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ON THE PARAMETRIZATION AND CONSTRUCTION OF NONLINEAR STABILIZING CONTROLLERS

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Abstract. Continuing on our previous papers, we specialize the parametrization of stabilizing controllers to the case of a stable nonlinear plant, and we obtain a nonlinear generalization of the Internal Model Control principle. Furthermore, based on the notions of a stable kernel and stable image representation of a nonlinear system, we derive two candidate stabilizing controllers for unstable nonlinear plants.

Key Words. nonlinear control systems, stabilizing controllers, parametrization

1. Introduction

In linear control theory the Youla parametrization of stabilizing controllers of a given linear plant has proved to be a very powerful tool in various control problems. In our previous paper [3] we have obtained an intrinsic generalization of the Youla parametrization to the nonlinear case. In fact, given a single stabilizing controller, the class of *all* nonlinear stabilizing controllers is being parametrized. A crucial notion in this approach is that of a stable kernel representation of a nonlinear system, generalizing (and in the linear case equivalent to) the notion of a left coprime factorization of a system.

In the present note we first explicitate the parametrization of stabilizing controllers for the special case of a *stable* nonlinear plant, where the given stabilizing controller can be taken to be the zero-system. In particular, we show that in this case the above parametrization of stabilizing controllers leads to a nonlinear version of Internal Model Control. Based on the notions of a stable kernel representation and a stable image representation of a nonlinear plant we propose in the last section two candidate stabilizing controllers for unstable plants.

2. An explicit parametrization of all stabilizing controllers of a stable plant

Consider a smooth nonlinear state space system (the plant), for simplicity given in affine form

$$G: \begin{array}{rcll} \dot{x} &=& f(x)+g(x)u, & u\in\Re^m \\ y &=& h(x), & y\in\Re^p \end{array} \tag{1}$$

where $x=(x_1,\cdots,x_n)$ are local coordinates for some n-dimensional state space manifold $\mathcal{X}$.

In our paper [3], see also [4], it has been shown how, given a single stabilizing controller K for G, the class of *all* stabilizing compensators may be parametrized. This result directly generalizes the well-known Youla parametrization of stabilizing *linear* controllers of a *linear* plant G to the nonlinear setting. In this section we wish to make this parametrization more explicit and transparable in the case the plant G is already stable, and so K may be taken to be the zero-compensator.

First we recall from [3] the following crucial notions. Consider an arbitrary state space system

$$\Sigma: \begin{array}{rcll} \dot{p} &=& F(p,v), & v\in\Re^k \\ z &=& H(p,v), & z\in\Re^\ell \end{array} \tag{2}$$

with inputs v, outputs z, and state p (belonging to some state space manifold $\mathcal{P}$). Denote the space of input signals for Σ by $\mathcal{V}$ (a subset of the space of (time-) functions from $[0,\infty)$ to $\Re^k$), and the space of output signals by $\mathcal{Z}$ (a subset of the space of functions from $[0,\infty)$ to $\Re^\ell$). In the next section we will take

$$\mathcal{V}=L_{2e}^k[0,\infty),\quad \mathcal{Z}=L_{2e}^\ell[0,\infty), \tag{3}$$

but this is not necessary yet at this level of generality. Write $\mathcal{V}$ as a disjoint union of a set of *stable* signals $\mathcal{V}^s$ including the zero signal, and a set of *unstable* signals $\mathcal{V}^u$, i.e.

$$\mathcal{V} = \mathcal{V}^s \cup \mathcal{V}^u, \quad \mathcal{V}^s \cap \mathcal{V}^u = \emptyset, \quad 0 \in \mathcal{V}^s \tag{4}$$

and similarly,

$$\mathcal{Z} = \mathcal{Z}^s \cup \mathcal{Z}^u, \quad \mathcal{Z}^s \cap \mathcal{Z}^u = \emptyset, \quad 0 \in \mathcal{Z}^s \tag{5}$$

(In case $\mathcal{V} = L_{2e}^k[0,\infty)$ we will take $\mathcal{V}^s = L_2^k[0,\infty)$ and $\mathcal{V}^u$ its complement; similarly for $\mathcal{Z}$.)

Definition 1 Σ is a *stable* system if for every initial condition $p_0 \in \mathcal{P}$ the input-output map associated to Σ maps $\mathcal{V}^s$ into $\mathcal{Z}^s$.

In [5] it has been shown that under appropriate technical conditions (see also Section 3) any plant G admits (at least locally around an equilibrium) a *stable kernel representation*:

Definition 2 Consider the plant G. A nonlinear system Σ

$$\begin{aligned} \dot{x} &= F(x,y,u), \quad u \in \Re^m, \quad y \in \Re^p \\ z &= H(x,y,u), \quad x \in \mathcal{X}, \quad z \in \Re^\ell \end{aligned} \tag{6}$$

with $\mathcal{U} = \mathcal{U}^s \cup \mathcal{U}^u, \mathcal{Y} = \mathcal{Y}^s \cup \mathcal{Y}^u, \mathcal{Z} = \mathcal{Z}^s \cup \mathcal{Z}^u$, is a stable kernel representation of G if

(i) For every initial condition $x_0 \in \mathcal{X}$ and every $u \in \mathcal{U}$ there exists a unique solution $y \in \mathcal{Y}$ to (6) with $z = 0$, which equals the output of (1) for the same initial condition x_0 and input u.
(ii) For every initial condition $x_0 \in \mathcal{X}$ and every $z \in \mathcal{Z}^s$ there exists a unique solution u, y to (6) with $u \in \mathcal{U}^s, y \in \mathcal{Y}^s$.

In shorthand notation a stable kernel representation for G will be denoted by $R_G : \mathcal{Y} \times \mathcal{U} \to \mathcal{Z}$.

Note that if the plant G is itself a stable system, then a stable kernel representation R_G of G is simply

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ z &= y - h(x) \end{aligned} \tag{7}$$

The class of controllers we wish to consider for G are stable kernel representations of smooth state space systems, i.e., controllers K with stable kernel representations

$$R_K : \mathcal{U} \times \mathcal{Y} \to \mathcal{Z}_K, \tag{8}$$

with a state space manifold (space of initial conditions) $\mathcal{X}_K$. The stability of the *closed-loop system*

$$\begin{cases} R_G(y,u) = 0 \\ R_K(u,y) = 0 \end{cases} \tag{9}$$

is defined in the following strong sense [3].

Definition 3 Let $R_G : \mathcal{Y} \times \mathcal{U} \to \mathcal{Z}$ be a stable kernel representation of G, and let $R_K : \mathcal{U} \times \mathcal{Y} \to \mathcal{Z}_K$, with state space $\mathcal{X}_K$, be a stable kernel representation of a controller K for G. The closed-loop system (9), denoted by $\{R_G, R_K\}$, is said to be *stable* if for all initial conditions $x^o \in \mathcal{X}, x_K^o \in \mathcal{X}_K$, and all $z \in \mathcal{Z}^s, z_K \in \mathcal{Z}_K^s$ there exists a unique solution $y \in \mathcal{Y}^s, u \in \mathcal{U}^s$ to

$$\begin{aligned} z &= R_G(y,u) \\ z_K &= R_K(u,y) \end{aligned} \tag{10}$$

Note that if the plant G is stable with stable kernel representation (7), and also the controller K is itself a *stable* system

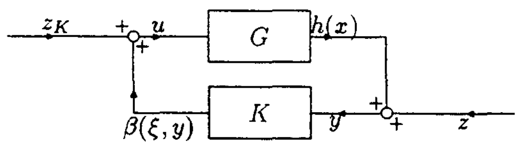
$$\begin{aligned} \dot{\xi} &= \alpha(\xi,y), \quad \xi \in \mathcal{X}_K \\ u &= \beta(\xi,y) \end{aligned} \tag{11}$$

with obvious stable kernel representation

$$\begin{aligned} \dot{\xi} &= \alpha(\xi,y) \\ z_K &= u - \beta(\xi,y) \end{aligned} \tag{12}$$

then the closed-loop system $\{R_G, R_K\}$ is stable if and only if for all initial conditions $x^o \in \mathcal{X}, \xi^o \in \mathcal{X}_K$, and all stable $z \in \mathcal{Z}^s, z_K \in \mathcal{Z}_K^s$, the signals y and u in Figure 1 are stable. This is a very classi-

Fig. 1.



cal notion of closed-loop stability, apart from the fact that usually the initial conditions x^o, ξ^o are taken to be *fixed*.
The basic idea in [3] is now the following. Consider a stable closed-loop system $\{R_G, R_K\}$, and consider two additional systems S and Q with stable kernel representations

$$\begin{aligned} R_S &: \mathcal{Z} \times \mathcal{Z}_K \to \mathcal{Z}_S \\ R_Q &: \mathcal{Z}_K \times \mathcal{Z} \to \mathcal{Z}_Q \end{aligned} \tag{13}$$

and initial condition spaces $\mathcal{X}_S, \mathcal{X}_Q$ respectively. Define new systems G_S and K_Q with stable kernel representations R_{G_S} and R_{K_Q} (in the signals y and u)

$$R_{G_S} \;:\; \mathcal{Y} \times \mathcal{U} \to \mathcal{Z}_S$$

$$R_{K_Q} \;:\; \mathcal{U} \times \mathcal{Y} \to \mathcal{Z}_Q$$

given as

$$\begin{aligned} z_S &= R_S(R_G(y,u), R_K(u,y)) \\ z_Q &= R_Q(R_K(u,y), R_G(y,u)) \end{aligned} \tag{14}$$

The main observation of [3] is that the closed-loop system $\{R_{G_S}, R_{K_Q}\}$ is stable if and only if the closed-loop system $\{R_S, R_Q\}$ is stable, and furthermore that all stabilizing controllers can be generated this way. This yields a *nonlinear Youla parametrization* of all stabilizing controllers (based on the given stabilizing controller K) by letting S to be the system 0 corresponding to a zero input-output map, i.e.

$$R_S(z, z_K) = R_0(z, z_K) = z. \tag{15}$$

In [3] it has been shown that the closed-loop system $\{R_0, R_Q\}$ is stable only if Q is a stable input-output system (from z to z_k):

$$Q: \begin{aligned} \dot{x}_Q &= F_Q(x_Q, z) \\ z_k &= H_Q(x_Q, z) \end{aligned} \tag{16}$$

Conversely, if Q is a stable input-output system then by taking the obvious stable kernel representation R_Q given as

$$\begin{aligned} \dot{x}_Q &= F_Q(x_Q, z) \\ z_Q &= u - H_Q(x_Q, z) \end{aligned} \tag{17}$$

it follows that $\{R_0, R_Q\}$ is stable *if and only if* Q is a stable input-output system.
Note that in this case (14) specializes to

$$\begin{aligned} z_S &= R_G(y,u) \\ z_Q &= R_Q(R_K(u,y), R_G(y,u)) \end{aligned} \tag{18}$$

Now, let us furthermore assume that the plant G is already stable with obvious stable kernel representation (7). Then, as above the zero-controller $K = 0$, with stable kernel representation $R_0(u,y) = u$, yields a stable closed-loop system $\{R_G, R_0\}$, while (18) further specializes to

$$\begin{aligned} z_S &= R_G(y,u) \\ z_Q &= R_Q(u, R_G(y,u)) \end{aligned} \tag{19}$$

It thus follows that the set of *all* stabilizing controllers for the stable plant G is given (in implicit form) as

$$0 = R_Q(u, R_G(y,u))$$

with R_Q given by (17). Since R_G is given by (7) the resulting stabilizing compensators are given in implicit form as

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x} + g(\hat{x})u, & \hat{x} \in X \\ \dot{x}_Q &= F_Q(x_Q, y - h(\hat{x})), & x_Q \in X_Q \\ u &= H_Q(x_Q, y - h(\hat{x})) \end{aligned} \tag{20}$$

and in explicit form as

$$K_Q: \begin{aligned} \dot{\hat{x}} &= f(\hat{x}) + g(\hat{x}) H_Q(x_Q, y - h(\hat{x})) \\ \dot{x}_Q &= F_Q(x_Q, y - h(\hat{x})) \\ & \hat{x} \in X, x_Q \in X_Q \end{aligned} \tag{21}$$

To be precise, it is shown in [3] that for every stable Q as in (16) the controller K_Q is stabilizing for G (i.e., the closed-loop system $\{R_G, R_{K_Q}\}$ is stable) whenever $\hat{x}(0) = x(0)$, and that moreover all stabilizing controllers may be generated in this way.
It follows that every stabilizing controller for G necessarily contains a *model* of G, namely

$$\dot{\hat{x}} = f(\hat{x}) + g(\hat{x})u, \hat{x} \in X$$

The signal flow diagram is given in Figure 2, and generalizes the concept of Internal Model Control (see [1]) to the nonlinear setting.

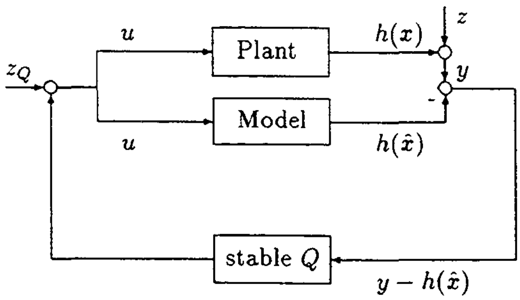


Fig. 2.

3. On the construction of stabilizing controllers

Consider the plant G, together with the Hamilton-Jacobi equations (in the unknowns V, resp. W)

$$\begin{aligned} & V_x(x)f(x) - \tfrac{1}{2}V_x(x)g(x)g^T(x)V_x^T(x) \\ & + \tfrac{1}{2}h^T(x)h(x) = 0 \end{aligned} \tag{22}$$

$$W_x(x)f(x) + \tfrac{1}{2}W_x(x)g(x)g^T(x)W_x^T(x) - \tfrac{1}{2}h^T(x)h(x) = 0 \tag{23}$$

with
$V_x(x)$ denoting the gradient $(\frac{\partial V}{\partial x_1}(x), \cdots, \frac{\partial V}{\partial x_n}(x))$ of the function $V(x)$, and similarly for $W_x(x)$. In [5] the following is proven. Suppose there exists a solution $W \geq 0$ to (23), and additionally assume there exists a solution $k(x)$ to

$$W_x(x)k(x) = h^T(x) \tag{24}$$

Then the system

$$R_G \left\{ \begin{array}{rcl} \dot{x} &=& f(x) - k(x)h(x) + g(x)u + k(x)y \\ z &=& y - h(x) \end{array} \right. \tag{25}$$

has finite L_2-gain from $\begin{bmatrix} y \\ u \end{bmatrix}$ to z; in fact the L_2-gain is equal to 1. Thus (25) constitutes a stable kernel representation of G (where we take signal spaces L_{2e} with stable part L_2).

On the other hand, suppose there exists a solution $V \geq 0$ to (22), then the system

$$I_G : \left\{ \begin{array}{rcl} \dot{x} &=& f(x) - g(x)g^T(x)V_x^T(x) + g(x)s \\ y &=& h(x) \\ u &=& -g^T(x)V_x^T(x) + s \end{array} \right. \tag{26}$$

has L_2-gain equal to 1 (from s to $\begin{bmatrix} y \\ u \end{bmatrix}$); in fact the system is *inner*. System (26) constitutes a *stable image representation* of G, since the set of input-output pairs generated by the driving signal s equals the input-output behavior of G.

In the linear case, R_G corresponds to the normalized left coprime factorization, while I_G corresponds to the normalized right coprime factorization.

A *right inverse* system to R_G is given by

$$R_G^{-1} : \left\{ \begin{array}{rcl} \dot{p} &=& f(p) - g(p)g^T(p)V_p^T(p) + k(p)\xi \\ u &=& -g^T(p)V_p^T(p) \\ y &=& h(p) + \xi \end{array} \right. \tag{27}$$

Indeed, if $p(0) = x(0)$, then the input-output map (from ξ to z) of $R_G \circ R_G^{-1}$ is the identity mapping. Furthermore, a *left inverse* system to I_G is given by

$$I_G^{-1} : \left\{ \begin{array}{rcl} \dot{p} &=& [f(p) - k(p)h(p)] + g(p)u + k(p)y \\ \zeta &=& g^T(p)V_p^T(p) + u \end{array} \right. \tag{28}$$

Indeed, if $p(0) = x(0)$, then the input-output map (from s to ζ) of $I_G^{-1} \circ I_G$ is the identity mapping. Now note that R_G^{-1} is an image representation of

$$K : \left\{ \begin{array}{rcl} \dot{p} &=& f(p) - g(p)g^T(p)V_p^T(p) \\ && -k(p)h(p) + k(p)y \\ u &=& -g^T(p)V_p^T(p) \end{array} \right. \tag{29}$$

while on the other hand I_G^{-1} is a kernel representation of this same system K!
Following linear theory, see e.g. [2], this strongly supports the idea that K is a "good" stabilizing controller for G. Note that K is the nonlinear version of the LQG controller; it is composed of the optimal state feedback (with regard to the cost criterion $\int_0^\infty (\| u \|^2 + \| y \|^2)$

$$u = -g^T(x)V_x^T(x), \tag{30}$$

with the actual state x replaced by the "optimal estimate" p of x, generated by the nonlinear observer

$$\dot{p} = f(p) + g(p)u + k(p)[y - h(p)] \tag{31}$$

(Indeed, in the linear case (31) is precisely the Kalman filter!) Since $R_K = I_G^{-1}$ the closed-loop system $\{R_G, R_K\}$ as in (10) is given in state space form as (see (25) and (28))

$$\begin{array}{rcl} \dot{x} &=& f(x) - k(x)h(x) + g(x)u + k(x)y \\ \dot{p} &=& f(p) - k(p)h(p) + g(p)u + k(p)y \\ z &=& y - h(x) \\ \xi &=& u + g^T(p)V_p^T(p) \end{array} \tag{32}$$

In order to investigate closed-loop stability in the sense of Definition 3 we *invert* the system (32) (by solving y and u) to obtain

$$\begin{array}{rcl} \dot{x} &=& f(x) - g(x)g^T(p)V_p^T(p) + g(x)\xi \\ && +k(x)z \\ \dot{p} &=& f(p) - k(p)[h(p) - h(x)] \\ && -g(p)g^T(p)V_p^T(p) + g(p)\xi + k(p)z \\ y &=& h(x) + z \\ u &=& -g^T(p)V_p^T(p) - \xi \end{array} \tag{33}$$

Following Definition 3 the closed-loop system

$\{R_G, R_K\}$ is stable if for every pair if initial conditions $x(0), p(0)$ of (33), and all stable signals z, ξ, the signals y, u produced by (33) are stable, i.e., $\{R_G, R_K\}$ is stable if (33) is a stable input-output system (from z, ξ to y, u).
Unfortunately the input-output stability of (33) is not easy to check in general. Note that for a *linear* plant $\dot{x} = Ax + Bu, y = Cx$, the matrix $k(x)$ will be a constant matrix K, and the error dynamics in $e := p - x$ is simply given as

$$\dot{e} = (A - KC)e \tag{34}$$

from which input-output stability immediately follows.

Remark 4 Suppose G has an equilibrium x_0, i.e., $f(x_0) = 0$ and without loss of generality $h(x_0) = 0$. Assume that the *linearization* G_L of G around x_0 is stabilizable and detectable. Then the linearization K_L of K around $p_0 = x_0$ equals the LQG controller for G_L, and thus the *linearized* closed-loop system of G and K is stable.

A different candidate stabilizing controller can be obtained as follows, generalizing an idea proposed in [6]. Again, consider the stable image representation I_G of G, and its left inverse I_G^{-1} given by (28). Now consider the control law (with v a new external input)

$$u = \tilde{u} + v - \zeta \tag{35}$$

$$\begin{aligned} \tilde{u} &= -g^T(\xi)V_\xi^T(\xi) + \zeta \\ \dot{\xi} &= f(\xi) - g(\xi)g^T(\xi)V_\xi^T(\xi) + g(\xi)\zeta, \\ &\quad \xi(0) = 0 \end{aligned} \tag{36}$$

where ζ is generated by I_G^{-1} for $p(0) = 0$. Since I_G^{-1} is the left inverse of I_G it follows that $\zeta(t) = s(t), t \geq 0$. Therefore, cf. (26), if $x(0) = 0$ then also $\tilde{u}(t) = u(t), t \geq 0$, yielding $v(t) = \zeta(t), t \geq 0$, and thus the input-output map from v to y (in closed-loop) is simply given as

$$\begin{aligned} \dot{x} &= f(x) - g(x)g^T(x)V_x^T(x) + g(x)v, \\ y &= h(x), x(0) = 0 \end{aligned} \tag{37}$$

which is *stable* by construction.

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