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Schaft, A.J. van der; Ball, J.A.

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Inner-outer factorization of nonlinear state space systems

A.J. van der Schaft *

J.A. Ball †

In a number of problems in analytic function theory the technique of inner-outer factorization has become a standard tool. In linear control theory inner-outer factorization (or more generally J -inner-outer factorization) of rational matrices has played an important role in the theory of \mathcal{H}_∞ optimal control. In a series of papers, see e.g. [3], [4], [1], Ball and Helton have investigated inner-outer factorization of nonlinear input-output operators and of nonlinear state space systems in discrete time. In the present note we will study inner-outer factorization of nonlinear state space systems in continuous time, using a quite different approach. Indeed our method will be a kind of "nonlinear spectral factorization" and concentrates on finding first the *outer* factor (instead of starting with the *inner* factor). An expanded version of this note, including all the proofs, will appear elsewhere [8].

Consider a (smooth) nonlinear system

$$\Sigma : \begin{cases} \dot{x} &= a(x) + b(x)u, & u \in \mathbf{R}^m \\ y &= c(x) + d(x)u, & y \in \mathbf{R}^p \end{cases} \quad (1)$$

where $x = (x_1, \dots, x_n)$ are local coordinates for the state space manifold M , with globally asymptotically stable equilibrium 0 (thus $a(0) = 0$). Without loss of generality we assume $c(0) = 0$. The problem of inner-outer factorization consists in finding a *lossless* nonlinear system Θ (the *inner* factor) and an *asymptotically stable* and *minimum phase* nonlinear system R (the *outer* factor), both of the same form as Σ , such that symbolically

$$\Sigma = \Theta \cdot R. \quad (2)$$

By this we mean that for every initial condition of Σ there exist initial conditions of Θ and R such that the input-output behavior of Σ equals the input-output behavior of the series interconnection of R followed by Θ .

Let us recall [9] that a nonlinear system (1) is called *lossless* with respect to the *supply rate* $\frac{1}{2} \|u\|^2 - \frac{1}{2} \|y\|^2$ if there exists a function $V(x) \geq 0$ (the *storage function*) such that

$$V(x(t_1)) - V(x(t_0)) = \frac{1}{2} \int_{t_0}^{t_1} (\|u(t)\|^2 - \|y(t)\|^2) dt \quad (3)$$

*Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, Fax 31-53-340733, e-mail twarjan@math.utwente.nl

†Department of Mathematics, Virginia Tech, Blacksburg, VA 24061, U.S.A.

for all t_0, t_1 and $u(\cdot)$, or equivalently, if V is C^1 ,

$$V_x(x) [a(x) + b(x)u] = \frac{1}{2} u^T u - \frac{1}{2} [c(x) + d(x)u]^T [c(x) + d(x)u] \quad (4)$$

for all x, u . ($V_x(x)$ denotes the row vector of partial derivatives of $V(x)$.) Taking $t_0 = 0$ and $t_1 = \infty$ in (3), it follows that (1) is L_2 -norm preserving. Furthermore, a nonlinear (1) is called *minimum phase* if 0 is a globally asymptotically stable equilibrium of its zero-dynamics [6].

Our approach for constructing the outer factor R runs as follows. First we consider the *Hamiltonian extension* of Σ [5]

$$\begin{cases} \dot{x} &= a(x) + b(x)u \\ \dot{p} &= - \left[\frac{\partial a}{\partial x}(x) + \frac{\partial b}{\partial x}(x)u \right]^T p - \frac{\partial c}{\partial x}(x)u_a - u^T \frac{\partial d}{\partial x}(x)u_a, \quad u_a \in \mathbf{R}^p \\ y &= c(x) + d(x)u, \\ y_a &= b^T(x)p + d^T(x)u_a, \quad y_a \in \mathbf{R}^m \end{cases} \quad (5)$$

which is Hamiltonian system living on T^*M (with coordinates (x, p)), having inputs (u, u_a) and outputs (y, y_a) . Imposing the interconnection $u_a = y$ to (5) leads to the Hamiltonian system

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p}(x, p, u) \\ \Sigma^* \Sigma : \dot{p} &= - \frac{\partial H}{\partial x}(x, p, u) \\ y_a &= \frac{\partial H}{\partial u}(x, p, u) \end{aligned} \quad (6)$$

with Hamiltonian function

$$H(x, p, u) = p^T [a(x) + b(x)u] + \frac{1}{2} [c(x) + d(x)u]^T [c(x) + d(x)u] \quad (7)$$

For a *linear* system (1) $\Sigma^* \Sigma$ reduces to the series interconnection of Σ and its adjoint linear system Σ^* , having transfer matrix $G^T(-s)G(s)$ ($G(s)$ being the transfer matrix of Σ). In the *linear* case spectral factorization using the canonical factorization theorem of [2] leads to the computation of the *unstable* eigenspace of $\Sigma^* \Sigma$ and the *stable* eigenspace of the inverse system $(\Sigma^* \Sigma)^{-1}$. In the general nonlinear case we do something similar by replacing the (un-)stable *eigenspace* by the (un-) stable *invariant manifold*. The unstable invariant manifold of $\Sigma^* \Sigma$ for $u = 0$ is trivial; it is simply the set $\{(x, p) \in T^*M | x = 0\}$, since Σ is asymptotically stable. The inverse system $(\Sigma^* \Sigma)^{-1}$ is easily computed under the standing assumption:

Assumption $E(x) := d^T(x)d(x)$ is invertible for all x .

(For the general case we refer to [8].) Indeed, $(\Sigma^*\Sigma)^{-1}$ is again a Hamiltonian system

$$\begin{aligned}\dot{x} &= \frac{\partial H^\times}{\partial p}(x, p, y_a) \\ \dot{p} &= -\frac{\partial H^\times}{\partial x}(x, p, y_a) \\ u &= -\frac{\partial H^\times}{\partial y_a}(x, p, y_a)\end{aligned}\tag{8}$$

with Hamiltonian H^\times obtained by Legendre transformation of H with respect to u and y_a , i.e.

$$\begin{aligned}H^\times(x, p, y_a) &= p^T [a(x) - b(x)E^{-1}(x)d^T(x)c(x)] + \\ &\frac{1}{2}c^T(x) [I_p - d(x)E^{-1}(x)d^T(x)] c(x) - \frac{1}{2}p^T b(x)E^{-1}(x)b^T(x)p \\ &+ [p^T b(x) + c^T(x)d(x)] E^{-1}(x)y_a - \frac{1}{2}y_a^T E^{-1}(x)y_a\end{aligned}\tag{9}$$

Computation of the stable invariant manifold of (8) for $y_a = 0$ leads to the Hamilton-Jacobi equation (see [7]) $H^\times(x, P_x^T(x), 0) = 0$, i.e.

$$\begin{aligned}P_x(x) [a(x) - b(x)E^{-1}(x)d^T(x)c(x)] + \frac{1}{2}c^T(x) [I_p - d(x)E^{-1}(x)d^T(x)] c(x) \\ - \frac{1}{2}P_x(x)b(x)E^{-1}(x)b^T(x)P_x^T(x) = 0, \quad P(0) = 0\end{aligned}\tag{10}$$

with the side condition

$$a(x) - b(x)E^{-1}(x) [d^T(x)c(x) + b^T(x)P_x^T(x)] \text{ asymptotically stable}\tag{11}$$

Since this is the Hamilton-Jacobi-Bellman equation of an optimal control problem with nonnegative costs it follows that $P(x) \geq 0$. Now define the canonical transformation $(x, p) \mapsto (x, \bar{p})$ with

$$p = \bar{p} + P_x^T(x)\tag{12}$$

Then in the new coordinates the stable invariant manifold of $(\Sigma^*\Sigma)^{-1}$ is simply the set $\{(x, \bar{p}) | \bar{p} = 0\}$, and the Hamiltonian H of $\Sigma^*\Sigma$ transforms into, using (10),

$$\begin{aligned}H(x, \bar{p} + P_x^T(x), u) &= \bar{p}^T [a(x) + b(x)u] + \frac{1}{2} [\bar{c}(x) + d(x)u]^T [\bar{c}(x) + d(x)u], \\ \bar{c}(x) &:= d(x)E^{-1}(x) [d^T(x)c(x) + b^T(x)P_x^T(x)]\end{aligned}\tag{13}$$

Comparing with (7) we see that $\Sigma^*\Sigma = \bar{\Sigma}^*\bar{\Sigma}$, where the newly defined system

$$\begin{aligned}\dot{x} &= a(x) + b(x)u \\ \bar{\Sigma} : \quad \dot{\bar{y}} &= \bar{c}(x) + d(x)u\end{aligned}\tag{14}$$

is asymptotically stable and *minimum phase* as follows from (11). (Premultiply the last equation of (14) for $\bar{y} = 0$ by $d^T(x)$ and solve for u .) Thus $\bar{\Sigma}$ is the outer factor R we are after!

The inner factor Θ is now easily obtained. Indeed a right factorization for Θ (with driving variable u) is

$$\dot{x} = a(x) + b(x)u$$

$$y = c(x) + d(x)u$$

$$\bar{y} = \bar{c}(x) + d(x)u$$

leading to the explicit input-output form

$$\Theta \begin{cases} \dot{x} = a(x) + b(x)E^{-1}(x) \left[-d^T(x)\bar{c}(x) + d^T(x)\bar{y} \right] \\ y = c(x) + d(x)E^{-1}(x) \left[-d^T(x)\bar{c}(x) + d^T(x)\bar{y} \right] \end{cases} \quad (15)$$

which is easily seen to be lossless with storage function V being given by the solution P of (10), (11).

For further extensions (including J inner-outer factorization) and applications we refer to [8].

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