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Pasumarthy, R.; Schaft, A.J. van der

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Stability and stabilization of mixed lumped and distributed parameter systems

R. Pasumarthy, A. J. van der Schaft

Department of Applied Mathematics,
University of Twente, The Netherlands
Email: R.Pasumarthy, a.j.vanderschaft@math.utwente.nl

Abstract—We study boundary control systems which are represented as port-Hamiltonian systems with respect to an infinite-dimensional Dirac structure. Interconnection of such systems with lumped parameter dissipative systems again defines a port-Hamiltonian system. Typical example is a power-drive consisting of a power converter, transmission line and electrical machine. We study the stability properties of such systems using the total Hamiltonian and other conserved quantities determined by the Dirac structure, and explore their control within this context.

1. Introduction

The key ingredient of any port-Hamiltonian system is a power-conserving interconnection structure (mathematically formalized by the geometric notion of a Dirac structure) linking the pairs of conjugate port variables of the various ports corresponding to energy storage (defined by a Hamiltonian function depending on energy variables), resistive effects, external interaction, etc. The interconnection of port-Hamiltonian systems defines a new port-Hamiltonian system with Dirac structure determined by the Dirac structures of the constituent parts. The port-Hamiltonian framework applies to lumped parameter, as well as to distributed parameter systems. This means that the interconnection of lumped parameter sub-systems with distributed parameter sub-systems defines a port-Hamiltonian system with infinite-dimensional Dirac structure determined by the finite-dimensional Dirac structures of the lumped parameter components and the infinite-dimensional Dirac structures of the distributed parameter components. Furthermore, the total Hamiltonian of the mixed interconnected system is the sum of the Hamiltonians of the components. This offers a natural starting point for the stability analysis of mixed systems. Furthermore, by appropriately selecting the Dirac structures of the control components, Casimirs for the mixed systems may be generated which may be used for shaping the total Hamiltonian to a Lyapunov function for a desired equilibrium.

2. Port-Hamiltonian systems and Dirac structures

It is well known [4, 5] that the notion of power conserving interconnections can be formulated by a geometric structure called a Dirac structure, which is a subspace of the space of flows and efforts. We briefly discuss these concepts both for finite as well as infinite-dimensional systems with scalar spatial variable. Refer [4, 5] for details.

2.1. Lumped parameter systems

To define the notion of Dirac structures for finite dimensional systems, we start with a space of power variables $\mathcal{F} \times \mathcal{F}^*$, for some linear space $\mathcal{F}$, with power defined by

$$P =< e \mid f >,\ (f, e) \in \mathcal{F} \times \mathcal{F}^*,$$

where $< e \mid f >$ denotes the duality product, that is, the linear functional $e \in \mathcal{F}^*$ acting on $f \in \mathcal{F}$. $\mathcal{F}$ is called the space of flows and $\mathcal{F}^*$ the space of efforts, with the power of a signal $(f, e) \in \mathcal{F} \times \mathcal{F}^*$ denoted as $< e \mid f >$.

There exists on $\mathcal{F} \times \mathcal{F}^*$ a canonically defined *bilinear form* $\ll, \gg$, defined as

$$\ll (f^a, e^a), (f^b, e^b) \gg \ :=< e^a \mid f^b > + < e^b \mid f^a >, \quad (1)$$
$$(f^a, e^a), (f^b, e^b) \in \mathcal{F} \times \mathcal{F}^*$$

Definition 1 *[4] A constant Dirac structure on $\mathcal{F} \times \mathcal{F}^*$ is a subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ such that $\mathcal{D} = \mathcal{D}^\perp$ with respect to the bilinear form (1).*

As an immediate corollary of the definition we see that for all $(f, e) \in \mathcal{D}$ we have that $< e \mid f >= 0$. Hence a Dirac structure defines a power conserving relation.

Consider a lumped parameter physical system given by power-conserving interconnection defined by a constant Dirac structure $\mathcal{D}$ and energy storing elements with energy variables x. For simplicity we assume that the energy variables are living in a linear space $\mathcal{X}$ although everything can be generalized to the case of manifolds. The constitutive relations of the energy storing elements are specified by their stored energy functions $H(x)$.

The space of flows is naturally partitioned as $\mathcal{F}_S \times \mathcal{F}$ with $f_S \in \mathcal{F}_S$, the flows corresponding to the energy storing elements and $f \in \mathcal{F}$ denoting the remaining flows (corresponding to ports/sources). Correspondingly, the space of effort variables is split as $\mathcal{F}_S^* \times \mathcal{F}^*$, with $e_S \in \mathcal{F}_S^*$ the efforts corresponding to the energy-storing elements and $e \in \mathcal{F}^*$ the remaining efforts. The Dirac structure $\mathcal{D}$ can then be

given in matrix kernel representation as

$$\mathcal{D} = \left\{ \begin{array}{c} (f_s, e_s, f, e) \in F_S \times \mathcal{F}_s^* \times \mathcal{F} \times \mathcal{F}^* \\ \mid F_S f_S + E_S e_S + F f + E e = 0 \end{array} \right\}, \\ E_S F_S^T + F_S E_S^T + E F^T + F E^T = 0 \\ \text{with } \operatorname{rank}[F_S \vdots E_S \vdots F \vdots E] = \dim(\mathcal{F}_S \times \mathcal{F}) \tag{2}$$

Now the flows of the energy storing elements are given by $\dot{x}$, and equated with $-f_S$ (the negative sign is included to have a consistent energy flow direction). The efforts e_S corresponding to the energy storing elements are given as $\frac{\partial H}{\partial x} = e_s$. Substituting these into (2) leads to the description of the physical system by the set of DAE's

$$F_S \dot{x}(t) = E_S \frac{\partial H}{\partial x}(x(t)) + F f(t) + E e(t) \tag{3}$$

with f, e the port power variables. The system of equations(3) is called a port-Hamiltonian system.

By the power conserving property of a Dirac structure it follows that any port-Hamiltonian system satisfies the energy balance

$$\frac{dH}{dt}(x(t)) = < \frac{\partial H}{\partial x}(x(t)) | \dot{x}(t) > = e^T(t) f(t)$$

which means that the increase in internal energy of the port-Hamiltonian system is equal to the externally supplied power.

2.2. Distributed parameter systems

In order to define a Dirac structure for an infinite-dimensional system we have to consider an infinite-dimensional function space. Through out the paper we will only consider systems with a 1-dimensional spatial domain. This function space will be defined as follows $\mathcal{F}^* = F_p^*(Z) \times F_q^*(Z) \times F_b^*$, with $F_p^*(Z)$, $F_q^*(Z)$ denoting the space of efforts of the two energy domains interacting with each other and F_b^* denoting the boundary efforts. The $\mathcal{F}$ is defined as the dual space of $\mathcal{F}^*$ with respect to the duality product

$$< (e_p, e_q, e_b), (f_p, f_q, f_b) > = \int_0^l [f_p e_p + f_q e_q] + e_b f_b \mid_0^l$$

Similar to the finite dimensional case, we define the bilinear form between two elements of $\mathcal{F} \times \mathcal{F}^*$ as

$$<< (f^1, e^1), (f^2, e^2) >> \triangleq \\ \int_0^l (e_p^1 f_p^2 + e_p^2 f_p^1 + e_q^2 f_q^1 + e_q^1 f_q^2) dz + (e_b^1 f_b^2 + e_b^1 f_b^2) \mid_0^l$$

Then the following system defines a port-Hamiltonian system

$$\begin{bmatrix} f_p \\ f_q \end{bmatrix} = \begin{bmatrix} 0 & d \\ d & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix}; \quad \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e_p \mid_{\partial Z} \\ e_q \mid_{\partial Z} \end{bmatrix} \tag{4}$$

with $\mid_{\partial Z}$ denoting the restriction to the boundary ∂Z.

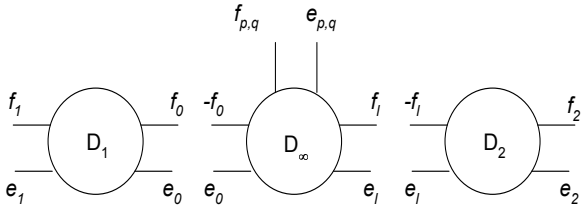

Figure 1: $\mathcal{D}_1 \parallel \mathcal{D}_\infty \parallel \mathcal{D}_2$

2.3. Systems with dissipation

Dissipation cam enter into port-Hamiltonian systems by terminating some of the ports by resistive relations. This can also be viewed as interconnecting a Dirac structure with a resistive relation given by

$$R_f f_R + R_e e_R = 0$$

where f_R, e_R respectively denote the flow and effort variables corresponding to the resistive elements and the square matrices R_f and R_e satisfy the symmetry and semi-positive definiteness condition

$$R_f R_e^T = R_e R_f^T \geq 0$$

The interconnection of the Dirac structure and the resistive relation then has the following property

$$(\mathcal{D} \parallel \mathcal{R})^\perp = \mathcal{D} \parallel -\mathcal{R}$$

where $-\mathcal{R}$ denotes the pseudo-resistive structure given by

$$R_f f_R - R_e e_R = 0$$

($-\mathcal{R}$ is called a pseudo-resistive structure since it corresponds to negative instead of positive resistance).

Similar property also holds for the case of distributed parameter systems where dissipation can enter into the system either through the spatial domain or by terminating the boundary or boundaries of the system by a resistive relation.

3. Interconnections of port-Hamiltonian systems

Mixed lumped and distributed parameter port-Hamiltonian systems arise from interconnections arise by interconnection of a distributed parameter system to a lumped parameter system either through the spatial domain or through the boundaries of the distributed parameter system (see [3] for example). Such interconnections of port-Hamiltonian systems represented by their respective Dirac structures can be studied as follows: Consider two Dirac structures $\mathcal{D}_1, \mathcal{D}_2$ (lumped parameter) defined respectively on the product spaces $\mathcal{F}_1 \times \mathcal{F}_0$ and $\mathcal{F}_1$ and $\mathcal{F}_0$ and a Stokes' Dirac structure $\mathcal{D}_\infty$ on the product space $\mathcal{F}_0 \times \mathcal{F}_p \times \mathcal{F}_q \times \mathcal{F}_l$, with $\mathcal{F}_0$ and $\mathcal{F}_l$ being linear spaces

(representing the space of boundary variables of the Stokes' Dirac structure) and $\mathcal{F}_{p,q}$ an infinite dimensional function space with p, q representing the two different physical energy domains interacting with each other.

Now we study interconnection of the two Dirac structures $\mathcal{D}_1$ and $\mathcal{D}_2$ interconnected to each other via the Stokes-Dirac structure $\mathcal{D}_\infty$. The total interconnection of $\mathcal{D}_1$, $\mathcal{D}_\infty$ and $\mathcal{D}_2$. The total interconnection of $\mathcal{D}_1$,$\mathcal{D}_\infty$ and $\mathcal{D}_2$ is defined such that f_0, e_0 are the shared flow and effort variables between $\mathcal{D}_1$ and $\mathcal{D}_\infty$ and f_l, e_l are the shared flow and effort variables between $\mathcal{D}_2$ and $\mathcal{D}_\infty$. Also see Figure(1).

This yields the following bilinear form on $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_{p,q} \times \mathcal{F}_{p,q}^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$:

$$\begin{aligned} &\ll (f_1^a, f_p^a, f_q^a, f_2^a, e_1^a, e_p^a, e_q^a, e_2^a), (f_1^b, f_p^b, f_q^b, f_2^b, e_1^b, e_p^b, e_q^b, e_2^b) \gg \\ &:= < e_1^b | f_1^a > + < e_1^a | f_1^b > + < e_2^a | f_2^b > + < e_2^b | f_2^a > \\ &+ \int_Z \left[e_p^a f_p^b + e_p^b f_p^a + e_q^b f_q^a + e_q^a f_q^b \right] \end{aligned} \tag{5}$$

Proposition 2 *[1] Let $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_\infty$ be Dirac structures as said above (defined respectively with respect to $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_0 \times \mathcal{F}_0^*$, $\mathcal{F}_l \times \mathcal{F}_l^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$ and $\mathcal{F}_0 \times \mathcal{F}_0^* \times \mathcal{F}_{p,q} \times \mathcal{F}_{p,q}^* \times \mathcal{F}_l \times \mathcal{F}_l^*$). Then $D = \mathcal{D}_1 \| \mathcal{D}_\infty \| \mathcal{D}_2$ is a Dirac structure defined with respect to the bilinear form on $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_{p,q} \times \mathcal{F}_{p,q}^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$ given by (5).*

So far in the composition we have considered systems without dissipation. At this point one might ask what if there is dissipation in the system? This is answered by the following corollary.

Corollary 3 *Let $\mathcal{D}_1 \parallel \mathcal{R}_1$, $\mathcal{D}_2 \parallel \mathcal{R}_2$ and $\mathcal{D}_\infty \parallel \mathcal{R}_\infty$ be Dirac structures as defined above interconnected to their respective resistive relations (representing their dissipation), then the composed system will again have a structure of the form $\mathcal{D} \parallel \mathcal{R}$ with the property that $(\mathcal{D} \parallel \mathcal{R})^\perp = \mathcal{D} \parallel -\mathcal{R}$ where $-\mathcal{R}$ is a pseudo resistive structure (corresponding to negative resistance). $\mathcal{D}$ is the composition of the individual Dirac structures and $\mathcal{R}$ is the composition of the individual resistances of the subsystems.*

4. Casimirs and Control

Casimirs are functions that are conserved quantities of the system for every Hamiltonian. They are completely characterized by the Dirac structure of the port-Hamiltonian system. A function $C : \mathcal{X} \to \mathbb{R}$ is a Casimir function of the autonomous port-Hamiltonian system (lumped parameter) with a given resistive relation to be any function $C : \mathcal{X} \to \mathbb{R}$ satisfying

$$(0, e, -f_R, e_R) \in \mathcal{D} \tag{6}$$

implying that

$$\frac{d}{dt} C = \frac{\partial^T C}{\partial x}(x(t))\dot{x}(t) = e^T f_p = 0 \tag{7}$$

Similarly in the case of a Stokes-Dirac (with zero boundary conditions) structure with a given resistive relation, a Casimir is any functional $C : \mathcal{X} \to \mathbb{R}$ which satisfies

$$(0, e_p, 0, e_q, -f_{Rp}, e_{Rp}, -f_{Rq}, e_{Rq}) \in \mathcal{D} \tag{8}$$

where f_{Rp}, f_{Rq} are the flow variables of the resistive elements corresponding to the p and q energy domains, similarly with the efforts. implying that

$$\frac{dC}{dt} = \int_Z e_p f_p + e_q f_q = 0$$

Remark 4 *In case of a finite dimensional system with dissipation we see that in case of definiteness of the resistive relation, if a function is a conserved quantity (a Casimir) for one resistive relation it actually needs to be a Casimir for all resistive relations; see [2] for a proof.*

Now let us address the question of designing a controller port-Hamiltonian system such that the controlled system has the desired stability properties. The controlled system satisfies

$$\frac{d}{dt}(H_p + H_c) \leq 0$$

In case x_* is not a minimum for H_p, then a possible strategy is that we generate Casimir functions $C(x, \xi)$ for the closed-loop system by appropriately choosing the controller port-Hamiltonian system. The resulting Lyapunov function is then given by the sum of the plant and controller Hamiltonians and the corresponding Casimir function,

$$V(x, \xi) := H_p(x) + H_c(\xi) + C(x, \xi)$$

The objective is then to construct a Lyapunov function such that V has a minimum at (x_*, ξ_*), with ξ_* still to be chosen. This strategy is based on finding all the achievable Casimirs of the closed-loop systems.

4.1. Achievable Casimirs

We consider here a case where the plant port-Hamiltonian system $\mathcal{D}_p$ is the interconnection of a distributed parameter port-Hamiltonian system with a lumped parameter system connected to one of its boundary and the controller system a lumped parameter port-Hamiltonian system connected to the other end of the distributed parameter system. Typical example is a power-drive consisting of a power converter, transmission line and electrical machine.

Following the theory of achievable Casimirs [1] and extending it to the case of systems with dissipation, the set of achievable Casimirs are *functionals* $C(x(t), \overline{q}(z, t))$ such that $\delta C(x(t), \overline{q}(z, t))$ belongs to the space

$$\begin{aligned} P_{Cas} = \{ &e_1, e_p, e_q \mid \exists D_c \text{ such that } \exists e_2 : \\ &(0, e_1, -f_R, e_R, 0, 0, e_p, e_q, -f_{Rp}, e_{Rp}, -f_{Rq}, e_{Rq}, 0, e_2, 0, 0) \\ &\in \mathcal{D}_p \parallel \mathcal{D}_c \} \end{aligned}$$

where $f_R = [f_{R1} \;\; f_{R2}]^T$, $e_R = [e_{R1} \;\; e_{R2}]^T$, with (f_{R1}, e_{R1}) denoting the flows and efforts variables of the dissipation term in the finite dimensional part of the plant Dirac structure, and (f_{R2}, e_{R2}) the flow and effort variables associated with the dissipation term in the controller Dirac structure (finite dimensional).

Similar to the case of systems without dissipation [1], the following proposition (which we state here without proof) characterizes the achievable Casimirs of the closed-loop system, regarded as functions of the plant state by characterization of the space P_{Cas}.

Proposition 5 *The space P_{Cas} defined above is equal to the space*

$$\tilde{P} = \{e_1, e_p, e_q \mid \exists (f_b, e_b) \text{ such that } (0, e_1, f_{R1}, e_{R1}, 0, 0, e_p, e_q, -f_{Rp}, e_{Rp}, -f_{Rq}, e_{Rq}, f_b, e_b) \in \mathcal{D}_p\}$$

where (f_b, e_b) are the boundary variables.

Example 6 *A simple example in this case would be to consider a plant system where we interconnect the distributed parameter system (for example a transmission line) at one end to a finite dimensional port-Hamiltonian system, the Dirac structure of which would be given as*

$$\begin{bmatrix} -\dot{x} \\ -\partial_t q \\ -\partial_t \phi \end{bmatrix} = \left(\begin{bmatrix} -J(x) & 0 & 0 \\ 0 & 0 & d \\ 0 & d & 0 \end{bmatrix} + \begin{bmatrix} R(x) & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & R \end{bmatrix} \right) \begin{bmatrix} \partial_x H \\ \delta_q H \\ \delta_\phi H \end{bmatrix} + \begin{bmatrix} -g(x) \\ 0 \\ 0 \end{bmatrix} \delta_q H \mid_0$$

$$\begin{bmatrix} f_l \\ e_l \end{bmatrix} = \begin{bmatrix} -\delta_\phi H \mid_l \\ \delta_q H \mid_l \end{bmatrix}; \quad \delta_\phi H \mid_0 = g^T(x) \partial_x H \tag{9}$$

The achievable Casimirs in this case are all functionals C such that

$$\begin{aligned} & J(x)\partial_x C + g(x)\delta_q C \mid_0 = 0 \\ & g^T(x)\partial_x C = 0 \\ & d\delta_\phi C - G\delta_q C = 0 \\ & d\delta_q C - R\delta_\phi C = 0 \end{aligned}$$

We see that the first two conditions correspond to the finite dimensional part of the plant sub-system where as the last two conditions correspond to the transmission line.

5. Stability analysis

We discuss stabilization of systems by the control by interconnection methodology. The aim is to generate Casimirs for the controlled system such that the closed-loop energy can be shaped in such a way that it has a minimum at the desired equilibrium. We define stability for the mixed lumped and distributed parameter system in the sense of Lyapunov as follows.

Definition 7 *The equilibrium point χ_* of a distributed parameter system is said to be stable in the sense of Lyapunov with respect to the norm $\| \cdot \|$, if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\| \chi(0) - \chi_* \| < \delta \implies \| \chi(t) - \chi_* \| < \epsilon$ for all $t > 0$, where $\chi(0)$ is the initial condition.*

The underlying proof for stability can be summarized in the following steps (similar to the case studied in [3]; see also the references therein).

- Defining the candidate Lyapunov function on the basis of the plant and controller Hamiltonians and the corresponding Casimir functions as

  $$V(\chi) = H(x) + H_c(\xi) + \mathcal{H}(\bar{q}(\cdot,\cdot)) + C(\chi)$$

  where $\chi = (x, \xi, \bar{q}(\cdot,\cdot))$, with H_c to be defined.
- Assigning equilibrium: The first-order conditions

  $$\nabla H_d(x) \triangleq \begin{bmatrix} \frac{\partial}{\partial x} H_d(\chi_*) \\ \frac{\partial}{\partial \xi} H_d(\chi_*) \\ \delta_M H_d(\chi_*) \\ \delta_E H_d(\chi_*) \end{bmatrix} \equiv 0$$

- The second-order conditions
  Introduce the nonlinear functional

  $$\mathcal{N}(\Delta\chi) = H_d(\chi_* + \Delta\chi) - H_d(\chi_*) \tag{10}$$

  proportional to the second variation of $H_d(\chi)$ in the sense that its Taylor expansion about $\Delta\chi$ is

  $$\mathcal{N}(\Delta\chi) \approx \frac{1}{2} \nabla^2 H_d(\chi_*)$$

  and determine the convexity conditions (with respect to a suitable norm) that the functional (10) must satisfy to assure that it is definite, i.e.

  $$c_1 \| \Delta\chi \|^2 \leq H_d(\chi_* + \Delta\chi - H_d(\chi_*) \leq c_2 \| \Delta\chi \|^2 \tag{11}$$

  with $\alpha, c_1, c_2 > 0$. A suitable norm in our case would be

  $$\| \Delta\chi \| = \left(| \Delta x |^2 + | \Delta\xi |^2 + \int_0^l \Delta q_M^2(z,t) dz + \int_0^l \Delta q_E^2(z,t) dz \right)^{\frac{1}{2}} \tag{12}$$

  with $| \cdot |$ the standard Euclidean norm.

Acknowledgments

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