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In a number of problems in analytic function theory the technique of inner-outer factorization has become a standard tool. In linear control theory inner-outer factorization (or more generally J-inner-outer factorization) of rational matrices has played an important role in the theory of $\mathcal{H}_\infty$ optimal control. In a series of papers, see e.g. [3], [4], [1], Ball and Helton have investigated inner-outer factorization of nonlinear input-output operators and of nonlinear state space systems in discrete time. In the present note we will study inner-outer factorization of nonlinear state space systems in continuous time, using a quite different approach. Indeed our method will be a kind of "nonlinear spectral factorization" and concentrates on finding first the *outer* factor (instead of starting with the *inner* factor). An expanded version of this note, including all the proofs, will appear elsewhere [8].

Consider a (smooth) nonlinear system

$$\Sigma : \begin{cases} \dot{x} &= a(x) + b(x)u, \quad u \in \mathbf{R}^m \\ y &= c(x) + d(x)u, \quad y \in \mathbf{R}^p \end{cases} \tag{1}$$

where $x = (x_1, \cdots, x_n)$ are local coordinates for the state space manifold M, with globally asymptotically stable equilibrium 0 (thus $a(0) = 0$). Without loss of generality we assume $c(0) = 0$. The problem of inner-outer factorization consists in finding a *lossless* nonlinear system Θ (the *inner* factor) and an *asymptotically stable* and *minimum phase* nonlinear system R (the *outer* factor), both of the same form as Σ, such that symbolically

$$\Sigma = \Theta \cdot R. \tag{2}$$

By this we mean that for every initial condition of Σ there exist initial conditions of Θ and R such that the input-output behavior of Σ equals the input-output behavior of the series interconnection of R followed by Θ.

Let us recall [9] that a nonlinear system (1) is called *lossless* with respect to the *supply rate* $\frac{1}{2} \parallel u \parallel^2 - \frac{1}{2} \parallel y \parallel^2$ if there exists a function $V(x) \geq 0$ (the *storage* function) such that

$$V(x(t_1)) - V(x(t_0)) = \frac{1}{2} \int_{t_0}^{t_1} (\parallel u(t) \parallel^2 - \parallel y(t) \parallel^2) dt \tag{3}$$

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for all t_0, t_1 and $u(\cdot)$, or equivalently, if V is C^1,

$$V_x(x)\left[a(x)+b(x)u\right] = \frac{1}{2}u^T u - \frac{1}{2}\left[c(x)+d(x)u\right]^T\left[c(x)+d(x)u\right] \tag{4}$$

for all x, u. ($V_x(x)$ denotes the row vector of partial derivatives of $V(x)$.) Taking $t_0 = 0$ and $t_1 = \infty$ in (3), it follows that (1) is *L_2-norm preserving.* Furthermore, a nonlinear (1) is called *minimum phase* if 0 is a globally asymptotically stable equilibrium of its zero-dynamics [6].

Our approach for constructing the outer factor R runs as follows. First we consider the *Hamiltonian extension* of Σ [5]

$$\begin{cases} \dot{x} &= a(x) + b(x)u \\ \dot{p} &= -\left[\frac{\partial a}{\partial x}(x) + \frac{\partial b}{\partial x}(x)u\right]^T p - \frac{\partial^T c}{\partial x}(x)u_a - u^T \frac{\partial^T d}{\partial x}(x)u_a, \quad u_a \in \mathbf{R}^p \end{cases}$$
$$\begin{cases} y &= c(x) + d(x)u, \\ y_a &= b^T(x)p + d^T(x)u_a, \quad y_a \in \mathbf{R}^m \end{cases} \tag{5}$$

which is Hamiltonian system living on T^*M (with coordinates (x,p)), having inputs (u, u_a) and outputs (y, y_a). Imposing the interconnection $u_a = y$ to (5) leads to the Hamiltonian system

$$\Sigma^*\Sigma: \quad \begin{aligned} \dot{x} &= \frac{\partial H}{\partial p}(x,p,u) \\ \dot{p} &= -\frac{\partial H}{\partial x}(x,p,u) \\ y_a &= \frac{\partial H}{\partial u}(x,p,u) \end{aligned} \tag{6}$$

with Hamiltonian function

$$H(x,p,u) = p^T\left[a(x)+b(x)u\right] + \frac{1}{2}\left[c(x)+d(x)u\right]^T\left[c(x)+d(x)u\right] \tag{7}$$

For a *linear* system (1) $\Sigma^*\Sigma$ reduces to the series interconnection of Σ and its adjoint linear system Σ^*, having transfer matrix $G^T(-s)G(s)$ ($G(s)$ being the transfer matrix of Σ). In the *linear* case spectral factorization using the canonical factorization theorem of [2] leads to the computation of the *unstable* eigenspace of $\Sigma^*\Sigma$ and the *stable* eigenspace of the inverse system $(\Sigma^*\Sigma)^{-1}$. In the general nonlinear case we do something similar by replacing the (un-)stable *eigenspace* by the (un-) stable *invariant manifold.* The unstable invariant manifold of $\Sigma^*\Sigma$ for $u = 0$ is trivial; it is simply the set $\{(x,p) \in T^*M | x = 0\}$, since Σ is asymptotically stable. The inverse system $(\Sigma^*\Sigma)^{-1}$ is easily computed under the standing assumption:

Assumption $E(x) := d^T(x)d(x)$ is invertible for all x.

(For the general case we refer to [8].) Indeed, $(\Sigma^*\Sigma)^{-1}$ is again a Hamiltonian system

$$\begin{aligned} \dot{x} &= \tfrac{\partial H^\times}{\partial p}(x, p, y_a) \\ \dot{p} &= -\tfrac{\partial H^\times}{\partial x}(x, p, y_a) \\ u &= -\tfrac{\partial H^\times}{\partial y_a}(x, p, y_a) \end{aligned} \tag{8}$$

with Hamiltonian $H^\times$ obtained by Legendre transformation of H with respect to u and y_a, i.e.

$$\begin{aligned} &H^\times(x, p, y_a) = p^T \left[a(x) - b(x)E^{-1}(x)d^T(x)c(x)\right] + \\ &\tfrac{1}{2}c^T(x)\left[I_p - d(x)E^{-1}(x)d^T(x)\right]c(x) - \tfrac{1}{2}p^T b(x)E^{-1}(x)b^T(x)p \\ &+ \left[p^T b(x) + c^T(x)d(x)\right]E^{-1}(x)y_a - \tfrac{1}{2}y_a^T E^{-1}(x)y_a \end{aligned} \tag{9}$$

Computation of the stable invariant manifold of (8) for $y_a = 0$ leads to the Hamilton-Jacobi equation (see [7]) $H^\times(x, P_x^T(x), 0) = 0$, i.e

$$\begin{aligned} &P_x(x)\left[a(x) - b(x)E^{-1}(x)d^T(x)c(x)\right] + \tfrac{1}{2}c^T(x)\left[I_p - d(x)E^{-1}(x)d^T(x)\right]c(x) \\ &-\tfrac{1}{2}P_x(x)b(x)E^{-1}(x)b^T(x)P_x^T(x) = 0, \quad P(0) = 0 \end{aligned} \tag{10}$$

with the side condition

$$a(x) - b(x)E^{-1}(x)\left[d^T(x)c(x) + b^T(x)P_x^T(x)\right] \ \text{asymptotically stable} \tag{11}$$

Since this is the Hamilton-Jacobi-Bellman equation of an optimal control problem with nonnegative costs it follows that $P(x) \geq 0$. Now define the canonical transformation $(x, p) \mapsto (x, \bar{p})$ with

$$p = \bar{p} + P_x^T(x) \tag{12}$$

Then in the new coordinates the stable invariant manifold of $(\Sigma^*\Sigma)^{-1}$ is simply the set $\{(x, \bar{p}) | \bar{p} = 0\}$, and the Hamiltonian H of $\Sigma^*\Sigma$ transforms into, using (10),

$$\begin{aligned} &H(x, \bar{p} + P_x^T(x), u) = \bar{p}^T[a(x) + b(x)u] + \tfrac{1}{2}[\bar{c}(x) + d(x)u]^T[\bar{c}(x) + d(x)u], \\ &\bar{c}(x) := d(x)E^{-1}(x)\left[d^T(x)c(x) + b^T(x)P_x^T(x)\right] \end{aligned} \tag{13}$$

Comparing with (7) we see that $\Sigma^*\Sigma = \bar{\Sigma}^*\bar{\Sigma}$, where the newly defined system

$$\bar{\Sigma}: \begin{aligned} \dot{x} &= a(x) + b(x)u \\ \bar{y} &= \bar{c}(x) + d(x)u \end{aligned} \tag{14}$$

is asymptotically stable and *minimum phase* as follows from (11). (Premultiply the last equation of (14) for $\bar{y} = 0$ by $d^T(x)$ and solve for u.) Thus $\bar{\Sigma}$ is the outer factor R we are after!

The inner factor Θ is now easily obtained. Indeed a right factorization for Θ (with driving variable u) is

$$\begin{aligned} \dot{x} &= a(x) + b(x)u \\ y &= c(x) + d(x)u \\ \bar{y} &= \bar{c}(x) + d(x)u \end{aligned}$$

leading to the explicit input-output form

$$\Theta \begin{cases} \dot{x} = a(x) + b(x)E^{-1}(x)\left[-d^T(x)\bar{c}(x) + d^T(x)\bar{y}\right] \\ y = c(x) + d(x)E^{-1}(x)\left[-d^T(x)\bar{c}(x) + d^T(x)\bar{y}\right] \end{cases} \tag{15}$$

which is easily seen to be lossless with storage function V being given by the solution P of (10), (11).

For further extensions (including J inner-outer factorization) and applications we refer to [8].

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