Bifurcations of attractors in 3D diffeomorphisms

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Chapter 3

Hénon-like strange attractors in a family of maps of the solid torus

3.1 Introduction

The research presented in this Chapter is motivated by the following question:

Are there maps having quasi-periodic Hénon-like attractors?

Numerical examples of quasi-periodic Hénon-like attractors are given in Chapters one and two. We begin by giving definitions of the concepts used here, mainly following the terminology in [42, 86, 120]. Consider a C^1 diffeomorphism $F : M \to M$, where M is an m-dimensional smooth manifold. A set $\mathscr{A} \subset M$ is called an *attractor* if $\mathscr{A}$ is a topologically transitive compact F-invariant set such that the stable set (basin of attraction) $W^s(\mathscr{A})$ has nonempty interior. We recall that an F-invariant set $\mathscr{A} \subset M$ is called topologically transitive if there exists a point $z \in \mathscr{A}$ such that the orbit $\mathrm{Orb}(z) = \{F^j(z)\}_{j\geq 0}$ of z under F is dense in $\mathscr{A}$. An attractor $\mathscr{A}$ is called *strange* if there exist constants $\kappa > 0$, $\lambda > 1$, a dense orbit $\mathrm{Orb}(z) \subset \mathscr{A}$ and a vector $v \in T_zM$ such that

$$\|DF^n(z)v\| \geq \kappa\lambda^n \qquad \text{for } n \geq 0. \tag{3.1}$$

Condition (3.1) means that the attractor $\mathscr{A}$ has a positive Lyapunov exponent on the dense orbit $\mathrm{Orb}(z)$. The attractor $\mathscr{A}$ is called *Hénon-like* [42, 86, 120] if there exist a saddle periodic orbit $\mathrm{Orb}(p) = \{s, F(p), \ldots, F^n(p)\}$, a point z in the unstable manifold $W^u(\mathrm{Orb}(p))$, constants $\kappa > 0$, $\lambda > 1$, and tangent vectors $v, w \in T_zM$, with $w \neq 0$, such that

$$\mathscr{A} = \operatorname{clos} W^u(\mathrm{Orb}(p)),$$

$\mathrm{Orb}(z)$ is dense in $\mathscr{A}$, equation (3.1) holds, and furthermore

$$\|DF^n(z)w\| \to 0 \qquad \text{as } n \to \pm\infty. \tag{3.2}$$

Hénon-like attractors are strange by (3.1), and are non-uniformly hyperbolic by (3.2). In particular, Hénon-like attractors contain *critical points*, that is, points belonging to a dense orbit for which a nonzero tangent vector w exists, which is contracted both by positive and by negative iteration of the derivative DF.

We say that the attractor $\mathscr{A}$ is *quasi-periodic Hénon-like* if there exist a quasi-periodic invariant circle $\mathscr{C}$ of saddle type, a point $p \in W^u(\mathscr{C})$, constants $\kappa > 0$, $\lambda > 1$, and a vector $v \in T_pM$ such that condition (3.1) holds, while

$$\mathscr{A} = \operatorname{clos} W^u(\mathscr{C}).$$

In the last decade several mathematical results have been obtained concerning the structure of strange attractors in families of maps. A basic example is provided by the Hénon attractor [58], occurring in the family of maps

$$H_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto (1 - ax^2 + y,\, bx), \tag{3.3}$$

where a and b are real parameters. Benedicks and Carleson [10, 11] proved that there exists a set of parameter values $\mathfrak{S}$, with positive Lebesgue measure, such that for all $(a, b) \in \mathfrak{S}$ the Hénon map $H_{a,b}$ (3.3) has an attractor coinciding with the closure $\operatorname{clos} W^u(p)$ of the unstable manifold of a saddle fixed point p. By using analogous ideas, strange attractors were proved to occur in parametrised families of maps, near homoclinic tangencies in two or higher dimensions [86, 96, 113, 120], and near tangencies in the saddle-node critical case [42]. See [127] for a general set-up to prove existence of strange attractors having one positive Lyapunov exponent. All strange attractors considered in the cited papers are Hénon-like, see the definition above. See [121] for a result concerning existence of strange attractors with two or more positive Lyapunov exponents.

In this Chapter we provide two partial answers to the question formulated at the beginning of this introduction. Our first result concerns the C^3-family of skew-product diffeomorphisms $T_{\alpha,\delta,a,\varepsilon}$, defined on the solid torus $\mathbb{R}^2 \times \mathbb{S}^1$, where $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, and given by

$$T_{\alpha,\delta,a,\varepsilon} : \mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{R}^2 \times \mathbb{S}^1, \qquad \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} 1 - ax^2 + \varepsilon f(a, x, y, \theta, \varepsilon, \alpha, \delta) \\ \varepsilon g(a, x, y, \theta, \varepsilon, \alpha, \delta) \\ \theta + \alpha + \delta \sin(2\pi\theta) \end{pmatrix}. \tag{3.4}$$

The restriction of (3.4) to $\mathbb{S}^1$ is the Arnol′d family of circle maps [4]:

$$A_{\alpha,\delta} : \mathbb{S}^1 \to \mathbb{S}^1, \quad \theta \mapsto \theta + \alpha + \delta \sin(2\pi\theta). \tag{3.5}$$

For $0 \le \delta < (1/2\pi)$ and $\alpha \in [0, 1]$, the map $A_{\alpha,\delta}$ is a diffeomorphism of the circle $\mathbb{S}^1$. There exist open subsets $\mathfrak{A}^{q/n}$ of the (α, δ) plane (Arnol′d tongues), such that the rotation number of $A_{\alpha,\delta}$ is q/n for all $(\alpha, \delta) \in \mathfrak{A}^{q/n}$.

The map (3.4) is a generalization of the planar Hénon-like families considered in [86, 120]. The latter are families of planar diffeomorphisms, which are C^3-small perturbations of the logistic family

$$Q_a : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto 1 - ax^2. \tag{3.6}$$

In $T_{\alpha,\delta,a,\varepsilon}$, the planar part also depends on the circle dynamics by the perturbative terms f and g. The only requirement on f and g is that their C^3-norms are bounded on compact sets. Occurrence of Hénon-like attractors is proved in the family $T_{\alpha,\delta,a,\varepsilon}$ for all parameter values belonging to a set of of positive (Lebesgue) measure. For all values in this set, the parameters (α, δ) are such that the dynamics of the Arnol′d family $A_{\alpha,\delta}$ (3.5) is of Morse-Smale type: there exist periodic points θ^s and θ^r in $\mathbb{S}^1$, such that θ^s is attracting and θ^r repelling for $A_{\alpha,\delta}$. The attractors $\mathscr{A}$ we obtain coincide with the closure of the one-dimensional unstable manifold

$$\mathscr{A} = \operatorname{clos} W^u\,(\mathrm{Orb}(p)),$$

where $p = (x_0, y_0, \theta^s) \in \mathbb{R}^2 \times \mathbb{S}^1$ belongs to a hyperbolic periodic orbit of saddle type. Occurrence of such attractors holds on a positive measure set of parameter values for all sufficiently C^3-small perturbations of f and g. This result is formulated in more detail in the next section.

A second situation is analysed subsequently. Fix $n > 0$ and let $K : \mathbb{R}^2 \to \mathbb{R}^2$ be a dissipative planar C^n-diffeomorphism having a hyperbolic saddle fixed point $p' \in \mathbb{R}^2$ with a transversal homoclinic point. Then it is well-known that the closure of the unstable manifold $\operatorname{clos} W^u(p')$ attracts an open set of points (initial states) [86, 95]. This result is here generalised to certain families of maps of the solid torus, having an invariant circle of saddle type. Let P_α be a family of diffeomorphisms of $\mathbb{R}^2 \times \mathbb{S}^1$ given by the product of a map K as above with a rigid rotation of angle α on $\mathbb{S}^1$, *i.e.*,

$$P_\alpha : \mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{R}^2 \times \mathbb{S}^1, \quad (x, y, \theta) \mapsto \big(K(x, y), \theta + \alpha\big).$$

The map P_α has the invariant saddle-like circle $\mathscr{C} = \{p'\} \times \mathbb{S}^1$. Then for any sufficiently C^2-small perturbation of P_α, the circle $\mathscr{C}$ persists as a normally hyperbolic C^1-manifold, and the invariant set $\operatorname{clos} W^u(\mathscr{C})$ attracts an open set of points, *i.e.*,

$$\operatorname{int}(W^s(\operatorname{clos} W^u(\mathscr{C}))) \neq \emptyset. \tag{3.7}$$

Notice that in general $\operatorname{clos} W^u(\mathscr{C})$ is not topologically transitive, as required in the definition of attractor we use. For example, $\operatorname{clos} W^u(\mathscr{C})$ might contain periodic attractors. Property (3.7) holds for an open set in the parameter space. However, by standard KAM arguments, the quasi-periodicity of $\mathscr{C}$ (which implies the transitivity of $\mathscr{C}$) generically is persistent only for a nowhere dense set of positive measure in the parameter space, see *e.g.* [17].

3.1.1 Hénon-like strange attractors in a family of skew product maps

We here formulate our main result about the family $T_{\alpha,\delta,a,\varepsilon}$ in (3.4). Throughout, the family $T_{\alpha,\delta,a,\varepsilon}$ is assumed to be C^3 in all variables and parameters. The parameter space is the set of all $(\alpha, \delta, a, \varepsilon) \in \mathbb{R}^4$ such that

$$\alpha \in [0, 1],\ \ \delta \in \big[0, 1/(2\pi)\big),\ \ a \in [0, 2],\ \ |\varepsilon| < 1. \tag{3.8}$$

Furthermore, we require the C^3-norm of f and g to be bounded on compact sets. We call such skew-product families *rotating Hénon-like*. For the statement of the result we need a few definitions and notations.

Definition 3.1. *Consider a map $M : J \to J$, where $J \subset \mathbb{R}$ is an interval.*

1. *The map M is called* topologically mixing *if for any open intervals J_1, $J_2 \subset J$ there exists n_0 such that*

$$M^n(J_1) \cap J_2 \neq \emptyset \quad \textit{for all } n \geq n_0.$$

2. *A point $p \in \mathbb{R}$ is* preperiodic *for M if there exists an $m \geq 2$ such that $M^m(p)$ is a periodic point of M.*

3. *For a given integer $n > 1$, denote by $\Phi(n)$ the set of all integers q such that q and n are relatively prime, where $1 \leq q < n$. If $n = 1$ put $\Phi(n) = \{1\}$.*

4. *The interval $K_a = [Q_a^2(0), Q_a(0)]$ is called the* core *or the* restrictive interval *of the logistic family Q_a (3.6).*

It is well-known that $Q_a([0,1]) = Q_a(K_a) = K_a$ for all a, where K_a is the core of Q_a (3.6), see *e.g.* [83].

Theorem 3.1. *Choose $a^* \in (0,2)$ such that the quadratic map Q_{a^*} in (3.6) is topologically mixing on its core $K = [1-a^*, 1]$ and its critical point $c = 0$ is preperiodic. Let $n \geq 1$ be an integer and p_0 be a (repelling) periodic point of the n-th iterate $Q_{a^*}^n$. Then there exist positive constants $\bar{\varepsilon}_n$, $\bar{a}_n$ and χ_n such that the following holds.*

1. *For all $(\alpha, \delta, a, \varepsilon)$ as in (3.8), with*

$$(\alpha, \delta) \in \cup_{q \in \Phi(n)} \operatorname{clos} \mathfrak{A}^{q/n}, \quad |a - a^*| < \bar{a}_n, \quad |\varepsilon| < \bar{\varepsilon}_n \tag{3.9}$$

the map $T_{\alpha,\delta,a,\varepsilon}$ has a saddle periodic point p such that the unstable manifold $W^u(\mathrm{Orb}(p))$ is one-dimensional.

2. *For all $(\alpha, \delta, \varepsilon)$ as in (3.9) there exists a set $\mathfrak{S}_{\alpha,\delta,\varepsilon}$ with*

$$\mathfrak{S}_{\alpha,\delta,\varepsilon} \subset [a^* - \bar{a}_n, a^* + \bar{a}_n], \qquad \operatorname{meas}(\mathfrak{S}) > \chi_n$$

such that for all $a \in \mathfrak{S}_{\alpha,\delta,\varepsilon}$ the closure $\operatorname{clos} W^u(\mathrm{Orb}(p))$ is a Hénon-like strange attractor of $T_{\alpha,\delta,a,\varepsilon}$.

Corollary 3.1. *The set of parameter values for which $T_{\alpha,\delta,a,\varepsilon}$ has a Hénon-like attractor contains the set*

$$\mathfrak{S} = \bigcup_{n \in \mathbb{N}} \left\{ (\alpha, \delta, a, \varepsilon) \;\mid\; (\alpha, \delta) \in \cup_{q \in \Phi(n)} \operatorname{clos} \mathfrak{A}^{q/n}, \quad |\varepsilon| < \bar{\varepsilon}_n, \quad a \in \mathfrak{S}_{\alpha,\delta,\varepsilon} \right\},$$

and the set $\mathfrak{S}$ has positive Lebesgue measure

$$\operatorname{meas}(\mathfrak{S}) \geq 2 \sum_{n=1}^{\infty} \bar{\varepsilon}_n \chi_n \sum_{q \in \Phi(n)} \operatorname{meas} \mathfrak{A}^{q/n}.$$

Our proof of Theorem 3.1 is given in Sec. 3.2. It is based on a result of Díaz-Rocha-Viana [42], and relies on the following facts:

1. For (α, δ) inside any tongue $\mathfrak{A}^{q/n}$, the asymptotic dynamics of $T_{\alpha,\delta,a,\varepsilon}$ is described by an $\mathcal{O}(\varepsilon)$-perturbation of the n-th iterate Q_a^n.

2. For all n the map Q_a^n is a generic n-modal family, in the sense of [42]. See the definition given in Sec. 3.2.

Two attractors occurring in the family

$$\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} 1-(a+\varepsilon\sin(2\pi\theta))x^2+y \\ bx \\ \theta+\alpha+\delta\sin(2\pi\theta) \end{pmatrix}, \tag{3.10}$$

are shown in Figure 3.1 (A) and (B), for (α, δ) in an Arnol′d tongue of period two and three, respectively. The Hénon-like character of these attractors remains conjectural for the specific parameter values considered. Notice that the family (3.10) takes the form (3.4) after a rescaling $y \mapsto \sqrt{|b|}y$ and by choosing $b = \mathcal{O}(\varepsilon)$.

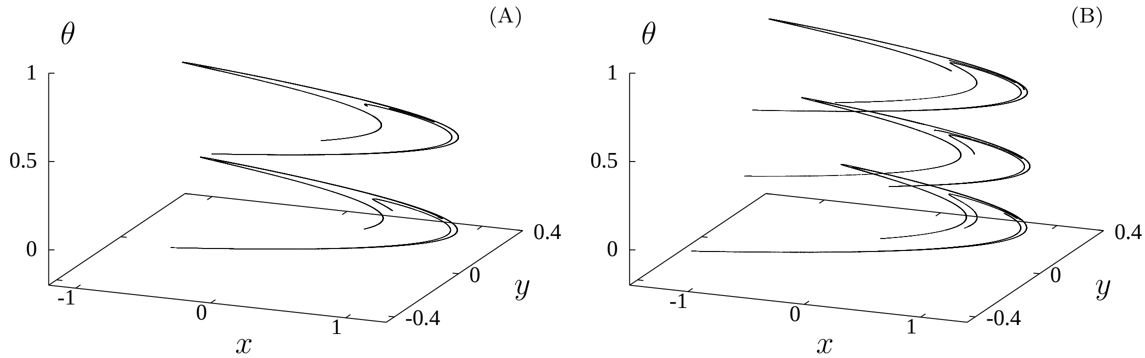

Figure 3.1: Attractors of the family in (3.10) for (α, δ) in Arnol′d tongues of periods two and three. (A) Parameters are fixed at $a = 1.3$, $b = 0.3$, $\varepsilon = 0.2$, $(\alpha, \delta) = (0.51, 0.116)$. (B) Same as (A) for $\alpha = 0.33793$.

A case which is not covered by Theorem 3.1 is when the dynamics of the forcing map $A_{\alpha,\delta}$ in (3.4) is quasi-periodic. In such a situation, by [11] it is straightforward that at $\varepsilon = 0$ Hénon-like strange attractors occur for a positive measure set of parameters (α, δ, a). To fix ideas, consider the family in (3.10). Choose a and b such that the Hénon map (3.3) has a strange attractor $\mathscr{A}'$, coinciding with the closure of the unstable manifold of a saddle fixed point p. According to [11], such (a, b) form a set of positive measure. Since at $\varepsilon = 0$ the dynamics of (3.10) on $\mathbb{R}^2$ is uncoupled from that on $\mathbb{S}^1$, map (3.10) has a strange attractor $\mathscr{A} = \mathscr{A}' \times \mathbb{S}^1$. Furthermore, $\mathscr{A}$ coincides with the closure of the unstable manifold of the quasi-periodic saddle-type invariant circle $\{p\} \times \mathbb{S}^1$. Numerical experiments (see Figure 3.2 (A)) suggest that attractors like $\mathscr{A}$ persist for small ε. Occurrence of the same kind of quasi-periodic Hénon-like strange attractors has been observed in several numerical studies. Compare [90] and the literature on strange nonchaotic attractors [49, 53, 65, 66, 68, 74, 91, 119]. In Chapter two of this thesis a diffeomorphism P of $\mathbb{R}^3 = \{x, y, z\}$ is studied (also see [24]). There we conjectured that the attractor $\mathscr{A}$ of P in Figure 3.2 (B) is contained inside the closure $\operatorname{clos} W^u(\mathscr{C})$ of the unstable manifold of a quasi-periodic invariant circle $\mathscr{C}$ of saddle type. A cross-section Σ of $\mathscr{A}$, magnified in Figure 3.3 left, suggests that the two-dimensional unstable manifold of $\mathscr{C}$ is folded onto itself, thereby creating a Hénon-like structure. To illustrate the dynamics inside $\mathscr{A}$ we computed the image $P(\Sigma)$. This yields a folded curve looking like a planar Hénon attractor.

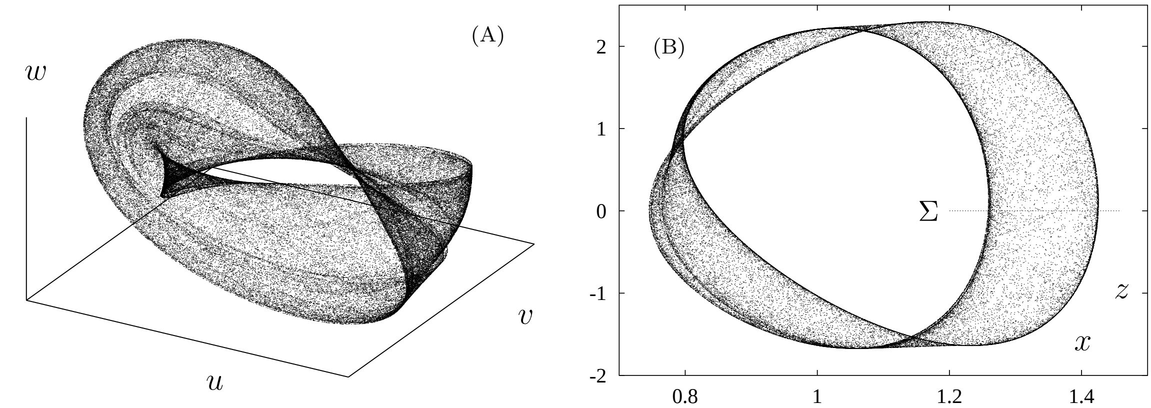

Figure 3.2: (A) Attractor of map (3.10) in the quasi-periodic case. Parameter values are fixed at $a = 1.85$, $b = -0.2$, $\delta = 0$, $\alpha = (\sqrt{5}-1)/2$, $\varepsilon = 0.1$. For a better visualisation of the folds, the plot is given in the variables (u, v, w), where $u = (r+4)\cos(\theta)$, $v = (r+4)\sin(\theta)$, with $r = x\cos(\theta) + 10y\sin(\theta)$, and $w = -x\sin(\theta) + 10y\cos(\theta)$. (B) Projection on the (x, z)-plane of a strange attractor of the three-dimensional Poincaré map $P_{F,G,\varepsilon}$ of Chapter two, also see [24].

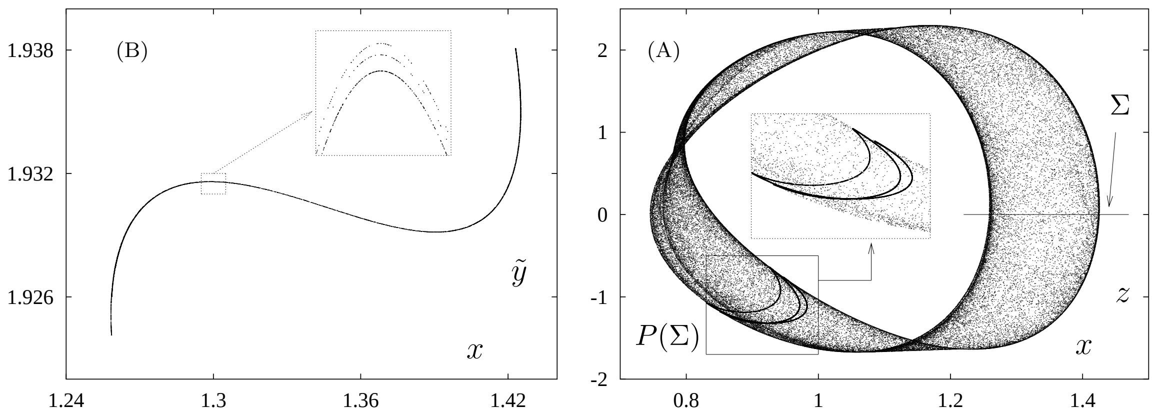

Figure 3.3: (A) Projection on $(x, \tilde{y})$, with $\tilde{y} = y - 0.133 * z$, of a slice Σ of the attractor $\mathscr{A}$ in Figure 3.2 (B). The slice Σ contains all points such that the distance from the plane $z = 0$ is less than 0.0001. (B) The attractor $\mathscr{A}$, with the slice Σ and the image $P(\Sigma)$ under the diffeomorphism P. The image $P(\Sigma)$ is magnified in the central box.

3.1.2 Homoclinic intersections of saddle invariant circles

Hénon-like attractors coincide with the closure $\operatorname{clos} W^u(\mathrm{Orb}(p))$ of the unstable manifold of a saddle periodic orbit. For the dissipative Hénon map (3.3), *i.e.*, for $|b| < 1$, under suitable hypotheses the Hénon attractor is *contained* in $\operatorname{clos} W^u(\mathrm{Orb}(p))$ [11, 86, 95]. We generalise this result to families of maps of the following type. Fix an integer $n \geq 2$ and let $K = (K_1, K_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be a dissipative (area contracting) C^n-diffeomorphism. Denote by $R_\alpha : \mathbb{S}^1 \to \mathbb{S}^1$ the rigid rotation $R_\alpha(\theta) = \theta + \alpha$. Consider the C^n-family of diffeomorphisms

$$P_{\alpha,\varepsilon} : \mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{R}^2 \times \mathbb{S}^1, \quad (x, y, \theta) \mapsto \big(K_1(x, y) + f(x, y, \theta, \alpha, \varepsilon),\\ K_2(x, y) + g(x, y, \theta, \alpha, \varepsilon),\\ \theta + \alpha + h(x, y, \theta, \alpha, \varepsilon)\big), \tag{3.11}$$

where $f = g = h = 0$ for $\varepsilon = 0$. A hyperbolic saddle fixed point p of K corresponds to an invariant circle $\mathscr{C}_{\alpha,0}$ of saddle type for the map $P_{\alpha,\varepsilon}$ at $\varepsilon = 0$. The circle $\mathscr{C}_{\alpha,0}$ is *normally hyperbolic* (see [60] for a definition), and, therefore, it is persistent under small perturbations. Notice that the perturbation in $P_{\alpha,\varepsilon}$ is of a more general type than in $T_{\alpha,\delta,a,\varepsilon}$, since no preservation of the skew-product structure is now required. A few basic results are summarised in the following proposition.

Proposition 3.2. *Suppose that K has a saddle fixed point $p = (x_0, y_0)$. Then for all $\alpha \in [0, 1]$ the map $P_{\alpha,0}$ has an invariant circle $\mathscr{C}_\alpha$ of saddle type. The manifold $\mathscr{C}_\alpha$ is r-normally hyperbolic for all integers r with $1 \le r \le n$. Moreover, for all $r < n$ there exists an $\varepsilon_r > 0$ such that for all $\varepsilon < \varepsilon_r$ and all $\alpha \in [0, 1]$, $P_{\alpha,\varepsilon}$ has a C^r-saddle invariant circle $\mathscr{C}_{\alpha,\varepsilon}$, C^r-close to $\mathscr{C}_{\alpha,0}$.*

Proof: The dynamics of $P_{\alpha,0}$ on $\mathscr{C}_{\alpha,0}$ is parallel with rotation number α. This implies that $\mathscr{C}_{\alpha,0}$ is an r-normally hyperbolic invariant manifold for all $r \le n$ and, therefore, it is C^n. So $\mathscr{C}_{\alpha,0}$, as well as its stable and unstable manifolds, is persistent under C^n-small perturbations. This directly follows from [60]. □

Proposition 3.2 allows us to construct a basin of attraction with nonempty interior for the invariant set $\operatorname{clos} W^u(\mathscr{C}_{\alpha,\varepsilon})$, provided that the one-dimensional unstable manifold $W^u(p)\mathbb{R}^2$ of the map K does not escape to infinity. For $(x, y, \theta) \in \mathbb{R}^2 \times \mathbb{S}^1$, denote by $\omega(x, y, \theta)$ the ω-limit set of (x, y, θ) under $P_{\alpha,\varepsilon}$.

Theorem 3.2. *Fix integers n and r such that $n \ge 2$ and $1 \le r < n$. Choose $\varepsilon < \varepsilon_r$ as in Proposition 3.2 and let $\alpha \in [0, 1]$. Suppose that $K : \mathbb{R}^2 \to \mathbb{R}^2$ is C^n and satisfies:*

1. *K has a saddle fixed point $p \in \mathbb{R}^2$ and a transversal homoclinic point $q \in W^s(p) \cap W^u(p)$.*
2. *The map K is uniformly dissipative: there exists $\kappa < 1$ such that $|\det(DK(x, y))| \le \kappa$ for all $(x, y) \in \mathbb{R}^2$.*
3. *$W^u(p)$ is contained in a bounded subset of $\mathbb{R}^2$.*

Then there exists an $\varepsilon^* < \varepsilon_r$ such that for all $\varepsilon < \varepsilon^*$ there exists an open, nonempty bounded set $U \subset \mathbb{R}^2 \times \mathbb{S}^1$ such that for all $(x, y, \theta) \in U$

$$\omega(x, y, \theta) \subset \operatorname{clos} W^u(\mathscr{C}_{\alpha,\varepsilon}). \tag{3.12}$$

Under the conditions of Theorem 3.2, the invariant set $\operatorname{clos} W^u(\mathscr{C}_{\alpha,\varepsilon})$ attracts all orbits with initial state in an open set U. This holds for an open set of ε-values. In general, however, $\operatorname{clos} W^u(\mathscr{C}_{\alpha,\varepsilon})$ is not an attractor in the sense of our definition (compare Sec. 3.1), since it might be non-topologically transitive. This occurse for example if $\operatorname{clos} W^u(\mathscr{C}_{\alpha,\varepsilon})$ contains a periodic attractor.

In the next theorem we prove that at least the circle $\mathscr{C}_{\alpha,\varepsilon}$ is quasi-periodic (and, hence, topologically transitive) for a set of parameter values of large relative measure.

Theorem 3.3. *Let $P_{\alpha,\varepsilon}$ be a C^n-family of diffeomorphisms as in* (3.11)*, where n is sufficiently large ($n \geq 5$ will do). Choose ε^* as in Theorem* 3.2*. Then for all $\varepsilon < \varepsilon^*$ sufficiently small the following holds.*

1. *There exists a set $D_\varepsilon \subset [0,1]$ with Lebesgue measure* $\mathrm{meas}(D_\varepsilon) > 0$ *such that for $\alpha \in D_\varepsilon$ the restriction of $P_{\alpha,\varepsilon}$ to the circle $\mathscr{C}_{\alpha,\varepsilon}$ is smoothly conjugate to an irrational rigid rotation.*

2. $\mathrm{meas}(D_\varepsilon)$ *tends to* 1 *for $\varepsilon \to 0$.*

Proofs of Theorems 3.2 and 3.3 are given in Sec. 3.3.

Remark 3.1. The quasi-periodicity of the dynamics inside $\mathscr{C}_{\alpha,\varepsilon}$ may have consequences for the dynamics in its stable and unstable manifolds. This is certainly the case if α is irrational and if the map $P_{\alpha,0}$ is perturbed within the class of skew-products, that is

$$P_{\alpha,\varepsilon}(x, y, \theta) = \big(K(x, y) + \varepsilon f(x, y, \theta, \alpha, \varepsilon), \theta + \alpha\big).$$

In this case, indeed, the dynamics inside $W^u(\mathscr{C}_{\alpha,\varepsilon})$ and $W^s(\mathscr{C}_{\alpha,\varepsilon})$ has a quasi-periodic component, given by a rotation over angle α, see [57]. More precisely, $W^u(\mathscr{C}_{\alpha,\varepsilon})$ can be parametrised as

$$W^u(\mathscr{C}_{\alpha,\varepsilon}) = \left\{(W(\theta, \eta), \theta) \;\mid\; \theta \in \mathbb{S}^1 \text{ and } \eta \in \mathbb{R}\right\},$$

where $W : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{R}^2$. Furthermore,

$$P_{\alpha,\varepsilon}(W(\theta, \eta), \theta) = \big(W(\theta + \alpha, N(\theta, \eta)), \theta + \alpha\big),$$

where $N : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{R}$. See [57] for rigorous statements and proofs.

3.2 Existence of Hénon-like attractors

Our proof of Theorem 3.1 is based on a result of Díaz-Rocha-Viana [42]. We begin by stating this result.

3.2.1 Perturbations of multimodal families

Two definitions from [42] are introduced now. For more information about the terminology, we refer to [83].

Definition 3.2. *Let $J \subset \mathbb{R}$ be a compact interval. Fix $d \geq 1$, $k \geq 3$, $a^* \in \mathbb{R}$, and an interval of parameter values $\mathfrak{U} = [a_-, a_+]$, with $a^* \in \mathrm{int}\,\mathfrak{U}$. A C^k-family of maps $M_a : J \to J$, with $a \in \mathfrak{U}$, is called a d-family if it satisfies the following conditions:*

1. Invariance: $M_{a^*}(J) \subset \mathrm{int}(J)$;

2. Nondegenerate critical points: M_{a^*} *has d critical points* $\{c_1, \ldots, c_d\} \overset{\text{def}}{=} \mathrm{Cr}\, M_{a^*}$ *that satisfy*

$$M''_{a^*}(c_i) \neq 0 \;\; \textit{for all } i \quad \textit{and} \quad M_{a^*}(c_i) \neq c_j \;\; \textit{for all } \; i, j;$$

3. Negative Schwarzian derivative: *$SM_{a^*} < 0$ for all $x \neq c_i$, where*

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2;$$

4. Topological mixing: *for any open intervals J_1, J_2 in the core of M_{a^*} there exists n_0 such that*

$$M_{a^*}^n(J_1) \cap J_2 \neq \emptyset \quad \textit{for all } n \geq n_0$$

(for the definition of core of a multimodal map, see e.g. [83]*);*

5. Preperiodicity: *for each $1 \leq i \leq d$ there exists m_i such that $p_i = M_{a^*}^{m_i}(c_i)$ is a (repelling) periodic point of M_{a^*};*

6. Genericity of unfolding: *For all $c_i \in \mathrm{Cr}\, M_{a^*}$, denote by $c_i(a)$ and $p_i(a)$ the continuations of c_i and p_i, respectively, for a close to a^*. Then*

$$\frac{\mathrm{d}}{\mathrm{d}a}\big(M_a^{m_i}(c_i(a)) - p_i(a)\big) \neq 0 \qquad \textit{at } a = a^*.$$

Next we introduce the notion of *η-perturbation* of a d-family M_a, with $a \in \mathfrak{U}$ and $d \geq 1$ fixed.

Definition 3.3. *Fix $\sigma > 0$ and consider the family $\overline{M}_a$ obtained by extending M_a as follows:*

$$\overline{M}_a : J \times I_\sigma \to J \times I_\sigma, \quad \overline{M}_a(x, y) \stackrel{\text{def}}{=} (M_a(x), 0). \tag{3.13}$$

Also denote by M the map

$$M : \mathfrak{U} \times J \times I_\sigma \to J \times I_\sigma, \quad M(a, x, y) \stackrel{\text{def}}{=} \overline{M}_a(x, y) = (M_a(x), 0).$$

Given a C^k-family of diffeomorphisms

$$G_a : J \times I_\sigma \to J \times I_\sigma, \qquad a \in J,$$

for a $k \geq 3$, denote by G its extension

$$G : \mathfrak{U} \times J \times I_\sigma \to J \times I_\sigma, \quad G(a, x, y) \stackrel{\text{def}}{=} G_a(x, y).$$

*Then G is called a η-*perturbation *of the d-family $\{M_a\}_a$ if*

$$\|M - G\|_{C^k} \leq \eta,$$

where $\|\cdot\|_{C^k}$ denotes the C^k-norm over $\mathfrak{U} \times J \times I_\sigma$.

The following result asserts that for η sufficiently small, any η-perturbation G_a of a d-family has a non-uniformly hyperbolic strange attractor for all parameter values a in a set $\mathfrak{S}$ of positive Lebesgue measure. See [10, 11, 86, 96, 120, 127] for similar results.

Proposition 3.3. **[42, Theorem 5.2]** *Let $\{M_a\}_a$ be a d-family and p a periodic point of M_{a^*}. Then there exist $\eta > 0$, $\bar{a}$ and $\chi > 0$ such that, given any η-perturbation $\{G_a\}_a$ of $\{M_a\}_a$ the following holds.*

1. *For all a with $|a - a^*| < \bar{a}$ the map G_a has a periodic point p_a which is the continuation of the periodic point $(p, 0)$ of the map $\overline{M}_a$ in* (3.13)*.*

2. *There exists a set $\mathfrak{S}$, contained in the interval $[a^* - \bar{a}, a^* + \bar{a}] \subset \mathfrak{U}$, with* $\mathrm{meas}(\mathfrak{S}) > \chi$*, such that for all $a \in \mathfrak{S}$ the set $\mathrm{clos}\, W^u(p_a)$ is a Hénon-like strange attractor of the map G_a.*

3.2.2 Strange attractors in rotating Hénon-like families

We here present a proof of Theorem 3.1. The argument is based on three facts. First, suppose that $a^* \in [0,2]$ is such that the quadratic family $Q_a(x) = 1 - ax^2$ in (3.6) is a d-family in the sense of Definition 3.2, with $d = 1$. Then for all $n \geq 1$ the family $M_a \stackrel{\text{def}}{=} Q_a^n$ given by the n-th iterate of Q_a is a d-family for some $d \leq 2^n$. Second, for all $\eta_1 > 0$, the composition of an η_1-perturbation of Q_a with an η_1-perturbation of Q_a^n is an η_2-perturbation of Q_a^{n+1}, where $\eta_2 = C(n)\eta_1$ and $C(n)$ is a positive constant depending on n. Third, for each $n > q \geq 1$ and for each $(\alpha, \delta) \in \mathfrak{A}^{q/n}$, the asymptotic dynamics of $T_{\alpha,\delta,a,\varepsilon}$ is described by a map that turns out to be an η-perturbation of the d-family M_a, with $\eta = \mathcal{O}(\varepsilon)$. Application of Proposition 3.3 then concludes the proof.

In the next lemma we show that M_a is a d-family. For each $\tilde{a} \in [0,2)$ there exists a $\beta > 0$ such that for all a with $a \in [0, \tilde{a}]$ the interval $J = [-1-\beta, 1+\beta] \subset \mathbb{R}$ satisfies $Q_a(J) \subset \operatorname{int}(J)$. In the sequel, it is always assumed that the family Q_a is defined on such an interval J, and that the values of a we consider are such that $Q_a(J) \subset \operatorname{int}(J)$.

Lemma 3.4. *Suppose $a^* \in [0,2) \stackrel{\text{def}}{=} \mathfrak{U}$ is such that the quadratic family*

$$Q_a : J \to J, \quad Q_a(x) = 1 - ax^2$$

satisfies hypotheses 4 and 5 of Definition 3.2. Then for all $n \geq 1$ there exists $d \geq 1$ such that the family

$$M_a : J \to J, \quad M_a \stackrel{\text{def}}{=} Q_a^n$$

is a d-family with $d \leq 2^n - 1$ critical points.

Proof. Take a^* as above. We first prove the case $n = 1$, that is, $Q_a : J_a \to J_a$ is a 1-family. Conditions 1, 2, 3 of Definition 3.2 are obviously satisfied by Q_a. Condition 6 will now be proved. By Conditions 4 and 5 (assumed by hypothesis), Q_{a^*} is a Misiurewicz map [84], *i.e.*, it has no periodic attractor and $c \notin \omega(c)$, where $c = 0$ is the critical point of Q_{a^*}. Moreover, by [83, Theorem 6.3] the map Q_{a^*} is Collet-Eckmann (see *e.g.* [83, Sec. V.4]), that is, there exist constants $\kappa > 0$ and $\lambda > 1$ such that

$$\left| DQ_{a^*}^j(Q_{a^*}(c)) \right| \geq \kappa \lambda^j \quad \text{for all } j \geq 0. \tag{3.14}$$

Therefore, according to [115, Theorem 3]

$$\lim_{n\to\infty} \frac{\frac{\mathrm{d}}{\mathrm{d}a} Q_a^n(c) \mid_{a=a^*}}{\frac{\mathrm{d}}{\mathrm{d}x} Q_{a^*}^{n-1}(Q_{a^*}(c))} > 0. \tag{3.15}$$

Assume $Q_{a^*}^k(c) = p$, with p periodic (and repelling) under Q_{a^*}. By $p(a)$ denote the continuation of p for a close to a^*. Then, for all n sufficiently large,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}a} Q_a^n(c) \mid_{a=a^*} &= \frac{\partial Q_a^{n-k}}{\partial a}(Q_{a^*}^k(c)) \mid_{a=a^*} + \frac{\partial Q_a^{n-k}}{\partial x}(Q_{a^*}^k(c)) \mid_{a=a^*} \frac{\mathrm{d}}{\mathrm{d}a} Q_a^k(c) \mid_{a=a^*} = \\
&= \frac{\partial}{\partial a} Q_a^{n-k}(p) \mid_{a=a^*} + \frac{\partial}{\partial x} Q_a^{n-k}(p) \mid_{a=a^*} \frac{\mathrm{d}}{\mathrm{d}a} \left[p(a) + Q_a^k(c) - p(a) \right] \mid_{a=a^*} = \\
&= \frac{\mathrm{d}}{\mathrm{d}a} \left(Q_a^{n-k}(p(a)) \right) + \frac{\partial}{\partial x} Q_{a^*}^{n-k}(p) \frac{\mathrm{d}}{\mathrm{d}a} \left[Q_a^k(c) - p(a) \right] \mid_{a=a^*} .
\end{aligned} \tag{3.16}$$

The point $Q_a^{n-k}(p(a))$ belongs to a hyperbolic periodic orbit, that varies smoothly with the parameter a. Therefore, its derivative with respect to a (which is the first term in the last equality) is uniformly bounded in n. On the other hand,

$$\frac{\mathrm{d}}{\mathrm{d}x}Q_{a^*}^{n-1}(Q_{a^*}(c)) = \frac{\partial}{\partial x}Q_{a^*}^{n-k}(p)\frac{\mathrm{d}}{\mathrm{d}x}Q_{a^*}^{k-1}(Q_{a^*}(c)).$$

Therefore, by (3.14), (3.15), and (3.16) we conclude that

$$0 < \lim_{n\to\infty} \frac{\frac{\mathrm{d}}{\mathrm{d}a}Q_a^n(c)\mid_{a=a^*}}{\frac{\mathrm{d}}{\mathrm{d}x}Q_{a^*}^{n-1}(Q_{a^*}(c))} = \frac{\frac{\mathrm{d}}{\mathrm{d}a}\left[Q_a^k(c)-p(a)\right]_{a=a^*}}{\frac{\mathrm{d}}{\mathrm{d}x}Q_{a^*}^{k-1}(Q_{a^*}(c))}. \tag{3.17}$$

This proves that Q_a satisfies Condition 6 of Definition 3.2.

We now show that the n-th iterate M_a of the quadratic map is a d-family for all $n > 1$ and for some $d \le 2^n$. For simplicity, we denote Q_{a^*} by Q for the rest of the proof. Condition 1 holds for M_{a^*} since it holds for Q_{a^*}. Condition 3 follows from the fact that the composition of maps with negative Schwarzian derivative also has negative Schwarzian derivative, see *e.g.* [83]. Condition 4 is obviously satisfied.

Condition 2 is now proved by induction on n, where the case $n = 1$ is obvious. Since Q is 2-to-1, the set $\operatorname{Cr} M_{a^*}$ of critical points of M_{a^*} has cardinality $d \le 2^n - 1$. Moreover,

$$\operatorname{Cr} M_{a^*} = Q^{-1}\left(\operatorname{Cr} Q^{n-1}\right) \cup \operatorname{Cr} Q = \bigcup_{j=0}^{n-1}(Q^{-j})(\operatorname{Cr} Q). \tag{3.18}$$

Suppose that Condition 2 holds for a given $n \ge 1$. We first show that

$$(Q^{n+1})''(x) \neq 0 \qquad \text{for all } \ x \in \operatorname{Cr} Q^{n+1}. \tag{3.19}$$

By (3.18), if $x \in \operatorname{Cr} Q^{n+1}$ then either $x = c$, or $Q(x) \in \operatorname{Cr} Q^n$. If $x = c$ then

$$(Q^{n+1})''(x) = (Q^n)'\left(Q(c)\right)\cdot(Q)''(c). \tag{3.20}$$

The second factor is nonzero. If the first factor is zero, then

$$0 = (Q^n)'\left(Q(c)\right) = Q'(Q^n(c))\dots Q'(Q(c)).$$

Therefore there exists j such that $Q^j(c) = c$, so that c is an attracting periodic point of Q. But this contradicts the fact that Q is Misiurewicz, so that (3.20) is nonzero. The other possibility is that $c \neq x$ and $Q(x) \in \operatorname{Cr} Q^n$. In this case,

$$(Q^{n+1})''(x) = (Q^n)''\left(Q(x)\right)\cdot Q'(x)^2,$$

which is nonzero. Indeed, $Q'(x) \neq 0$, otherwise $x = c$. Moreover $(Q^n)''\left(Q(x)\right) \neq 0$ by the induction hypotheses since the critical points of Q^n are nondegenerate. This proves (3.19), from which the first part of Condition 2 follows.

We now prove, again arguing by contradiction, that

$$Q^{n+1}(x) \neq y \qquad \text{for all } \ x, y \in \operatorname{Cr} Q^{n+1}.$$

Suppose that there exist $x, y \in \operatorname{Cr} Q^{n+1}$ such that $Q^{n+1}(x) = y$. By (3.18) there exist i and j such that $Q^i(x) = Q^j(y) = c$, where $0 \le i, j \le n$. This would imply that

$$Q^{n+1+j-i}(c) = Q^j(Q^{n+1}(x)) = Q^j(y) = c,$$

with $n+1+j-i \ge 1$ and, therefore, c would be an attracting periodic point of Q, which is impossible since Q is Misiurewicz. Condition 2 is proved.

To prove Condition 5, fix $y \in \operatorname{Cr} M_{a^*}$ and $j \ge 0$ such that $Q^j(y) = c$. Since c is preperiodic for Q by hypothesis, there exists $k \ge 1$ such that $Q^{j+k}(y) = p$, where p is periodic under Q with period $u \ge 1$. The orbit of y under M_{a^*} is, except for a finite number of initial iterates, a subset of the orbit of p under Q. This shows that y is preperiodic for M_{a^*}.

To prove Condition 6, take $y \in \operatorname{Cr} M_{a^*}$, j, u, k and $p \in J$ as in the proof of Condition 5. Then there exist integers l and m, with $0 \le l < u$ and $m \ge 1$, such that

$$M_{a^*}^m(y) = Q^{k+l}(c) = Q^l(p) \in \operatorname{Orb}_Q(p). \tag{3.21}$$

By Condition 5 (assumed by hypothesis) and by (3.21), the point $z = Q^l(p)$ is periodic (and repelling) under M_{a^*}. Denote by $y(a)$, $z(a)$, and $p(a)$ the continuations of y, z, and p, respectively, for a close to a^*. In particular,

$$Q_a^j(y(a)) = c \quad \text{and} \quad Q_a^l(p(a)) = z(a).$$

We have to show that

$$\frac{\mathrm{d}}{\mathrm{d}a}\left[M_a^m(y(a)) - z(a)\right]\Big|_{a=a^*} \neq 0. \tag{3.22}$$

By the chain rule we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}a}\, Q_a^{l+k}(c)\Big|_{a=a^*} &= \frac{\partial Q_a^l}{\partial a}\,(Q_a^k(c))\Big|_{a=a^*} + \frac{\partial Q_a^l}{\partial x}\,(Q_a^k(c))\Big|_{a=a^*}\, \frac{\mathrm{d}Q_a^k}{\mathrm{d}a}\,(c)\Big|_{a=a^*} = \\
&= \frac{\partial Q_{a^*}^l}{\partial a}(p) + \frac{\partial Q_{a^*}^l}{\partial x}(p)\, \frac{\mathrm{d}Q_{a^*}^k}{\mathrm{d}a}(c), \\
\frac{\mathrm{d}}{\mathrm{d}a}\, Q_a^l(p(a))\Big|_{a=a^*} &= \frac{\partial Q_{a^*}^l}{\partial a}(p) + \frac{\partial Q_{a^*}^l}{\partial x}(p)\, \frac{\mathrm{d}}{\mathrm{d}a} p(a^*),
\end{aligned}$$

where $p = p(a^*) = Q_{a^*}^k(c)$. Therefore,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}a}\left[M_a^m(y(a)) - z(a)\right]\Big|_{a=a^*} &= \frac{\mathrm{d}}{\mathrm{d}a}\left[Q_a^{k+l}(c) - Q_a^l(p(a))\right]\Big|_{a=a^*} = \\
&= \frac{\partial Q_{a^*}^l}{\partial x}(p)\, \frac{\mathrm{d}}{\mathrm{d}a}\left[Q_a^k(c) - p(a)\right]\Big|_{a=a^*}.
\end{aligned}$$

The factor $\frac{\mathrm{d}}{\mathrm{d}a}\left[Q_a^k(c(a)) - p(a)\right]\Big|_{a=a^*}$ is nonzero by (3.17). The same holds for the other factor, otherwise p would be an attracting periodic point of Q_{a^*}. This proves inequality (3.22). □

In the next lemma we show that the composition of a small perturbation of the map $\overline{Q}_a(x, y) = (Q_a(x), 0)$ (we use here the notation of Definition 3.3) with a small perturbation of $\overline{Q_a^n}(x, y) = (Q_a^n(x), 0)$ yields a small perturbation of $\overline{Q_a^{n+1}}(x, y)$. As in Definition 3.3, denote by $Q, Q^n : [0, 2] \times J \times I \to J \times I$ the functions $Q(a, x, y) = (Q_a(x), 0)$ and $Q^n(a, x, y) = (Q_a^n(x), 0)$, respectively.

Lemma 3.5. *For each $\eta > 0$ there exists a $\zeta > 0$ such that for all $F, G : [0,2]\times J\times I \to J \times I$ such that*

$$\|G - Q\|_{C^3} < \zeta \quad \textit{and} \quad \|F - Q^n\|_{C^3} < \zeta, \tag{3.23}$$

we have

$$\left\|G \circ F - Q^{n+1}\right\|_{C^3} < \eta. \tag{3.24}$$

Proof. Write

$$G(a,x,y) = \begin{pmatrix} Q_a(x) + g_1(a,x,y) \\ g_2(a,x,y) \end{pmatrix} \quad \text{and} \quad F(a,x,y) = \begin{pmatrix} Q_a^n(x) + f_1(a,x,y) \\ f_2(a,x,y) \end{pmatrix}.$$

Then

$$G \circ F(a,x,y) - \begin{pmatrix} Q_a^{n+1}(x) \\ 0 \end{pmatrix} = \begin{pmatrix} -2a(f_1(a,x,y))^2 - 2af_1(a,x,y)Q_a^n(x) + g_1\big(a, \tilde f_1(a,x,y), f_2(a,x,y)\big) \\ g_2\big(a, \tilde f_1(a,x,y), f_2(a,x,y)\big) \end{pmatrix},$$

where $\tilde f_1(a,x,y) = Q_a^n(x) + f_1(a,x,y)$. The C^3-norm of the terms $-2a(f_1(a,x,y))^2$ and $-2af_1(a,x,y)Q_a^n(x)$ is bounded by a constant times the C^3-norm of f_1. We now estimate the norm of $\tilde g_1$, defined by

$$\tilde g_1(x_0,x_1,x_2) = g_1(a, \tilde f_1(a,x,y), f_2(a,x,y)).$$

Denote $x_0 = a$, $x_1 = x$, and $x_2 = y$. Then any second order derivative of $\tilde g_1$ is a sum of terms of the following type:

$$\frac{\partial^2 g_1}{\partial x_j x_k}\frac{\partial \tilde f_k}{\partial x_l}, \quad \frac{\partial g_1}{\partial x_k}\frac{\partial^2 \tilde f_k}{\partial x_j x_l},$$

where we put $\tilde f_2 = f_2$ to simplify the notation. For the third order derivatives a similar property holds. Since the C^3-norm of $\tilde f_k$ is bounded, we get that each term in the third order derivative of $\tilde g_1$ is bounded by a constant times the C^3-norm of the g_j. This concludes the proof. □

***Proof of Theorem* 3.1.** The theorem will be first proved for $a^* < 2$. The case $a^* = 2$ follows by choosing another value $\bar a^* < 2$ sufficiently close to 2. Fix $a^* \in [0,2)$ verifying the hypotheses of Lemma 3.4. To begin with, we consider the case $(\alpha,\delta) \in \operatorname{int}\mathfrak{A}^1$, the interior of the tongue of period one. Then the Arnol′d family $A_{\alpha,\delta}$ on $\mathbb{S}^1$ has two hyperbolic fixed points θ_1^s (attracting) and θ_1^r (repelling), see [41, Sec. 1.14]. The θ-coordinate of both points depends on the choice of $(\alpha,\delta) \in \operatorname{int}\mathfrak{A}^1$. So for all $\theta \in \mathbb{S}^1$ with $\theta \neq \theta_1^r$, the orbit of θ under $A_{\alpha,\delta}$ converges to θ_1^s. This means that the manifold

$$\Theta_1 = \left\{(x,y,\theta) \in \mathbb{R}^2 \times \mathbb{S}^1 \mid \theta = \theta_1^s\right\} \subset \mathbb{R}^2 \times \mathbb{S}^1$$

is invariant and attracting under $T_{\alpha,\delta,a,\varepsilon}$. Denote by $G_{a,1}$ the restriction of $T_{\alpha,\delta,a,\varepsilon}$ to Θ_1:

$$G_{a,1} : \Theta_1 \to \Theta_1, \qquad (x,y,\theta_1^s) \mapsto (1 - ax^2 + \varepsilon f_1,\ \varepsilon g_1,\ \theta_1^s),$$

where $f_1 = f(a, x, y, \theta_1^s, \alpha, \delta)$ and similarly for g_1. Since $Q_{a^*}(J) \subset \mathrm{int}(J)$, there exists a constant $\sigma > 0$ such that for all ε sufficiently small and all a close enough to a^*,

$$\begin{aligned} G_{a,1}(J \times I_\sigma \times \{\theta_1^s\}) &\subset \mathrm{int}(J \times I_\sigma \times \{\theta_1^s\}) \quad \text{and} \\ T_{\alpha,\delta,a,\varepsilon}\left(J \times I_\sigma \times (\mathbb{S}^1 \setminus \{\theta_1^r\})\right) &\subset \mathrm{int}\left(J \times I_\sigma \times (\mathbb{S}^1 \setminus \{\theta_1^r\})\right). \end{aligned} \tag{3.25}$$

Since Θ_1 is diffeomorphic to $\mathbb{R}^2$, we consider $G_{a,1}$ as a map of $\mathbb{R}^2$. Then $G_{a,1}$, is an η-perturbation of the quadratic family $Q_a(x)$, where $\eta = \mathcal{O}(\varepsilon)$. We now apply Proposition 3.3 to the family $G_{a,1}$. Let p_0 be a periodic point of M_{a^*}. For all ε sufficiently small there exists a constant $\bar{a} > 0$ and a set $\mathfrak{S}$ of positive Lebesgue measure, contained in the interval $[a^* - \bar{a}, a^* + \bar{a}]$, such that the following holds. For all $a \in [a^* - \bar{a}, a^* + \bar{a}]$, $G_{a,1}$ has a saddle periodic point $\bar{p}$ which is the continuation of the point p_0. Furthermore, for all $a \in \mathfrak{S}$ the closure $\widetilde{\mathscr{A}} = \mathrm{clos}\, W^u(\bar{p})$ is a Hénon-like strange attractor of $G_{a,1}$ contained inside Θ_1. The point $p = (\bar{p}, \theta_1^s)$ is a saddle periodic point of the map $T_{\alpha,\delta,a,\varepsilon}$, and $W^u(p) = W^u(\bar{p}) \times \{\theta_1^s\}$. Therefore $\mathscr{A} = \mathrm{clos}\, W^u(p) = \widetilde{\mathscr{A}} \times \{\theta_1^s\}$. Moreover, the basin of attraction of $\mathrm{clos}\, W^u(p)$ has nonempty interior in $\mathbb{R}^2 \times \mathbb{S}^1$ because of (3.25). This proves the claim for $(\alpha, \delta) \in \mathrm{int}\, \mathfrak{A}^1$.

We pass to the case of higher period tongues. Suppose that $(\alpha, \delta) \in \mathrm{int}\, \mathfrak{A}^{q/n}$, with $n > q \geq 1$. Then $A_{\alpha,\delta}$ has (at least) two hyperbolic periodic orbits

$$\begin{aligned} \mathrm{Orb}(\theta_1^s) &= \{\theta_1^s, \theta_2^s, \dots, \theta_n^s\} \quad \text{attracting, and} \\ \mathrm{Orb}(\theta_1^r) &= \{\theta_1^r, \theta_2^r, \dots, \theta_n^r\} \quad \text{repelling.} \end{aligned}$$

For $j = 1, \dots, n$, denote by Θ_j the manifold

$$\Theta_j = \left\{(x, y, \theta) \in \mathbb{R}^2 \times \mathbb{S}^1) \mid \theta = \theta_j^s\right\},$$

and define maps G_j as the restriction of $T_{\alpha,\delta,a,\varepsilon}$ to Θ_j:

$$\begin{aligned} &G_j : \Theta_j \to \Theta_{j+1} \qquad \text{for } j = 1, \dots, n-1 \\ &G_n : \Theta_n \to \Theta_1, \qquad \text{where} \\ &(x,\ y,\ \theta_1^s) \overset{G_j}{\mapsto} (Q_a(x) + \varepsilon f_j,\ \varepsilon g_j,\ \theta_{j+1}^s), \qquad \text{for } j = 1, \dots, n-1 \\ &(x,\ y,\ \theta_n^s) \overset{G_n}{\mapsto} (Q_a(x) + \varepsilon f_n,\ \varepsilon g_n,\ \theta_1^s). \end{aligned}$$

Here, $f_j = f(a, x, y, \theta_j^s, \alpha, \delta)$. The manifold Θ_1 is invariant and attracting under the n-th iterate of the map $T_{\alpha,\delta,a,\varepsilon}$. For all (x, y, θ) in the complement of the set

$$\{(x, y, \theta) \mid \theta \in \mathrm{Orb}(\theta_1^r)\},$$

the asymptotic dynamics is given by the map

$$G_{a,1,\dots n} \overset{\text{def}}{=} G_n \circ G_{n-1} \circ \dots \circ G_1.$$

Notice that each of the G_j's is an η_j-perturbation of the family Q_a in the sense of Definition 3.3, where $\eta_j = B\varepsilon$ and B can be chosen uniform on θ_j^s (and, therefore, on (α, δ)).

Let p_0 be a periodic point of $M_a \stackrel{\text{def}}{=} Q_a^n$. Then $(p_0, 0)$ is a saddle periodic point for the map $\overline{M}_a$ defined as in (3.13). Take η, $\bar{a}$, and χ as in Proposition 3.3. By inductive application of Lemma 3.5 there exists an $\bar{\varepsilon} > 0$ depending on η and n such that

$$\|G_{a,1,\dots n} - Q^n\|_{C^3} < \eta,$$

for all $(\alpha, \delta) \in \operatorname{int} \mathfrak{A}^{q/n}$ and all $|\varepsilon| < \bar{\varepsilon}$. That is, $G_{a,1,\dots n}$ is an η-perturbation of M_a for all q with $1 \le q < n$ and all $(\alpha, \delta, a, \varepsilon)$ with

$$(\alpha, \delta) \in \mathfrak{A}^{q/n}, \quad \varepsilon \in [-\bar{\varepsilon}, \bar{\varepsilon}].$$

By Proposition 3.3 there exist an $\bar{a} > 0$ and a set $\mathfrak{S}$ contained in the interval $[a^* - \bar{a}, a^* + \bar{a}]$ such that $\operatorname{meas}(\mathfrak{S}) \ge \chi$ and the following holds. For all $a \in [a^* - \bar{a}, a^* + \bar{a}]$ the map $G_{a,1,\dots,n}$ has a periodic point $\bar{p}_a$ which is the continuation of the periodic point $(p_0, 0)$ of $\overline{M}_a$. Moreover, for all $a \in \mathfrak{S}$ the closure $\widetilde{\mathscr{A}} = \operatorname{clos} W^u(\bar{p}_a)$ is a Hénon-like strange attractor of $G_{a,1,\dots,n}$, contained inside Θ_1.

To finish the proof, observe that $p_a = (\bar{p}_a, \theta_1^s)$ is a saddle periodic point of $T_{\alpha,\delta,a,\varepsilon}$. The set $\mathscr{A} = \operatorname{clos} W^u(p_a)$ is compact and invariant under $T_{\alpha,\delta,a,\varepsilon}$, where

$$\mathscr{A} = \big(\widetilde{\mathscr{A}} \times \{\theta_1^s\}\big) \cup T_{\alpha,\delta,a,\varepsilon}\big(\widetilde{\mathscr{A}} \times \{\theta_1^s\}\big) \cup \dots \cup T^{n-1}_{\alpha,\delta,a,\varepsilon}\big(\widetilde{\mathscr{A}} \times \{\theta_1^s\}\big).$$

To show that $\mathscr{A}$ has a dense orbit, suppose that the orbit of $z = (x_0, y_0, \theta_1^s)$ under $G_{a,1,\dots n}$ is dense in $\widetilde{\mathscr{A}}$. Then given $\eta > 0$ and a point

$$q = T^j_{\alpha,\delta,a,\varepsilon}(q') \in T^j_{\alpha,\delta,a,\varepsilon}(\widetilde{\mathscr{A}} \times \{\theta_1^s\}), \quad \text{with } \; 1 \le j \le n-1,$$

there exists $m > 0$ such that $\operatorname{dist}(G^m_{a,1,\dots n}(z), q') < \eta$. By continuity of $T^j_{\alpha,\delta,a,\varepsilon}$, for all $\varrho > 0$ there exists $\eta > 0$ such that

$$\operatorname{dist}(T^j_{\alpha,\delta,a,\varepsilon}(q''), T^j_{\alpha,\delta,a,\varepsilon}(q')) < \varrho \quad \text{for all } q'' \text{ with } \quad \operatorname{dist}(q'', q') < \eta.$$

We conclude that for all $\varrho > 0$ there exists $m > 0$ such that

$$\operatorname{dist}(T^j_{\alpha,\delta,a,\varepsilon}(G^m_{a,1,\dots n}(z)), T^j_{\alpha,\delta,a,\varepsilon}(q')) = \operatorname{dist}(T^{j+mn}_{\alpha,\delta,a,\varepsilon}(z), q) < \varrho.$$

This proves that the orbit of z under $T_{\alpha,\delta,a,\varepsilon}$ is dense in $\mathscr{A}$. Properties (3.1) and (3.2) will now be proved. Since $G_{a,1,\dots,n} = T^n_{\alpha,\delta,a,\varepsilon}$ on Θ_1, for any $m \in \mathbb{N}$ and any $z \in \mathscr{A}$ we have

$$DT^m_{\alpha,\delta,a,\varepsilon}(z) = DT^r_{\alpha,\delta,a,\varepsilon}(G^s_{a,1,\dots,n}(z)) DG^s_{a,1,\dots,n}(z),$$

where $s = m \mod n$ and $r = m - s$. Take $z \in \mathscr{A}$ having a dense orbit and $v = (v_x, v_y, 0) \in T_z\mathscr{A}$ such that $\left\|DG^s_{a,1,\dots,n}(z)v\right\| \ge \kappa\lambda^s$ for all s, where $\kappa > 0$ and $\lambda > 1$ are constants. Since $T^r_{\alpha,\delta,a,\varepsilon}$ is a diffeomorphism for all $r = 1, \dots, s-1$ and $G^s_{a,1,\dots,n}(z)$ belongs to the compact set $\mathscr{A}$ for all $s \in \mathbb{N}$, then there exists a constant $c > 0$ such that

$$\left\|DT^m_{\alpha,\delta,a,\varepsilon}(z)v\right\| = \left\|DT^r_{\alpha,\delta,a,\varepsilon}(G^s_{a,1,\dots,n}(z)) DG^s_{a,1,\dots,n}(z)v\right\| \ge c\left\|DG^s_{a,1,\dots,n}(z)v\right\|,$$

where c is uniform in r. This proves property (3.1). Property (3.1) is proved similarly. This shows that the closure $\operatorname{clos} W^u(p_a)$ is a Hénon-like strange attractor of $T_{\alpha,\delta,a,\varepsilon}$.

Remark 3.2. At the boundary of a tongue $\mathfrak{A}^{q/n}$ the Arnol′d family $A_{\alpha,\delta}$ has a saddle-node periodic point θ_1. However, the basin of attraction of $\operatorname{Orb}\theta_1$ still has nonempty interior, so that the above conclusions hold for all (α, δ) in the closure $\operatorname{clos} \mathfrak{A}^{q/n}$.

$\square$

3.3 Basins of attraction and invariant circles

In this section we give proofs of Theorems Theorem 3.2 and Theorem 3.3. The setting of the problem is now briefly recalled, also see Sec. 3.1.2. Suppose $K : \mathbb{R}^2 \to \mathbb{R}^2$ is a dissipative (area contracting) C^n-diffeomorphism, where $n \geq 2$. Define the direct product map

$$P_{\alpha,0} : \mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{R}^2 \times \mathbb{S}^1, \quad (x, y, \theta) \mapsto (K(x, y), \theta + \alpha), \tag{3.26}$$

where $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. We consider C^n-small perturbations $P_{\alpha,\varepsilon}$ of $P_{\alpha,0}$, to be written as

$$\begin{aligned} P_{\alpha,\varepsilon} &: \mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{R}^2 \times \mathbb{S}^1, \\ &(x, y, \theta) \mapsto \big(K(x, y) + \varepsilon f_\varepsilon(x, y, \theta, \alpha), \theta + \alpha + \varepsilon g_\varepsilon(x, y, \theta, \alpha)\big), \end{aligned}$$

see (3.11). Here the dependence of $P_{\alpha,\varepsilon}$ on the parameters (α, ε) is C^n. In general $P_{\alpha,\varepsilon}$ has not a skew-product structure such as the rotating Hénon map (3.4). We assume that K has a saddle fixed point $p = (x_0, y_0)$. This corresponds to an invariant circle $\mathscr{C}_\alpha$ of saddle type of $P_{\alpha,\varepsilon}$ at $\varepsilon = 0$. Normal hyperbolicity of $\mathscr{C}_\alpha$ guarantees its persistence under small perturbations, see Proposition 3.2. Our proof of Theorem 3.3 is based on a version of the KAM Theorem holding for finite differentiability. We begin by proving Theorem 3.2

3.3.1 Basins of attraction: The Tangerman-Szewc argument generalised

Let $K : \mathbb{R}^2 \to \mathbb{R}^2$ be a dissipative diffeomorphism having a saddle fixed point $p = (x_0, y_0)$. Suppose the stable and unstable manifolds $W^s(p)$ and $W^u(p)$ intersect transversally at the homoclinic point $q \in W^s(p) \cap W^u(p)$, see Figure 3.4. Also assume that $W^u(p)$ is bounded as a subset of $\mathbb{R}^2$. The Tangerman-Szewc Theorem

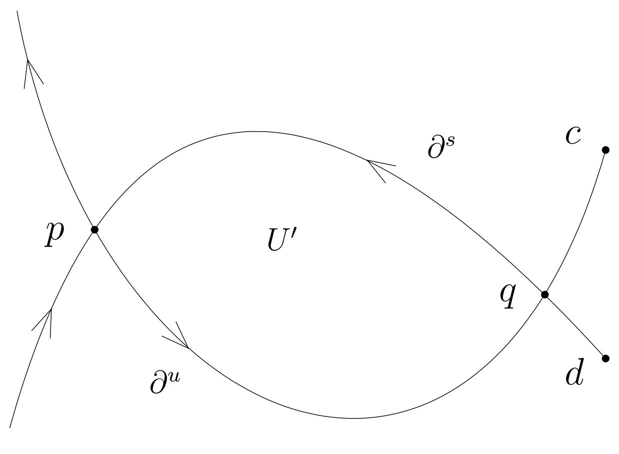

Figure 3.4: Segments ∂^s and ∂^u of the stable and unstable manifold, respectively, of a saddle fixed point p bound a region U, see text for more explanation.

states that the basin of attraction of the closure of $W^u(p)$ contains the open region

U' bounded by the two arcs $\partial^s \subset W^s(p)$ and $\partial^u \subset W^u(p)$ with extremes p and q, see Figure 3.4. This argument is by now standard, see *e.g.* [95] Appendix 3. It is also used to prove existence of strange attractors close to homoclinic tangencies of a saddle fixed point of a dissipative diffeomorphism, cf. [86, 120, 127].

We first prove Theorem 3.2 for $\varepsilon = 0$. This is a straightforward generalisation of the above Tangerman-Szewc Theorem. For small ε, the result is obtained by using persistence of normally hyperbolic invariant manifolds [60] and two transversality lemmas.

Proof of Theorem 3.2. Consider the circle $\mathscr{C}_\alpha = \mathscr{C}_{\alpha,0}$, invariant under map $P_{\alpha,0}$ in (3.26). The manifolds $W^u(\mathscr{C}_\alpha)$ and $W^s(\mathscr{C}_\alpha)$ are given by $W^u(p) \times \mathbb{S}^1$ and $W^s(p) \times \mathbb{S}^1$, respectively. They intersect transversally at a circle $\mathscr{H} = \{q\} \times \mathbb{S}^1$, consisting of points homoclinic to $\mathscr{C}_\alpha$. Consider the two arcs $\partial^s \subset W^s(p)$ and $\partial^u \subset W^u(p)$ with extremes p and q (Figure 3.4).They bound an open set $U' \subset \mathbb{R}^2$. Define D^s and D^u to be the portions of stable, and unstable manifold of $\mathscr{C}_\alpha$, respectively, given by

$$D^s = \partial^s \times \mathbb{S}^1 \subset W^s(\mathscr{C}_\alpha) \qquad \text{and} \qquad D^u = \partial^u \times \mathbb{S}^1 \subset W^u(\mathscr{C}_\alpha).$$

Both surfaces D^s and D^u are compact, and their union forms the boundary of the open region $U = U' \times \mathbb{S}^1$, which is topologically a solid torus.

The volume of U decreases under iteration of $P_{\alpha,0}$. Denoting by meas$(\cdot)$ the Lebesgue measure both on $\mathbb{R}^2$ and on $\mathbb{R}^2 \times \mathbb{S}^1$, due to Condition 2 in Theorem 3.2 we have

$$\operatorname{meas}(P^n_{\alpha,0}(U)) = 2\pi \int_{K^n(U')} dxdy = 2\pi \int_{U'} |\det DK^n|\, dxdy \le 2\pi\kappa^n \operatorname{meas}(U').$$

This implies that the forward evolution of every point $(x, y, \theta) \in U$ approaches the boundary of $P^n_{\alpha,0}(U)$:

$$\operatorname{dist}\bigl(P^n_{\alpha,0}(x, y, \theta), \partial P^n_{\alpha,0}(U)\bigr) \to 0 \quad \text{as } n \to +\infty.$$

Indeed, suppose that this does not hold. Then there exists a $\varrho > 0$ such that for all n there exists $N > n$ such that the ball with centre $P^N_{\alpha,0}(x, y, \theta)$ and radius $\varrho > 0$ is contained inside $P^N_{\alpha,0}(U)$. But this would contradict the fact that $\operatorname{meas}(P^n_{\alpha,0}(U)) \to 0$ as $n \to +\infty$.

The boundary of $P^n_{\alpha,0}(U)$ also consists of two portions of stable and unstable manifold of $\mathscr{C}$:

$$\partial P^n_{\alpha,0}(U) = P^n_{\alpha,0}(D^s) \cup P^n_{\alpha,0}(D^u).$$

The diameter of $P^n_{\alpha,0}(D^s)$ tends to zero as $n \to +\infty$, because all points in D^s are attracted to the circle $\mathscr{C}_\alpha$. Since $W^u(\mathscr{C}_\alpha)$ is bounded, all evolutions starting in U are bounded and approach $W^u(\mathscr{C}_\alpha)$, that is,

$$\operatorname{dist}(P^n_{\alpha,0}(x, y, \theta), P^n_{\alpha,0}(D^u)) \to 0 \quad \text{as } n \to +\infty$$

for all $(x, y, \theta) \in U$. This implies that $\omega(x, y, \theta) \subset \operatorname{clos} W^u(\mathscr{C}_\alpha)$ for all $(x, y, \theta) \in U$.

To extend this result to small perturbations $P_{\alpha,\varepsilon}$ of F_α, the following transversality lemmas are used.

Lemma 3.6. [94, 103] *Consider a map $f : V \to M$, where V and M are C^r-differentiable manifolds and f is C^r. Suppose V is compact, $W \subset M$ is a closed C^r-submanifold and f is transversal to W at V (notation: $f \pitchfork W$). Then $f^{-1}(W)$ is a C^r-submanifold of codimension* $\operatorname{codim}_V(f^{-1}(W)) = \operatorname{codim}_M(W)$. *Further suppose that there is a neighbourhood of $f(\partial_V) \cup \partial_W$ disjoint from $f(V) \cap W$, where ∂_V and ∂_W are the boundaries of V and W. Then any map $g : V \to M$, sufficiently C^r-close to f, is also transversal to W, and the two submanifolds $g^{-1}(W)$ and $f^{-1}(W)$ are diffeomorphic.*

Lemma 3.7. [59] *Let V_1, V_2, and M be C^r-differentiable manifolds and consider two diffeomorphisms $f_i : V_i \to M$, $i = 1, 2$. Then $f_1 \pitchfork f_2$ if and only if $f_1 \times f_2 \pitchfork \Delta$, where $f_1 \times f_2 : V_1 \times V_2 \to M \times M$ is the product map and $\Delta \subset M \times M$ is the diagonal: $\Delta = \{(y, y) \mid y \in M\}$.*

Fix $r \in \mathbb{N}$ and take $\varepsilon < \varepsilon_r$, where ε_r is given in Proposition 3.2. Then the map $P_{\alpha,\varepsilon}$ has an r-normally hyperbolic invariant circle $\mathscr{C}_{\alpha,\varepsilon}$ of saddle type. Furthermore, the manifolds $W^u(\mathscr{C}_{\alpha,\varepsilon})$, $W^s(\mathscr{C}_{\alpha,\varepsilon})$, and $\mathscr{C}_{\alpha,\varepsilon}$ are C^r-close to $W^u(\mathscr{C}_\alpha)$, $W^s(\mathscr{C}_\alpha)$, and $\mathscr{C}_\alpha$. We now show that the two manifolds $W^u(\mathscr{C}_{\alpha,\varepsilon})$, $W^s(\mathscr{C}_{\alpha,\varepsilon})$ still intersect transversally. To apply Lemma 3.6 we restrict to two suitable compact subsets $A^u \subset W^u(\mathscr{C}_\alpha)$ and $A^s \subset W^s(\mathscr{C}_\alpha)$ as follows. Consider the segments $\overline{pc} \subset W^u(p)$ and $\overline{pd} \subset W^s(p)$ in Figure 3.4. Define

$$A^u = \overline{pc} \times \mathbb{S}^1, \qquad A^s = \overline{pd} \times \mathbb{S}^1.$$

In this way, the circle $\mathscr{H}$ is the intersection of the manifolds A^u and A^s, bounded away from their boundaries. Consider the inclusions $i : A^u \to M$ and $j : A^s \to M$. By the closeness of $W^u(\mathscr{C}_\alpha)$ to $W^u(\mathscr{C}_{\alpha,\varepsilon})$ there exists a C^r-diffeomorphism $h : A^u \to A^u_\varepsilon \subset W^u(\mathscr{C}_{\alpha,\varepsilon})$ such that the map i is C^r-close to $i_\varepsilon \circ h$, where $i_\varepsilon : A^u_\varepsilon \to M$ is the inclusion [94]. Similarly, there exists a diffeomorphism $k : A^s \to A^s_\varepsilon \subset W^s(\mathscr{C}_{\alpha,\varepsilon})$ such that the map j is C^r-close $j_\varepsilon \circ k$, where $j_\varepsilon : A^s_\varepsilon \to M$ is the inclusion. By Lemma 3.7 the map $i \times j : A^u \times A^s \to M \times M$ is transversal to the diagonal Δ. For ε small, the map $(i_\varepsilon \circ h) \times (j_\varepsilon \circ k) : A^u \times A^s \to M \times M$ is C^r-close to $i \times j$:

$$\begin{array}{ccc} A^u \times A^s & \xrightarrow{i \times j} & M \times M \\ {\scriptstyle h\times k}\big\downarrow & & \\ A^u_\varepsilon \times A^s_\varepsilon & \xrightarrow{i_\varepsilon \times j_\varepsilon} & M \times M. \end{array}$$

Since Δ is closed and $A^u \times A^s$ is compact, Lemma 3.6 implies that there exists an ε^*, with $0 < \varepsilon^* < \varepsilon_r$, such that $(i_\varepsilon \circ h) \times (j_\varepsilon \circ k) \pitchfork \Delta$ for $\varepsilon < \varepsilon^*$. Furthermore, the submanifolds

$$(i \times j)^{-1}(\Delta) \quad \text{and} \quad \big((i_\varepsilon \circ h) \times (j_\varepsilon \circ k)\big)^{-1}(\Delta)$$

are diffeomorphic. We also have that $\big((i_\varepsilon \circ h) \times (j_\varepsilon \circ k)\big)^{-1}(\Delta)$ is diffeomorphic to $A^u_\varepsilon \cap A^s_\varepsilon$, and $(i \times j)^{-1}(\Delta) = A^u \cap A^s = \mathscr{H}$.

This shows that the intersection $\mathscr{H}_\varepsilon = A^u_\varepsilon \cap A^s_\varepsilon$ is diffeomorphic to $\mathscr{H}$. Define D^u_ε as the part of $W^u(\mathscr{C}_{\alpha,\varepsilon})$ bounded by the invariant circle $\mathscr{C}_{\alpha,\varepsilon}$ and the circle of homoclinic points $\mathscr{H}_\varepsilon$. Define $D^s_\varepsilon = k(D^s)$ similarly. Then the manifolds $D^u_\varepsilon \subset W^u(\mathscr{C}_{\alpha,\varepsilon})$ and $D^s_\varepsilon \subset W^s(\mathscr{C}_{\alpha,\varepsilon})$ form the boundary of an open region $U \subset M$ homeomorphic to a torus. By the closeness of the perturbed manifolds $W^s(\mathscr{C}_{\alpha,\varepsilon})$ and $W^u(\mathscr{C}_{\alpha,\varepsilon})$ to the unperturbed $W^s(\mathscr{C})$ and $W^u(\mathscr{C})$, both U and $W^u(\mathscr{C}_{\alpha,\varepsilon})$ are bounded. Also notice that $P_{\alpha,\varepsilon}$ is dissipative: by taking ε^* small enough, we ensure that $|\det(DF(x,y,\theta))| < \tilde{c} < 1$ for all $\varepsilon < \varepsilon^*$ and (x,y,θ) in U. Therefore, all forward evolutions beginning at points $(x,y,\theta) \in U$ remain bounded. Like in the first part of the proof, one has

$$\omega(x,y,\theta) \subset \operatorname{clos} W^u(\mathscr{C}_{\alpha,\varepsilon})$$

for all $(x,y,\theta) \in U$, $\alpha \in [0,1]$ and $\varepsilon < \varepsilon^*$. □

3.3.2 Quasi-periodic invariant circles

So far, we did not discuss the dynamics in the saddle invariant circle $\mathscr{C}_{\alpha,\varepsilon}$ of map $P_{\alpha,\varepsilon}$ in (3.11). Generically, the dynamics on $\mathscr{C}_{\alpha,\varepsilon}$ is of Morse-Smale type. In this case, the circle consists of the union of the unstable manifold of some periodic saddle. Theorem 3.3 describes a complementary case, for which the dynamics is quasi-periodic. Fix $\tau > 2$ and define the set of Diophantine frequencies D_γ by

$$D_\gamma = \left\{ \alpha \in [0,1] \;\mid\; \left|\alpha - \frac{p}{q}\right| \geq \gamma q^{-\tau} \quad \text{for all } p,\, q \in \mathbb{N},\, q \neq 0 \right\}, \tag{3.27}$$

where $\gamma > 0$. Since we will apply a dissipative version of the KAM theorem in the case of finite differentiability (see [17, 18]), a certain amount of smoothness of the circle $\mathscr{C}_{\alpha,\varepsilon}$ is needed, depending on the Diophantine condition specified in (3.27). Therefore we require that the perturbed family of maps $P_{\alpha,\varepsilon}$ is C^n, for n large enough.

Proof of Theorem 3.3. Consider map F_α in (3.26), and let $p = (x_0, y_0)$ be a saddle fixed point of the dissipative diffeomorphism K. The invariant circle $\mathscr{C}_{\alpha,0}$ of F_α can be seen as a graph over $\mathbb{S}^1$:

$$\mathscr{C}_{\alpha,0} = \left\{ (\theta, x_0, y_0) \in \mathbb{R}^2 \times \mathbb{S}^1 \;\mid\; \theta \in \mathbb{S}^1 \right\}.$$

Fix $r \in \mathbb{N}$ large enough and $\varepsilon < \varepsilon_r$, where ε_r is taken as in Proposition 3.2. By the C^r-closeness of $\mathscr{C}_{\alpha,0}$ and $\mathscr{C}_{\alpha,\varepsilon}$ (Proposition 3.2), the circle $\mathscr{C}_{\alpha,\varepsilon}$ of $P_{\alpha,\varepsilon}$ can be written as a C^r-graph over $\mathbb{S}^1$:

$$\mathscr{C}_{\alpha,\varepsilon} = \left\{ (\theta, x_\varepsilon(\theta), y_\varepsilon(\theta)) \in \mathbb{R}^2 \times \mathbb{S}^1 \;\mid\; \theta \in \mathbb{S}^1 \right\}, \tag{3.28}$$

where $x_\varepsilon : \mathbb{S}^1 \to \mathbb{R}$, $x_\varepsilon(\theta) = x_0 + \mathcal{O}(\varepsilon)$, and similarly for $y_\varepsilon(\theta)$. So the restriction of $P_{\alpha,\varepsilon}$ to $\mathscr{C}_{\alpha,\varepsilon}$ has the following form

$$P_{\alpha,\varepsilon}\big|_{\mathscr{C}_{\alpha,\varepsilon}} : \mathscr{C}_{\alpha,\varepsilon} \to \mathscr{C}_{\alpha,\varepsilon}, \qquad P_{\alpha,\varepsilon}(\theta) = \theta + \alpha + \varepsilon g_\varepsilon(x_0, y_0, \theta, \alpha) + \mathcal{O}(\varepsilon^2).$$

By (3.28), we may consider $P_{\alpha,\varepsilon}$ as a map on $\mathbb{S}^1$. Fix $\gamma > 0$, $\tau > 3$ and take D_γ as in (3.27). For $\alpha \in D_\gamma$, the map $P_{\alpha,\varepsilon}$ can be averaged repeatedly over the circle, putting the θ-dependency into terms of higher order in ε, compare [23, Proposition 2.7] and [29, Sec. 4]. After such changes of variables, $P_{\alpha,\varepsilon}$ is brought into the normal form

$$P_{\alpha,\varepsilon}(\theta) = \theta + \alpha + c(\alpha,\varepsilon) + \mathcal{O}(\varepsilon^{r+1}).$$

In fact, it is convenient to consider α as a variable, and to define the cylinder maps

$$\begin{aligned} P_\varepsilon : \mathbb{S}^1 \times [0,1] \to \mathbb{S}^1 \times [0,1], &\qquad P_\varepsilon(\theta,\alpha) = (P_{\alpha,\varepsilon}(\theta),\alpha) \\ R : \mathbb{S}^1 \times [0,1] \to \mathbb{S}^1 \times [0,1], &\qquad R(\theta,\alpha) = (R_\alpha(\theta),\alpha), \end{aligned}$$

where $R_\alpha : \mathbb{S}^1 \to \mathbb{S}^1$ is the rigid rotation of an angle α. We now apply a finite differentiability version of the KAM Theorem to the family of diffeomorphisms P_ε, see *e.g.* [17, 18]. There exists an integer m with $1 \leq m < r$ and a C^m-map

$$\Phi_\varepsilon : \mathbb{S}^1 \times [0,1] \to \mathbb{S}^1 \times [0,1], \quad \Phi_\varepsilon(\theta,\alpha) = (\theta + \varepsilon A(\theta,\alpha,\varepsilon), \alpha + \varepsilon B(\alpha,\varepsilon)), \tag{3.29}$$

such that the restriction of Φ_ε to $\mathbb{S}^1 \times D_\gamma$ makes the following diagram commute:

$$\begin{array}{ccc} \mathbb{S}^1 \times D_\gamma & \xrightarrow{R} & \mathbb{S}^1 \times D_\gamma \\ {\scriptstyle \Phi_\varepsilon}\big\uparrow & & {\scriptstyle \Phi_\varepsilon}\big\uparrow \\ \mathbb{S}^1 \times D_\gamma & \xrightarrow{P_\varepsilon} & \mathbb{S}^1 \times D_\gamma. \end{array}$$

The differentiability of Φ_ε restricted to $\mathbb{S}^1 \times D_\gamma$ is of Whitney type. Since $P_{\alpha,\varepsilon}|_{\mathscr{C}_{\alpha,\varepsilon}}$ is C^m-conjugate to a rigid rotation on $\mathbb{S}^1$, the circle $\mathscr{C}_{\alpha,\varepsilon}$ is in fact C^m. This proves parts 1 and 2 of the Theorem.

Furthermore, the constant γ in (3.27) can be taken equal to ε^r. This gives that the measure of the complement of D_γ in $[0,1]$ is of order ε^r as $\varepsilon \to 0$. $\square$

3.4 Overview and future research

In this Chapter we prove that Hénon-like strange attractors occur in a family $T_{\alpha,\delta,a,\varepsilon}$ of diffeomorphisms of the solid torus $\mathbb{R}^2 \times \mathbb{S}^1$. The family $T_{\alpha,\delta,a,\varepsilon}$ is a perturbation of the quadratic family $Q_a(x) = 1 - ax^2$, and has a skew-product structure over $\mathbb{S}^1$. The strange attractors we obtain coincide with the closure of the one-dimensional unstable manifold $W^u(\mathrm{Orb}(p))$, where $\mathrm{Orb}(p)$ is a hyperbolic periodic orbit of saddle type which is a sink for the restriction of $T_{\alpha,\delta,a,\varepsilon}$ to $\mathbb{S}^1$.

In a slightly different context, we show that the invariant set $\operatorname{clos} W^u(\mathscr{C})$ attracts an open set of points, where $\mathscr{C}$ is an invariant circle of saddle type. This is proved for a family $P_{\alpha,\varepsilon}$ of diffeomorphisms of $\mathbb{R}^2 \times \mathbb{S}^1$, obtained as follows. We first consider the direct product of a rigid rotation on $\mathbb{S}^1$ with a diffeomorphism of $\mathbb{R}^2$ with a saddle fixed point having a point of transversal homoclinic intersection. Then $P_{\alpha,\varepsilon}$ is a perturbation of this product map.

Future research will focus on scenario's in which $\mathscr{A} = \operatorname{clos} W^u(\mathscr{C})$ is a strange attractor, *i.e.*, it is topologically transitive and has a dense orbit with a positive Lyapunov exponent, where $\mathscr{C}$ is a quasi-periodic invariant circle of saddle type. Attractors having these properties are called quasi-periodic Hénon-like attractors, see Chapter two and compare the numerical examples in Figures 3.3 and 3.5. Similar types of attractors have been found in several numerical studies [65, 66, 74, 91, 119].

We like to mention three other points of interest related to the above problem.

1. Consider first the strange attractors we obtain for the map $T_{\alpha,\delta,a,\varepsilon}$ (3.4), see Theorem 3.1 and compare Figure 3.1. An open problem is to characterize the bifurcations occurring when approaching the boundary of a resonance region for the dynamics in $\mathbb{S}^1$ (Arnol′d tongue). This corresponds to the transition from Hénon-like to quasi-periodic Hénon-like attractors.

2. Secondly, the transition between the strange attractors obtained in Theorem 3.1 for the map $T_{\alpha,\delta,a,\varepsilon}$ (3.4), and the attractors for maps which are perturbations of $T_{\alpha,\delta,a,\varepsilon}$ where the skew-product structure is slightly perturbed, *e.g.*, by adding terms depending on (x, y) to the angular dynamics.

3. Finally, the transition from attractors of $T_{\alpha,\delta,a,\varepsilon}$ (3.4), for which $\delta = 0$ and α irrational to attractors of maps which are perturbations of $T_{\alpha,\delta,a,\varepsilon}$ as in the previous item.

In all cases homoclinic bifurcations [95] are likely to play a fundamental role.

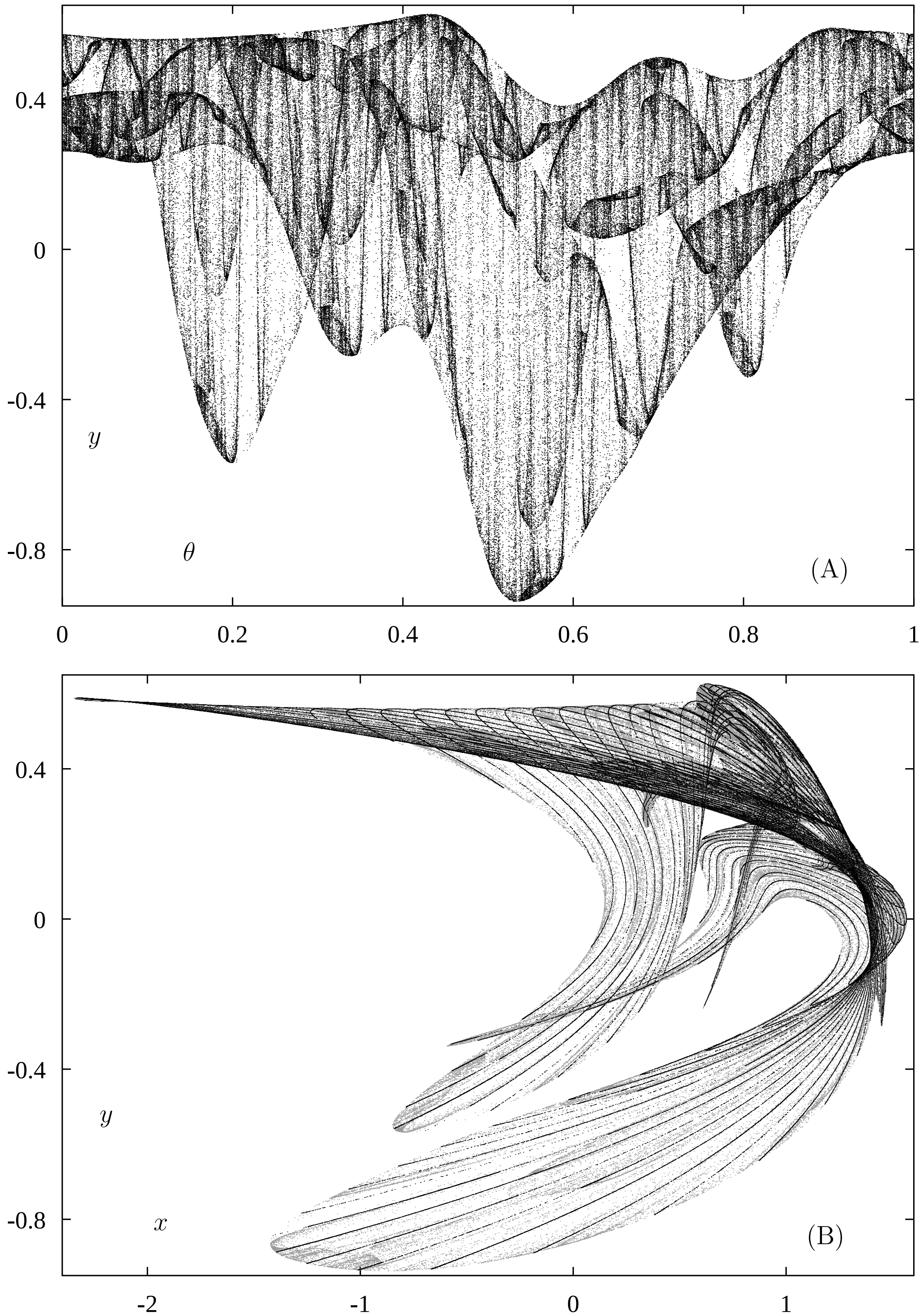

Figure 3.5: (A) Attractor of map (3.10) in the quasi-periodic case. Parameter values are fixed at $a = 0.8$, $b = 0.4$, $\delta = 0$, $\alpha = (\sqrt{5}-1)/2$, $\varepsilon = 0.7$, initial conditions $x_0 = 1.5$, $y_0 = 0$, $\theta_0 = 0$. Projection on the (θ, y)-plane. (B) Same as (A), with projection on the (x, y)-plane in the background (in grey) with 'slices' of the attractor $2\pi\theta \in [0.1 \times j, 0.1 \times j + 0.001]$, for $j = 0, 1, \ldots, 62$ (in black).