_{\texttt{PLvCC}}$ if and only if there exists Ψ such that (3.5) is satisfied.

For the sake of completeness, we compute the resultant dynamics in the (q, P) coordinates for a general full rank matrix Ψ which may or may not satisfy the condition 3.5. We state the following Lemma.

Lemma 3.4. *For a general full rank matrix Ψ, the transformed dynamics in the coordinates (q, P) is given by*

$$\begin{pmatrix} \dot{q} \\ \dot{P} \end{pmatrix} = \begin{bmatrix} 0 & \Psi \\ -\Psi^\top & \mathcal{J} \end{bmatrix} \begin{pmatrix} \nabla_q \bar{H} \\ \nabla_P \bar{H} \end{pmatrix} + \begin{pmatrix} 0 \\ \Psi^\top G \end{pmatrix} u, \tag{3.14}$$

with the new energy function being

$$\bar{H}(q, P) := \frac{1}{2} P^\top (\Psi^\top M \Psi)^{-1} P + U(q),$$

and the jk element of the skew–symmetric matrix $\mathcal{J} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ being given by

$$\mathcal{J}_{jk}(q, p) = -p^\top [\Psi_j, \Psi_k]. \tag{3.15}$$

Proof. The expression for $\dot{q}$ is obtained in a straightforward manner. We now compute the expression for $\dot{P}$. The proof proceeds as follows (also refer to [93]). We first note that $\mathbf{D}_\Psi$ can be written as

$$\begin{aligned} \mathbf{D}_\Psi &= \sum_{i=1}^{n} [(\Psi^\top e_i)(p^\top \nabla_{q_i} \Psi) - (p^\top \nabla_{q_i} \Psi)^\top (e_i^\top \Psi)] \nabla_P \bar{H} \\ &\quad + \frac{1}{2} \sum_{i=1}^{n} (\Psi^\top e_i)[P^\top \nabla_{q_i}((\Psi^\top M \Psi)^{-1}) P] \\ &= -\mathcal{J} \nabla_P \bar{H} + \frac{1}{2} \sum_{i=1}^{n} (\Psi^\top e_i)[P^\top \nabla_{q_i}((\Psi^\top M \Psi)^{-1}) P], \end{aligned}$$

where we used the following identity

$$\begin{aligned} \nabla_q \{\frac{1}{2} p^\top M^{-1} p\} &= \sum_{i=1}^{n} e_i \Big\{ p^\top (\nabla_{q_i} \Psi) \Psi^{-1} M^{-1} p + \frac{1}{2} p^\top \Psi \nabla_{q_i}((\Psi^\top M \Psi)^{-1}) \Psi^\top p \Big\} \\ &= \sum_{i=1}^{n} e_i \Big\{ p^\top (\nabla_{q_i} \Psi)(\Psi^\top M \Psi)^{-1} P + \frac{1}{2} P^\top (\nabla_{q_i} \Psi)((\Psi^\top M \Psi)^{-1}) P \Big\} \\ &= \sum_{i=1}^{n} e_i \Big\{ p^\top (\nabla_{q_i} \Psi) \nabla_P \bar{H} + \frac{1}{2} P^\top \nabla_{q_i}((\Psi^\top M \Psi)^{-1}) P \Big\}, \end{aligned}$$

and the definition of $\mathcal{J}$ given by (3.3) in Lemma 1. Replacing the expression of $\mathbf{D}_\Psi$ in (3.8) we finally obtain

$$\begin{aligned} \dot{P} &= \mathcal{J} \nabla_P \bar{H} - \frac{1}{2} \sum_{i=1}^{n} (\Psi^\top e_i)[P^\top \nabla_{q_i}((\Psi^\top M \Psi)^{-1}) P] - \Psi^\top (\nabla U - G u) \\ &= \mathcal{J} \nabla_P \bar{H} - \Psi^\top \nabla_q \bar{H} + \Psi^\top G u, \end{aligned}$$

which corresponds to (3.14). □

The following remarks are in order.

Remark 3.5. Equation (3.6) may be alternatively viewed as a family of operators, parameterized by Ψ, mapping $\mathbb{R}_P^{n\times n}$ into $\mathbb{R}^{n\times n}$. The system is PLvCC if there exists Ψ such that, upon the action of this operator, its mass matrix is mapped into a skew-symmetric matrix.

Remark 3.6. The results obtained in Section 3.2 can be alternatively expressed in a more compact form using Poisson brackets. Please refer to [73] for further details.

We thus characterized the class of mechanical systems which can be partially linearized via a change of coordinates i.e., showed that they belong to the set $\mathcal{S}_{\mathtt{PLvCC}}$. In the next section, we give a physical interpretation for the set $\mathcal{S}_{\mathtt{PLvCC}}$.

3.3 How Large is the Set $\mathcal{S}_{\mathtt{PLvCC}}$?

A natural question that arises at this point is: For what kind of inertia matrices M is the condition (3.5) satisfied? Providing a complete answer is equivalent to characterizing the solvability of the PDE's (3.5), (3.6) in the unknown function Ψ, which appears to be a daunting task. However, it turns out that this set contains some interesting subsets that have a clear physical interpretation and in some cases also a differential geometric interpretation. Some of these sets have been studied in the literature, which we now briefly review in this section.

3.3.1 Four Subsets of the Set $\mathcal{S}_{\mathtt{PLvCC}}$

To get a better understanding of the condition (3.5), we present four sets of non–decreasing cardinality (displayed in Figure 3.1), and show them to be subsets of $\mathcal{S}_{\mathtt{PLvCC}}$. Three of them are well–known, but the fourth (and far more interesting) one does not seem to have been reported in the literature. We now introduce the following important definitions that will be used repeatedly in the sequel.

Definition 3.7. *A full rank matrix $T:\mathbb{R}^n \to \mathbb{R}^{n\times n}$ is said to be a factor of M^{-1} if*

$$M^{-1}(q) = T(q)T^\top(q). \tag{3.16}$$

Definition 3.8. *The sets of inertia matrices $\mathcal{S}_{\mathtt{CI}}$, $\mathcal{S}_{\mathtt{ZCS}}$, $\mathcal{S}_{\mathtt{ZRS}}$, $\mathcal{S}_{\mathtt{T}}$ are defined as follows.*

(i) (Constant inertia)

$$\mathcal{S}_{\mathtt{CI}} := \{M \in \mathbb{R}_P^{n\times n} \mid M_{ij} = \text{constant},\ i,j \in \bar{n}\},$$

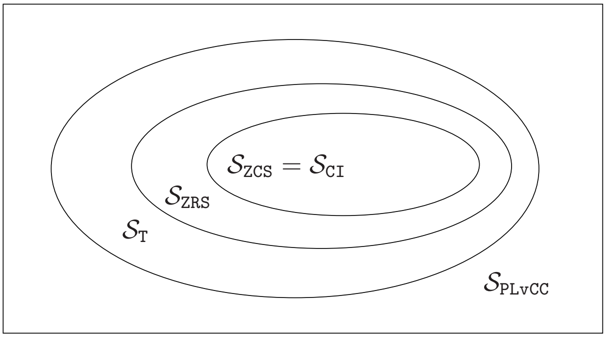

Figure 3.1: Sets of inertia matrices with non-decreasing cardinality

(ii) (Zero Christoffel symbols)

$$\mathcal{S}_{\mathtt{ZCS}} := \{M \in \mathbb{R}_P^{n\times n} \mid C_{ijk} = 0,\ i,j,k \in \bar{n}\},$$

where $C_{ijk} : \mathbb{R}^n \to \mathbb{R}$ *are the Christoffel symbols of the first kind defined in (2.4) for a given inertia matrix* M.

(iii) (Zero Riemann symbols)

$$\mathcal{S}_{\mathtt{ZRS}} := \{M \in \mathbb{R}_P^{n\times n} \mid R_{ijlk} = 0,\ i,j,l,k \in \bar{n}\},$$

with $R_{ijlk} : \mathbb{R}^n \to \mathbb{R}$ *are the Riemann symbols given by*

$$\begin{aligned} R_{ijlk}(q) := \frac{1}{2}\left[\nabla^2_{q_j q_l} M_{ik} + \nabla^2_{q_i q_k} M_{jl} - \nabla^2_{q_j q_k} M_{il} - \nabla^2_{q_i q_l} M_{jk}\right] \\ + \sum_{a,b=1}^{n} (M^{-1})_{ab}\left[C_{jla}C_{ikb} - C_{ila}C_{jkb}\right]. \end{aligned} \tag{3.17}$$

(iv) (Skew–symmetry condition)

$$\begin{aligned} \mathcal{S}_{\mathtt{T}} := \Big\{ M \in \mathbb{R}_P^{n\times n} \mid M^{-1} \text{ admits a factor } T \text{ such that} \\ \sum_{j=1}^{n}[T_i, T_j]T_j^\top = -\left[\sum_{j=1}^{n}[T_i, T_j]T_j^\top\right]^\top, i \in \bar{n} \Big\}. \end{aligned} \tag{3.18}$$

We next give the following proposition which proves the non-decreasing cardinality of the above defined sets as also indicated in Figure 3.1.

Proposition 3.9. *The sets of inertia matrices in Definition 3.8 satisfy*

$$\mathcal{S}_{\mathtt{CI}} = \mathcal{S}_{\mathtt{ZCS}} \subset \mathcal{S}_{\mathtt{ZRS}} \subset \mathcal{S}_{\mathtt{T}} \subseteq \mathcal{S}_{\mathtt{PLvCC}},$$

where the inclusion $\mathcal{S}_{\mathtt{ZCS}} \subset \mathcal{S}_{\mathtt{ZRS}}$ *is* strict *for every* $n > 1$*, and the inclusion* $\mathcal{S}_{\mathtt{ZRS}} \subset \mathcal{S}_{\mathtt{T}}$ *is* strict *for every* $n > 2$*.*

Proof. • ($\mathcal{S}_{\mathtt{CI}} = \mathcal{S}_{\mathtt{ZCS}}$) The fact that $[M \in \mathcal{S}_{\mathtt{CI}} \Rightarrow M \in \mathcal{S}_{\mathtt{ZCS}}]$ follows trivially from the definition of the Christoffel symbols in (2.4). The proof for $[M \in \mathcal{S}_{\mathtt{ZCS}} \Rightarrow M \in \mathcal{S}_{\mathtt{CI}}]$ can be worked out as follows.

Assume all Christoffel symbols are identically equal to zero, that is,

$$C_{ijk}(q) := \frac{1}{2}\left\{\nabla_{q_j} M_{ik} + \nabla_{q_i} M_{jk} - \nabla_{q_k} M_{ij}\right\} \equiv 0, \tag{3.19}$$

for all q. The equations above define $\frac{1}{2}\{n^3 + n^2\}$ PDE's where we used the fact that $C_{ijk} = C_{jik}$. First consider the case when $i = j = k$. This gives the n equations,

$$C_{iii} = \nabla_{q_i} M_{ii} = 0, \tag{3.20}$$

for all $0 \le i \le n$. Next consider the case $i = k \ne j$. Then for every $0 \le i, j \le n$, $i \ne j$ we get the $\frac{n(n-1)}{2}$ equations,

$$C_{iji} = \nabla_{q_j} M_{ii} = 0. \tag{3.21}$$

Thus, from (3.20), (3.21) we conclude that all the diagonal entries of the inertia matrix are constant. Next, we choose $i = j \ne k$ and impose $C_{iik}(q) = 0$ which gives the $n(n-1)$ equations

$$C_{iik} = \nabla_{q_i} M_{ik} = 0, \tag{3.22}$$

for every $0 \le i, k \le n$ and $i \ne k$. But, we also know that $M_{ik} = M_{ki}$ and hence from (3.22) we conclude that,

$$\nabla_{q_i} M_{ik} = \nabla_{q_k} M_{ik} = 0, \tag{3.23}$$

for all $0 \le i, k \le n$, $i \ne k$. At this point we can conclude from (3.22), (3.23) that each off-diagonal element M_{ij} is such that

$$\nabla_{q_i} M_{ij} = \nabla_{q_j} M_{ij} = 0. \tag{3.24}$$

Next, we choose $j = k \ne i$ and impose $C_{ikk}(q) = 0$ which gives the $\frac{n(n-1)}{2}$ equations

$$C_{ikk} = \nabla_{q_k} M_{ik} + \nabla_{q_i} M_{kk} - \nabla_{q_k} M_{ik} \equiv 0. \tag{3.25}$$

Hence, these $\frac{n(n-1)}{2}$ equations in (3.25) identically become equal to zero. Finally, we consider $i \ne j \ne k$ and get $\frac{n(n-1)(n-2)}{2}$ equations in total which are of the form,

$$C_{ijk} = \frac{1}{2}\left\{\nabla_{q_j} M_{ik} + \nabla_{q_i} M_{jk} - \nabla_{q_k} M_{ij}\right\} = 0. \tag{3.26}$$

From (3.26), we obtain that

$$C_{ijk} + C_{ikj} = \nabla_{q_i} M_{jk} = 0, \tag{3.27}$$

which holds for all $i \neq j \neq k$. Thus, from (3.24) and (3.27) we can conclude that each off-diagonal element M_{ij} is constant, establishing the claim.

- ($\mathcal{S}_{\text{ZCS}} \subset \mathcal{S}_{\text{ZRS}}$) The proof that $[M \in \mathcal{S}_{\text{ZCS}} \Rightarrow M \in \mathcal{S}_{\text{ZRS}}]$ follows from the identity $\mathcal{S}_{\text{CI}} = \mathcal{S}_{\text{ZCS}}$ and the definition of the Riemann symbols (3.17). To show that the inclusion is strict, we first recall a well known characterization of the set $\mathcal{S}_{\text{ZRS}}$, which may be found in [11], [79], [80] (also refer to Proposition 3.13). The Riemann symbols given by (3.17), corresponding to an inertia matrix M become identically equal to zero if and only if the matrix M admits a factorization $M^{-1} = TT^\top$ such that the Lie brackets of the columns of the matrix T are identically equal to zero. That is,

$$\begin{aligned} R_{ijlk} = 0,\ i,j,l,k \in \bar{n} \ \Leftrightarrow\ & M^{-1} \text{ admits a factor } T \\ & \text{such that } \ [T_i, T_j] = 0,\ i,j \in \bar{n}. \end{aligned} \tag{3.28}$$

We now present the physical example of the inverted pendulum on cart which was introduced in Chapter 1. We show that its inertia matrix belongs to the set $\mathcal{S}_{\text{ZRS}}$ but its Crystoffel symbols do not identically become equal to zero which concludes the proof. Indeed, consider the inertia matrix given by (1.6). Next, for any given positive definite matrix M, it is always possible to find a uniquely defined lower triangular Cholesky factorization, T of M^{-1} satisfying (3.16) and such that its diagonal entries are positive. Please refer to Corollary 7.2.9 of [36] and [39] for a discussion on this. For the mass matrix in (1.6), we have

$$T = \begin{bmatrix} \frac{\sqrt{m_3}}{\sqrt{m_3 - b^2\cos^2 q_1}} & 0 \\ \frac{-b\cos q_1}{\sqrt{m_3}\sqrt{m_3 - b^2\cos^2 q_1}} & \frac{1}{\sqrt{m_3}} \end{bmatrix}, \tag{3.29}$$

and it can be easily verified that $[T_1, T_2] = 0$. Hence, from (3.28), the matrix in (1.6) has zero Riemann symbols. We next compute the Christoffel symbols for M and obtain that $C_{112} = -b\sin q_1$, while the rest of the symbols are identically zero. Thus, the inclusion $\mathcal{S}_{\text{ZCS}} \subset \mathcal{S}_{\text{ZRS}}$ is strict.

- ($\mathcal{S}_{\text{ZRS}} \subset \mathcal{S}_{\text{T}}$) If the columns of T commute, that is, if $[T_i, T_j] = 0$, it is clear that the skew–symmetry condition (3.18) is satisfied. Hence, by using the equivalence (3.28), the claim $[M \in \mathcal{S}_{\text{ZRS}} \Rightarrow M \in \mathcal{S}_{\text{T}}]$ follows in a straight forward manner.

We now proceed to prove that, for $n > 2$, the converse implication is not true, which shows that the inclusion is strict. First, we prove that for $n \leq 2$ the sets are the same. For $n = 1$ the equivalence is, of course, trivial. For

$n = 2$ this can be easily shown as follows. The skew–symmetry condition (3.18) yields two equations of the form

$$[T_1, T_2]d^\top = \begin{bmatrix} 0 & \tilde{\alpha} \\ -\tilde{\alpha} & 0 \end{bmatrix}, \; \tilde{\alpha} \in \mathbb{R}, \tag{3.30}$$

for $d = T_1, T_2$, respectively which have a solution if and only if $[T_1, T_2] = 0$ or $d = 0$. The proof follows by noting that since T is full rank, $[T_1, T_2] = 0$ for (3.30) to hold true.

We now consider the case $n = 3$ and construct an inertia matrix $M \in \mathcal{S}_{\mathrm{T}}$ such that $M \notin \mathcal{S}_{\mathrm{ZRS}}$. Let M be a 3×3 symmetric positive definite matrix and let T be its factorization. First, we observe that the condition

$$[T_1, T_2] = T_3, \quad [T_2, T_3] = T_1, \quad [T_3, T_1] = T_2 \tag{3.31}$$

is sufficient to prove the skew–symmetry condition in (3.18). We can see that the condition (3.31) is satisfied by the vectors $T_i = A_i x$ where $x \in \mathbb{R}^3$ and $A_i \in \mathbb{R}^{3\times 3}$ are the rotation matrices given as

$$A_i = \begin{bmatrix} 0 & \Omega_{3i} & \Omega_{2i} \\ -\Omega_{3i} & 0 & \Omega_{1i} \\ -\Omega_{2i} & -\Omega_{1i} & 0 \end{bmatrix}, \; i \in \{1, 2, 3\},$$

where $\Omega_{jk} := e_j^\top e_k$. However, the resulting matrix $T = [T_1|T_2|T_3]$ has zero determinant and hence cannot qualify as a factor of M^{-1}.

To complete the example we invoke some concepts from Lie group theory (refer to [50], [79]). The first observation is that the matrices A_i are tangent vectors at the identity point of the Lie group $SO(3)$ and, furthermore, form a basis for its associated Lie algebra $so(3)$. We then extend these vectors to left–invariant vector fields on the group $SO(3)$ using a push–forward of the left multiplication map $L_g(h) = gh$, where $g, h \in SO(3)$. The push–forward is defined as $(L_g)_*(A_i) = gA_i$, where g is taken to be the product matrix $R(x) = R_1(x)R_2(x)R_3(x)$ with,

$$R_1 = \begin{bmatrix} \cos x_1 & \sin x_1 & 0 \\ -\sin x_1 & \cos x_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos x_2 & \sin x_2 \\ 0 & -\sin x_2 & \cos x_2 \end{bmatrix}$$

$$R_3 = \begin{bmatrix} \cos x_3 & \sin x_3 & 0 \\ -\sin x_3 & \cos x_3 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is a parametrization (using the Euler angles) of $SO(3)$. The question is then to find the vectors $\tilde{T}_i$, whose push–forward by R_*, that is $R_*(\tilde{T}_i)$, will equal $(L_R)_*(A_i)$. This leads to the following set of equations

$$(\nabla_{x_1} R)\tilde{T}_{i1}(x) + (\nabla_{x_2} R)\tilde{T}_{i2}(x) + (\nabla_{x_3} R)\tilde{T}_{i3}(x) = R(x)A_i, \quad i = 1, 2, 3.$$

By solving these equations, we obtain the matrix $\tilde{T}$

$$\tilde{T}(x) = \begin{bmatrix} -\sin(x_1)\cot(x_2) & -\cos(x_1)\cot(x_2) & 1 \\ \cos(x_1) & -\sin(x_1) & 0 \\ \sin(x_1)\mathrm{cosec}(x_2) & \cos(x_1)\mathrm{cosec}(x_2) & 0 \end{bmatrix}.$$

Some simple computations show that the matrix $\tilde{T}$ has full rank (almost everywhere) and verifies (3.31). But, the matrix $\tilde{T}$ above is singular at the point zero and hence is not full rank. This singularity can be removed by introducing an homeomorphism $F : \mathbb{R} \times (0, \pi) \times \mathbb{R} \to \mathbb{R}^3 : x \mapsto q$. For instance, $F(x) = [x_1, \tan(x_2 - \frac{\pi}{2}), x_3]^\top$, which has an inverse map $F^I : \mathbb{R}^3 \to \mathbb{R}^3$, $F^I(q) = [q_1, \frac{\pi}{2} + \tan^{-1}(q_2), q_3]^\top$. Now, upon defining the transformed vectors,

$$T_i(q) = [(\nabla F(x))\tilde{T}_i(x)]_{x=F^I(q)},\ i = 1, 2, 3,$$

we get after some simple calculations that

$$T = \begin{bmatrix} \sin(q_1)q_2 & \cos(q_1)q_2 & 1 \\ (1+q_2^2)\cos(q_1) & -(1+q_2^2)\sin(q_1) & 0 \\ \sqrt{1+q_2^2}\sin(q_1) & \sqrt{1+q_2^2}\cos(q_1) & 0 \end{bmatrix}, \tag{3.32}$$

which also verifies (3.31). We then obtain $M^{-1} = T(q)T^\top(q)$ as

$$M^{-1} = \begin{bmatrix} 1+q_2^2 & 0 & q_2\sqrt{1+q_2^2} \\ 0 & (1+q_2^2)^2 & 0 \\ q_2\sqrt{1+q_2^2} & 0 & 1+q_2^2 \end{bmatrix}. \tag{3.33}$$

Now, computing the Riemann symbols and recalling that, because of the symmetries of the tensor, only $R_{1212}, R_{1213}, R_{1223}, R_{1313}, R_{1323}, R_{2323}$ need to be calculated, one can verify that $R_{1212}, R_{1323}, R_{2323} \neq 0$ for all q and $R_{1223} \neq 0$ for $q_2 \neq 0$, and hence conclude that $M \notin \mathcal{S}_{\texttt{ZRS}}$. Since, T satisfies (3.31), each of the matrices $[T_1, T_2]T_2^\top + [T_1, T_3]T_3^\top$, $[T_2, T_1]T_1^\top + [T_2, T_3]T_3^\top$ and $[T_3, T_1]T_1^\top + [T_2, T_3]T_3^\top$ are skew symmetric as desired. This completes the proof.

- ($\mathcal{S}_{\texttt{T}} \subseteq \mathcal{S}_{\texttt{PLvCC}}$) We can see that, replacing $\Psi = T$ in (3.6), the second right term vanishes and we get

$$\mathcal{B}_{(i)} = \sum_{j=1}^{n} [T_i, T_j]T_j^\top. \tag{3.34}$$

and hence, condition (3.5) is satisfied with $\Psi = T$. Now, the skew-symmetry condition (3.18) and (3.34) ensure that the condition (3.5) is satisfied. Hence the proof follows. □

Remark 3.10. The case $M \in \mathcal{S}_{\text{ZRS}}$ has been extensively studied in analytical mechanics and has a deep geometric significance (refer to [19], [79] and also look at Theorem 2.36 in [87]). The property has been exploited in the context of linearization (refer to [11], [80]). It can be shown using Riemannian geometry that if $M \in \mathcal{S}_{\text{ZRS}}$ the system is said to be *Euclidean* [11], where the qualifier stems from the fact that the system is diffeomorphic to a “linear double integrator”. In the next section, we discuss in more detail about the physical and geometric significance of this set and also explain about Euclidean systems.

3.3.2 Physical Interpretation of the Sets $\mathcal{S}_{\text{ZCS}}, \mathcal{S}_{\text{ZRS}}, \mathcal{S}_{\text{T}}$

In this section, we throw some light on the classes of physical systems for which the mass matrix belongs to the sets defined in proposition 3.9. As shown in the proof of Proposition 3.3, condition (3.5) holds if and only if the mapping $\mathbf{D}_\Psi$, defined in (3.9), identically vanishes. We will therefore use this test to answer the questions. An additional motivation to analyze $\mathbf{D}_\Psi$ is that it allows to establish some connections of our work with the existing literature.

We first relate the key mapping $\mathbf{D}_\Psi$, defined in (3.9), with the matrices M and C in the Euler-Lagrange model (2.3). For this, we define the vector function $\tilde{\mathbf{D}}_\Psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ as $\tilde{\mathbf{D}}_\Psi(q, \dot{q}) := \mathbf{D}_\Psi(q, M(q)\dot{q})$. Now, from (3.9) we get

$$\begin{aligned} \tilde{\mathbf{D}}_\Psi &= \Psi^\top \nabla_q \{\frac{1}{2}\dot{q}^\top M \dot{q}\} - \dot{\Psi}^\top M \dot{q}, \\ &= [\Psi^\top C - \frac{d}{dt}(\Psi^\top M)]\dot{q}, \end{aligned} \tag{3.35}$$

where, to obtain the second identity, we have used the property (2.6).

The Set $\mathcal{S}_{\text{ZCS}}$

Proposition 3.11. *The following statements are equivalent.*

(i) $M \in \mathcal{S}_{\text{ZCS}}$.

(ii) Condition (3.5) holds for any constant Ψ.

(iii) The Coriolis and centrifugal forces $C(q, \dot{q})\dot{q}$ *equal zero.*

Moreover, if $M \in \mathcal{S}_{\text{ZCS}}$*, and we take* $\Psi = M^{-1}$*, the transformed dynamics (3.7) become*

$$\dot{q} = P, \ \dot{P} = -M^{-1}(\nabla U - Gu). \tag{3.36}$$

Proof. The equivalence between (i) and (iii) follows directly from the property (2.7). Now, from (3.9), it is clear that (ii) is true if and only if

$$\nabla_q\{\frac{1}{2}p^\top M^{-1}p\} = 0,$$

which is equivalent to $M \in \mathcal{S}_{\mathtt{CI}}$. The proof is completed by recalling from Proposition 3.9 that $\mathcal{S}_{\mathtt{ZCS}} = \mathcal{S}_{\mathtt{CI}}$.

The proof of (3.36) follows by noting that, $\Psi = M^{-1}$ gives $P = \dot{q}$ and $\dot{P}$ is obtained by replacing $\mathbf{D}_{M^{-1}} = 0$ in (3.8). □

Remark 3.12. In [25] an observer was designed for Lagrangian systems to estimate $\dot{q}$, under the following sufficient condition for linearizability. Define a change of coordinates of the form $(q, v) = (q, \mathcal{E}(q)\dot{q})$, with $\mathcal{E} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ full rank. Then

$$\begin{aligned} \dot{v} &= \dot{\mathcal{E}}\dot{q} + \mathcal{E}\ddot{q} \\ &= (\dot{\mathcal{E}} - \mathcal{E}M^{-1}C)\mathcal{E}^{-1}v - \mathcal{E}M^{-1}(\nabla U - Gu), \end{aligned} \tag{3.37}$$

where (2.3) was used to get the last equation. It is clear that the dynamics becomes linear in v if

$$\dot{\mathcal{E}} = \mathcal{E}M^{-1}C. \tag{3.38}$$

(Of course, this conclusion also follows from (3.35) setting $\Psi = M^{-1}\mathcal{E}^\top$). Condition (3.38) is imposed in [25], which besides being obviously stronger than condition (3.5), does not seem to admit any geometric or system theoretic interpretation.

The Set $\mathcal{S}_{\mathtt{ZRS}}$

Proposition 3.13. *The following statements are equivalent:*

(i) $M \in \mathcal{S}_{\mathtt{ZRS}}$.

(ii) There exists a matrix T which is a factor of M^{-1}, that is, $M^{-1} = TT^\top$ and a mapping $Q : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\nabla Q(q) = T^{-1}(q). \tag{3.39}$$

(iii) The system is Euclidean.

Proof. We first prove the equivalence between (i) and (ii). For this, we denote $\mathcal{C}_q \subset \mathbb{R}^n$ as the n-dimensional manifold defined by the configuration space of the generalized position coordinates q. We then recall from (3.28) that $M \in \mathcal{S}_{\mathtt{ZRS}}$ if and only if there exists a full rank factorization T of M^{-1} such that

$$[T_i, T_j] = 0,\ i, j \in \bar{n},\ q \in \mathcal{C}_q. \tag{3.40}$$

Next, each $T_i(q),\ i \in \bar{n}$ would be a vector field acting on the manifold $\mathcal{C}_q$. Since, the matrix $T(q)$ has a full rank for all q, its columns are linearly independent. We now assume that:

- the columns of $T(q)$ satisfy (3.40).
- the n vector fields $T_i,\ i \in \bar{n}$ are complete, that is, the integral curves of the vector fields exist for all times t.

Then, from [79] (also refer to Theorem 2.36 in [58]), we know that there exists a coordinate chart for $\mathcal{C}_q$ given by the coordinates $\bar{q} = Q(q)$ for some $Q : \mathbb{R}^n \to \mathbb{R}^n$ such that, the vector fields in the new coordinates satisfy $T_i(\bar{q}) = e_i$ where e_i denotes the i^{th} natural basis vector of $\mathbb{R}^n$. Further, this coordinate chart would be global and hence the mapping from q to Q is bijective for all $q \in \mathcal{C}_q$.

We next invoke the fact that the vector fields transform in a covariant fashion [79] under such coordinate changes which means

$$T_i(\bar{q}) = \nabla Q(q) T_i(q) = e_i. \tag{3.41}$$

Subsequently, we perform the following computations

$$\begin{aligned} \nabla Q(q) T_i(q) &= e_i, & (3.42)\\ => \ \nabla Q(q)\left[T_1(q)|T_2(q)|...|T_n(q)\right] &= I_{n\times n}, & (3.43)\\ => \ \nabla Q(q) &= T^{-1}(q). & (3.44) \end{aligned}$$

Thus, we can conclude that if the columns of $T(q)$ commute, then $T^{-1}(q)$ is the Jacobian of some vector $S(q)$ and is thus integrable. We now assume that $T^{-1}(q) = \nabla_q S$ for some vector $Q : \mathbb{R}^n \to \mathbb{R}^n$. We then easily obtain, $\nabla Q(q) T_i(q) = e_i$ which implies that there exists a set of coordinates $\bar{q} = Q(q)$ such that in those coordinates, the columns of T assume the form $T_i(\bar{q}) = \nabla_{\bar{q}_i}$. We once again invoke Theorem 2.36 in [58] and conclude that the columns of $T(q)$ commute among each other. Hence, the proof follows.

We next prove the equivalence between (i) and (iii). Firstly, if $M \in \mathcal{S}_{\mathtt{ZRS}}$, then there exists a mapping $Q : \mathbb{R}^n \to \mathbb{R}^n$ such that (3.39) is satisfied and further condition (3.5) holds with $\Psi = T$ (where $TT^\top = M^{-1}$), that is, $\mathbf{D}_T = 0$. Consider now the coordinates (Q, P) where $P = T^\top p$. We compute

$$\dot{Q} = (\nabla Q)\dot{q} = T^{-1}\dot{q} = T^\top p = P,$$

where we have used (3.39). Next, from (3.8) and using the fact that $\mathbf{D}_T = 0$ we obtain the dynamics of $\dot{P}$ as

$$\dot{P} = T^\top\{Gu - \nabla U\} = -\nabla_Q \tilde{V}(Q) + \tilde{G}(Q)u. \tag{3.45}$$

where $\tilde{G}(Q) := T^\top(Q^I(Q))G(Q^I(Q))$, $\tilde{V}(Q) := V(Q^I(Q))$, with $Q^I : \mathbb{R}^n \to \mathbb{R}^n$ a right inverse of $Q(q)$, that is, $Q(Q^I(x)) = x$ for all $x \in \mathbb{R}^n$. Hence, the system is Euclidean [11] as there exists a canonical transformation given by Q, P such that the system in the new coordinates has a constant inertia matrix and moreover, if the potential energy $U = 0$, the system dynamics is given as a double integrator $\ddot{Q} = \tilde{G}(Q)u = v$. $\square$

The Set $\mathcal{S}_{\mathrm{T}}$

Proposition 3.14. *For any matrix T, factor of M^{-1}, the following statements are equivalent:*

(i) $M \in \mathcal{S}_{\mathrm{T}}$.

(ii) Condition (3.5) holds with $\Psi = T$, that is, $\mathbf{D}_T = 0$.

Further, if $M \in \mathcal{S}_{\mathrm{T}}$, the transformed dynamics takes the form

$$\dot{q} = TP, \ \dot{P} = -T^\top (\nabla U - Gu) \, . \tag{3.46}$$

Proof. The evaluation of the matrices $\mathcal{B}_{(i)}$, defined in (3.6), for $\Psi = T$ is given in (3.34). Now, from (3.13) we conclude that $\mathbf{D}_T = 0$ if and only if these matrices are skew symmetric, which is precisely the condition for $M \in \mathcal{S}_{\mathrm{T}}$.

The last claim is established by noting that, $\Psi = T$ gives $\dot{q} = TP$ and $\dot{P}$ is obtained by replacing $\mathbf{D}_T = 0$ in (3.8). $\square$

3.3.3 The Set $\mathcal{S}_{\mathrm{PLvCC}}$

In this subsection, we present an interesting example of the robotic leg system which is not Euclidean but we show that its inertia matrix belongs to $\mathcal{S}_{\mathrm{PLvCC}}$. We then consider the classical ball and beam system and show that it does not belong to $\mathcal{S}_{\mathrm{PLvCC}}$.

Example 3.15. *The Robotic Leg ($M \in \mathcal{S}_{\mathrm{PLvCC}}$ but $M \notin \mathcal{S}_{\mathrm{ZRS}}$): We consider the robotic leg example [22] depicted in Figure 3.2. The system consists of a rigid body that is pinned to a fixed point on the ground at its center of mass. The body can rotate about this fixed point and has a moment of inertia J about the axis of rotation. The body has an extensible massless leg which is attached at the fixed point and the leg has a point mass m present at its tip. The coordinate ψ represents the angle of the body, θ represents the angle made by the extensible leg with the fixed horizontal axis in an inertial reference frame and r denotes the extension of the leg which is assumed to be strictly positive. Further, η_1 represents the torque acting at the point of rotation which controls the angle between the body and the leg and η_2 represents the force that controls the extension of the leg.*

We thus have a 3 degree of freedom mechanical system whose kinetic energy is given by $KE = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}J\dot{\psi}^2$. Letting $q = (r, \theta, \psi)$, we subsequently obtain the inertia matrix as

$$M = \mathit{diag}\{m, mq_1^2, J\}, \tag{3.47}$$

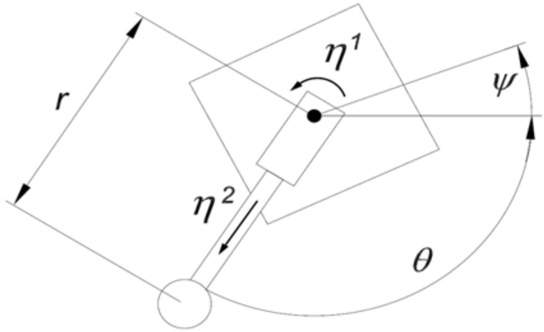

Figure 3.2: Robotic leg, where we denote $q := (r, \theta, \psi)$

where $q_1 \geq \epsilon > 0$. Firstly, the only non–zero Christoffel symbols for M are $C_{122} = -C_{221} = mq_1$ which implies that $M \notin \mathcal{S}_{\mathtt{ZCS}}$. Furthermore, the Riemann symbol $R_{1212} = m \neq 0$ implies that $M \notin \mathcal{S}_{\mathtt{ZRS}}$. We will now prove that $M \in \mathcal{S}_{\mathtt{PLvCC}}$ provided

$$q \in \mathcal{C} := \{q_1 \geq \epsilon > 0,\ q_2 \neq i\pi,\ i \in \mathcal{Z}_+\}.$$

Indeed, some lengthy but straightforward calculations, prove that the matrix

$$\Psi(q_1, q_2) := \begin{bmatrix} \sin(q_2) & \sin(q_2) & 0 \\ \frac{1}{q_1}\cos(q_2) + \kappa & \frac{1}{q_1}\cos(q_2) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \kappa \neq 0 \tag{3.48}$$

which is well–defined and full–rank for all $q \in \mathcal{C}$, ensures $\mathbf{D}_\Psi = 0$ for the inertia matrix (3.47). It should be pointed out that (3.48) was obtained by solving the PDEs (3.5), (3.6) for the inertia matrix (3.47).

Remark 3.16. Although not yet proven, some preliminary calculations lead us to conjecture that $M \notin \mathcal{S}_{\mathtt{T}}$. Notice that the "natural" choice for the factor T, namely

$$T = \text{diag}\{\frac{1}{\sqrt{m}}, \frac{1}{q_1\sqrt{m}}, \frac{1}{\sqrt{J}}\},$$

does not satisfy the condition $[T_1, T_2]T_2^\top = -\left([T_1, T_2]T_2^\top\right)^\top$.

Example 3.17. *The Ball-Beam System ($M \notin \mathcal{S}_{\mathtt{PLvCC}}$): It is interesting to note that, in spite of the similarities with the robotic leg, the classical ball–and–beam system [62] is not PLvCC. The system consists of the ball whose position along the beam is described by the coordinate q_1, the angle made by the beam with the horizontal axis is denoted by q_2 and a torque u acts on the beam and controls its angular position. Then, the inertia matrix of the ball–and–beam is $M =$ diag$\{1, \ell^2 + q_1^2\}$, where $\ell > 0$ is the*

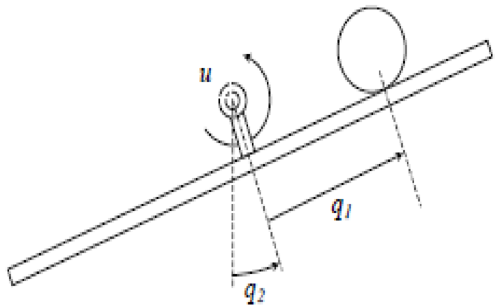

Figure 3.3: Ball-Beam System

length of the beam, and $q \in \{|q_1| \leq \ell\ \}$. *The PDEs (3.5), (3.6) for the ball–and–beam system are*

$$\nabla_{q_1}\Psi_{11}=0,\ \nabla_{q_2}\Psi_{11}+(\ell^2+q_1^2)\nabla_{q_1}\Psi_{21}=0,\ \nabla_{q_2}\Psi_{21}=\frac{-q_1}{\ell^2+q_1^2}\Psi_{11}.$$

The first and third PDE's together imply

$$\Psi_{21}(q_1,q_2)=\frac{q_1}{\ell^2+q_1^2}\tilde{\Psi}_{21}(q_2)+\kappa, \tag{3.49}$$

where $\Psi_{11}(q_2)=-\nabla_{q_2}\tilde{\Psi}_{21}$. *Next, using (3.49) together with the second PDE yields the ODE*

$$\nabla^2\tilde{\Psi}_{21}(q_2)=\frac{\ell^2-q_1^2}{\ell^2+q_1^2}\tilde{\Psi}_{21}(q_2),$$

that clearly does not admit a solution.

3.3.4 A Globally Exponentially Convergent Reduced Order I&I Observer for PLvCC Systems

In this section, we construct a globally exponentially convergent reduced order observer for PLvCC systems using the Immersion and Invariance principle introduced in Chapter 2. Proceeding on the similar lines, we define the observer for the system (3.1).

Definition 3.18. *The dynamical system*

$$\dot{\eta}=\Upsilon(q,\eta,u), \tag{3.50}$$

with $\eta\in\mathbb{R}^n$, *is called a reduced order I&I observer for the system (3.1) if there exists a full rank matrix* $\Psi:\mathbb{R}^n\rightarrow\mathbb{R}^{n\times n}$ *and a vector function* $\beta:\mathbb{R}^n\rightarrow\mathbb{R}^n$, *such that the manifold*

$$\mathcal{M}:=\{(\eta,q,p):\beta(q)=\eta+\Psi^\top(q)p\}\subset\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n \tag{3.51}$$

is invariant and attractive with respect to the system (3.1), (3.50). The asymptotic estimate of p*, denoted by* $\hat{p}$*, is then given by*

$$\hat{p} = \Psi^{-\top}(\beta - \eta).$$

Remark 3.19. The manifold $\mathcal{M}$ in (3.51) is a particular case of (2.53) which we saw in chapter 2. More particularly, we can draw parallels from [8], where the manifold is defined as $\{(\eta, q, p) : \tilde{\beta}(q, \eta) = \Psi^\top(q)p\}$, with $\tilde{\beta} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. From (3.51), we can see that the function $\tilde{\beta}(q, \eta) = \beta(q) - \eta$ in our case and thus is an affine function of η. It is clear that by considering a more general manifold expression, it is possible in principle to handle a larger class of systems. However, it can be shown that for PLvCC systems, the choice of β in (3.51) is without loss of generality. Please see Remark 3.22 for some additional relationships between both observers.

Before presenting the proposed observer design, we need the following assumption.

Assumption 1. *There exists a mapping* $\beta : \mathbb{R}^n \to \mathbb{R}^n$ *satisfying the matrix inequality*

$$\mathcal{Q}\mathcal{A}(q) + \mathcal{A}^\top(q)\mathcal{Q} \geq \epsilon I_n, \tag{3.52}$$

uniformly in q*, for some* $\epsilon > 0$ *and some constant matrix* $\mathcal{Q} \in \mathbb{R}_P^{n \times n}$*, where*

$$\mathcal{A}(q) := \nabla\beta(q)[\Psi^\top(q)M(q)]^{-1}. \tag{3.53}$$

Proposition 3.20. *Consider the mechanical system (3.1). Assume* $M \in \mathcal{S}_{\text{PLvCC}}$ *with a matrix* Ψ *whose inverse is uniformly bounded and that there exists a mapping* β *satisfying Assumption 1. Then, the dynamical system*

$$\begin{aligned} \dot{\eta} &= \nabla\beta(q)(\Psi^\top M)^{-1}(\beta - \eta) + \Psi^\top(\nabla U - Gu) \\ \hat{p} &= \Psi^{-\top}(\beta - \eta) \end{aligned} \tag{3.54}$$

is a globally exponentially convergent reduced order I&I observer—with the estimation error verifying

$$|\hat{p}(t) - p(t)| \leq \bar{\alpha} \exp^{-\bar{\rho}t} |\hat{p}(0) - p(0)|,$$

for some $\bar{\alpha}, \bar{\rho} > 0$*, where* $|\cdot|$ *is the Euclidean norm.*

Proof. By following the I&I procedure in Chapter 2, we prove that the manifold $\mathcal{M}$, defined in (3.51), is attractive and invariant by showing that the off–the–manifold coordinate

$$z = \beta - \eta - \Psi^\top p \ = \beta - \eta - P, \tag{3.55}$$

verifies: (i) $(z(0) = 0 \Rightarrow z(t) = 0)$ for all $t \geq 0$, and (ii) $z(t)$ asymptotically (actually, exponentially) converges to zero. Note that from (3.51) we get that dist$\{(\eta, q, p), \mathcal{M}\} = 0$ if and only if $z = 0$.

To obtain the dynamics of z, we differentiate (3.55) to get

$$\begin{aligned} \dot{z} &= \dot{\beta} - \dot{\eta} - \dot{P} \\ &= (\nabla\beta)M^{-1}p - (\nabla\beta)(\Psi^\top M)^{-1}(\beta - \eta) \\ &= -\mathcal{A}z \end{aligned} \tag{3.56}$$

where (3.7) and (3.54) are used for the second identity while the third one is obtained invoking (3.53) and (3.55).

The manifold $\mathcal{M}$ is clearly positively invariant. To establish global exponential attractivity of $\mathcal{M}$, consider the Lyapunov function

$$\mathcal{V}(z) = \frac{1}{2} z^\top \mathcal{Q} z. \tag{3.57}$$

Conditions (3.52) and (3.56) ensure that

$$\dot{\mathcal{V}} \leq -\frac{\epsilon}{\bar{\lambda}(\mathcal{Q})}\mathcal{V}, \tag{3.58}$$

with $\bar{\lambda}(\mathcal{Q})$ denoting the maximum eigenvalue of $\mathcal{Q}$, which proves, after some basic bounding, the global exponential convergence to zero of z. Exponential convergence of $\hat{p} - p$ is concluded invoking uniform boundedness of Ψ^{-1}. □

Remark 3.21. Assumption 1 may be rephrased as follows. Instead of assuming the existence of β we assume that there exists a mapping $\mathcal{N} : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ such that (3.52) holds with

$$\mathcal{A}(q) = \mathcal{N}(q)[\Psi^\top(q)M(q)]^{-1}, \tag{3.59}$$

and

$$\nabla\mathcal{N}^j = \left(\nabla\mathcal{N}^j\right)^\top, \quad j \in \bar{n}. \tag{3.60}$$

The latter (integrability) condition ensures, from Poincaré's Lemma, that there exists a β such that

$$\nabla\beta = \mathcal{N}. \tag{3.61}$$

That is, the problem reduces to the solution of the PDE (3.60), subject to the inequality constraint (3.52), (3.59).

In section 3.4, we propose a step–by–step procedure to compute $\mathcal{N}$ for the special choice of $\Psi = T$ with T being the lower triangular Cholesky factorization of M^{-1}.

3.3.5 Implications on observer design for the sets $\mathcal{S}_{\mathtt{ZCS}}, \mathcal{S}_{\mathtt{ZRS}}, \mathcal{S}_{\mathtt{T}}$

We now have a look at the various sets of inertia matrices which were defined previously and study the implications on observer design for these sets.

i) $M \in \mathcal{S}_{\mathtt{ZCS}}$: This is the case of constant inertia matrix. From the definition of $\mathcal{A}$ in (3.59) we see that when $\Psi = M^{-1}$, the Assumption 1 is satisfied with any constant matrix $\mathcal{N}$ such that $-\mathcal{N}$ is a Hurwitz matrix. Further, the construction of β from (3.61) is trivial and the observer error dynamics is linear, namely $\dot{z} = -\mathcal{N}z$. For instance, selecting $\mathcal{N} = I_n$ the observer takes the simple form

$$\dot{\eta} = q - \eta + M^{-1}\left(\nabla U - Gu\right),\ \hat{p} = M(q - \eta).$$

The reason why this basic construction works can be easily explained recalling (3.36) of Proposition 3.11.

ii) $M \in \mathcal{S}_{\mathtt{ZRS}}$: In this case, the dynamics (in principle) can be expressed in the coordinates (Q, P) as (3.45) for which the $I\&I$ observer construction procedure becomes trivial as in the case for $M \in \mathcal{S}_{\mathtt{ZCS}}$. However, to obtain the representation (3.45), it is necessary to solve the PDE (3.39) in order to obtain the expression for Q as a function of q which severely restricts the practical applicability of the approach. Indeed, the explicit solution of this PDE may sometimes not be possible which happens in the case of the classical inverted pendulum on the cart system. In Section 3.6, this system is shown to be Euclidean but, as indicated in [11], the PDE (3.39) leads to an elliptic integral of the second kind that does not admit a closed form solution.

However, as we have seen, it is more facile (or rather convenient) to express Euclidean systems in the partially linearized form in the context of observer design. Indeed, if we choose the coordinates as $(q, T^\top(q)p)$ (T given by (3.16)), we obtain the partially linearized system (3.46) with $P = T^\top q$. Further, from (3.53) we see that when $\Psi = T$ we get $\mathcal{A} = \mathcal{N}T$. In Section 3.4 we propose to take T to be the lower triangular Cholesky factor of M^{-1}, and present a systematic procedure to design $\mathcal{N}$ in order to satisfy Assumption 1.

iii) $M \in \mathcal{S}_{\mathtt{T}}$: Similar to the systems where $M \in \mathcal{S}_{\mathtt{ZRS}}$, we get in this case $\mathcal{A} = \mathcal{N}T$ and thus the procedure for construction of $\mathcal{N}$ in Section 3.4 is applicable.

Remark 3.22. Some connections between our observer and the one proposed in [8] may be established at this point. Towards this end, we refer to the function $\tilde{\mathbf{D}}_\Psi$ defined in (3.35) and evaluate it for $\Psi = T$ (T given by (3.16)) to obtain

$$\tilde{\mathbf{D}}_T = [T^\top C - \frac{d}{dt}T^{-1}]\dot{q} =: \bar{C}(q, \dot{q})\dot{q}.$$

It can be shown that the matrix $\bar{C}(q,\dot{q})T(q)$ is linear in $\dot{q}$ and furthermore, by invoking (2.6) we can also prove that it is skew–symmetric. These properties are used in [8] to generate an error dynamics of the form

$$\dot{z} = [\Gamma(q,\eta) - \bar{C}(q,\dot{q})T(q)]z,$$

where $\Gamma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a matrix that can be shaped by selecting the function $\tilde{\beta}$. A constructive solution has been given in [8] for some particular cases of systems with $n = 2$, namely diagonal inertia matrix (with possibly unbounded elements) and inertia matrix with bounded elements. Some recent calculations show that this technique can be extended beyond these cases, but the need to explicitly solve the integrals that define $\tilde{\beta}$ make this more of an "existence result", than an actual constructive procedure. Of course, it may be argued that the analysis done in this chapter (that aims at eliminating the term $\mathbf{D}_\Psi$), although leading to the explicit identification of some PDEs to be solved, is also not constructive—given our inability to guarantee their solution in general.

3.4 A Constructive Procedure for $\mathcal{N}$

In this Section we present a simple algorithm to construct a matrix $\mathcal{N}$ that satisfies Assumption 1 for the selection $\Psi = T$ (T given by (3.16)). The starting point of the procedure is to choose T as the lower triangular Cholesky factorization of the mass matrix M. The idea is then to construct a matrix $\mathcal{N}$ such that, on one hand, $\mathcal{N}T$ is *diagonal* with positive diagonal entries and, on the other hand, $\mathcal{N}$ is "trivially" integrated (refer to 3.62) in the sense of Remark 3.21. The first condition will ensure (3.52) of Assumption 1, while the second one guarantees (3.60). As expected, the construction involves the solution of some PDEs that we show can be easily solved for several examples of practical interest.

To enhance readability we present the algorithm first for the simplest case when the inertia matrix depends on one coordinate, for which the differential equations to be solved are ODEs. We then consider the case of dependence on two coordinates, and deal with PDEs. We then generalize the procedure for the case where the inertia matrix can depend on k coordinates where $k \in \bar{n}$.

3.4.1 Procedure for Computing $\mathcal{N}$ when M Depends on a Single Coordinate

Without loss of generality we assume that M is a function of q_1. We propose the following form for the matrix $\mathcal{N}$

$$\mathcal{N} = \Lambda + \nabla_q\{\phi(q_1) + \psi(q_1)q\}, \tag{3.62}$$

where $\Lambda > 0$ is an $n \times n$ constant diagonal matrix, $\phi : \mathbb{R} \to \mathbb{R}^n$ is a function to be defined that depends only on q_1 and verifies $e_1^\top \phi = 0$ and $\psi : \mathbb{R} \to \mathbb{R}^{n\times n}$, also to be defined, depends only on q_1 and verifies $e_i^\top \psi e_j = 0$ for all $j \geq i$ and $\psi e_1 = 0$.

Given the proposed form of $\mathcal{N}$, condition (ii) of Assumption 1 is trivially satisfied and β can be immediately computed as

$$\beta = \Lambda q + \phi(q_1) + \psi(q_1) q. \tag{3.63}$$

Moreover, $\mathcal{N}$ is lower triangular and hence, $N(q) := \mathcal{N}(q)T(q)$ is also lower triangular since the matrix T is lower triangular. Further, each diagonal entry of N is given as $N_{ii} = \Lambda_{ii}T_{ii} > 0$. To satisfy the positivity condition (i) of Assumption 1 our strategy will be to find ψ and ϕ such that the matrix N becomes *diagonal*.

The matrix N has the following form:

$$\left[\begin{array}{c|c|c|c|c|c}
\Lambda_{11}T_{11} & 0 & 0 & 0 & 0....... & 0 \\
\frac{\partial\phi_2}{\partial q_1}T_{11} + \Lambda_{22}T_{21} & \Lambda_{22}T_{22} & 0 & 0 & 0....... & 0 \\
\begin{array}{c}\{\frac{\partial\phi_3}{\partial q_1} + \frac{\partial\psi_{32}}{\partial q_1}q_2\}T_{11} \\ +\psi_{32}T_{21} + \Lambda_{33}T_{31}\end{array} & \psi_{32}T_{22} + \Lambda_{33}T_{32} & \Lambda_{33}T_{33} & 0 & 0........ & 0 \\
\begin{array}{c}\{\frac{\partial\phi_4}{\partial q_1} + \frac{\partial\psi_{42}}{\partial q_1}q_2 + \frac{\partial\psi_{43}}{\partial q_1}q_3\}T_{11} \\ +\psi_{42}T_{21} + \psi_{43}T_{31} + \Lambda_{44}T_{41}\end{array} & \psi_{42}T_{22} + \psi_{43}T_{32} + \Lambda_{44}T_{42} & \psi_{43}T_{33} + \Lambda_{44}T_{43} & \Lambda_{44}T_{44} & 0....... & 0 \\
 & & & & & \\
.. & .. & .. & .. & 0....... & \Lambda_{nn}T_{nn}
\end{array}\right]$$

The algorithm proceeds along the following steps:

1. For every $i \geq 3$, solve $N_{i,i-1} = 0$ to obtain the function $\psi_{i,i-1}$. For example, we compute $\psi_{32} = -\frac{\Lambda_{33}T_{32}}{T_{22}}$, $\psi_{43} = -\frac{\Lambda_{44}T_{43}}{T_{33}}$ and so on. Note that the terms $T_{ii} > 0$ for every $i \in \bar{n}$.
2. For every $i \geq 4$, solve $N_{i,i-2} = 0$ using the function $\psi_{i,i-1}$ obtained in step 1 to get $\psi_{i,i-2}$. For example, $\psi_{42} = \frac{\Lambda_{44}}{T_{22}T_{33}}\{T_{43}T_{32} - T_{42}T_{33}\}$.
3. Proceed in this manner until $i = n$ to complete the computation of ψ.
4. Solve the ordinary differential equations $N_{i1} = 0, 2 \leq i \leq n$ and compute the vector ϕ. For example, the function ϕ_2 is obtained by solving

the ODE $\frac{\partial \phi}{\partial q_1} = -\frac{\Lambda_{22} T_{21}(q_1)}{T_{11}(q_1)}$. We can continue in this manner, use the (now known) function ψ and compute the function ϕ by solving ODE's.

The elements of the matrix Λ can be chosen freely and it suffices to just ensure that they are positive constants. Finally, after having computed $\mathcal{N}$, we obtain β from (3.63). In the case when the dimension of the mechanical system is $n \leq 2$, the matrix ψ is not needed and we skip the first three steps.

We now illustrate this procedure on two physical examples.

Inverted Pendulum on a Cart [24], [81]

We construct a velocity observer for the inverted pendulum on a cart system whose inertia matrix is given in (1.6) and the lower triangular Cholesky factor in (3.29). As shown before $[T_1, T_2] = 0$ and thus condition (3.5) is satisfied. We now proceed to construct $\mathcal{N}$ by following the above algorithm and accordingly set it as

$$\mathcal{N} = \begin{bmatrix} \Lambda_{11} & 0 \\ 0 & \Lambda_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \nabla_{q_1}\phi_2 & 0 \end{bmatrix},$$

where $\Lambda_{ii} > 0$. We next solve the ordinary differential equation, $N_{21} = 0$, which is of the form

$$\nabla_{q_1}\phi_2 = \frac{\Lambda_{22} b}{m_3}\cos(q_1)$$

and hence $\phi_2 = \frac{\Lambda_{22} b}{m_3}\sin(q_1)$. Thus, we obtain

$$\beta = \begin{bmatrix} \Lambda_{11} q_1 \\ \Lambda_{22}(q_2 + \frac{b}{m_3}\sin(q_1)) \end{bmatrix}. \tag{3.64}$$

Some simulation results of this example are presented in Section 3.5.

3-Link Underactuated Planar Manipulator [3], [38]

The system shown in figure 3.4 is a 3-link underactuated planar manipulator with the first two joints being prismatic and actuated while the third joint is a revolute joint and is unactuated. We let (q_2, q_3) denote the horizontal and vertical positions of the third joint from the origin and q_1 denotes the orientation of the third link with respect to the horizontal axis. Further, m_1, m_2 and m_3 denote the masses of the links, L denotes the distance of the center of mass of the third link from the third joint and I denotes the moment of inertia of the third link about the third joint. The kinetic energy of the system is given by $KE = \frac{1}{2}m_x\dot{r}_x^2 + \frac{1}{2}m_y\dot{r}_y^2 + \frac{1}{2}I\dot{\theta}^2 - m_3 l\sin\theta\dot{\theta}\dot{r}_x + m_3 l\cos\theta\dot{\theta}\dot{r}_y$ where $m_x = m_1 + m_2 + m_3$ and $m_y = m_2 + m_3$. Thus, the moment of inertia of the

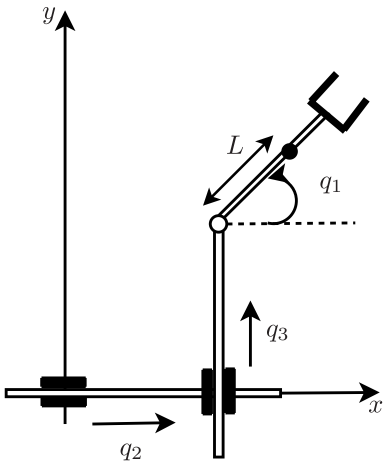

Figure 3.4: 3-Link underactuated planar manipulator.

system is given by

$$M^{-1} = \frac{1}{F^2}\begin{bmatrix} 1 & \frac{m_3L}{m_x}\sin q_1 & -\frac{m_3L}{m_y}\cos q_1 \\ \frac{m_3L}{m_x}\sin q_1 & \frac{m_yI+m_3^2L^2}{m_xm_y}\cos^2 q_1 & -\frac{m_3L^2}{m_xm_y}\sin q_1\cos q_1 \\ -\frac{m_3L}{m_y}\cos q_1 & -\frac{m_3L^2}{m_xm_y}\sin q_1\cos q_1 & \frac{m_xI-m_3^2L^2}{m_xm_y}\sin^2 q_1 \end{bmatrix}$$

where $F(q) := \sqrt{1 - \frac{m_3^2L^2}{m_y}\cos^2 q_1 - \frac{m_3^2L^2}{m_x}\sin^2 q_1}$. We compute the lower triangular Cholesky factorization as

$$T = \begin{bmatrix} \frac{1}{F} & 0 & 0 \\ \frac{m_3L}{m_xF}\sin q_1 & \frac{1}{\sqrt{m_x}} & 0 \\ -\frac{m_3L}{m_yF}\cos q_1 & 0 & \frac{1}{\sqrt{m_y}} \end{bmatrix}.$$

We can easily check that the columns of T commute thus the system is Euclidean. Following the procedure described above we set $\mathcal{N}$ as

$$\mathcal{N} = \begin{bmatrix} \Lambda_{11} & 0 & 0 \\ 0 & \Lambda_{22} & 0 \\ 0 & 0 & \Lambda_{33} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \nabla_{q_1}\phi_2 & 0 & 0 \\ \nabla_{q_1}\phi_3 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \nabla_{q_1}\psi_{32}q_2 & \psi_{32} & 0 \end{bmatrix},$$

where $\Lambda_{ii} > 0$. We first solve $N_{32} = 0$ to obtain $\psi_{32} = 0$. We next solve $N_{31} = 0$ and get $\phi_3 = \frac{\Lambda_{33}m_3L}{m_y}\sin q_1$. We finally solve $N_{21} = 0$ to obtain

$\phi_2 = \frac{\Lambda_{22} m_3 L}{m_x} \cos q_1$. We finally get

$$\beta = \begin{bmatrix} \Lambda_{11} q_1 \\ \Lambda_{22}(q_2 + \frac{m_3 L}{m_x} \cos q_1) \\ \Lambda_{33}(q_3 + \frac{m_3 L}{m_y} \sin q_1) \end{bmatrix}.$$

3.4.2 Procedure for Computing $\mathcal{N}$ when the Mass Matrix Depends on Two Coordinates

Without loss of generality we assume that the mass matrix depends on q_1 and q_2. Next we propose the following form for $\mathcal{N}$ given as,

$$\mathcal{N} = \Lambda + \nabla_q \{\phi(q_1, q_2) + \psi(q_1, q_2) q\} \tag{3.65}$$

where $\Lambda > 0$ is an $n \times n$ constant diagonal matrix, $\phi : \mathbb{R}^2 \to \mathbb{R}^n$ depends only on q_1, q_2 and verifies $e_1^\top \phi = 0$, $\psi : \mathbb{R}^2 \to \mathbb{R}^{n \times n}$ depends only on q_1, q_2 and verifies $e_i^\top \psi e_j = 0$ for all $j \geq i$ and $\psi e_1 = \psi e_2 = 0$.

Again, condition (ii) of Assumption 1 is trivially satisfied and β can be immediately computed as

$$\beta = \Lambda q + \phi(q_1, q_2) + \psi(q_1, q_2) q. \tag{3.66}$$

Moreover, $\mathcal{N}$ is lower triangular and hence, $N := \mathcal{N} T$ is also lower triangular. Further, each diagonal entry of N is given as $N_{ii} = \Lambda_{ii} T_{ii} > 0$. To satisfy the positivity condition (i) of Assumption 1 our strategy will be to find ψ and ϕ to render N *diagonal*.

The matrix N has the following form:

$$\left[\begin{array}{c|c|c|c|c|c}
\Lambda_{11} T_{11} & 0 & 0 & 0 & 0\ldots\ldots & 0 \\
\frac{\partial \phi_2}{\partial q_1} T_{11} + \{\Lambda_{22} + \frac{\partial \phi_2}{\partial q_2}\} T_{21} & \{\Lambda_{22} + \frac{\partial \phi_2}{\partial q_2}\} T_{22} & 0 & 0 & 0\ldots\ldots & 0 \\
\frac{\partial \phi_3}{\partial q_1} T_{11} + \frac{\partial \phi_3}{\partial q_2} T_{21} + \Lambda_{33} T_{31} & \frac{\partial \phi_3}{\partial q_2} T_{22} + \Lambda_{33} T_{32} & \Lambda_{33} T_{33} & 0 & 0\ldots\ldots & 0 \\
\begin{array}{c} \{\frac{\partial \phi_4}{\partial q_1} + \frac{\partial \psi_{43}}{\partial q_1} q_3\} T_{11} \\ +\{\frac{\partial \phi_4}{\partial q_2} + \frac{\partial \psi_{43}}{\partial q_2} q_3\} T_{21} \\ +\psi_{43} T_{31} + \Lambda_{44} T_{41} \end{array} & \begin{array}{c} \{\frac{\partial \phi_4}{\partial q_2} + \frac{\partial \psi_{43}}{\partial q_2} q_3\} T_{22} \\ +\psi_{43} T_{32} \\ +\Lambda_{44} T_{42} \end{array} & \begin{array}{c} \psi_{43} T_{33} \\ +\Lambda_{44} T_{43} \end{array} & \Lambda_{44} T_{44} & 0\ldots\ldots & 0 \\
\begin{array}{c} \{\frac{\partial \phi_5}{\partial q_1} + (\frac{\partial \psi_{53}}{\partial q_1} + \frac{\partial \psi_{54}}{\partial q_1}) q_3\} T_{11} \\ +\{\frac{\partial \phi_5}{\partial q_2} + (\frac{\partial \psi_{53}}{\partial q_2} + \frac{\partial \psi_{54}}{\partial q_2}) q_3\} T_{21} \\ +\psi_{53} T_{31} + \psi_{54} T_{41} + \Lambda_{55} T_{51} \end{array} & \begin{array}{c} \{\frac{\partial \phi_5}{\partial q_2} + (\frac{\partial \psi_{53}}{\partial q_2} + \frac{\partial \psi_{54}}{\partial q_2}) q_3\} T_{22} \\ +\psi_{53} T_{32} + \psi_{54} T_{42} \\ +\Lambda_{55} T_{52} \end{array} & \begin{array}{c} \psi_{53} T_{33} \\ +\psi_{54} T_{43} \\ +\Lambda_{55} T_{53} \end{array} & \begin{array}{c} \psi_{54} T_{44} \\ +\Lambda_{55} T_{54} \end{array} & \Lambda_{55} T_{55} & 0 \\
\\
.. & .. & .. & .. & \ldots\ldots\ldots & \Lambda_{nn} T_{nn}
\end{array}\right]$$

The algorithm proceeds along the following steps:
1. For every $i \geq 4$, solve $N_{i,i-1} = 0$ to obtain the function $\psi_{i,i-1}$. As before, we compute $\psi_{43} = -\frac{\Lambda_{44}T_{43}}{T_{33}}$, $\psi_{54} = -\frac{\Lambda_{55}T_{54}}{T_{44}}$ and so on.
2. For every $i \geq 5$, solve $N_{i,i-2} = 0$ by using the function $\psi_{i,i-1}$ obtained in step 1 to get $\psi_{i,i-2}$. For example, $\psi_{53} = \frac{\Lambda_{55}}{T_{33}T_{44}}\{T_{54}T_{43} - T_{53}T_{44}\}$.
3. Proceed in this manner until $i = n$ to complete the computation of ψ.
4. Solve the equations $N_{22} > 0$ and $N_{21} = 0$ to determine the function ϕ_2. Note that the equation $N_{21} = 0$ leads to a PDE which needs to be solved along with the inequality $N_{22} > 0$ to yield ϕ_2. In the previous case of dependence on a single coordinate, all the equations were ODEs.
5. Solve the partial differential equations $N_{ij} = 0,\ 3 \leq i \leq n,\ j = 1, 2$ and compute the matrix ϕ. Note once again that, unlike the previous case of dependence on a single coordinate, here we encounter PDE's. However, the solvability of these PDE's can be easily verified and in case of the existence of a solution, the residual step involves computing a set of first integrals in order to obtain the function ϕ.

As an illustration, consider the equations $N_{32} = 0$ and $N_{31} = 0$. They together yield

$$\frac{\partial \phi_3}{\partial q_2} = -\Lambda_{33}\frac{T_{32}}{T_{22}}, \tag{3.67}$$

$$\frac{\partial \phi_3}{\partial q_1} = \frac{\Lambda_{33}}{T_{11}T_{22}}\{T_{21}T_{32} - T_{22}T_{31}\}. \tag{3.68}$$

A solution to (3.67), (3.68) would exist if and only if

$$\frac{\partial}{\partial q_1}\{-\Lambda_{33}\frac{T_{32}}{T_{22}}\} = \frac{\partial}{\partial q_2}\{\frac{\Lambda_{33}}{T_{11}T_{22}}\{T_{21}T_{32} - T_{22}T_{31}\}\}, \tag{3.69}$$

which can be verified in a straightforward manner as the terms of the matrix T are known. In case a solution exists, then the next step is a simple integration of (3.67), (3.68) to obtain the function ϕ_3.

The elements of the matrix Λ can be chosen freely and it suffices to just ensure that they are positive constants. Finally, after having computed $\mathcal{N}$, we obtain β from (3.66).

We now illustrate this procedure for a four degree of freedom mechanical system.

Planar Redundant Manipulator with one elastic degree of freedom [10], [71]

This is an interesting example of a four degree of freedom underactuated mechanical system whose mass matrix depends on two coordinates. In the figure shown, there is a base body of mass M and rotational inertia I which can translate and rotate freely in the plane and contains a massless arm of length

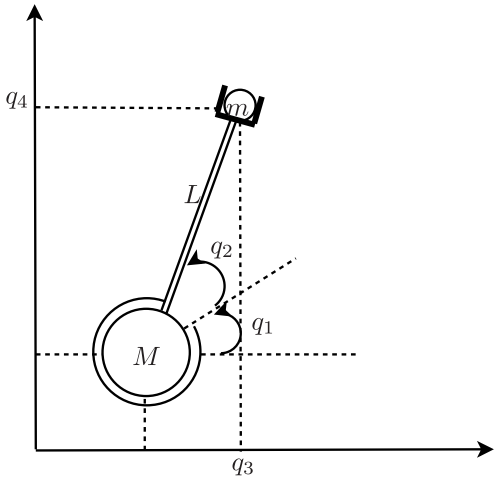

Figure 3.5: Planar redundant manipulator with one elastic degree of freedom.

L at the tip of which the end-effector of mass m is attached. The base body is connected to the massless arm by a linear torsional spring whose extension is denoted by the coordinate q_2. The coordinates (q_3, q_4) represent the position of the end effector while q_1 denotes the angle made by the base body with the fixed horizontal axis. The base body is actuated by forces in the horizontal and vertical directions and has a torque that controls its rotation in the plane. However, the elastic joint is assumed to be unexcited which makes the system underactuated and hence interesting from a control perspective.

Thus, the kinetic energy of the system is given by $KE = \frac{1}{2}(M+m)(\dot{q}_3^2 + \dot{q}_4^2) + \frac{1}{2}I\dot{q}_1^2 + \frac{1}{2}ML^2\{\dot{q}_1 + \dot{q}_2\}^2 + ML\{\dot{q}_1 + \dot{q}_2\}\{\dot{q}_3 \sin(q_1+q_2) - \dot{q}_4 \cos(q_1+q_2)\}$ and subsequently the inertia matrix is given as

$$M^{-1} = \begin{bmatrix} \frac{1}{I} & -\frac{1}{I} & 0 & 0 \\ -\frac{1}{I} & \frac{M+m}{MmL^2} + \frac{1}{I} & -\frac{1}{mL}\sin(q_1+q_2) & \frac{1}{mL}\cos(q_1+q_2) \\ 0 & -\frac{1}{mL}\sin(q_1+q_2) & \frac{1}{M+m} & 0 \\ 0 & \frac{1}{M+m} & 0 & \frac{1}{M+m} \end{bmatrix}.$$

We compute the lower triangular cholesky factorization, T of $M^{-1}(q)$ as

$$T = \begin{bmatrix} \frac{1}{\sqrt{I}} & 0 & 0 & 0 \\ -\frac{1}{\sqrt{I}} & \frac{\sqrt{M+m}}{\sqrt{MmL}} & 0 & 0 \\ 0 & -\sqrt{\frac{M}{m}}\frac{1}{\sqrt{M+m}}\sin(q_1+q_2) & \frac{1}{\sqrt{M+m}} & 0 \\ 0 & \sqrt{\frac{M}{m}}\frac{1}{\sqrt{M+m}}\cos(q_1+q_2) & 0 & \frac{1}{\sqrt{M+m}} \end{bmatrix}.$$

We can again easily check that the columns of T commute among each other thus satisfying condition (3.5). We let the matrix $\mathcal{N}$ be given as,

$$\begin{aligned}\mathcal{N} \;=\;& \begin{bmatrix} \Lambda_{11} & 0 & 0 & 0 \\ 0 & \Lambda_{22} & 0 & 0 \\ 0 & 0 & \Lambda_{33} & 0 \\ 0 & 0 & 0 & \Lambda_{44} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ \nabla_{q_1}\phi_2 & \nabla_{q_2}\phi_2 & 0 & 0 \\ \nabla_{q_1}\phi_3 & \nabla_{q_2}\phi_3 & 0 & 0 \\ \nabla_{q_1}\phi_4 & \nabla_{q_2}\phi_4 & 0 & 0 \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \nabla_{q_1}\psi_{43}q_3 & \nabla_{q_2}\psi_{43}q_3 & \psi_{43} & 0 \end{bmatrix}, \end{aligned} \tag{3.70}$$

where $\Lambda_{ii} > 0$. From $N_{21} = 0$, we get $\nabla_{q_1}\phi_2 = \nabla_{q_2}\phi_2$ and from $N_{22} > 0$, we get $\nabla_{q_1}\phi_2 > 0$. Thus, we let $\phi_2 = k(q_1 + q_2)$ where $k > 0$. We now solve $N_{43} = 0$ to obtain $\psi_{43} = 0$. We then solve $N_{42} = 0$ to get

$$\phi_4 = -\frac{ML\Lambda_{44}}{M+m}\sin(q_1+q_2) + g(q_1).$$

Finally, from $N_{41} = 0$, we get $\nabla_{q_1}\phi_4 = \nabla_{q_2}\phi_4$ and hence we can set $g = 0$. We next solve $N_{32} = 0$ to obtain

$$\phi_3 = -\frac{ML\Lambda_{33}}{M+m}\cos(q_1+q_2) + f(q_1).$$

Next, from $N_{31} = 0$, we get $\nabla_{q_1}\phi_3 = \nabla_{q_2}\phi_3$ and hence we can set $f = 0$. We finally get

$$\beta = \begin{bmatrix} \Lambda_{11}q_1 \\ \Lambda_{22}q_2 + k(q_1+q_2) \\ \Lambda_{33}(q_3 - \frac{ML}{M+m}\cos(q_1+q_2)) \\ \Lambda_{44}(q_4 - \frac{ML}{M+m}\sin(q_1+q_2)) \end{bmatrix}.$$

One can check that the condition (3.69) holds true for this example.

3.4.3 Procedure for Computing $\mathcal{N}$ when the Mass Matrix Depends on k Coordinates, $k \in \bar{n}$

Without loss of generality we assume that the mass matrix depends on the first k coordinates q_1, q_2, ... q_k. Next we propose the following form for $\mathcal{N}$ given as,

$$\mathcal{N} = \Lambda + \nabla_q\{\phi(q_1, q_2, \ldots.., q_k) + \psi(q_1, q_2, \ldots.., q_k)q\} \tag{3.71}$$

where $\Lambda > 0$ is an $n \times n$ constant diagonal matrix, $\phi : \mathbb{R}^k \to \mathbb{R}^n$ and verifies $e_1^\top\phi = 0$, $\psi : \mathbb{R}^k \to \mathbb{R}^{n\times n}$ and verifies $e_i^\top\psi e_j = 0$ for all $j \geq i$ and $\psi e_1 = \psi e_2 = \psi e_3 = \ldots.. = \psi e_k = 0$. The matrix $\mathcal{N}$ is then given as below with $k+2 \leq i \leq n$.

As before, condition (ii) of Assumption 1 is trivially satisfied and β can be immediately computed as

$$\beta = \Lambda q + \phi(q_1, q_2, \ldots.., q_k) + \psi(q_1, q_2, \ldots.., q_k)q. \tag{3.72}$$

$$
\left[\begin{array}{c|c|c|c|c|c|c|c|c}
\Lambda_{11} & 0 & 0.. & 0 & 0 & 0 & 0 & 0.. & 0 \\
\frac{\partial\phi_2}{\partial q_1} & \Lambda_{22}+\frac{\partial\phi_2}{\partial q_2} & \frac{\partial\phi_2}{\partial q_3}.. & \frac{\partial\phi_2}{\partial q_k} & 0 & 0.. & 0 & 0 & 0 \\
\frac{\partial\phi_3}{\partial q_1} & \frac{\partial\phi_3}{\partial q_2} & \Lambda_{33}+\frac{\partial\phi_3}{\partial q_3}.. & \frac{\partial\phi_3}{\partial q_k} & 0 & 0.. & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\partial\phi_k}{\partial q_1} & \frac{\partial\phi_k}{\partial q_2} & \frac{\partial\phi_k}{\partial q_3}.. & \Lambda_{kk}+\frac{\partial\phi_k}{\partial q_k} & 0 & 0.. & 0 & 0 & 0 \\
\frac{\partial\phi_{k+1}}{\partial q_1} & \frac{\partial\phi_{k+1}}{\partial q_2} & \frac{\partial\phi_{k+1}}{\partial q_3}.. & \frac{\partial\phi_{k+1}}{\partial q_k} & \Lambda_{k+1,k+1} & 0.. & 0 & 0 & 0 \\
\begin{array}{c}\frac{\partial\phi_{k+2}}{\partial q_1}+\\ \frac{\partial\psi_{k+2,k+1}}{\partial q_1}q_{k+1}\end{array} & \begin{array}{c}\frac{\partial\phi_{k+2}}{\partial q_2}+\\ \frac{\partial\psi_{k+2,k+1}}{\partial q_2}q_{k+1}\end{array} & \begin{array}{c}\frac{\partial\phi_{k+2}}{\partial q_3}+\\ \frac{\partial\psi_{k+2,k+1}}{\partial q_3}q_{k+1}..\end{array} & \begin{array}{c}\frac{\partial\phi_{k+2}}{\partial q_k}+\\ \frac{\partial\psi_{k+2,k+1}}{\partial q_{k+1}}q_{k+1}\end{array} & \psi_{k+2,k+1} & \Lambda_{k+2,k+2} & 0 & 0.. & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\begin{array}{c}\frac{\partial\phi_i}{\partial q_1}+\\ \sum_{j=k+1}^{i-1}\frac{\partial\psi_{i,j}}{\partial q_1}q_j\end{array} & \begin{array}{c}\frac{\partial\phi_i}{\partial q_2}+\\ \sum_{j=k+1}^{i-1}\frac{\partial\psi_{i,j}}{\partial q_2}q_j\end{array} & \begin{array}{c}\frac{\partial\phi_i}{\partial q_3}+\\ \sum_{j=k+1}^{i-1}\frac{\partial\psi_{i,j}}{\partial q_3}q_j..\end{array} & \begin{array}{c}\frac{\partial\phi_i}{\partial q_k}+\\ \sum_{j=k+1}^{i-1}\frac{\partial\psi_{i,j}}{\partial q_k}q_j\end{array} & \psi_{i,k+1} & \psi_{i,k+2}.. & \Lambda_{i,i} & 0.. & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
.. & .. & .. & .. & .... & .. & .. & .. & \Lambda_{nn}T_{nn}
\end{array}\right]
$$

We can see that for the cases, $k = 1, 2$, the matrix $\mathcal{N}$ assumes the form (3.62) and (3.65) respectively. Moreover for these cases, $\mathcal{N}$ is lower triangular and hence is $N = \mathcal{N}T$ but, for $k > 2$, the $k \times k$ upper left block of the matrix $\mathcal{N}$ (and subsequently N) is clearly not lower triangular which makes the algorithm more complicated. The algorithm proceeds along the following steps:
1. First compute $\tilde{N} = \frac{1}{2}\{N + N^\top\}$,
2. For every $i \geq k + 2$, solve $\tilde{N}_{i,i-1} = 0$ to obtain the function $\psi_{i,i-1}$.
3. For every $i \geq k + 3$, solve $\tilde{N}_{i,i-2} = 0$ by using the function $\psi_{i,i-1}$ obtained in step 1 to get $\psi_{i,i-2}$.
4. Proceed in this manner until $i = n$ to complete the computation of ψ.
5. Solve the inequalities $\tilde{N}_{ii} > 0$ for all $2 \leq i \leq k$ and the partial differential equations $\tilde{N}_{ij} = 0$ for all $1 \leq j < i \leq k$ to determine the functions ϕ_l, for all $l = 2, ...k$.
6. Solve the partial differential equations $\tilde{N}_{ij} = 0, k + 1 \leq i \leq n,\ 1 \leq j \leq k$ and compute the matrix ϕ.

The elements of the matrix Λ can be chosen freely and it suffices to just ensure that they are positive constants. Finally, after having computed $\mathcal{N}$, we obtain β from (3.72).

Remark 3.23. Step 5 is the difficult one as it involves solving $(k-1)$ inequalities and $\frac{(k)(k-1)}{2}$ partial differential equations with the number of unknowns being $k(k-1)$. We can see that for $k = 2$, the number of equations (inequalities and equalities together) is same as the number of unknowns and we thus get an exact solution, but for $k > 2$ we have more equations than unknowns. Hence, it could be possible that we can get more than one solution for the functions ϕ_i for $i = 2, ...k$. The steps $2, 3$ in the algorithm, which involve solving a set of algebraic equations and step 6 that involves some simpler PDEs (as also seen in the $k = 2$ case), are relatively straightforward.

Remark 3.24. If $k = n$, then the matrix $\psi = 0$ from our construction. In that case, we would have to follow only step 5 of the algorithm. Hence (as expected), the larger the value of k, more PDEs need to be solved and the complication of the algorithm increases.

3.4.4 Computation of $\mathcal{N}$ for a general non-Cholesky factorization of the inertia matrix

The constructive procedure to compute $\mathcal{N}$ given above proceeds from the Cholesky factorization of the matrix M^{-1}. It may happen that this particular factorization does not satisfy the skew–symmetry condition of Proposition 3.9 but another factorization does—this is so for the mass matrix (3.33). Moreover, the inertia matrix may not admit a suitable factorization, *i.e.*, one that satisfies the skew–symmetry condition, but we may be able to find a matrix Ψ that verifies the most general condition (3.5).

To compute $\mathcal{N}$ we can, of course, combine the two conditions of Assumption 1 to obtain, directly in terms of β, the differential inequality

$$(\nabla\beta)M^{-1}\Psi^{-\top} + \left((\nabla\beta)M^{-1}\Psi^{-\top}\right)^{\top} \geq \epsilon I_n,$$

but it seems difficult to even establish conditions for existence of solutions to this inequality. Alternatively, we can fix "candidate" matrices $\mathcal{N}$ that already satisfy the integrability condition (3.60) and concentrate on the inequality (3.52). Obviously, the first natural candidates are constant matrices. Another useful option is to fix the ij element of $\mathcal{N}$ to be of the form

$$\mathcal{N}_{ij}(q) = a^1_{i1}(q_1)a^2_{i2}(q_2)\cdots\frac{da^j_{ij}(q_j)}{dq_j}\cdots a^n_{in}(q_n)$$

for some free functions $a^j_{ij} : \mathbb{R} \to \mathbb{R}$—it is easy to see that (3.60) will hold for the resulting $\mathcal{N}$.

Example 3.25. *We now show how the above construction works for the mass matrix (3.33) with the (non–Cholesky) factorization (3.32) that we repeat here for ease of reference*

$$T = \begin{bmatrix} q_2\sin(q_1) & q_2\cos(q_1) & 1 \\ (1+q_2^2)\cos(q_1) & -(1+q_2^2)\sin(q_1) & 0 \\ \sqrt{1+q_2^2}\sin(q_1) & \sqrt{1+q_2^2}\cos(q_1) & 0 \end{bmatrix}.$$

We recall that, as shown in Proposition 3.9, the columns of this matrix do not commute, however, it verifies the skew–symmetry condition. For the sake of illustration, we select the desired operating point to be $q_\star = 0$. *We now Consider the matrix* T *given above and the matrix*

$$\mathcal{N} = \begin{bmatrix} 0 & \frac{\tilde{\lambda}}{\sqrt{1+q_2^2}} & 0 \\ 0 & 0 & \tilde{\lambda} \\ \cos(q_1) & 0 & \end{bmatrix},$$

with $\tilde{\lambda} > 0$. *We aim to show that* $\mathcal{N}T + T^\top \mathcal{N}^\top > 0$ *for all* q *in the set*

$$\{q \in \mathbb{R}^3 \mid \frac{-\pi}{2} + \kappa \leq q_1 \leq \frac{\pi}{2} - \kappa,\ 4\,\tilde{\lambda} > |q_2|\},$$

where $\kappa > 0$ *is an* arbitrarily small *constant. For this, We compute* $\mathcal{N}T + T^\top \mathcal{N}^\top$ *as*

$$\begin{bmatrix} 2\tilde{\lambda}\sqrt{1+q_2^2}\cos(q_1) & 0 & \frac{1}{2}\sin(2q_1)q_2 \\ 0 & 2\tilde{\lambda}\sqrt{1+q_2^2}\cos(q_1) & \cos^2(q_1)q_2 \\ \frac{1}{2}\sin(2q_1)q_2 & \cos^2(q_1)q_2 & 2\cos(q_1) \end{bmatrix}.$$

The determinant of this matrix equals

$$2\tilde{\lambda}(1+q_2^2)c^3(q_1)[4\tilde{\lambda} - \frac{q_2^2}{\sqrt{1+q_2^2}}],$$

from which the claim follows immediately.

3.5 Asymptotic Stability of IDA–PBC Designs with I&I Observers

In this section, we study the stability properties of the combination of the Interconnection and Damping assignment passivity-based controller (IDA-PBC) introduced in chapters 1 and 2 with the I&I observer derived in the previous section. In particular, we show that the measurement of momenta, p, required in IDA-PBC, can be replaced by its estimated signal $\hat{p}$, preserving asymptotic stability of the desired equilibrium.

Remark 3.26. In [1] a similar property is established for an IDA–PBC controller with a different I&I observer for the case of systems with under-actuation degree one written in *Spong's normal form* [82]—see Section 6 of [1]. However, to transform a mechanical system to Spong's normal form it is necessary in general to feed–back the full state and hence the result is not applicable for the problem at hand. For example, in the case of inverted pendulum on cart, the feed-back consists of velocity measurements (refer to [84]).

Even though global exponential convergence of the I&I observer has been established and, furthermore, mechanical systems are linear in u, the proof of this claim, in its global formulation, is non–trivial for the following reasons. First, the control law of IDA–PBC is quadratic in p and will, in general, depend on all the elements of this vector. Second, non–positivity of the Lyapunov function derivative is obtained in IDA–PBC via damping injection, more precisely by feeding–back the passive output (for instance $y = G^\top(q)\nabla H(q,p)$ is the passive output for (3.1)) which is a function only of the actuated components of p, that is, the elements in the image of the input matrix G. Consequently, when p is replaced by their estimates the derivative of the (state–feedback) Lyapunov function will contain sign indefinite terms. While classical perturbation arguments allow to conclude local asymptotic stability, to establish the global version some particular properties of cascaded systems must be invoked.

For the sake of brevity the IDA–PBC methodology is not reviewed here (refer to Chapters 1 and 2) while only the key equations needed for the analysis are given. The objective in IDA–PBC is to assign to the closed–loop the energy function

$$H_d(q,p) = \frac{1}{2}p^\top M_d^{-1}(q)p + V_d(q) - V_d(q_\star)$$

where $M_d = M_d^\top \in \mathbb{R}_P^{n\times n}$, V_d are the desired inertia matrix and potential energy function, respectively, and $q_\star$ is the desired position, by preserving the mechanical structure of the system. This is achieved imposing the closed–loop dynamics

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{bmatrix} 0 & M^{-1}M_d \\ -M_dM^{-1} & J_2 - GK_vG^\top \end{bmatrix} \begin{pmatrix} \nabla_q H_d \\ \nabla_p H_d \end{pmatrix}, \tag{3.73}$$

where $K_v = K_v^\top \in \mathbb{R}_P^{n\times n}$ is a damping injection matrix and $J_2(q,p)$ is a skew–symmetric matrix of the form

$$\begin{bmatrix} 0 & p^\top\alpha_1(q) & p^\top\alpha_2(q) & \dots & p^\top\alpha_{n-1}(q) \\ -p^\top\alpha_1(q) & 0 & p^\top\alpha_n(q) & \dots & p^\top\alpha_{2n-3}(q) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -p^\top\alpha_{n-1}(q) & -p^\top\alpha_{2n-3}(q) & \dots & & 0 \end{bmatrix} \tag{3.74}$$

where $\alpha_i : \mathbb{R}^n \to \mathbb{R}^n,\ i = 1,\dots,\frac{n}{2}(n-1)$, are free functions.

If $q_\star = \arg\min V_d(q)$ then $(q_\star, 0)$ is a stable equilibrium of the closed loop with Lyapunov function H_d clearly verifying

$$\dot{H}_d = -p^\top M_d^{-1}GK_vG^\top M_d^{-1}p \le -c_1|\bar{p}|^2,$$

where, to simplify the notation in the sequel, we have defined the function

$$\bar{p}(q,p) := G^\top(q)M_d^{-1}(q)p, \tag{3.75}$$

and use the convention of denoting with c_i an (often unspecified) positive constant—in this case $c_1 := \bar{\lambda}_m\{K_v\}$. Stability will be asymptotic if $\bar{p}$ is a *detectable* output (refer to footnote 1 in Chapter 1) for the closed–loop system (3.73).

The full–state measurement IDA–PBC is given by

$$u(q,p) = (G^\top G)^{-1} G^\top \left(\nabla_q H - M_d M^{-1} \nabla_q H_d + J_2 M_d^{-1} p\right) - K_v \bar{p}, \tag{3.76}$$

which, as shown in [1], may be written in the form

$$u(q,p) = u_0(q) + \begin{bmatrix} p^\top A_1(q) p \\ \vdots \\ p^\top A_m(q) p \end{bmatrix} - K_v \bar{p}, \tag{3.77}$$

where the vector $u_0 : \mathbb{R}^n \to \mathbb{R}^n$ and the matrices $A_i : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ are functions of q. As will be shown below, establishing boundedness of $A_i,\ i = 1, ...m$, will be critical for our analysis. Towards this end, we center our attention on the quadratic terms in p of (3.76) stemming from $\nabla_q H$ and $\nabla_q H_d$ and introduce the following assumption 2. Moreover, from (3.74) it is clear that the term $J_2 M_d^{-1} p$ is also quadratic in p. It will be shown below that Assumption 2 allows to establish a suitable bound for this term as well.

Assumption 2. *The matrices $\nabla_{q_i} M$, $\nabla_{q_i} M_d$ and G are bounded.*

Proposition 3.27. *Consider the system (3.1) and assume $M \in \mathcal{S}_{\mathtt{PLvCC}}$. Define the* position feedback *controller as $u = u(q, \hat{p})$ with $\hat{p}$ an estimate of p generated by the I&I observer (3.54). Assume $\bar{p}(q,p)$ in (3.75) is a* detectable *output for the closed–loop system (3.73) and that Assumption 1 is satisfied. Then there exists a neighborhood of the point $(q^\star, 0, \beta(q^\star))$ such that all trajectories of the closed–loop system starting in this neighborhood are bounded and satisfy*

$$\lim_{t\to\infty} (q(t), p(t), \eta(t)) = (q^\star, 0, \beta(q^\star)).$$

Furthermore, if Assumption 2 holds and the full state–feedback controller (3.77) ensures global asymptotic stability then the neighborhood is the whole space $\mathbb{R}^{3n}$, thus boundedness and convergence are global.

Proof. To carry out the proof the overall system is written as a cascade interconnection of the observer error subsystem $\dot{z} = -\mathcal{A} z$ and the full state–feedback dynamics (3.73). We first write $u(q,\hat{p}) = u(q,p) + \chi(q,p,z)$ where we define

$$\begin{aligned} \chi(q,p,z) := \sum_{i=1}^{m} \left[z^\top \Psi^{-1} A_i \Psi^{-\top} z + z^\top \Psi^{-1} (A_i + A_i^\top) p \right] e_i \\ -K_v G^\top M_d^{-1} \Psi^{-\top} z. \end{aligned} \tag{3.78}$$

The overall system can then be written in the cascaded form

$$\begin{aligned} \begin{pmatrix} \dot q \\ \dot p \end{pmatrix} &= \begin{bmatrix} 0 & M^{-1}M_d \\ -M_d M^{-1} & J_2 - GK_vG^\top \end{bmatrix} \begin{pmatrix} \nabla_q H_d \\ \nabla_p H_d \end{pmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix} \chi \\ \dot z &= -\mathcal{A} z \end{aligned} \tag{3.79}$$

Note that the system with $\chi = 0$ is asymptotically stable. Furthermore, the disturbance term is such that

$$G(q)\chi(q,p,0) = 0.$$

Invoking well–known results of asymptotic stability of cascaded systems [75] completes the proof of local asymptotic stability. To establish the global claim we invoke the fundamental result of [74], see also [78], and see that the proof will be completed if we can establish boundedness of the trajectories $(q(t), p(t))$. Computing the time derivative of H_d along the trajectories of (3.79) we get the bound

$$\dot H_d \leq -c_1|\bar p|^2 + |\bar p||G\chi|. \tag{3.80}$$

From the expression above it is clear that the key step to prove boundedness of trajectories is to establish a suitable bound for $|G\chi|$. At this point Assumption 2 is imposed. Comparing (3.76) with (3.77) we observe that the matrices A_i will be bounded if Assumption 2 holds and J_2 may be bounded as $\|J_2\| \leq c_2|p|$. Now, from the IDA–PBC procedure we have that J_2 satisfies the so–called kinetic energy PDE

$$G^\perp\{\nabla_q(p^\top M^{-1}p) - M_d M^{-1}\nabla_q(p^\top M_d^{-1}p) + 2J_2 M_d^{-1}p\}{=}0.$$

Comparing in this equation the terms which are quadratic in p and (3.74) we conclude that, under Assumption 2, J_2 will satisfy the bound above and the matrices A_i are also bounded.

From the previous discussion, and boundedness of z, we get the bound $|G\chi| \leq |z|(c_2 + c_3|p|)$, which replaced in (3.80) yields

$$\dot H_d \leq -c_1|\bar p|^2 + |\bar p||z|(c_2 + c_3|p|). \tag{3.81}$$

Now, invoking standard (Young's inequality[1]) arguments we get

$$|\bar p||z| \leq \frac{c_1}{c_2}|\bar p|^2 + \frac{c_2}{4c_1}|z|^2.$$

[1] The Young's inequality argument states that if a, b are nonnegative real numbers and p, q, ϵ are positive real numbers, then the following holds true

$$ab \leq \frac{a^p}{\epsilon p} + \frac{\epsilon b^q}{q}. \tag{3.82}$$

We replace this bound in the second right hand term in (3.81) to get

$$\begin{aligned}\dot{H}_d &\leq -c_1|\bar{p}|^2 + [\frac{c_1}{c_2}|\bar{p}|^2 + \frac{c_2}{4c_1}|z|^2]c_2 + c_3|\bar{p}||z||p| \\ &\leq \frac{c_2^2}{4c_1}|z|^2 + c_5|z||p|^2,\end{aligned}$$

where we have used the bound of $|\bar{p}| \leq c_4|p|$ to define $c_5 := c_3c_4$. Now, let us consider the non-negative function

$$W(q,p,z) := H_d(q,p) + \frac{c_2^2\bar{\lambda}(\mathcal{Q})}{4c_1\epsilon}\mathcal{V}(z),$$

where $\mathcal{V}(z)$ is given in (3.57), which as shown in the proof of Proposition 1 verifies (3.58). Finally, evaluating the derivative of W we get

$$\dot{W} \leq c_5|z||p|^2 \ \leq \frac{2c_5}{\bar{\lambda}(M_d)}|z|W, \tag{3.83}$$

where we have used the bounds $W \geq H_d \geq \frac{1}{2}\bar{\lambda}(M_d)|p|^2$ to obtain the last inequality. Since z is clearly an *integrable* function, invoking the Comparison Lemma [46], we immediately conclude boundedness of W and, consequently, boundedness of the trajectories $(q(t), p(t))$ and complete the proof. □

Remark 3.28. In the context of passivity based stabilization of mechanical systems, we point the reader to the recent article [29] which considers the IDA-PBC problem for mechanical systems by using only position measurements and designs a dynamic controller for the same. However, the mechanical systems considered in [29] are assumed to be fully actuated while we do not impose a restriction on the actuation.

3.6 Simulation Results

The theoretical results of the previous sections have been verified through simulations of the inverted pendulum example. The dynamical equations for this system are given by (3.1), (4.32) with

$$M^{-1} = \frac{1}{m_3 - b^2\cos^2 q_1}\begin{bmatrix} m_3 & -b\cos q_1 \\ * & 1 \end{bmatrix}, V = a\cos q_1$$

$$G = e_2, \ \ a = \frac{g}{l}, \ \ b = \frac{1}{l}, \ \ m_3 = \frac{M+m}{ml^2},$$

where q_1 denotes the pendulum angle with respect to the upright vertical, q_2 the cart position, m and l are, respectively, the mass and length of the pendulum, M is the mass of the cart and g is the gravitational acceleration. The equilibrium to be stabilized is the upward position of the pendulum ($q_{1*} = 0$) with the cart placed in any desired location (arbitrary q_{2*}).

The detailed expressions of the full–state IDA–PBC, given by (3.76), may be found in [93]. We proved the separation principle for the I & I reduced-order observer and the IDA-PBC full–state feedback controller in Section 3.5. We now numerically verify our proposed observer design and the *certainty–equivalence* implementation of the full–state feedback IDA-PBC controller design with the I & I observer for the inverted pendulum on cart example.

Firstly, the estimate of the momenta p given by $\hat{p}$, is generated by the I&I observer design (refer to Section 3.4) and is given as

$$\begin{aligned}
\dot{\eta}_1 &= \frac{\Lambda_{11}\sqrt{m_3}}{\sqrt{m_3 - b^2\cos^2 q_1}}(\beta_1 - \eta_1) - \frac{a\sqrt{m_3}\sin q_1}{\sqrt{m_3 - b^2\cos^2 q_1}} \\
&\quad + \frac{b\cos q_1}{\sqrt{m_3}\sqrt{m_3 - b^2\cos^2 q_1}} u \\
\dot{\eta}_2 &= \frac{\Lambda_{22}}{\sqrt{m_3}}(\beta_2 - \eta_2) - \frac{1}{\sqrt{m_3}} u \\
\hat{p}_1 &= \frac{\sqrt{m_3 - b^2\cos^2 q_1}}{\sqrt{m_3}}(\beta_1 - \eta_1) + \frac{b\cos q_1}{\sqrt{m_3}}(\beta_2 - \eta_2) \\
\hat{p}_2 &= \sqrt{m_3}(\beta_2 - \eta_2)
\end{aligned}$$

with β given by (3.64). The observer error dynamics takes the form

$$\begin{aligned}
\dot{z}_1 &= -\frac{\Lambda_{11}\sqrt{m_3}}{\sqrt{m_3 - b^2\cos^2 q_1}} z_1 \\
\dot{z}_2 &= -\frac{\Lambda_{22}}{\sqrt{m_3}} z_2,
\end{aligned}$$

from which it is clear that the rate of convergence is (essentially) determined by the constant Λ_{11} and Λ_{22}. Next, the "*certainty–equivalence*" controller is obtained by replacing p by $\hat{p}$ in the IDA-PBC full–state control law. We do not give here the expression for the IDA-PBC controller but, as mentioned before, it has been directly taken from [93]. Having obtained the closed-loop dynamical equations, we now perform the simulations.

The values of the system and controller parameters, as well as the initial conditions, are shown in Table 5.2. The initial conditions of the observer states $(\eta_1(0), \eta_2(0))$ are chosen so that the initial estimate $\hat{p}(0) = 0$, that is, no prior knowledge for the initial momentum.

Simulation results are shown for the open–loop system, *i.e.*, $u = 0$, in Fig. 3.6. To reveal the role of the observer tuning gains, the time histories of z are depicted for $\Lambda_{11} = \Lambda_{22}$ for the values 1 and 10. Fig. 5.5 shows the behavior of the system in closed loop with the IDA–PBC controller with full–state feedback and observer–based feedback. As it can be seen, the trajectories of the observer–based feedback system show an almost identical behavior with the trajectories of the full–state feedback system, concluding the effectiveness of the proposed scheme.

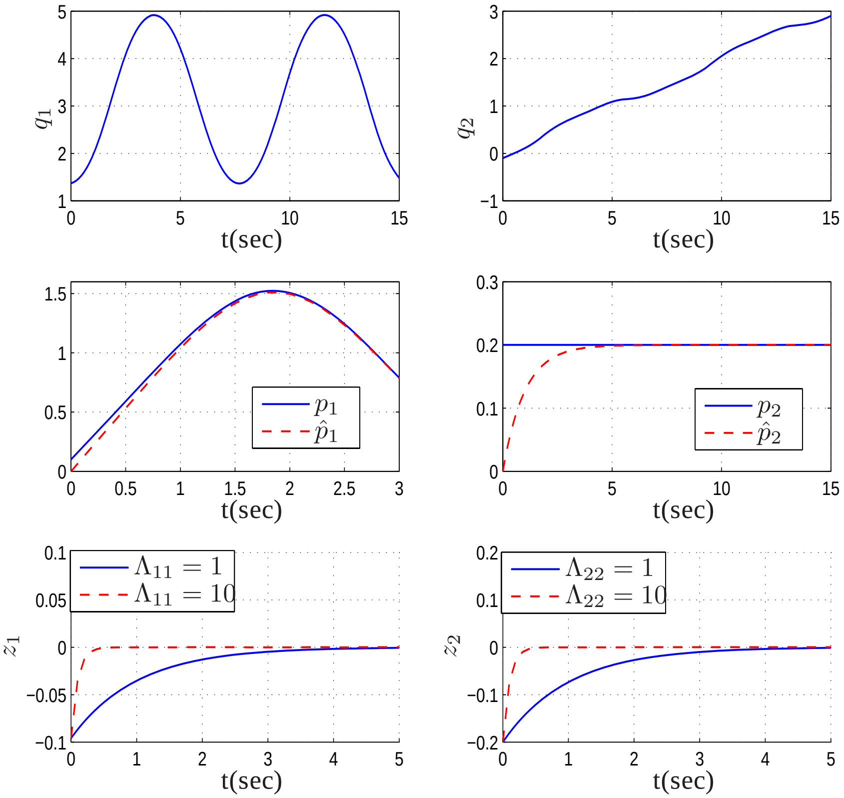

Figure 3.6: I & I observer for the open–loop system ($u = 0$).

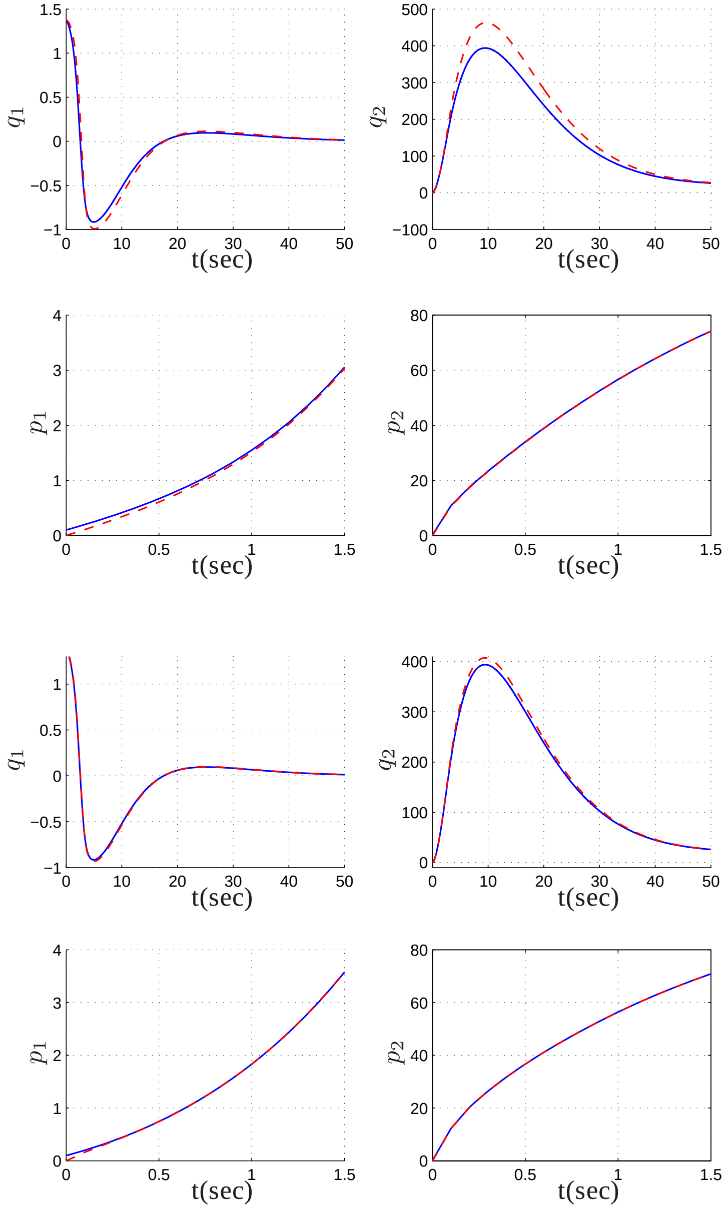

Figure 3.7: Full–state (solid line) and observer-based (dashed line) IDA–PBC for $\Lambda_{11} = \Lambda_{22} = 1$ and $\Lambda_{11} = \Lambda_{22} = 10$

$q_{2*} = 20$	$q_1(0) = \frac{\pi}{2} - 0.2$
$a = m_3 = 1$	$q_2(0) = -0.1$
$b = 1/g$	$p_1(0) = 0.1$
$\Lambda_{11} = \Lambda_{22} = 1$	$p_2(0) = 0.2$
$K_v = k = m_{22}^0 = 0.01$	$\eta_1(0) = \Lambda_{11} q_1(0)$
$P = 1$	$\eta_2(0) = \Lambda_{22}(q_2(0) + \frac{b \sin(q_1(0))}{m_3})$

Table 3.1: Simulation parameters for the inverted pendulum example

3.7 Concluding Remarks

We have identified a special class of mechanical systems for which a globally exponentially stable reduced order observer can be designed. The class consists of all the systems that can be rendered linear in (the unmeasurable) momenta via a (partial) change of coordinates $P = \Psi^\top(q)p$ and is characterized by (the solvability of) a set of PDEs. A detailed analysis of the class is carried out and it is shown to contain many interesting practical examples and is much larger than the class reported in the literature in the context of observer design and linearization. It is also proven that, under a very weak assumption, the observer can be used in conjunction with a globally asymptotically stabilizing full state–feedback IDA–PBC preserving *global* stability.