Geometric approximation of curves and singularities of secant maps

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Summary

In this thesis we study: (i) geometric approximation of curves in the plane and in space, and (ii) singularities of secant maps of immersed surfaces from a geometric perspective.

Geometric Curve Approximation.

Geometric modeling is a branch of applied mathematics and computational geometry that studies methods and algorithms for the mathematical description of shapes. The shapes studied are mostly two or three dimensional, although many of the tools and principles can be applied to sets in any finite dimension. Today geometric modeling is done with computers and for computer-based applications.

Curve modeling is a part of geometric modeling and in the current thesis our focus is on problems related to approximation of parametric curves in the plane with conic arcs and biarcs, and in space with bihelical arcs. Biarcs are curves formed by joining two circular arcs in a tangent continuous fashion. Similarly, a bihelical arc is formed by joining two circular helices in a tangent continuous manner. Two curves are said to join in a tangent continuous fashion at a point if they meet at a point and the derivative of both curves at that point are parallel to each other.

Approximating a parametric curve with some spline curve is always based on some metric with which we measure how close the original curve and its approximation are to each other. A spline curve in general is defined piecewise by some special class of functions. In this thesis we consider tangent continuous splines, e.g., a conic spline is made up of piecewise conic arcs, where two consecutive conic arcs join in a tangent continuous manner. In our study we primarily consider the Hausdorff distance and, in case of parabolic and conic arcs, we also consider approximation with respect to the symmetric

difference distance. The Hausdorff distance in case of plane curves is defined as the maximum of the distance function between two curves: For every point on a given parametric curve we consider the Frenet-Serret frame and consider the distance between the given point and the point of intersection of the normal line at that point with the approximating curve. The symmetric difference distance between two curves sharing the endpoints is the area enclosed by the curves.

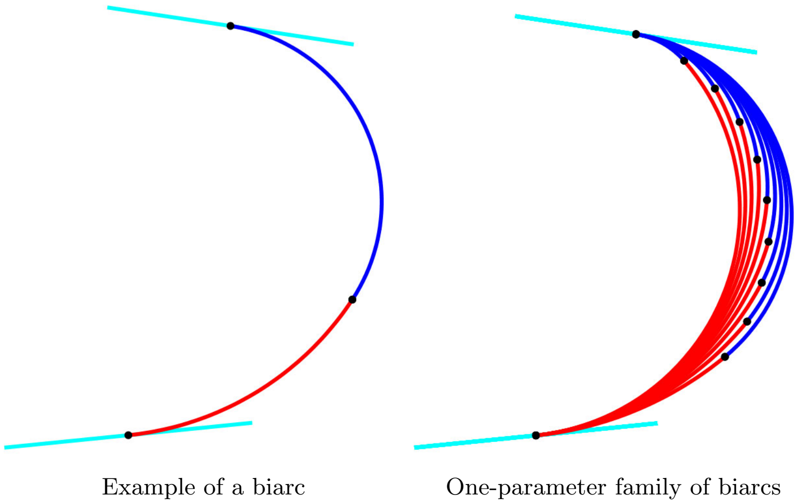
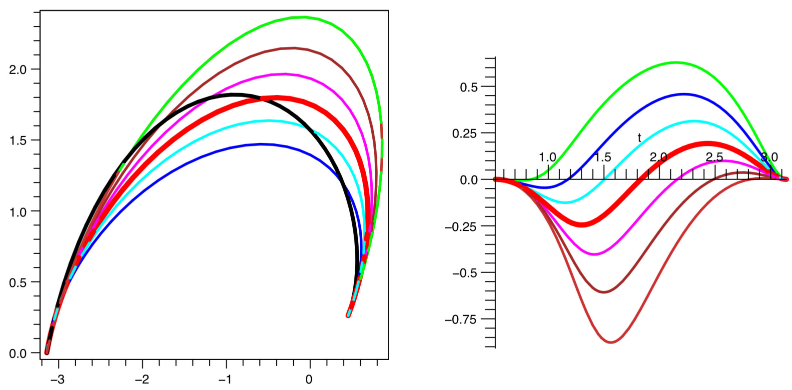


Figure 6.1: *Example of a biarc formed by two circular arcs and a one-parameter family of bitangent biarcs.*

Complexity. Our goal is to compute the complexity (minimum number of elements) of approximation of a sufficiently smooth curve, with non-vanishing curvature, in the plane with biarc, parabolic and conic splines. We also determine the complexity of approximation of space curves with bihelix splines. Circles are the only curves in plane with non-zero constant curvature, conics are the only curves in plane with constant affine curvature, where affine curvature is a differential invariant arising in affine differential geometry and is a notion similar to curvature. Furthermore, circular helices are curves in space with constant curvature and constant torsion. We exploit these properties of the curves in our computation of the Hausdorff distance and show that the approximation error improves by one order of magnitude in going from biarc splines (third order) to parabolic splines (fourth order) to conic splines (fifth order). We also compute the leading term in the approximation error, and, hence, in the complexity of the approximating spline. Consequently the complexity can be expressed in terms of differential geometric quantities like curvature and affine curvature and for:

- biarc splines depends on the derivative of the curvature,
- parabolic splines depends on the affine curvature, and for conic splines depends on the derivative of the affine curvature, and for
- bihelical splines depends on the derivative of the curvature.



One-parameter family of conics Family of displacement functions

Figure 6.2: *One-parameter family of bitangent conics and their corresponding displacement functions.*

Geometry of Approximation. We consider the approximation of spiral arcs with bitangent biarcs. Spirals are curves in the plane with monotonically increasing/decreasing curvature. A bitangent biarc to a spiral arc is tangent to a spiral arc at its endpoints. We prove that there exists a one-parameter family of biarcs bitangent to a spiral arc. Furthermore, we show that in this one-parameter family of biarcs there exists a unique biarc which is closest to the curve with respect to the Hausdorff distance. Similar to the notion of spirals, there is the notion of affine spirals in the plane. Affine spirals are curves with monotonically increasing/decreasing affine curvature. We show that there exists a one-parameter family of bitangent conic arcs to a given affine spiral arc. Moreover, the distance function of this one-parameter family is bimodal as shown in Figure 6.2.

We prove that there exists a unique bitangent conic which is closest to the given affine spiral with respect to the Hausdorff distance. In this case the bimodal displacement function is equioscillatory in nature as shown by the red curve in Figure 6.2. In case of biarcs as well as conics we also prove that the displacement function and hence the Hausdorff distance function is monotonic, i.e., if we reduce the length of the given spiral or affine spiral arc the Hausdorff distance of its best approximating biarc or conic arc keeps decreasing respectively. Due to these properties of the approximating biarc and conic spline we are able to design an algorithm for computing a best approximating biarc or conic spline of a given spiral or affine spiral curve. In case of approximation of regular space curves, we prove a general result in Chapter 4 regarding the Hausdorff distance between a regular space curve and an approximating G^1-spline consisting of two smooth arcs which are tangent to each other at one of their endpoints (junction point) and the spline formed by these two smooth arcs is bitangent to the given space curve. Imposing the condition that the derivative of curvature of this kind of spline curve is zero at the endpoints shared with the curve, we obtain a lower bound on the Hausdorff distance between a curve and such a bitangent spline. Furthermore, we obtain an asymptotic characterization of the junction point and tangent at the junction point (junction tangent) for a bihelix spline where the lower bound of the Hausdorff distance is obtained. We present an algorithm using these characterizations of the junction point and the junction tangent and conclude that the experimentally measured complexity of the bihelix spline matches its theoretical complexity almost exactly. Nonetheless, in this case the problem of finding a bitangent bihelix which is closest to the curve is complex and still remains open.

Singularities of Secant Maps

Singularity theory is a far-reaching generalization of the study of functions at its critical points. It provide some useful tools for the study of the geometry of curves and surfaces locally i.e., properties around a small neighborhood of a point lying on a curve or surface. Given a function $f : I \to \mathbb{R}$, x_0 is called its critical point if $f'(x_0) = 0$. A critical point or a singular point gives crucial information about the shape of the graph $f(x)$. The maximal or minimal points of a function are its nondegenerate critical points. Moreover x_0 is a degenerate critical point if $f'(x_0) = f''(x_0) = 0$. In Whitney's theory functions are replaced by mappings, i.e., collections of several functions of several variables. Generically all critical points are non-degenerate, but degenerate critical points may occur in generic families of functions. The problem considered in this thesis is an extension of Bruce's work where he studies the singularities of secant maps of curves in space. Given a space curve γ, which is a regular, smooth embedding from $\mathbb{R}$ to $\mathbb{R}^3$, the projectivized secant map $\hat{S} : \mathbb{R} \times \mathbb{R} \to \mathbb{P}^2$ maps a pair of distinct points t_1 and t_2 in $\mathbb{R}$ to an unoriented line onto a real projective space $\mathbb{P}^2$. Furthermore, it maps two non-distinct points onto the corresponding unoriented tangent line. The goal is then to study the local behavior of the secant map for points on the diagonal $\Delta = \{(t, t) \,|\, t \in \mathbb{R}\}$. In this thesis we study the secant maps and projectivized secant maps of a surface immersed in $\mathbb{R}^n$, where $n \geq 3$. A secant map acting on two points p, q on a surface, maps them to the line $p - q$. We discuss in detail the singularities of such maps and its projectivized counterpart in this thesis along the points on the diagonal.