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Alessandro Macchelli[†], Arjan J. van der Schaft[‡], Claudio Melchiorri[†]

[†]CASY–DEIS, University of Bologna
viale Risorgimento 2, 40136 Bologna (Italy)
{amacchelli,cmelchiorri}@deis.unibo.it

[‡]Toegepaste Wiskunde, Universiteit Twente,
Postbus 217, 7500AE Enschede (The Netherlands)
a.j.vanderschaft@math.utwente.nl

Abstract

In this paper, some new results concerning the modeling and control of distributed parameter systems in port Hamiltonian form are presented. The classical finite dimensional port Hamiltonian formulation of a dynamical system is generalized in order to cope with the distributed parameter and multi-variable case. The resulting class of infinite dimensional systems is quite general, thus allowing the description of several physical phenomena, such as heat conduction, piezoelectricity and elasticity. Furthermore, classical PDEs can be rewritten within this framework. The key point is the generalization of the notion of finite dimensional Dirac structure in order to deal with an infinite dimensional space of power variables. In this way, also in the distributed parameter case, the variation of total energy within the spatial domain of the system can be related to the power flow through the boundary. Since this relation deeply relies on the Stokes theorem, these structures are called Stokes–Dirac structures. As far as concerns the control problem, it seems natural that also finite dimensional control methodologies developed for finite dimensional port Hamiltonian systems can be extended in order to cope with infinite dimensional systems. In this paper, the *control by interconnection and energy shaping* methodology is applied to the stabilization problem of a distributed parameter system by means of a finite dimensional controller interconnected to its boundary. The key point is the generalization of the definition of *Casimir function* to the *hybrid* case, that is the dynamical system to be considered results from the power conserving interconnection of an infinite and a finite dimensional part. A simple application concerning the stabilization of the one-dimensional heat equation is presented.

1 Introduction

Following the same ideas behind the bond graph formalism [17], a finite dimensional physical system can be modeled as the result of the interconnection of a small set of atomic elements, each of them characterized by a particular energetic behavior (e.g. energy storing, dissipation or conversion). Each element can interact with the environment by means of a *port*, that is a couple of input and output signals whose combination gives the *power flow*. The network structure allows a power exchange between these components and describes the power flows within the system and between the system and the environment. This network can be mathematically described by means of a Dirac structure [1, 2, 11, 22], generalization of the well-known Kirchoff laws of circuit theory, [12].

Once the Dirac structure is defined, the dynamics of the system is specified when the space of energy (state) variables and the energy (Hamiltonian) function are given. The port Hamiltonian formalism [11, 22] is based on these ideas and allows the description of a wide class of finite dimensional non-linear systems, such as mechanical, electro-mechanical, hydraulic and chemical ones.

The port Hamiltonian representation of a finite dimensional system has been recently extended in order to cope with the infinite dimensional case, [23], thus generalizing the *classical* Hamiltonian formulation of a distributed parameter system which is a well-established mathematical result, [14, 21]. From the network modeling perspective, the dynamics of an infinite dimensional system with spatial domain $\mathcal{Z}$ and boundary $\partial\mathcal{Z}$ is the result of the interaction among (at least) two energy domains within $\mathcal{Z}$ and/or between the system and its environment through $\partial\mathcal{Z}$. This interaction is mathematically described by a generalization of the Dirac structure to the distributed parameter case. Since this new class of power conserving interconnection deeply relies on the Stokes theorem, we speak about Stokes–Dirac structure.

In [23], a simple Stokes–Dirac structure has been introduced and it has been shown that it is can be the starting point for the description in port Hamiltonian form of the telegrapher equation, of Maxwell's equations and of the vibrating string equation. Moreover, in [13], this Stokes-Dirac structure has been modified in order to model fluid dynamical systems and in [3, 9] to model the Timoshenko beam equation. In any case, it is not completely clear how a general formulation of a multi-variable distributed parameter system within the port Hamiltonian formalism could be obtained.

In this paper, some new results in this direction are presented. In particular, a novel class of Dirac structures over an infinite dimensional space of power variables are introduced. The interconnection, damping and input/output matrices are replaced by matrix differential operators which are assumed to be constant, that is no explicit dependence on the state (energy) *variables* is considered. As in finite dimensions, given the Stokes–Dirac structure, the model of the system easily follows once the Hamiltonian function is specified. The resulting class of infinite dimensional systems in port Hamiltonian form is quite general, thus allowing the interpretation of classical PDEs within this framework and the description of several physical phenomena, as the heat conduction, piezo electricity and elasticity.

From the control perspective, one of the main advantages in adopting the port Hamiltonian approach in both the finite either the infinite dimensional case is that the energy (Hamiltonian) function, which is usually a *good* Lyapunov function, explicitly appears in the dynamics of the system. Given a desired state of equilibrium, if the Hamiltonian of the system assumes its minimum at this configuration, then asymptotic stability can be assured by introducing a dissipative effect with the controller. In this way, energy decreases until the minimum of energy or, equivalently, the desired equilibrium configuration is reached. This control methodology is called control by *damping injection*, [20, 22].

On the other hand, if the Hamiltonian function of the system does not assume its minimum in the desired equilibrium state, it is necessary to *shape* the open-loop energy function and to introduce a new minimum in the desired configuration. The idea is to interconnect a controller to the plant and to choose the Hamiltonian of the regulator in order to properly shape the total (closed-loop) energy function. It is important to note that, in general, there is no *a priori* relation between the state of the plant and the state of the controller, so it is not immediate how the controller Hamiltonian function can be chosen in order to correctly shape the total energy. This problem can be solved by choosing the structure of the controller, i.e. its interconnection, damping and input/output matrices, in such a way that the state of the closed-loop system is constrained on certain subspace *independently* of the energy function of both the plant either the controller. Equivalently, this can be done by introducing a set of *Casimir functions* in the system, [2, 10]. Under some technical hypothesis, then, it is possible to introduce an *intrinsic* non-linear state feedback law that will be used in order to choose the energy function of the controller so that the closed-loop Hamiltonian can be properly shaped. Note that, under these hypothesis, this energy function depends on the state variables of the plant. This control methodology is called *invariant function method* or, within the framework of port Hamiltonian systems, control by *interconnection and energy shaping* and it is deeply discussed in [2, 10] and also in [15, 16] for the stabilization of non-linear port Hamiltonian systems.

In this paper, the control by interconnection and energy shaping is extended and applied to the regulation problem of an infinite dimensional system by means of a finite dimensional controller that can act on the system by exchanging power through the boundary. Some preliminary results in this direction have already been presented in [8, 19] where the infinite dimensional system is given by a set of transmission lines, while an application to stabilization of the Timoshenko beam has been discussed in [7, 9]. The main result concerns the necessary and sufficient conditions for a real-valued function defined over the closed-loop state space to be a structural invariant (Casimir function) for the controlled system which is an *hybrid* system since it results from the power conserving interconnection of an infinite and of a finite dimensional system. Once these conditions are deduced, by choosing a proper family of Casimir functions, the control by interconnection and energy shaping methodology can be applied as in the finite dimensional case. In this way, the open-loop energy function can be shaped by introducing a new minimum at the desired equilibrium configuration.

This work is organized as follows. After a short background concerning finite dimensional Dirac structures and port Hamiltonian system in Sect. 2, the infinite dimensional Stokes–Dirac structures are introduced in Sect. 3 and the corresponding port Hamiltonian formulation of multi-variable infinite dimensional system (mdpH systems) is discussed in Sect. 4. In Sect. 5, some simple examples are presented, the Harry–Dym equation, a *classical* nonlinear PDE, the heat equation and the general elasticity equation. Then, in Sect. 6, a short introduction on the control by interconnection and energy shaping for finite dimensional port Hamiltonian systems is given and, then, the boundary control by interconnection for infinite dimensional systems is discussed in Sect. 7. Necessary and sufficient conditions for the existence of Casimir functions in the closed loop system are deduced and their applications in the energy shaping procedure is described. Finally, a simple example concerning the boundary stabilization of the heat equation is discussed in Sect. 8, while conclusions are presented in Sect. 9.

2 Dirac structures and finite dimensional port Hamiltonian systems

2.1 Background on Dirac structures

The interconnection of physical system basically is power exchange. In order to mathematically model these phenomena, it is necessary to give a definition of power and to introduce a proper set of tools that will be useful to treat and describe the network structure behind a physical system.

Consider an n-dimensional linear space $\mathcal{F}$ and denote by $\mathcal{E} \equiv \mathcal{F}^*$ its dual, that is the space of linear operator $e : \mathcal{F} \to \mathbb{R}$. The elements belonging to $\mathcal{F}$ are called *flows* (e.g. velocities and currents), while the elements in $\mathcal{E}$ are called *efforts* (i.e. forces and voltages). Flows and efforts are the *port variables*, that is the input/output signals, whose combination gives the power flowing inside the physical system. The space $\mathcal{F} \times \mathcal{E}$ is called space of power variables.

Given an effort $e \in \mathcal{E}$ and a flow $f \in \mathcal{F}$, define the associated power P as

$$P := \langle e, f \rangle = e(f) \quad (\in \mathbb{R})$$

where $\langle \cdot, \cdot \rangle$ is the *dual product* between f and e. Based on the dual product, the following linear operator is well-defined.

Definition 2.1 (+pairing operator). Consider the space of power variables $\mathcal{F} \times \mathcal{E}$. The following symmetric bilinear form is well-defined:

$$\ll (f_1, e_1), (f_2, e_2) \gg \, := \langle e_1, f_2 \rangle + \langle e_2, f_1 \rangle \tag{1}$$

with $(f_i, e_i) \in \mathcal{F} \times \mathcal{E}$, $i = 1, 2$; $\ll \cdot, \cdot \gg$ is called +pairing operator.

Consider a linear subspace $\mathbb{S} \subset \mathcal{F} \times \mathcal{E}$ of dimension m and denote by $\mathbb{S}^\perp$ its orthogonal complement with respect to the +pairing operator (1), which is again a linear subspace of $\mathcal{F} \times \mathcal{E}$ with dimension $2n - m$ since (1) is a non-degenerate form. Based on the +pairing operator (1), it is possible to give the fundamental definition of Dirac structure, that is the basic mathematical tool that is used to describe the interconnection structure between physical systems.

Definition 2.2 (Dirac structure). Consider the space of power variables $\mathcal{F} \times \mathcal{E}$ and the symmetric bilinear form (1). A (constant) Dirac structure on $\mathcal{F}$ is a linear subspace $\mathbb{D} \subset \mathcal{F} \times \mathcal{E}$ such that

$$\mathbb{D} = \mathbb{D}^\perp$$

Note 2.1. It is possible to prove that the dimension of a Dirac structure $\mathbb{D}$ on an n-dimensional space $\mathcal{F}$ is equal to n. This result is related to an interesting property of physical systems. Consider, for example, the interconnection of electrical networks: it is well known that it is not possible to impose both currents and voltages. By generalization, a physical interconnection cannot determine both the flow either the effort.

Moreover, suppose that $(f, e) \in \mathbb{D}$; from (1), we have that

$$0 = \ll (f, e), (f, e) \gg = 2 \langle e, f \rangle$$

Then, it can be deduced that, for every $(f, e) \in \mathbb{D}$,

$$\langle e, f \rangle = 0$$

or, equivalently, that every Dirac structure $\mathbb{D}$ on $\mathcal{F}$ defines a power-conserving relation between power variables $(f, e) \in \mathcal{F} \times \mathcal{E}$.

With the following proposition, a quite general class of Dirac structures is introduced, [22].

Proposition 2.1. *Consider the space of power variables $\mathcal{F} \times \mathcal{E}$ and denote by $\mathcal{X}$ an n-dimensional space, the space of energy variables. Suppose that $\mathcal{F} := (\mathcal{F}_s, \mathcal{F}_r, \mathcal{F}_e)$ and that $\mathcal{E} := (\mathcal{E}_s, \mathcal{E}_r, \mathcal{E}_e)$, with $\dim \mathcal{F}_s = \dim \mathcal{E}_s = n$, $\dim \mathcal{F}_r = \dim \mathcal{E}_r = n_r$ and $\dim \mathcal{F}_e = \dim \mathcal{E}_e = m$. Moreover, denote by $J(x)$ a skew-symmetric matrix of dimension n and by $G_r(x)$ and $G(x)$ two matrices of dimension $n_r \times n$ and $m \times n$ respectively. Then, the set*

$$\mathbb{D} := \{\ (f_s, f_r, f_e, e_s, e_r, e_e) \in \mathcal{F} \times \mathcal{E} \ \mid \ \begin{aligned} f_s &= -\,J(x)e_s - G_r(x)f_r - G(x)f_e \\ e_r &= G_r^{\mathrm{T}}(x)e_s \\ e_e &= G^{\mathrm{T}}(x)e_s \ \} \end{aligned} \tag{2}$$

is a Dirac structure on $\mathcal{F}$

Note 2.2. In Def. 2.2, the pairs (f_s, e_s) and (f_r, e_r) represent the port variables of the *storage* and *dissipative* elements respectively, while (f_e, e_e) are the port variables through which the *environment* can exchange power with the system. Given the interconnection structure (2), the dynamics of the system can be specified once the port behavior of the energy storage elements is specified and when the dissipative ports are *terminated*.

2.2 Finite dimensional port Hamiltonian systems

The Dirac structure introduced in Def. 2.2 is quite general. Based on that, a general formulation of non-linear system in port Hamiltonian form can be easily given. As discussed in Note 2.2, a dynamical system can be interpreted as the result of the combination of the Dirac structure (2) with the port behavior of the energy storing and of the dissipative elements.

Under the same hypothesis of Prop. 2.1, denote by $H : \mathcal{X} \to \mathbb{R}$ a real valued function bounded from below defined over the space of energy variables $\mathcal{X}$. Then, define the port behavior of the energy storing elements as:

$$f_s = -\dot{x} \qquad\qquad e_s = \frac{\partial H}{\partial x} \tag{3}$$

where the minus sign is necessary in order to have a consistency in the power flow. If restricted to the linear case, dissipative effects can be taken into account by imposing the following relation on the variables (f_r, e_r) of the Dirac structure (2):

$$f_r = -Y_r e_r \tag{4}$$

where $Y_r = Y_r^{\mathrm{T}} \geq 0$. By substitution of (3) and (4) in (2), the representation of a port Hamiltonian system with dissipation can be deduced [11, 22] and the following definition makes sense.

Definition 2.3 (port Hamiltonian systems). Denote by $\mathcal{X}$ an n-dimensional space of state (energy) variables and by $H : \mathcal{X} \to \mathbb{R}$ a scalar energy function (Hamiltonian) bounded from below. Denote by

$\mathcal{U} \equiv \mathcal{F}_e$ an m-dimensional (linear) space of input variables and by its dual $\mathcal{Y} \equiv \mathcal{E}_e$ the space of output variables. Then,

$$\begin{cases} \dot{x} &= [J(x) - R(x)]\dfrac{\partial H}{\partial x} + G(x)u \\ y &= G^{\mathrm{T}}(x)\dfrac{\partial H}{\partial x} \end{cases} \tag{5}$$

with $J(x) = J^{\mathrm{T}}(x)$, $R(x) = R^{\mathrm{T}}(x) \geq 0$ and $G(x)$ matrices of proper dimensions, is a port Hamiltonian system with dissipation. The $n \times n$ matrices J and R are called *interconnection* and *damping* matrix respectively.

Note 2.3. Given a dynamical system in port Hamiltonian from (5), the variation of internal energy equals the dissipated power plus the power provided to the system by the environment, that is:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{\partial^{\mathrm{T}} H}{\partial x} R(x) \frac{\partial H}{\partial x} + y^{\mathrm{T}} u \leq y^{\mathrm{T}} u$$

This relation expresses a fundamental property of port Hamiltonian systems, their *passivity*. Roughly speaking, the internal energy of the unforced system ($u = 0$) is non-increasing along system trajectories or, if the port variable are closed on a dissipative element, that is a relation similar to (4) is imposed between u and y, then the energy function is always a decreasing function. If the definition of Lyapunov stability is recalled, together with the sufficient condition for the stability of an equilibrium point, then it can be deduced that the Hamiltonian is a good candidate for being a Lyapunov function.

3 Power conserving interconnections in infinite dimensions

3.1 Constant matrix differential operators

In the finite dimensional formulation (5) of a port Hamiltonian system, an important role is played by the interconnection, damping and input matrices. These operators are strictly related to the properties of the Dirac structure defining the power flows within the dynamical system and between the system and its environment. In infinite dimensions, these *objects* are generalized and they are mathematically described by matrix differential operators. In this paper, only the constant case is taken into account. In the finite dimensional framework, this means that the dependence on the x variable of the elements of the Dirac structure (2) is neglected.

Denote by $\mathcal{Z}$ a compact subset of $\mathbb{R}^d$ representing the spatial domain of the distributed parameter system. Then, denote by $\mathcal{U}$ and $\mathcal{V}$ two sets of *smooth* functions from $\mathcal{Z}$ to $\mathbb{R}^{q_u}$ and $\mathbb{R}^{q_v}$ respectively.

Definition 3.1 (constant matrix differential operator). A constant matrix differential operator of order N is a map L from $\mathcal{U}$ to $\mathcal{V}$ such that, given $u = (u^1, \ldots, u^{q_u}) \in \mathcal{U}$ and $v = (v^1, \ldots, v^{q_v}) \in \mathcal{V}$

$$v = Lu \quad \iff \quad v^b := \sum_{\#\alpha=0}^{N} P^{\alpha}_{a,b} D^{\alpha} u^a \tag{6}$$

where $\alpha := \{\alpha_1, \ldots, \alpha_d\}$ is a multi-index of order $\#\alpha := \sum_{i=1}^{d} \alpha_i$, P^α are a set of constant $q_u \times q_v$ matrices and $D^\alpha := \partial_{z_1}^{\alpha_1} \cdots \partial_{z_d}^{\alpha_d}$ is an operator resulting from a combination of spatial derivatives. Note that, in (6), the sum is intended over all the possible multi-indexes α with order 0 to N and, implicitly, on a from 1 to q.

Definition 3.2 (*formal* adjoint). Consider the constant matrix differential operator (6). Its formal adjoint is the map L^* from $\mathcal{V}$ to $\mathcal{U}$ such that

$$u = L^* v \quad \iff \quad u^b := \sum_{\#\alpha=0}^{N} (-1)^{\#\alpha} P^{\alpha}_{b,a} D^{\alpha} v^a \tag{7}$$

Definition 3.3 (skew-adjoint differential operator). Denote by J a constant matrix differential operator. Then, J is *skew-adjoint* if and only if

$$J = -J^*$$

Note 3.1. It is easy to prove that, L is a skew-adjoint matrix differential operator if and only if

$$P^{\alpha}_{a,b} = (-1)^{\#\alpha+1} P^{\alpha}_{b,a}$$

for every multi-index α from order 0 to N.

An important relation between a differential operator and its adjoint is expressed by the following lemma, which generalizes an analogous result presented in [18] to the multi variable case. As it will be discussed in Sect. 3.2, this result is fundamental in the definition of Stokes–Dirac structure and, basically, it generalizes the well-known integration by parts formula.

Lemma 3.1. *Consider a matrix differential operator L and denote by L^* its formal adjoint. Then, for every functions $u \in \mathcal{U}$ and $v \in \mathcal{V}$, we have that*

$$\int_{\mathcal{Z}} \left[v^{\mathrm{T}} L u - u^{\mathrm{T}} L^* v \right] \, \mathrm{dV} = \int_{\partial \mathcal{Z}} B_L(u, v) \cdot \mathrm{dA} \tag{8}$$

where B_L is a differential operator induced on $\partial\mathcal{Z}$ by L.

Note 3.2. Given $u \in \mathcal{U}$ and $v \in \mathcal{V}$, from the Stokes' theorem, it is well known that relation (8) can be equivalently written as

$$v^{\mathrm{T}} L u - u^{\mathrm{T}} L^* v = \operatorname{div} B_L(u, v)$$

that is $v^{\mathrm{T}} L u - u^{\mathrm{T}} L^* v$ can be expressed in divergence form. From (6) and (7), we have that

$$v^{\mathrm{T}} L u - u^{\mathrm{T}} L^* v = \sum_{\#\alpha=0}^{N} P^{\alpha}_{a,b} \left[(D^{\alpha} u^a) \, v^b - (-1)^{\#\alpha} \left(D^{\alpha} v^b \right) u^a \right] \tag{9}$$

whose divergence form is

$$\sum_{\#\beta=1} D^{\beta} \sum_{\alpha \geq \beta}^{N} \sum_{\gamma \leq \alpha-\beta} (-1)^{\#\gamma} P^{\alpha}_{a,b} \left(D^{\gamma} v^b \right) \left(D^{\alpha-\beta-\gamma} u^a \right) \tag{10}$$

in which the first sum is extended to all the multi index β of order 1.

Note 3.3. It is important to note that B_L is a constant differential operator. The quantity $B_L(u, v)$ is a constant linear combination of the functions u and v together with their spatial derivatives up to a certain order and depending on L. Consequently, denote by $B_{\mathcal{Z}}$ an operator providing a vector with all the spatial derivatives in (10) and by B^i_L, $i = 1, \ldots, d$, a set of constant square matrices of a certain order given by a proper combinations of all the P^{α} matrices. Then, (8) can be equivalently written as

$$\int_{\mathcal{Z}} \left[v^{\mathrm{T}} L u - u^{\mathrm{T}} L^* v \right] \, \mathrm{dV} = \int_{\partial \mathcal{Z}} B^{\mathrm{T}}_{\mathcal{Z}}(u) \left[B^1_L B_{\mathcal{Z}}(v) \ \cdots \ B^d_L B_{\mathcal{Z}}(v) \right] \cdot \mathrm{dA}$$

or, with some abuse in notation, as

$$\int_{\mathcal{Z}} \left[v^{\mathrm{T}} L u - u^{\mathrm{T}} L^* v \right] \, \mathrm{dV} = \int_{\partial \mathcal{Z}} B_L(B_{\mathcal{Z}}(u), B_{\mathcal{Z}}(v)) \cdot \mathrm{dA}$$

The last representation is the one that will be more often used in the remaining part of this paper.

Corollary 3.2. *Consider a skew-adjoint matrix differential operator J. Then, for every functions $u \in \mathcal{U}$ and $v \in \mathcal{V}$ with $q_u = q_v$, we have that*

$$\int_{\mathcal{Z}} \left[v^{\mathrm{T}} J u + u^{\mathrm{T}} J v \right] \, \mathrm{dV} = \int_{\partial \mathcal{Z}} B_J(u, v) \cdot \mathrm{dA} \tag{11}$$

where B_J is a symmetric operator on $\partial\mathcal{Z}$ depending on the differential operator J.

Proof. It is immediate from Def. 3.3 and the previous lemma. ■

Note 3.4. From Note 3.3, relation (11) can be alternatively written as

$$\begin{aligned}\int_{\mathcal{Z}} \left[v^{\mathrm{T}} J u + u^{\mathrm{T}} J v\right] \, \mathrm{dV} &= \int_{\partial\mathcal{Z}} B^{\mathrm{T}}_{\mathcal{Z}}(u) \left[B^1_J B_{\mathcal{Z}}(v) \ \ \cdots \ \ B^d_J B_{\mathcal{Z}}(v)\right] \cdot \mathrm{dA} \\ &= \int_{\partial\mathcal{Z}} B_J(B_{\mathcal{Z}}(u), B_{\mathcal{Z}}(v)) \cdot \mathrm{dA}\end{aligned}$$

3.2 Constant Stokes–Dirac structures

As in finite dimensions, the definition of a power conserving interconnection structure is possible once the notion of power is properly introduced. Denote by $\mathcal{F}$ the space of flows and assume that $\mathcal{F}$ is the space of *smooth* functions from the compact set $\mathcal{Z} \subset \mathbb{R}^d$ to $\mathbb{R}^q$. As far as concerns the space of efforts $\mathcal{E}$, assume for simplicity that $\mathcal{E} \equiv \mathcal{F}$. Then, given $f = (f^1, \dots, f^q) \in \mathcal{F}$ and $e = (e^1, \dots, e^q) \in \mathcal{E}$, define the dual product as follows:

$$\langle e, f \rangle := \int_{\mathcal{Z}} \sum_{i=1}^{q} e^i f^i \, \mathrm{dV} = \int_{\mathcal{Z}} e^{\mathrm{T}} f \, \mathrm{dV}$$

From Def. 2.1, the +pairing operator on $\mathcal{F} \times \mathcal{E}$ is given by

$$\ll (f_1, e_1), (f_2, e_2) \gg \, := \int_{\mathcal{Z}} \left[e_1^{\mathrm{T}} f_2 + e_2^{\mathrm{T}} f_1\right] \, \mathrm{dV}$$

where $(f_1, e_1), (f_2, e_2) \in \mathcal{F} \times \mathcal{E}$.

Denote by J a skew-adjoint differential operator and consider the following subset of the space of power variables:

$$\tilde{\mathbb{D}} := \{ \ (f, e) \in \mathcal{F} \times \mathcal{E} \ \ | \ \ f = -J e \ \} \tag{12}$$

Then, for every $(f_i, e_i) \in \tilde{\mathbb{D}}$, $i = 1, 2$, we have that

$$\begin{aligned}\ll (f_1, e_1), (f_2, e_2) \gg &= \int_{\mathcal{Z}} \left[e_1^{\mathrm{T}} f_2 + e_2^{\mathrm{T}} f_1\right] \, \mathrm{dV} = - \int_{\mathcal{Z}} \left[e_1^{\mathrm{T}} J e_2 + e_2^{\mathrm{T}} J e_1\right] \, \mathrm{dV} \\ &= - \int_{\partial\mathcal{Z}} B_J(e_1, e_2) \cdot \mathrm{dA}\end{aligned} \tag{13}$$

If only the elements of $\tilde{\mathbb{D}}$ with compact support on $\mathcal{Z}$ are considered, then the resulting subset of $\mathcal{F} \times \mathcal{E}$ is a Stokes–Dirac structure on $\mathcal{F}$, as it can be directly deduced from Def. 2.2 since the integral over $\partial\mathcal{Z}$ is equal to 0. In general, when an exchange of power between system and environment through the boundary of the spatial domain is present, (12) is not a Stokes–Dirac structure because also the boundary terms have to be taken into account. These boundary terms are the restriction of the efforts and their spatial derivatives on $\partial\mathcal{Z}$.

Denote by $w := B_{\mathcal{Z}}(e)$ the boundary terms, where $B_{\mathcal{Z}}$ the operator providing the restriction on $\partial\mathcal{Z}$ of the effort e and of its spatial derivatives of *proper* order as discussed in Note 3.3 and Note 3.4. In this way, it is possible to write (with some abuse in notation):

$$\int_{\partial\mathcal{Z}} B_J(e_1, e_2) \cdot \mathrm{dA} = \int_{\partial\mathcal{Z}} B_J(w_1, w_2) \cdot \mathrm{dA}$$

with $w_i = B_{\mathcal{Z}}(e_i)$, $i = 1, 2$ and where, in the last integral, B_J is the operator based on the square constant matrices B^i_J introduced in Note 3.4, $i = 1, \dots, d$. Furthermore, based on $B_{\mathcal{Z}}$, the following set representing the space of boundary conditions can be introduced:

$$\mathcal{W} := \{ w \mid w = B_{\mathcal{Z}}(e), \ \ \forall \, e \in \mathcal{E} \} \tag{14}$$

Then, the following proposition can be proved.

Proposition 3.3. *Consider the extended space of power variables $\mathcal{F} \times \mathcal{E} \times \mathcal{W}$ and denote by J a skew-adjoint differential operator. Then, the following subset*

$$\mathbb{D}_J := \{ \ (f, e, w) \in \mathcal{F} \times \mathcal{E} \times \mathcal{W} \ \mid \ f = -Je, \ w = B_{\mathcal{Z}}(e) \ \} \tag{15}$$

is a Stokes–Dirac structure on $\mathcal{F}$ with respect to the pairing

$$\ll (f_1, e_1, w_1), (f_2, e_2, w_2) \gg_J \ := \int_{\mathcal{Z}} \left[e_1^{\mathrm{T}} f_2 + e_2^{\mathrm{T}} f_1 \right] \mathrm{dV} + \int_{\partial\mathcal{Z}} B_J(w_1, w_2) \cdot \mathrm{dA} \tag{16}$$

where B_J is a symmetric operator on $\partial\mathcal{Z}$.

Proof. The proof is immediate from (13) and (14). The symmetry of B_J follows from Corollary 3.2. ■

Note 3.5. From the properties of a Stokes–Dirac structure, summarized in Note 2.1 for the finite dimensional case, if $(f, e, w) \in \mathbb{D}$, then $\ll (f, e, w), (f, e, w) \gg_J = 0$, that is

$$-\int_{\mathcal{Z}} e^{\mathrm{T}} f \, \mathrm{dV} = \frac{1}{2} \int_{\partial\mathcal{Z}} B_J(w, w) \cdot \mathrm{dA}$$

This relation, beside expressing the power conservation property of the Stokes–Dirac structure, is able to relate the variation of internal energy (the integral on the spatial domain $\mathcal{Z}$) with the power flowing inside the domain through the boundary (the integral on $\partial\mathcal{Z}$).

The Stokes–Dirac structure introduced in Prop. 3.3 is developed around a skew-adjoint differential operator which *induces* a non-degenerate differential operator on the boundary. In finite dimensions, this situation can be obtained by assuming $G_r = G = 0$ in the Dirac structure of Prop. 2.1, that is by assuming that the power conserving network interconnects only a set of energy storing elements. It is interesting to completely generalize the result of Prop. 2.1 to the distributed parameter case or, equivalently, to properly modify the Stokes–Dirac structure (15) of Prop. 3.3 in order to take into account dissipative effects and an interaction between system and environment along the spatial domain $\mathcal{Z}$ and not only through the boundary $\partial\mathcal{Z}$. The last situation can be encountered, for example, in the case of Maxwell's equations when a current density different from 0 is present, [5, 23].

Theorem 3.4 (constant Stokes–Dirac structure). *Denote by $\mathcal{Z} \subset \mathbb{R}^d$ a compact set and by $\mathcal{F} = (\mathcal{F}_s, \mathcal{F}_r, \mathcal{F}_d)$ a space of vector values smooth functions on $\mathcal{Z}$, the space of flows. For simplicity, suppose that $\mathcal{E} = (\mathcal{E}_s, \mathcal{E}_r, \mathcal{E}_d) \equiv \mathcal{F}$ is the space of efforts. Moreover, assume that J, G_r and G_d are constant matrix differential operator such that $J : \mathcal{E}_s \to \mathcal{F}_s$ and $J = -J^*$, $G_r : \mathcal{F}_r \to \mathcal{F}_s$ and $G_d : \mathcal{F}_d \to \mathcal{F}_s$. Then,*

$$\mathbb{D} := \{ \ (f, e, w) \in \mathcal{F} \times \mathcal{E} \times \mathcal{W} \ \mid \ \begin{aligned} & f_s = -Je_s - G_r f_r - G_d f_d \\ & e_r = G_r^* e_s \\ & e_d = G_d^* e_s \\ & w = B_{\mathcal{Z}}(e_s, f_r, f_d) \ \} \end{aligned} \tag{17}$$

is a Stokes–Dirac structure with respect to the pairing

$$\begin{aligned} \ll (f_1, e_1, w_1), (f_2, e_2, w_2) \gg_{\{J, G_r, G_d\}} \ := & \int_{\mathcal{Z}} \left[e_1^{\mathrm{T}} f_2 + e_2^{\mathrm{T}} f_1 \right] \mathrm{dV} \\ & + \int_{\partial\mathcal{Z}} B_{\{J, G_r, G_d\}}(w_1, w_2) \cdot \mathrm{dA} \end{aligned} \tag{18}$$

where $B_{\mathcal{Z}}$ is the analogous of the boundary operator of Prop. 3.3 and $B_{\{J, G_r, G_d\}}$ is the boundary differential operator induced by J, G_r and G_d on $\partial\mathcal{Z}$.

Proof. Consider $(f_i, e_i) \in \mathcal{F} \times \mathcal{E}$, $i = 1, 2$. Then,

$$\begin{aligned}
\int_{\mathcal{Z}} \left[e_1^{\mathrm{T}} f_2 + e_2^{\mathrm{T}} f_1\right] \mathrm{dV} &= \int_{\mathcal{Z}} \left[e_{s,1}^{\mathrm{T}} f_{s,2} + e_{s,2}^{\mathrm{T}} f_{s,1} + e_{r,1}^{\mathrm{T}} f_{r,2} + e_{r,2}^{\mathrm{T}} f_{r,1} + e_{d,1}^{\mathrm{T}} f_{d,2} + e_{d,2}^{\mathrm{T}} f_{d,1}\right] \mathrm{dV} \\
&= -\int_{\mathcal{Z}} \left[e_{s,1}^{\mathrm{T}} \left(J e_{s,2} + G_r f_{r,2} + G_d f_{d,2}\right) + e_{s,2}^{\mathrm{T}} \left(J e_{s,1} + G_r f_{r,1} + G_d f_{d,1}\right)\right] \mathrm{dV} \\
&\quad + \int_{\mathcal{Z}} \left[f_{r,2}^{\mathrm{T}} G_r^* e_{s,1} + f_{r,1}^{\mathrm{T}} G_r^* e_{s,2} + f_{d,2}^{\mathrm{T}} G_d^* e_{s,1} + f_{d,1}^{\mathrm{T}} G_d^* e_{s,2}\right] \mathrm{dV} \\
&= -\int_{\mathcal{Z}} \left[e_{s,1}^{\mathrm{T}} J e_{s,2} + e_{s,2}^{\mathrm{T}} J e_{s,1}\right] \mathrm{dV} \\
&\quad - \int_{\mathcal{Z}} \left[\left(e_{s,1}^{\mathrm{T}} G_r f_{r,2} - f_{r,2}^{\mathrm{T}} G_r^* e_{s,1}\right) + \left(e_{s,2}^{\mathrm{T}} G_r f_{r,1} - f_{r,1}^{\mathrm{T}} G_r^* e_{s,2}\right)\right] \mathrm{dV} \\
&\quad - \int_{\mathcal{Z}} \left[\left(e_{s,1}^{\mathrm{T}} G_d f_{d,2} - f_{d,2}^{\mathrm{T}} G_d^* e_{s,1}\right) + \left(e_{s,2}^{\mathrm{T}} G_d f_{d,1} - f_{d,1}^{\mathrm{T}} G_d^* e_{s,2}\right)\right] \mathrm{dV}
\end{aligned}$$

From Lemma 3.1 and its Corollary 3.2, all the quantities under integration can be expressed in divergence form, that is as the divergence of some differential form which is non-degenerate. In particular, denote by B_J, B_{G_r}, $B_{-G_r^*}$, B_{G_d} and $B_{-G_d^*}$ the differential operators induced on $\partial\mathcal{Z}$ by J, G_r and G_d and their adjoint. Then,

$$\begin{aligned}
\int_{\mathcal{Z}} \left[e_1^{\mathrm{T}} f_2 + e_2^{\mathrm{T}} f_1\right] \mathrm{dV} &= -\int_{\partial\mathcal{Z}} e_{s,1}^{\mathrm{T}} \left[B_J^1 e_{s,2} \ \cdots \ B_J^d e_{s,2}\right] \cdot \mathrm{dA} \\
&\quad - \int_{\partial\mathcal{Z}} \left\{e_{s,1}^{\mathrm{T}} \left[B_{G_r}^1 f_{r,2} \ \cdots \ B_{G_r}^d f_{r,2}\right] + f_{r,1}^{\mathrm{T}} \left[B_{-G_r^*}^1 e_{s,2} \ \cdots \ B_{-G_r^*}^d e_{s,2}\right]\right\} \cdot \mathrm{dA} \\
&\quad - \int_{\partial\mathcal{Z}} \left\{e_{s,1}^{\mathrm{T}} \left[B_{G_d}^1 f_{d,2} \ \cdots \ B_{G_d}^d f_{d,2}\right] + f_{d,1}^{\mathrm{T}} \left[B_{-G_d^*}^1 e_{s,2} \ \cdots \ B_{-G_d^*}^d e_{s,2}\right]\right\} \cdot \mathrm{dA}
\end{aligned}$$

If $w_i = (e_{s,i}, f_{r,i}, f_{d,i})$, $i = 1, 2$, and

$$B_{\{J,\, G_r,\, G_d\}}^i = \begin{pmatrix} B_J^i & B_{G_r}^i & B_{G_d}^i \\ B_{-G_r^*}^i & 0 & 0 \\ B_{-G_d^*}^i & 0 & 0 \end{pmatrix}$$

with $i = 1, \ldots, d$, then it is possible to write that

$$\int_{\mathcal{Z}} \left[e_1^{\mathrm{T}} f_2 + e_2^{\mathrm{T}} f_1\right] \mathrm{dV} + \int_{\partial\mathcal{Z}} w_1^{\mathrm{T}} \left[B_{\{J,\, G_r,\, G_d\}}^1 w_2 \ \cdots \ B_{\{J,\, G_r,\, G_d\}}^d w_2\right] \cdot \mathrm{dA} = 0$$

or, in a more compact way (see Note 3.3), that

$$\int_{\mathcal{Z}} \left[e_1^{\mathrm{T}} f_2 + e_2^{\mathrm{T}} f_1\right] \mathrm{dV} + \int_{\partial\mathcal{Z}} B_{\{J,\, G_r,\, G_d\}}(w_1, w_2) \cdot \mathrm{dA} = 0$$

which, beside providing the expression (18) of the pairing $\ll \cdot, \cdot \gg_{\{J,\, G_r,\, G_d\}}$, proves that the set defined in (17) is Stokes–Dirac structure on $\mathcal{F}$ with respect to the bilinear form (18). ■

Note 3.6. The previous theorem is the generalization of the result presented in Prop. 2.1 to the constant infinite dimensional case. It is possible, eventually, to introduce the dependence on the energy *variables* and their spatial derivatives in the differential operators J, G_r and G_d. The result is the definition of nonlinear and *state* modulated Dirac structure in infinite dimensions. The way in which this result can be obtained relies on the generalization to the nonlinear case of matrix differential operator and, in particular, of the result expressed by Lemma 3.1.

Note 3.7. Suppose that $(f, e, w) \in \mathbb{D}$. From (18), we have that

$$-\int_{\mathcal{Z}} e_s^{\mathrm{T}} f_s = \int_{\mathcal{Z}} e_r^{\mathrm{T}} f_r \,\mathrm{dV} + \int_{\mathcal{Z}} e_d^{\mathrm{T}} f_d \,\mathrm{dV} + \frac{1}{2} \int_{\partial\mathcal{Z}} B_{\{J,\, G_r,\, G_d\}}(w_1, w_2) \cdot \mathrm{dA} \tag{19}$$

This relation, which is a direct consequence of the definition of Dirac structure, expresses the property that the variation of internal energy is equal to the sum of the dissipated power with the power provided to the system through the domain $\mathcal{Z}$ and the boundary $\partial\mathcal{Z}$.

Note 3.8. The boundary differential operator $B_{\{J,G_r,G_d\}}$ introduced in (29) is symmetric. In fact, from Corollary 3.2 and relation (28), but see also Prop. 3.3, it is induced on the boundary by the operator $\mathcal{J}$ on $\mathcal{Z}$ defined in the Dirac structure (28) as follows:

$$\begin{bmatrix} f_s \\ e_r \\ e_d \end{bmatrix} = -\underbrace{\begin{bmatrix} J & G_r & G_d \\ -G_r^* & 0 & 0 \\ -G_d^* & 0 & 0 \end{bmatrix}}_{\mathcal{J}} \begin{bmatrix} e_s \\ f_r \\ f_d \end{bmatrix}$$

which is skew-adjoint in the sense of Def. 3.3.

4 Multi-variable infinite dimensional port Hamiltonian systems

4.1 General definition

As in finite dimensions, the dynamics of a distributed parameter system can be obtained from its Stokes–Dirac structure once the power ports are terminated on the corresponding elements, that is the input/output behavior of the *components* are specified.

Denote by $\mathcal{X}$ the space of smooth real valued functions on $[0,+\infty)\times\mathcal{Z}$ representing the space of energy *configuration.* The total energy is a functional $\mathcal{H}:\mathcal{X}\to\mathbb{R}$ such that

$$\mathcal{H}(x) = \int_{\mathcal{Z}} H(z,x)\,\mathrm{dV}$$

where H is the energy density. As proposed in [23], the port behavior of the energy storing element is given by

$$f_s = -\frac{\partial x}{\partial t} \qquad\qquad e_s = \delta_x\mathcal{H} \tag{20}$$

where $\delta_x\mathcal{H}$ is the variational derivative of the Hamiltonian with respect to the energy configuration. Linear dissipation can be introduced by imposing that

$$f_r = -Y_r e_r, \quad \text{with} \quad \int_{\mathcal{Z}} e_r^{\mathrm{T}} Y_r e_r\,\mathrm{dV} \geq 0 \tag{21}$$

where $Y_r:\mathcal{E}_r\to\mathcal{F}_r$ is a linear operator. If $\tilde{B}_{\mathcal{Z}}$ is the boundary operator introduced in (17), from (20) we have that

$$\tilde{B}_{\mathcal{Z}}(e_s,f_r,f_d) = \tilde{B}_{\mathcal{Z}}(e_s,-Y_rG_r^*e_s,f_d) =: B_{\mathcal{Z}}(e_s,f_d) \tag{22}$$

and then the boundary terms can be computed as $w = B_{\mathcal{Z}}(e_s,f_d)$. Consequently, taking into account (17), (20), (21) and (22), the following definition makes sense.

Definition 4.1 (mdpH system). Denote by $\mathcal{X}$ the space of vector value smooth functions on $[0,+\infty)\times\mathcal{Z}$ (energy configurations), by $\mathcal{F}_d$ the space of vector value smooth functions on $\mathcal{Z}$ (distributed flows) and assume that $\mathcal{E}_d\equiv\mathcal{F}_d$ is its dual (distributed efforts) and by $\mathcal{W}$ the space of vector value smooth functions on $\partial\mathcal{Z}$ representing the boundary terms. Moreover, denote by J a skew-adjoint differential operator, by G_d a differential operator and by $B_{\mathcal{Z}}$ the boundary operator defined in (22). If $\mathcal{H}:\mathcal{X}\to\mathbb{R}$ is the Hamiltonian function, the general formulation of a multi-variable distributed port Hamiltonian system with constant Stokes–Dirac structure is

$$\begin{cases} \dfrac{\partial x}{\partial t} &=& (J-R)\,\delta_x\mathcal{H} + G_d f_d \\ e_d &=& G_d^*\,\delta_x\mathcal{H} \\ w &=& B_{\mathcal{Z}}(\delta_x\mathcal{H},f_d) \end{cases} \tag{23}$$

where $R := G_rY_rG_r^*$ is a differential operator taking into account energy dissipation and $(f_d,e_d)\in\mathcal{F}_d\times\mathcal{E}_d$.

Note 4.1. It is important to note that there is no *a priori* distinction between flows and efforts in the boundary terms w. These variables result from the restriction on $\partial\mathcal{Z}$ of the variational derivative of $\mathcal{H}$ and of its spatial derivatives and, consequently, they are not characterized by an explicit physical meaning. In other words, given a generic multi-variable distributed port Hamiltonian system, the classical structure of power port, i.e. a couple of signals (flow and effort) whose *combination* gives the power flow, has been lost on the boundary. Only if the boundary operator $B_{\{J,G_r,G_d\}}$ has a particular structure, it is possible to split the boundary variable w into two components, that is into a flow and an effort.

Note 4.2. Consider the operator $B_{\{J,G_r,G_d\}}$ introduced in Theorem 3.4, which is symmetric, as discussed in Note 3.8. As in (22) for the boundary operator $B_{\mathcal{Z}}$, it is possible to define a new operator $B_{\{J,R\}}$ on $\partial\mathcal{Z}$ based on $B_{\{J,G_r,G_d\}}$ and taking into account relation (21), which introduces and describes the dissipative effects in the system. Since $f_{r,i} = -Y_r G_r^* e_{s,i}$, $i = 1,2$, it is possible to define this new operator as follows:

$$B_{\{J,R\}}(w_1,w_2) := B_{\{J,G_r,G_d\}}(\tilde{B}_{\mathcal{Z}}(e_{s,1}, -Y_r G_r^* e_{s,1}, f_{d,1}), \tilde{B}_{\mathcal{Z}}(e_{s,2}, -Y_r G_r^* e_{s,2}, f_{d,2}))$$

where $w_i = B_{\mathcal{Z}}(e_{s,i}, f_{d,i})$, $i = 1,2$, as introduced in (22). From the symmetry property of $B_{\{J,G_r,G_d\}}$, we deduce that also $B_{\{J,R\}}$ is symmetric.

Proposition 4.1. *Consider the mdpH system (23). Then, the following energy balance inequality holds:*

$$\begin{aligned}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} &= -\int_{\mathcal{Z}} (\delta_x\mathcal{H})^{\mathrm{T}} R\,\delta_x\mathcal{H}\,\mathrm{dV} + \int_{\mathcal{Z}} e_d^{\mathrm{T}} f_d\,\mathrm{dV} + \frac{1}{2}\int_{\partial\mathcal{Z}} B_{\{J,R\}}(w,w)\cdot\mathrm{dA} \\ &\leq \int_{\mathcal{Z}} e_d^{\mathrm{T}} f_d\,\mathrm{dV} + \frac{1}{2}\int_{\partial\mathcal{Z}} B_{\{J,R\}}(w,w)\cdot\mathrm{dA}\end{aligned} \tag{24}$$

Proof. From (20), we have that

$$-\int_{\mathcal{Z}} e_s^{\mathrm{T}} f_s\,\mathrm{dV} = \int_{\mathcal{Z}} (\delta_x\mathcal{H})^{\mathrm{T}} \frac{\partial x}{\partial t}\,\mathrm{dV} = \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t}$$

Then, (24) is immediate from (19), (21) and Note 4.2. ■

Note 4.3. Relation (24) expresses an obvious property of physical systems, that is the variation of internal energy is less or equal (if no dissipation is present) to the power provided to the system. In the case of distributed parameter system, the power can flow inside the system either through the boundary and/or the spatial domain.

5 Simple examples

5.1 Harry–Dym equation

The Harry–Dym equation is

$$\frac{\partial x}{\partial t} = \frac{\partial^3}{\partial z^3}\left(x^{-1/2}\right) \tag{25}$$

Denote by $\mathcal{Z} = [0,1]$ the spatial domain and by $\mathcal{X} = L^2([0,+\infty)\times\mathcal{Z})$ the space of energy configurations. The differential operator $J = \frac{\partial^3}{\partial z^3}$ is skew-adjoint and, then, it is possible to define a Stokes–Dirac structure based on J as discussed in Prop. 3.3. We give the following proposition.

Proposition 5.1. *Denote by $\mathcal{Z} = [0,1]$ the spatial domain and by $\mathcal{F} = L^2(\mathcal{Z})$ the space of flows and assume that $\mathcal{E} \equiv \mathcal{F}$ is the space of efforts. Then*

$$\mathbb{D}_{HD} := \left\{\ (f,e,w) \in \mathcal{F}\times\mathcal{E}\times\mathcal{W}\ \mid\ f = \partial_z^3 e,\ w = B_{\mathcal{Z}}(e) = (e\,|_{\partial\mathcal{Z}}, \partial_z e\,|_{\partial\mathcal{Z}}, \partial_z^2 e\,|_{\partial\mathcal{Z}})\ \right\}$$

is a Stokes–Dirac structure with respect to the pairing

$$\ll (f_1,e_1,w_1),(f_2,e_2,w_2) \gg_{HD} := \int_0^1 [e_1 f_2 + e_2 f_1]\,\mathrm{dz} + w_1^{\mathrm{T}} B_J w_2\Big|_0^1$$

with

$$B_J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and $\mathcal{W} = \mathbb{R}^3$.

Proof. Since ∂_z^3 is a skew-adjoint differential operator, from Prop. Prop. 3.3 we deduce that it can define a Stokes–Dirac structure. Then, it is necessary only to compute $B_\mathcal{Z}$ and B_J. Given $(f_i, e_i) \in \mathcal{F} \times \mathcal{E}$, $i = 1, 2$, we have that

$$\begin{aligned} e_1 f_2 + e_2 f_1 &= -\left(e_1 \frac{\partial^3 e_2}{\partial z^3} + e_2 \frac{\partial^3 e_1}{\partial z^3}\right) \\ &= -\frac{\partial}{\partial z}\left(e_1 \frac{\partial^2 e_2}{\partial z^2} - \frac{\partial e_1}{\partial z}\frac{\partial e_2}{\partial z} + e_2 \frac{\partial^2 e_1}{\partial z^2}\right) \end{aligned}$$

which gives $B_\mathcal{Z}$ and B_J thus concluding the proof. ■

The mdpH formulation of the Harry–Dym equation is completed once the Hamiltonian function is specified. In this case, we have that

$$\mathcal{H}(x) := 2 \int_0^1 x^{1/2}(z)\, \mathrm{d}z$$

then (25) can be obtained if, as in (20), we assume that $f = -\dot{x}$ and $e = \delta_x \mathcal{H} = x^{-1/2}$. Clearly, the following energy balance relation holds:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \left[\delta_x \mathcal{H} \frac{\partial^2 \delta_x \mathcal{H}}{\partial z^2} - \frac{1}{2}\left(\frac{\partial \delta_x \mathcal{H}}{\partial z}\right)^2\right]_0^1$$

Note that, in this case, it is not possible to define a pair of flow and effort variables on the boundary of the spatial domain (see Note 4.1) and that the model is nonlinear.

5.2 Heat equation

The one-dimensional heat equation is

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial z^2} \tag{26}$$

This system is not Hamiltonian in the classical sense [4], but it can be written in mdpH form.

Denote by $\mathcal{Z} = [0, 1]$ the spatial domain and by $\mathcal{X} = L^2([0, +\infty) \times \mathcal{Z})$ the space of energy configurations. The differential operator $R = \frac{\partial^2}{\partial z^2}$ is *not* skew-adjoint and, then, it is not possible to refer to the result of Prop. 3.3 in order to define a Stokes–Dirac structure.

Define the energy $\mathcal{H}$ of the system as

$$\mathcal{H}(x) = \frac{1}{2} \int_0^1 x^2(z)\, \mathrm{d}z \tag{27}$$

and then

$$\begin{aligned} \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} &= \int_\mathcal{Z} x\dot{x}\, \mathrm{d}z = \int_\mathcal{Z} x \frac{\partial^2 x}{\partial z^2}\, \mathrm{d}z = \int_\mathcal{Z} \frac{\partial}{\partial z}\left(x \frac{\partial x}{\partial z}\right) \mathrm{d}z - \int_\mathcal{Z} \left(\frac{\partial x}{\partial z}\right)^2 \mathrm{d}z \\ &\leq x \frac{\partial x}{\partial z}\bigg|_0^L \end{aligned} \tag{28}$$

This relation can be interpreted as an energy balance equation: the variation of internal energy is less or equal to the power provided to the system through the boundary. In this way, the diffusion phenomenum modeled by (26) can be described as pure dissipation. Clearly, a mdpH formulation of (26) is possible only once a proper Stokes–Dirac structure is determined.

We give the following proposition:

Proposition 5.2. *Denote by $\mathcal{Z} = [0,1]$ the spatial domain and by $\mathcal{F} = (L^2(\mathcal{Z}))^2$ the space of flows and suppose that $\mathcal{E} \equiv \mathcal{F}$ is the space of efforts. Then, the set*

$$\mathbb{D}_H := \{ \, (f_s, f_r, e_s, e_r, w) \in \mathcal{F} \times \mathcal{E} \times \mathcal{W} \;\; | \;\; \begin{aligned} f_s &= -\,\partial_z f_r \\ e_r &= -\,\partial_z e_s \\ w &= (e_s\,|_{\partial\mathcal{Z}}, f_r\,|_{\partial\mathcal{Z}}) \,\} \end{aligned} \tag{29}$$

is a Stokes–Dirac structure on $\mathcal{F}$ with respect to the pairing

$$\ll (f_1, e_1, w_1), (f_2, e_2, w_2) \gg_H = \int_{\mathcal{Z}} \left[e_1^{\mathrm{T}} f_2 + e_2^{\mathrm{T}} f_1 \right] \,\mathrm{dz} + w_1^{\mathrm{T}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} w_2 \Big|_0^1 \tag{30}$$

where $\mathcal{W} = \mathbb{R}^2$.

Proof. The proof can be found in [23] since (29) is the same Stokes–Dirac structure of the telegrapher equation or, equivalently, it can be deduced from Theorem 3.4 if $J = 0$, $G_r = \partial_z$ and $G_d = 0$. ■

The heat equation (26) can be obtained from the Stokes–Dirac structure by imposing that $f_s = -\dot{x}$ and $e_s = \delta_x \mathcal{H} = x$, where the Hamiltonian function is given in (27). Moreover, it is necessary to properly terminate the resistive port (f_r, e_r) in (29) by supposing that

$$f_r = -e_r$$

Finally, the energy balance relation (28) can be obtained from (30) since given

$$(f_s, f_r, e_s, e_r; w) = \left(-\dot{x}, \frac{\partial \delta_x \mathcal{H}}{\partial z}, \delta_x \mathcal{H}, -\frac{\partial \delta_x \mathcal{H}}{\partial z}; \, \delta_x \mathcal{H}\,|_{\partial\mathcal{Z}} \,, \, \frac{\partial \delta_x \mathcal{H}}{\partial z} \Big|_{\partial\mathcal{Z}} \right) \in \mathbb{D}_H$$

then $\ll (f_s, f_r, e_s, e_r; w), (f_s, f_r, e_s, e_r; w) \gg_H = 0$.

5.3 General elasticity equation

Denote by $\mathcal{Z}$ a spatial domain of dimension p and by $u \in \mathcal{Z} \to \mathbb{R}^q$ a vector valued smooth function. The general elasticity equation [14] is

$$\frac{\partial^2 u^\alpha}{\partial t^2} = \sum_{i=1}^{p} \frac{\partial}{\partial x_i} \left(\frac{\partial W}{\partial u^\alpha_{x_i}} \right), \qquad \text{with} \;\; \alpha = 1, \ldots, q \;\;\; \text{and} \;\; u^\alpha_{x_i} = \frac{\partial u^\alpha}{\partial x_i}, \tag{31}$$

where $W(z, \nabla u^1, \ldots, \nabla u^q)$ is a potential. The energy (Hamiltonian) of the system is given by

$$\mathcal{H}(\dot{u}; \nabla u^1, \ldots, \nabla u^q) = \int_{\mathcal{Z}} \left[\frac{1}{2} \|\dot{u}\|^2 + W(z, \nabla u^1, \ldots, \nabla u^q) \right] \mathrm{dV} \tag{32}$$

thus suggesting that the energy variables are given by $u^\alpha_{x_i}$ and $\dot{u}^\alpha$, with $\alpha = 1, \ldots, q$ and $i = 1, \ldots, p$. The flow and effort variables are assumed to be

$$\begin{aligned} f &= [f_{1,1} \cdots f_{1,p} \cdots f_{q,1} \cdots f_{q,p}; \, f^1 \cdots f^q]^{\mathrm{T}} \\ e &= [e_{1,1} \cdots e_{1,p} \cdots e_{q,1} \cdots e_{q,p}; \, e^1 \cdots e^q]^{\mathrm{T}} \end{aligned}$$

and, then, the corresponding spaces of flows and effort are $\mathcal{F} = (L^2(\mathcal{Z}))^{q(p+1)} \equiv \mathcal{E}$. Since no dissipative effect is present, the Stokes–Dirac structure corresponding to (31) should have the structure (15) of Prop. 3.3. In particular, only the skew-adjoint matrix differential operator J has to be computed, since both the operators $B_{\mathcal{Z}}$ and B_J, together with the space of boundary variables $\mathcal{W}$, follow automatically once J is specified. Furthermore, (16) is given by

$$\ll (f_1, e_1, w_1), (f_2, e_2, w_2) \gg_{GE} = \int_{\mathcal{Z}} \left[e_1^{\mathrm{T}} f_2 + e_2^{\mathrm{T}} f_1 \right] \mathrm{dV} + \int_{\partial\mathcal{Z}} B_J(w_1, w_2) \cdot \mathrm{dA}$$

More precisely, once the *correct* differential operator J is determined, the bilinear form B_J results from $\langle e, f\rangle$ if expressed in terms of the boundary variables. In this simpler way, also $B_{\mathcal{Z}}$ is computed.

Assume that J is characterized by the following structure:

$$J = \begin{bmatrix} 0 & \tilde{J} \\ -\tilde{J}^* & 0 \end{bmatrix}, \qquad \text{with} \quad \tilde{J} = \begin{bmatrix} \partial_{x_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \partial_{x_p} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \partial_{x_1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \partial_{x_p} \end{bmatrix}$$

Clearly, J is skew-adjoint and then, from Prop. 3.3, it is possible to define the corresponding Stokes–Dirac structure $\mathbb{D}_{GE}$. Given $(f, e) \in \mathcal{F} \times \mathcal{E}$, the associated power is equal to

$$\begin{aligned} -\int_{\mathcal{Z}} \mathrm{e}^{\mathrm{T}} f \, \mathrm{dV} &= \int_{\mathcal{Z}} \left(\sum_{i=1}^{q} \sum_{j=1}^{p} e_{ij} f_i j + \sum_{i=1}^{q} e^i f^j \right) \mathrm{dV} \\ &= \int_{\mathcal{Z}} \left(\sum_{i=1}^{q} \sum_{j=1}^{p} e_{ij} \frac{\partial e^i}{\partial x_j} + \sum_{i=1}^{q} e^i \sum_{j=1}^{p} \frac{\partial e_{ij}}{\partial x_j} \right) \mathrm{dV} \\ &= \int_{\mathcal{Z}} \sum_{i=1}^{q} \sum_{j=1}^{p} \left(e_{ij} \frac{\partial e^i}{\partial x_j} + e^i \frac{\partial e_{ij}}{\partial x_j} \right) \mathrm{dV} = \int_{\mathcal{Z}} \sum_{i=1}^{q} \sum_{j=1}^{p} \frac{\partial}{\partial x_j} \left(e_{ij} e^i \right) \mathrm{dV} \\ &= \int_{\mathcal{Z}} \sum_{j=1}^{p} \frac{\partial}{\partial x_j} \left(\sum_{i=1}^{q} e_{ij} e^i \right) \mathrm{dV} \end{aligned} \tag{33}$$

where the last quantity under integration is expressed in divergence form. Consequently, in (15) it makes sense to assume

$$w = B_{\mathcal{Z}}(e) = e \,|_{\partial \mathcal{Z}}, \quad \text{with } \mathcal{W} = \{ w \mid w = B_{\mathcal{Z}}(e), \ \forall\, e \in \mathcal{E} \} \subset (L^2(\partial \mathcal{D}))^p, \tag{34}$$

and

$$\frac{1}{2} B_J(w, w) = \left[\sum_{i=1}^{q} e_{i1} e^i \ \cdots \ \sum_{i=1}^{q} e_{ip} e^i \right] \tag{35}$$

From (33) and (35), if $(f, e, w) \in \mathcal{F} \times \mathcal{E} \times \mathcal{W}$, the following relation expressing the power conservation property holds:

$$\int_{\mathcal{Z}} \mathrm{e}^{\mathrm{T}} f \, \mathrm{dV} + \frac{1}{2} \int_{\partial \mathcal{Z}} B_J(w, w) \cdot \mathrm{dA} = 0 \tag{36}$$

where

$$\frac{1}{2} \int_{\partial \mathcal{Z}} B_J(w, w) \cdot \mathrm{dA} = \int_{\partial \mathcal{Z}} \left[\sum_{i=1}^{q} e_{i1} e^i \ \cdots \ \sum_{i=1}^{q} e_{ip} e^i \right] \cdot \mathrm{dA}$$

In (34), write the boundary terms w as $(e_{b,1}, \dots, e_{b,q}, f_{b,1}, \dots, f_{b,q})$, where

$$\begin{cases} f_{b,i} = e^i \,|_{\partial \mathcal{Z}} \\ e_{b,i} = [e_{1i} \,|_{\partial \mathcal{Z}} \ \cdots \ e_{pi} \,|_{\partial \mathcal{Z}}]^{\mathrm{T}} \end{cases}, \quad i = 1, \dots, q \tag{37}$$

Then,

$$\frac{1}{2} B_J(w, w) = \sum_{i=1}^{q} e_{b,i}^{\mathrm{T}} f_{b,i}$$

that is, the boundary power flow can be expressed as a combination of boundary variables that are suitable of a physical interpretation.

Since (32) is the Hamiltonian function, the dynamics of the system can be obtained by defining

$$\begin{cases} f_{ij} = -\dfrac{\partial u^i_{x_j}}{\partial t} \\ f^i = -\dfrac{\partial \dot{u}^i}{\partial t} \end{cases} \quad \text{and} \quad \begin{cases} e_{ij} = \delta_{u^i_{x_j}} \mathcal{H} \ \left(= \dfrac{\partial W}{\partial u^i_{x_j}} \right) \\ e^i = \delta_{\dot{u}^i} \mathcal{H} \ \left(= \dot{u}^i \right) \end{cases}$$

The boundary terms defined in (37) become

$$\begin{cases} f_{b,i} = \dot{u}^i \big|_{\partial \mathcal{Z}} \\ e_{b,i} = \left[\dfrac{\partial W}{\partial u^i_{x_1}} \bigg|_{\partial \mathcal{Z}} \ \cdots \ \dfrac{\partial W}{\partial u^i_{x_p}} \bigg|_{\partial \mathcal{Z}} \right]^{\mathrm{T}}, \end{cases} \qquad i = 1, \ldots, q$$

and the power balance relation (36)

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \int_{\partial \mathcal{Z}} \sum_{i=1}^{q} \underbrace{\dot{u}^i \big|_{\partial \mathcal{Z}}}_{v_i} \underbrace{\left[\frac{\partial W}{\partial u^i_{x_1}} \bigg|_{\partial \mathcal{Z}} \ \cdots \ \frac{\partial W}{\partial u^i_{x_p}} \bigg|_{\partial \mathcal{Z}} \right]}_{F_i} \cdot \mathrm{dA}$$

where v_i represent the i-th component of the speed of a point on $\partial \mathcal{Z}$ and F_i the i-th component of the force acting on $\partial \mathcal{Z}$ along the normal to the boundary surface. Note that forces and velocities are quantities in duality, whose dual product gives power: this is the physical interpretation of the boundary variables introduced in (37).

Note that the general elasticity equation can be also written in Hamiltonian form as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} u \\ \dot{u} \end{bmatrix} = \begin{bmatrix} 0_q & I_q \\ -I_q & 0_q \end{bmatrix} \begin{bmatrix} \delta_u \mathcal{H} \\ \delta_{\dot{u}} \mathcal{H} \end{bmatrix} \tag{38}$$

with Hamiltonian

$$\mathcal{H}(u, \dot{u}) = \int_{\mathcal{Z}} \left[\frac{1}{2} \|\dot{u}\|^2 + W(z, \nabla u) \right] \mathrm{dV}$$

depending on u and $\dot{u}$, as in classic mechanics, and where (see [14])

$$\delta_u \mathcal{H} = -\left[\sum_{i=1}^{p} \frac{\partial}{\partial x_i} \frac{\partial W}{\partial u^1_{x_i}} \ \cdots \ \sum_{i=1}^{p} \frac{\partial}{\partial x_i} \frac{\partial W}{\partial u^q_{x_i}} \right]^{\mathrm{T}} \tag{39}$$

The resulting Hamiltonian formulation is quite similar to the classical Hamiltonian description of a finite dimensional mechanical system. From (38), the differential operator defining the Stokes–Dirac structure of this formulation of the general elasticity equation is given by

$$J = -\begin{bmatrix} 0 & -I_q \\ I_q & 0 \end{bmatrix}$$

that is a skew-symmetric matrix and then a skew-adjoint differential operator of order 0. Consequently, the power conservation property of a Dirac structure is satisfied without taking into account the boundary terms, thus implying that in (38) the boundary conditions are not clearly specified by the (differential) operators involved in the definition of the Stokes–Dirac structure.

In conclusion, it is important to understand what are the differences between the proposed mdpH formulation of (31) and the *classical* one presented in (38). Formally, both the Hamiltonian descriptions are equivalent to (31), but relation (39) is valid only under the assumption of *zero* boundary conditions and, then, *zero* boundary conditions are implicitly assumed in (38). As discussed in Sect. 1, this is the strongest limitation in the classical Hamiltonian description of a distributed parameter systems. This limitation can be removed within the mdpH systems framework: basically, in this case, it is enough to choose the *right* state variables, that is to express the Hamiltonian of the system as a function of *physical* energy variables.

6 Control by interconnection in finite dimensions

Consider the finite dimensional port Hamiltonian system (5) that has to be asymptotically stabilized around the configuration $x^* \in \mathcal{X}$ by means of the following dynamical controller in port Hamiltonian form:

$$\begin{cases} \dot{x}_c &= [J_c(x_c) - R_c(x_c)] \dfrac{\partial H_c}{\partial x_c} + G_c(x_c) u_c \\ y_c &= G_c^{\mathrm{T}}(x_c) \dfrac{\partial H_c}{\partial x_c} \end{cases} \tag{40}$$

Denote by $\mathcal{X}_c$ the controller state space, with $\dim \mathcal{X}_c = n_c$, and by $H_c : \mathcal{X}_c \to \mathbb{R}$ the Hamiltonian function, bounded from below. Moreover, suppose that $J_c(x_c) = -J_c^{\mathrm{T}}(x_c)$ and $R_c(x_c) = R_c^{\mathrm{T}}(x_c)$ and that $\dim \mathcal{U}_c = \dim \mathcal{Y}_c = m$.

If systems (5) and (40) are interconnected in power conserving way, that is if

$$\begin{cases} u &= -y_c \\ y &= u_c \end{cases} \tag{41}$$

the resulting dynamics is given by the following autonomous port Hamiltonian systems, with state space $\mathcal{X} \times \mathcal{X}_c$ and Hamiltonian $H + H_c$:

$$\begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} J(x) - R(x) & -G(x) G_c^{\mathrm{T}}(x_c) \\ G_c(x_c) G^{\mathrm{T}}(x) & J_c(x_c) - R_c(x_c) \end{bmatrix} \begin{bmatrix} \partial_x H \\ \partial_{x_c} H_c \end{bmatrix} \tag{42}$$

Given a generic port Hamiltonian system, it is possible to give the following fundamental definition of *structural invariant* or, equivalently, of Casimir function, [2, 10, 22].

Definition 6.1 (Casimir function). Consider the port Hamiltonian system (5) with state space $\mathcal{X}$ and Hamiltonian function $H : \mathcal{X} \to \mathbb{R}$. A function $\mathcal{C} : \mathcal{X} \to \mathbb{R}$ is a Casimir function for (5) if and only if

$$\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}t} = 0$$

for every possible choice of Hamiltonian H.

From Def. 6.1, a scalar function $\mathcal{C} : \mathcal{X} \times \mathcal{X}_c \to \mathbb{R}$ is a Casimir function for (42) if and only if the following relations are satisfied:

$$\frac{\partial^{\mathrm{T}} \mathcal{C}}{\partial x} (J - R) + \frac{\partial^{\mathrm{T}} \mathcal{C}}{\partial x_c} G_c G^{\mathrm{T}} = 0 \tag{43a}$$

$$\frac{\partial^{\mathrm{T}} \mathcal{C}}{\partial x_c} (J_c - R_c) - \frac{\partial^{\mathrm{T}} \mathcal{C}}{\partial x} G G_c^{\mathrm{T}} = 0 \tag{43b}$$

These conditions are direct consequence of the interconnection law (41).

The existence of Casimir functions for the closed-loop system (42) plays an important role in the control by interconnection and energy shaping methodology. If $x^* \in \mathcal{X}$ is the desired equilibrium configuration for (5), asymptotic stability in x^* can be achieved by properly choosing the Hamiltonian function of (40) in order to shape the closed-loop energy $H + H_c$ so that a (possibly) global minimum in the desired equilibrium configuration can be introduced. It is important to note that there is no relation between the state of the controller and the state of the system to be controlled. Then, it is not clear how the controller energy, which is freely assignable, has to be chosen in order to solve the regulation problem.

A possible solution can be to constrain the state of the closed-loop system (42) on a certain subspace of $\mathcal{X} \times \mathcal{X}_c$, for example given by:

$$\Omega_c := \{(x, x_c) \in \mathcal{X} \times \mathcal{X}_c \,|\, x_c = S(x) + c\}$$

where $c \in \mathbb{R}^{n_c}$ and $S : \mathcal{X} \to \mathcal{X}_c$ is a function to be computed. In other words, we are looking for a set of Casimir functions $\mathcal{C}_i : \mathcal{X} \times \mathcal{X}_c \to \mathbb{R}$, $i = 1, \ldots, n_c$ for the closed-loop system (42) such that

$$\mathcal{C}_i(x, x_c) := S_i(x) - x_{c,i} \tag{44}$$

where $[S_1(x), \ldots, S_{n_c}(x)]^{\mathrm{T}} = S(x)$. Due to the nature of a Casimir function, it is possible to introduce an *intrinsic* non-linear state feedback law that will be used in order to choose the energy function of the controller so that the closed-loop Hamiltonian can be properly shaped. Note that, under these hypothesis, this energy function depends on the state variables of system (5). This control methodology is called *invariant function method*, [2, 10].

From (43), the set of functions (44) are Casimir functions for (42) if and only if

$$\begin{aligned} -\frac{\partial^{\mathrm{T}} S}{\partial x} G G_c^{\mathrm{T}} &= J_c - R_c \\ \frac{\partial^{\mathrm{T}} S}{\partial x}(J - R) &= G_c G^{\mathrm{T}} \end{aligned}$$

Then, the following proposition can be proved, [16, 22].

Proposition 6.1. *The functions $\mathcal{C}_i$, $i = 1, \ldots, n_c$, defined in (44) are Casimir functions for the system (42) if and only if the following conditions are satisfied:*

$$\frac{\partial^{\mathrm{T}} S}{\partial x} J(x) \frac{\partial S}{\partial x} = J_c(x_c) \tag{45a}$$

$$R(x) \frac{\partial S}{\partial x} = 0 \tag{45b}$$

$$R_c(x_c) = 0 \tag{45c}$$

$$\frac{\partial^{\mathrm{T}} S}{\partial x} J(x) = G_c(x_c) G^{\mathrm{T}}(x) \tag{45d}$$

Suppose that (45) are satisfied. Then, from (44), the state variables of the controller are *robustly* related to the state variable of the system to be stabilized since

$$x_{c,i} = S_i(x) + c_i, \quad i = 1, \ldots, n_c \tag{46}$$

with $c_i \in \mathbb{R}$ depending on the initial conditions. Moreover, the closed-loop dynamics (42) evolves on the foliation induced by the level sets

$$\mathcal{L}_{\mathcal{C}_i}^{c_i} = \{(x, x_c) \in \mathcal{X} \times \mathcal{X}_c \,|\, x_{c,i} = S_i(x) + c_i\} \tag{47}$$

with $i = 1, \ldots, n_c$, which can be expressed as a function of the x coordinate. If conditions (45b) and (45d) are taken into account, the reduced dynamics of (42) on these level sets is given by

$$\begin{aligned} \dot{x} &= [J(x) - R(x)] \frac{\partial H}{\partial x} - G(x) G_c^{\mathrm{T}}(x_c) \frac{\partial H_c}{\partial x_c} \\ &= [J(x) - R(x)] \left(\frac{\partial H}{\partial x} + \frac{\partial S}{\partial x} \frac{\partial H_c}{\partial x_c} \right) \end{aligned} \tag{48}$$

From (46), we have that $H_c(x_c) \equiv H_c(S(x) + c)$: the controller energy function is finally dependent from x_b through the non-linear feedback action $S(\cdot)$. If

$$H_d(x) := H(x) + H_c(S(x) + c) \tag{49}$$

then (48) can be written as

$$\dot{x} = (J - R) \left(\frac{\partial H}{\partial x} + \frac{\partial S}{\partial x} \frac{\partial H_c}{\partial x_c} \right) = (J - R) \frac{\partial H_d}{\partial x} \tag{50}$$

In conclusion, the following proposition has been proved, [16, 22].

Proposition 6.2. *Consider the closed-loop port Hamiltonian system (42) and suppose that the vector function $S(x) = [S_1(x), \ldots, S_{n_c}(x)]^{\mathrm{T}}$ satisfies conditions (45). Then, the reduced dynamics on the level sets (47) is given by (50), where the closed-loop energy function H_d is given by (49).*

By properly choosing the controller energy function H_c, it is possible to shape the closed-loop energy function H_d defined in (49) so that a new minimum in x^* is introduced. Then, the desired configuration can be reached with the dynamics given by (50).

7 Boundary control by interconnection of mdpH systems

7.1 Introduction

In this section, the control by interconnection and energy shaping, discussed in Sect. 6 for the finite dimensional case, is generalized to distributed parameter systems in port Hamiltonian form. In particular, it is shown how it is possible to shape the open loop energy function of a distributed parameter system by interconnecting a finite dimensional controller to its boundary. The structure of the controller has to be chosen so that a proper set of structural invariants (Casimir functions) are introduced in the closed loop system. In this way, the energy *variables* of the distributed parameter system can be robustly related to the state variables of the controller, thus introducing an implicit state feedback law. Then, the energy (Hamiltonian) function of the controller, which is freely assignable, can be chosen in order to introduce a new (possibly global) minimum at the desired configuration and, by damping injection, this new configuration can be reached.

7.2 Existence of Casimir functions

Consider the following multi-variable distributed port Hamiltonian system with spatial domain $\mathcal{Z} \subset \mathbb{R}^d$ (closed and compact):

$$\begin{cases} \dfrac{\partial x}{\partial t} & = & (J - R)\, \delta_x \mathcal{H} \\ w & = & B_{\mathcal{Z}}(\delta_x \mathcal{H}) \end{cases} \tag{51}$$

where $x \in \mathcal{X}$ is the configuration variable, $w \in \mathcal{W}$ are the boundary terms defined by the boundary operator $B_{\mathcal{Z}}$, $\mathcal{H} : \mathcal{X} \to \mathbb{R}$ is the Hamiltonian function, J is a skew adjoint differential operator and R is a self-adjoint differential operator taking into account the dissipative effects. Both $\mathcal{X}$ either $\mathcal{W}$ are spaces of vector value smooth functions of proper dimension. From Prop. 4.1, the following energy balance relation holds:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = -\int_{\mathcal{Z}} (\delta_x \mathcal{H})^{\mathrm{T}} R \delta_x \mathcal{H}\, \mathrm{dV} + \frac{1}{2}\int_{\partial\mathcal{Z}} B_{\{J,\,R\}}(w,w) \cdot \mathrm{dA} \leq \frac{1}{2}\int_{\partial\mathcal{Z}} B_{\{J,\,R\}}(w,w) \cdot \mathrm{dA} \tag{52}$$

Suppose that (51) has to be stabilized in the configuration $x^* \in \mathcal{X}$ by means of the finite dimensional controller (40) that has to be interconnected to the system (51) in power conserving way. Then, relation (41) has to be generalized in order to deal with a situation in which the *power port* of the system to be stabilized is not a finite dimensional vector space. A possible solution can be the following. Denote by $\Psi_u(z)$ and $\Psi_y(z)$ a couple of matrices depending eventually on $z \in \partial\mathcal{Z}$ and suppose that it is possible to write the boundary terms in (51) as follows:

$$w = \Psi_u u_c - \Psi_y y_c \tag{53}$$

The interconnection law expressed in (53) is power conserving if and only if

$$y_c^{\mathrm{T}} u_c + \frac{1}{2}\int_{\partial\mathcal{Z}} B_{\{J,\,R\}}(w,w) \cdot \mathrm{dA} = 0$$

where, from (53), we have that

$$\begin{aligned} &\int_{\partial\mathcal{Z}} B_{\{J,\,R\}}(w,w) \cdot \mathrm{dA} = \\ &\quad = \int_{\partial\mathcal{Z}} \left[B_{\{J,\,R\}}(\Psi_u u_c, \Psi_u u_c) + B_{\{J,\,R\}}(\Psi_y y_c, \Psi_y y_c) - 2B_{\{J,\,R\}}(\Psi_u u_c, \Psi_y y_c)\right] \cdot \mathrm{dA} \\ &\quad = \sum_{i,j=1}^{m} \left[\int_{\partial\mathcal{Z}} B_{\{J,\,R\}}(\Psi_{u,i}, \Psi_{u,j}) \cdot \mathrm{dA}\right] u_{c,i} u_{c,j} + \sum_{i,j=1}^{m} \left[\int_{\partial\mathcal{Z}} B_{\{J,\,R\}}(\Psi_{y,i}, \Psi_{y,j}) \cdot \mathrm{dA}\right] y_{c,i} y_{c,j} \\ &\quad\quad - 2\sum_{i,j=1}^{m} \left[\int_{\partial\mathcal{Z}} B_{\{J,\,R\}}(\Psi_{u,i}, \Psi_{y,j}) \cdot \mathrm{dA}\right] u_{c,i} y_{c,j} \end{aligned}$$

and, then, relation (53) can be satisfied if and only if

$$\int_{\partial\mathcal{Z}} B_{\{J,\,R\}}(\Psi_{u,i},\Psi_{u,j})\cdot \mathrm{dA} = \int_{\partial\mathcal{Z}} B_{\{J,\,R\}}(\Psi_{y,i},\Psi_{y,j})\cdot \mathrm{dA} \;=\; 0 \tag{54a}$$

$$\int_{\partial\mathcal{Z}} B_{\{J,\,R\}}(\Psi_{u,i},\Psi_{y,j})\cdot \mathrm{dA} \;=\; \delta_{ij} \tag{54b}$$

for every i, $j = 1,\dots,m$ and where δ is the Kronecker symbol. Note that, given $w \in \mathcal{W}$

$$u_{c,i} = \int_{\partial\mathcal{Z}} B_{\{J,\,R\}}(w,\Psi_{y,i})\cdot \mathrm{dA} \quad\Longleftrightarrow\quad u_c = \mathcal{B}^y_{\{J,\,R\}}(w) \tag{55a}$$

and

$$y_{c,i} = -\int_{\partial\mathcal{Z}} B_{\{J,\,R\}}(w,\Psi_{u,i})\cdot \mathrm{dA} \quad\Longleftrightarrow\quad y_c = -\mathcal{B}^u_{\{J,\,R\}}(w) \tag{55b}$$

where the *linear* operators $\mathcal{B}^y_{\{J,\,R\}} : \mathcal{W} \to \mathcal{U}_c$ and $\mathcal{B}^u_{\{J,\,R\}} : \mathcal{W} \to \mathcal{Y}_c$ are defined as:

$$\begin{aligned}
\mathcal{B}^u_{\{J,\,R\}}(w) &:= \left[\int_{\partial\mathcal{Z}} B_{\{J,\,R\}}(\Psi_{u,1},w)\cdot \mathrm{dA} \;\;\cdots\;\; \int_{\partial\mathcal{Z}} B_{\{J,\,R\}}(\Psi_{u,m},w)\cdot \mathrm{dA}\right]^{\mathrm{T}}\\
\mathcal{B}^y_{\{J,\,R\}}(w) &:= \left[\int_{\partial\mathcal{Z}} B_{\{J,\,R\}}(\Psi_{y,1},w)\cdot \mathrm{dA} \;\;\cdots\;\; \int_{\partial\mathcal{Z}} B_{\{J,\,R\}}(\Psi_{y,m},w)\cdot \mathrm{dA}\right]^{\mathrm{T}}
\end{aligned}$$

Consider a function $\mathcal{C} : \mathcal{X} \times \mathcal{X}_c \to \mathbb{R}$ defined over the state space of the closed loop system resulting from the power conserving interconnection (53) of (51) and (40). From Def. 6.1, we can say that $\mathcal{C}$ is a Casimir function if and only if:

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}t} &= \frac{\partial^{\mathrm{T}}\mathcal{C}}{\partial x_c}\dot{x}_c + \int_{\mathcal{Z}} (\delta_x\mathcal{C})^{\mathrm{T}}\dot{x}\,\mathrm{dV}\\
&= \frac{\partial^{\mathrm{T}}\mathcal{C}}{\partial x_c}(J_c - R_c)\frac{\partial H_c}{\partial x_c} + \frac{\partial^{\mathrm{T}}\mathcal{C}}{\partial x_c}G_c\mathcal{B}^y_{\{J,\,R\}}(w) + \int_{\mathcal{Z}} (\delta_x\mathcal{C})^{\mathrm{T}}(J-R)\,\delta_x\mathcal{H}\,\mathrm{dV}\\
&= 0
\end{aligned}$$

for every Hamiltonian functions $\mathcal{H}$ and H_c, where u_c is expressed as a function of the boundary terms as in (55). Since J and R are a skew adjoint and a self adjoint differential operator respectively, we have that:

$$(\delta_x\mathcal{C})^{\mathrm{T}}(J-R)\,\delta_x\mathcal{H} = -(\delta_x\mathcal{H})^{\mathrm{T}}(J+R)\,\delta_x\mathcal{C} + \operatorname{div} B_{\{J,\,R\}}(B_{\mathcal{Z}}(\delta_x\mathcal{C}),w)$$

and then

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}t} =& -\frac{\partial^{\mathrm{T}} H_c}{\partial x_c}(J_c + R_c)\frac{\partial\mathcal{C}}{\partial x_c} - \int_{\mathcal{Z}} (\delta_x\mathcal{H})^{\mathrm{T}}(J+R)\,\delta_x\mathcal{C}\,\mathrm{dV}\\
&+ \int_{\partial\mathcal{Z}} B_{\{J,\,R\}}\left(B_{\mathcal{Z}}(\delta_x\mathcal{C}) + \Psi_y G_c^{\mathrm{T}}\frac{\partial\mathcal{C}}{\partial x_c}, w\right)\cdot \mathrm{dA}
\end{aligned}$$

that has to be 0 for every Hamiltonian function of the closed loop system. This is true if

$$(J_c + R_c)\frac{\partial\mathcal{C}}{\partial x_c} \;=\; 0 \tag{56a}$$

$$(J+R)\,\delta_x\mathcal{C} \;=\; 0 \quad (\text{on } \mathcal{Z}) \tag{56b}$$

$$B_{\mathcal{Z}}(\delta_x\mathcal{C}) + \Psi_y G_c^{\mathrm{T}}\frac{\partial\mathcal{C}}{\partial x_c} \;=\; 0 \quad (\text{on } \partial\mathcal{Z}) \tag{56c}$$

where, from (55), (56c) can be written as:

$$\mathcal{B}^y_{\{J,\,R\}}(B_{\mathcal{Z}}(\delta_x\mathcal{C})) = 0 \tag{57a}$$

and

$$G_c^{\mathrm{T}}\frac{\partial\mathcal{C}}{\partial x_c} = -\mathcal{B}^u_{\{J,\,R\}}(B_{\mathcal{Z}}(\delta_x\mathcal{C})) \tag{57b}$$

In conclusion, the following proposition has been proved.

Proposition 7.1. *Consider the closed loop system resulting from the power conserving interconnection (53) of the infinite dimensional system (51) with the finite dimensional controller (40). Denote by $\mathcal{X}$ and $\mathcal{X}_c$ the state space of the distributed parameter system and of the controller respectively. Then, a real value function $\mathcal{C} : \mathcal{X} \times \mathcal{X}_c \to \mathbb{R}$ is a Casimir function for the closed loop system with respect to the interconnection law (53) if and only the set of conditions (56) are satisfied, where (56c) can be written in integral form as in (57).*

Note 7.1. The set of necessary and sufficient conditions (56) concerning the existence of structural invariants in the closed loop system are the generalization of the analogous conditions (43) in the finite dimensional case. In the hybrid case described in Prop. 7.1, the structural invariants have to satisfy the PDE (56a) in the controller variables and the PDE (56b) in the spatial variable of the distributed parameter system providing the variational derivative of the candidate Casimir function with respect to the configuration variable. The connection between these PDEs is given by (56c) that relates the boundary conditions required for the solution of (56a) and (56b) and that deeply depends on the (power conserving) interconnection law (53).

7.3 Energy shaping via structural invariants

As discussed in finite dimensions, the existence of a particular class of Casimir functions in the controlled system can be of great interest in the energy shaping procedure. In this situation, it is the energy function H_c of the controller that plays an important role since the closed loop Hamiltonian function is $\mathcal{H}_{cl} = \mathcal{H} + H_c$. On the other hand, there is no relation between the state of the controller and the configuration of the distributed parameter system and then it is not clear how the controller energy has to be chosen in order to introduce a minimum at the desired configuration $x^* \in \mathcal{X}$.

A possible solution can be to choose the structure of the controller (40) in order to introduce a set of $\bar{n} \leq n_c$ structural invariants in the closed loop system in the form

$$\mathcal{C}_i(x, x_c) = \mathcal{S}_i(x) - x_{c,i}, \text{ where } \mathcal{S}_i(x) = \int_{\mathcal{Z}} S_i(z, x) \, \mathrm{dV} \tag{58}$$

and $i = 1, \ldots, \bar{n}$. These functions are Casimir function for the closed loop system if and only if the set of conditions (56) are satisfied. In particular, denote by $\bar{J}_c$, $\bar{R}_c$ and $\bar{G}_c$ the sub-matrices of the interconnection, damping and input matrices of (40) corresponding to the first $\bar{n}$ state variables and define $\mathcal{S} : \mathcal{X} \times \mathcal{X}_c \to \mathbb{R}^{\bar{n}}$ as $\mathcal{S} = [\, \mathcal{S}_1 \ \cdots \ \mathcal{S}_{\bar{n}} \,]^{\mathrm{T}}$. Then, from (56a), (56b) and (57), we obtain the following set of conditions:

$$\bar{J}_c + \bar{R}_c = 0 \tag{59a}$$

$$(J + R)\, \delta_x \mathcal{S}_i = 0 \qquad (i = 1, \ldots, \bar{n}) \tag{59b}$$

$$\mathcal{B}^y_{\{J,\,R\}}(B_{\mathcal{Z}}(\delta_x \mathcal{S}_i)) = 0 \qquad (i = 1, \ldots, \bar{n}) \tag{59c}$$

$$\left[\mathcal{B}^y_{\{J,\,R\}}(B_{\mathcal{Z}}(\delta_x \mathcal{S}_1)) \ \cdots \ \mathcal{B}^u_{\{J,\,R\}}(B_{\mathcal{Z}}(\delta_x \mathcal{S}_{\bar{n}})) \right] = \bar{G}_c^{\mathrm{T}} \tag{59d}$$

Note that, from (59a), it is necessary that $\bar{J}_c = \bar{R}_c = 0$, while (59d) gives the expression of the input sub-matrix $\bar{G}_c$. Clearly, $\bar{G}_c$ depends on $\mathcal{S}$ that can be deduced from the solution of the PDE (59b) whose boundary conditions have to be chosen in such a way that (59c) is satisfied. If the set of conditions (59) can be satisfied, then the closed loop Hamiltonian function becomes

$$\begin{aligned} \mathcal{H}_{cl}(x, x_c) &= \mathcal{H}(x) + H_c(x_{c,1}, \ldots, x_{c,n_c}) \\ &= \mathcal{H}(x) + H_c(\mathcal{S}_1(x), \ldots, \mathcal{S}_{\bar{n}}(x), \ldots, x_{c,n_c}) \end{aligned} \tag{60}$$

thus depending explicitly on the configuration variable of the distributed parameter system.

If $\bar{n} = n_c$, then (58) are Casimir functions of the closed loop system if and only if the controller structure is chosen as follows:

$$J_c = R_c = 0 \tag{61a}$$

$$G_c^{\mathrm{T}} = \left[\mathcal{B}^y_{\{J,\,R\}}(B_{\mathcal{Z}}(\delta_x \mathcal{S}_1)) \ \cdots \ \mathcal{B}^y_{\{J,\,R\}}(B_{\mathcal{Z}}(\delta_x \mathcal{S}_{n_c})) \right] \tag{61b}$$

where the functionals $\mathcal{S}_i$, $i = 1, \ldots, n_c$, results from the solution of the PDEs (59b) with boundary conditions satisfying (59c). Note that, from (61a), the controller reduces to a set of nonlinear integrators, since, in this case, (40) becomes:

$$\begin{cases} \dot{x}_c &= G_c u_c \\ y_c &= G_c^{\mathrm{T}} \dfrac{\partial H_c}{\partial x_c} \end{cases}$$

with $u_c = \mathcal{B}^u_{\{J,\,R\}}(w)$ and H_c freely assignable. In this case, the closed loop Hamiltonian becomes

$$\mathcal{H}_{cl}(x, x_c) = \mathcal{H}(x) + H_c(\mathcal{S}_1(x), \ldots, \mathcal{S}_{n_c}(x))$$

and then it is only a function of the configuration variable of the distributed parameter system. By properly choosing the controller energy function, it is possible to introduce a minimum at the desired equilibrium configuration that can be reached is some dissipative effect is present in the system. In particular, if in (51) $R = 0$, that is no dissipative/diffusion phenomena are present in the infinite dimensional plant, it is convenient to chose the controller structure in order to have $\bar{n} < n_c$ Casimir function in the form (58) and then to introduce energy dissipation by acting on the remaining energy variables.

It is interesting to investigate what are the configuration in which the mdpH system can be stabilized by means of the proposed control technique. Suppose that a set of $\bar{n} \leq n_c$ Casimir functions in the form (58) have been introduced in the closed loop system, that is the interconnection, damping and input matrices of the controller (40) are chosen as follows:

$$J_c = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{J}_c \end{bmatrix} \qquad R_c = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{R}_c \end{bmatrix} \qquad G_c = \begin{bmatrix} \bar{G}_c \\ \tilde{G}_c \end{bmatrix} \tag{62}$$

where $\bar{G}_c$ is given in (59d) and $\tilde{J}_c = -\tilde{J}_c^{\mathrm{T}}$ and $\tilde{R}_c = \tilde{R}_c^{\mathrm{T}} \geq 0$ are freely assignable. Denote by $\bar{x}_c$ the first $\bar{n}$ controller state variables and by $\tilde{x}_c$ the remaining ones, that is $x_c = [\,\bar{x}_c,\, \tilde{x}_c\,]^{\mathrm{T}}$. Since $x^* \in \mathcal{X}$ is the desired equilibrium configuration for (51), from (58) define $\bar{x}_c^*$ as $\bar{x}_{c,i}^* := \mathcal{S}_i(x^*)$, $i = 1, \ldots, \bar{n}$. The existence of a set of Casimir functions in the form (58) allows to shape the open loop energy function $\mathcal{H}$ of (40) in such a way that x^* becomes a minimum of the closed loop Hamiltonian function (60). In particular, it is necessary that $(x^*, \tilde{x}_c^*)$ becomes a critical point of $\mathcal{H}_{cl}$, that is

$$\nabla \mathcal{H}_{cl} = \begin{bmatrix} \delta_x \mathcal{H} + \displaystyle\sum_{i=1}^{\bar{n}} \frac{\partial H_c}{\partial x_{c,i}} \delta_x \mathcal{S}_i \\ \dfrac{\partial H_c}{\partial \tilde{x}_c} \end{bmatrix} = 0 \tag{63a}$$

for $x = x^*$ and some $\tilde{x}_c = \tilde{x}_c^*$. Note that, if x^* satisfies the first condition in (63a), then it is an equilibrium configuration of $\dot{x} = (J - R)\,\delta_x \mathcal{H}$. In fact, from condition (59b), we have that:

$$0 = (J - R)\left(\delta_x \mathcal{H}(x^*) + \sum_{i=1}^{\bar{n}} \frac{\partial H_c}{\partial x_{c,i}} (\bar{x}_c^*, \tilde{x}_c^*)\, \delta_x \mathcal{S}_i(x^*) \right) = (J - R)\,\delta_x \mathcal{H}(x^*)$$

Furthermore, the same condition assures that input and output signals of the controller are compatible with the boundary conditions $w^* := B_{\mathcal{Z}}(\delta_x \mathcal{H}(x^*))$ at the desired equilibrium. In particular, from (59c) and (63a), it is possible to deduce that $u_c = 0$ when $x = x^*$. In order to have a minimum in the desired configuration, it is also necessary that:

$$\nabla^2 \mathcal{H}_{cl}(x^*, \tilde{x}_c^*) \geq 0 \tag{63b}$$

It can be deduced that the Hamiltonian function of the controller (40) has to be chosen in such a way that the couple of relations (63) is satisfied. Moreover, it is easy to prove that the following energy balance relation holds:

$$\frac{\mathrm{d}\mathcal{H}_{cl}}{\mathrm{d}t} = -\int_{\mathcal{Z}} (\delta_x \mathcal{H})^{\mathrm{T}} R\, \delta_x \mathcal{H}\, \mathrm{dV} - \frac{\partial^{\mathrm{T}} H_c}{\partial \tilde{x}_c} \tilde{R}_c \frac{\partial H_c}{\partial \tilde{x}_c} \leq 0 \tag{64}$$

If the hypothesis of the La Salle's theorem in infinite dimensions hold (see [6]), from (64) we deduce that the trajectories of the system approach the maximal invariant contained in the set defined by the following equations:

$$R\,\delta_x\mathcal{H} = 0 \tag{65a}$$

$$\tilde{R}_c\frac{\partial H_c}{\partial \tilde{x}_c} = 0 \tag{65b}$$

Consequently, $(x^*, \tilde{x}_c^*)$ is asymptotically stable if it is the only invariant solution compatible with (65). Note that a desired equilibrium configuration for the mdpH system has to necessarily satisfy the couple of relations (65), which is a system of integro–differential equations in the x and $\tilde{x}_c$ variables. In particular, relation (65b) defines the boundary conditions that the set of PDEs (65a) have to satisfy. It is important to note that the fundamental problem of the existence of solution for the closed loop system, which is an *hybrid* system, has not been approached in the previous discussion.

8 Example: stabilization of the heat equation

Consider the heat equation (26) that can be written in mdpH form as:

$$\begin{cases} \dfrac{\partial x}{\partial t} = \dfrac{\partial^2}{\partial z^2}\delta_x\mathcal{H} \\ w = \left.\begin{bmatrix} \delta_x\mathcal{H} \\ \frac{\partial}{\partial z}\delta_x\mathcal{H} \end{bmatrix}\right|_{\{0,\,1\}} \end{cases} \tag{66}$$

where $\mathcal{Z} = [\,0,\,1]$ is the spatial domain, $\mathcal{X} = L^2(\mathcal{Z})$ is the space of energy variables,

$$\mathcal{H}(x) = \frac{1}{2}\int_0^1 x^2\,\mathrm{dz}$$

is the Hamiltonian function and

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is the constant matrix representing the operator which gives the power through the boundary as in (52). Denote by $x^* \in \mathcal{X}$ a desired equilibrium configuration of (66) that the following one dimensional controller should render asymptotically stable.

$$\begin{cases} \dot{x}_c = G_c u_c \\ y_c = G_c^{\mathrm{T}}\dfrac{\partial H_c}{\partial x_c} \end{cases} \tag{67}$$

Suppose that $x_c \in \mathbb{R}$ and that $u_c, y_c \in \mathbb{R}^2$. From (28), (66) and (67) are interconnected in a power conserving way if

$$u_c = \begin{bmatrix} \frac{\partial}{\partial z}\delta_x\mathcal{H}(0) \\ \frac{\partial}{\partial z}\delta_x\mathcal{H}(1) \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial z}(0) \\ \frac{\partial x}{\partial z}(1) \end{bmatrix} \quad \text{and} \quad y_c = \begin{bmatrix} x(0) \\ -x(1) \end{bmatrix} \tag{68}$$

since, in this way,

$$\frac{1}{2}\int_{\partial\mathcal{Z}} w^{\mathrm{T}} B w \cdot \mathrm{dA} = x(1)\frac{\partial x}{\partial z}(1) - x(0)\frac{\partial x}{\partial z}(0) = -y_c^{\mathrm{T}} u_c$$

The first step in the control by interconnection and energy shaping is to choose the controller structure in order to have a set of structural invariants in the form (58). In this case, since the controller is a dynamical system of order 1, it is necessary to determine G_c such that the function

$$\mathcal{C}(x, x_c) = x_c - \mathcal{S}(x) = x_c - \int_0^1 S(z, x)\mathrm{dz} \tag{69}$$

is a Casimir function for the closed loop system, that is $\dot{\mathcal{C}} = 0$ for every $\mathcal{H}$ and H_c. We have that

$$\begin{aligned}\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}t} =& G_c \begin{bmatrix} \partial_z x(0) \\ \partial_z x(1) \end{bmatrix} - \int_0^1 (\delta_x \mathcal{S}) \frac{\partial^2 x}{\partial z^2}\,\mathrm{dz} \\ =& \left(G_c + [\, \delta_x \mathcal{S}(0) \quad -\delta_x \mathcal{S}(1) \,]\right) \begin{bmatrix} \partial_z x(0) \\ \partial_z x(1) \end{bmatrix} - \int_0^1 x \frac{\partial^2}{\partial z^2} (\delta_x \mathcal{S}) \mathrm{dz} \\ & - [\, \partial_z \delta_x \mathcal{S}(0) \quad \partial_z \delta_x \mathcal{S}(1) \,]\, G_c^{\mathrm{T}} \frac{\partial H_c}{\partial x_c}\end{aligned}$$

Consequently, (69) is a Casimir function for the controlled system if and only if

$$\frac{\partial^2}{\partial z^2} \delta_x \mathcal{S} = 0 \tag{70a}$$

and

$$\begin{cases} G_c + [\, \delta_x \mathcal{S}(0) \quad -\delta_x \mathcal{S}(1) \,] &= 0 \\ [\, \partial_z \delta_x \mathcal{S}(0) \quad \partial_z \delta_x \mathcal{S}(1) \,]\, G_c^{\mathrm{T}} &= 0 \end{cases} \tag{70b}$$

Condition (70a) provides the admissible functionals $\mathcal{S}$, while (70b) the input matrix G_c of the controller. From (70a), $\delta_x \mathcal{S} = az + b$, with $a, b \in \mathbb{R}$, while, in order to satisfy (70b), it is necessary that $a = 0$ and it is possible to choose $b = 1$. Consequently,

$$G_c = \begin{bmatrix} -1 & 1 \end{bmatrix} \tag{71}$$

and

$$\mathcal{C}(x, x_c) = x_c - \int_0^1 x(z)\,\mathrm{dz} \tag{72}$$

is a Casimir function for the closed loop system. From (71), the controller (67) becomes

$$\begin{cases} \dot{x}_c &= \dfrac{\partial x}{\partial z}(1) - \dfrac{\partial x}{\partial z}(0) \\ y_c &= \begin{bmatrix} -\partial_{x_c} H_c \\ \partial_{x_c} H_c \end{bmatrix} \end{cases}$$

and then, from (68),

$$x(0) = x(1) = -\frac{\partial H_c}{\partial x_c} \tag{73}$$

that is the controller acts on the system by imposing the same *temperature* on both the extremities of the infinite dimensional system. Moreover, the controller internal energy changes, that is $\dot{x}_c \neq 0$, only if there is a difference in the gradient of temperature at the extremities of the domain. As a consequence, the controller can stabilize the distributed parameter system only in the configurations for which the temperature is constant along the domain, that is $x^*(z) = x^*$ for every $z \in \mathcal{Z}$. Under the hypothesis that the initial configuration of the system is known, from (72) and from the properties of the Casimir functions, we have that

$$x_c = x_c(x) = \int_0^1 x(z)\,\mathrm{dz}$$

for the closed loop system. Define $x_c^* = x_c(x^*) = x^*$. The configuration x^* is asymptotically stable if the controller Hamiltonian is $H_c(x_c) = -x_c x_c^* - x_c x^*$. In fact, if $\mathcal{H}_{cl}(x, x_c) = \mathcal{H}(x) + H_c(x_c)$ is the energy function of the closed loop system, taking into account (73), we have that

$$\begin{aligned}\frac{\mathrm{d}\mathcal{H}_{cl}}{\mathrm{d}t} &= -\int_0^1 \left(\frac{\partial x}{\partial z}\right)^2 \mathrm{dz} + x(1)\frac{\partial x}{\partial z}(1) - x(0)\frac{\partial x}{\partial z}(0) - x_c^* \left(\frac{\partial x}{\partial z}(1) - \frac{\partial x}{\partial z}(0)\right) \\ &= -\int_0^1 \left(\frac{\partial x}{\partial z}\right)^2 \mathrm{dz} \leq 0\end{aligned}$$

Then, $\dot{\mathcal{H}}_{cl} = 0$ if $\partial_z x = 0$ on $\mathcal{Z}$, that is if $x(z)$ is constant on $\mathcal{Z}$. Since $x(0) = x(1) = x^*$, the only admissible configuration is x^* that results to be asymptotically stable. The asymptotic stability of x^* can be alternatively proved by looking at the expression of the closed loop Hamiltonian. We have that

$$\begin{aligned}\mathcal{H}_{cl}(x, x_c) &= \frac{1}{2}\int_0^1 x^2\,\mathrm{dz} - x_c x^* = \frac{1}{2}\int_0^1 x^2\,\mathrm{dz} - x^*\int_0^1 x\,\mathrm{dz}\\ &= \frac{1}{2}\int_0^1 (x - x^*)^2\,\mathrm{dz} + \mathrm{cons.}\end{aligned}$$

which has a global minimum in x^*.

9 Conclusions

In this paper, the classical finite dimensional port Hamiltonian formulation of a dynamical system is generalized in order to cope with the distributed parameter and multi-variable case and some new results concerning modeling and control of distributed parameter systems in port Hamiltonian form have been presented. In this way, the description of several physical phenomena, such as heat conduction, is now possible within this new port-based framework. The central result is the generalization of the notion of finite dimensional Dirac structure to the distributed parameter case in order to deal with an infinite dimensional space of power variables. Moreover, a novel technique for the boundary control of distributed parameter systems in port Hamiltonian form has been developed by extending the well known control by interconnection and energy shaping methodology. The basic result is the generalization of the conditions for obtaining a particular set of *Casimir function* to the *hybrid* case, that is the dynamical system to be considered results from the power conserving interconnection of an infinite dimensional system (the plant) and of a finite dimensional one (the controller). A simple application concerning the stabilization of the one-dimensional heat equation has been presented.

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References

[1] T. J. Courant. Dirac manifolds. *Trans. American Math. Soc. 319*, pages 631–661, 1990.

[2] M. Dalsmo and A. J. van der Schaft. On representation and integrability of mathematical structures in energy-conserving physical systems. *SIAM J. Control and Optimization*, (37):54–91, 1999.

[3] G. Golo, V. Talasila, and A. J. van der Schaft. A Hamiltonian formulation of the Timoshenko beam model. In *Proc. of Mechatronics 2002*. University of Twente, June 2002.

[4] A. Gomberoff and S. A. Hojman. Non-standard construction of Hamiltonian structures. *J. Phys.*, A30:5077–5084, 1997.

[5] R. S. Ingarden and A. Jamiolkowsky. *Classical Electrodynamics*. PWN-Polish Sc. Publ., Warszawa, Elsevier, 1985.

[6] Z. H. Luo, B. Z. Guo, and O. Morgul. *Stability and stabilization of infinite dimensional systems with applications*. Springer–Verlag, London, 1999.

[7] A. Macchelli and C. Melchiorri. Control by interconnection and energy shaping of the Timoshenko beam. Accepted for publication on the Journal of Mathematical and Computer Modelling of Dynamical Systems (MCMDS), 2003.

[8] A. Macchelli and C. Melchiorri. Control by interconnection of mixed port Hamiltonian systems. Submitted to the IEEE Transaction on Automatic Control, 2003.

[9] A. Macchelli and C. Melchiorri. Modeling and control of the Timoshenko beam. The distributed port Hamiltonian approach. Accepted for publication on the SIAM Journal on Control and Optimization, 2003.

[10] J. E. Marsden and T. S. Ratiu. *Introduction to mechanics and symmetry.* Springer, New York, 1994.

[11] B. M. Maschke and A. J. van der Schaft. Port controlled Hamiltonian systems: modeling origins and system theoretic properties. In *Proceedings of the third Conference on nonlinear control systems (NOLCOS)*, 1992.

[12] B. M. Maschke and A. J. van der Schaft. Interconnection of systems: the network paradigm. In *Proc. 35rd IEEE Conf. on Decision and Control*, pages 207–212, Dec. 11–13 1996.

[13] B. M. Maschke and A. J. van der Schaft. Fluid dynamical systems as Hamiltonian boundary control systems. In *Proc. of the 40th IEEE Conference on Decision and Control*, volume 5, pages 4497–4502, 2001.

[14] P. J. Olver. *Application of Lie groups to differential equations.* Springer–Verlag, 1993.

[15] R. Ortega, A. J. van der Schaft, I. Mareels, and B. M. Maschke. Putting energy back in control. *IEEE Control System Magazine*, pages 18–33, April 2001.

[16] R. Ortega, A. J. van der Schaft, B. M. Maschke, and G. Escobar. Energy–shaping of port–controlled Hamiltonian systems by interconnection. In *Proc. IEEE Conf. Dec. and Control*, Phoenix, AZ, December 1999.

[17] H. M. Paynter. *Analysis and design of engineering systems.* The M.I.T. Press, Cambridge, Massachusetts, 1961.

[18] M. Renardy and R. C. Rogers. *An Introduction to Partial Differential Equations.* Number 13 in Texts in Applied Mathematics. Springer Verlag, 2nd edition, 2004.

[19] H. Rodriguez, A. J. van der Schaft, and R. Ortega. On stabilization of nonlinear distributed parameter port-controlled Hamiltonian systems via energy shaping. In *Proc. of the 40th IEEE Conference on Decision and Control*, volume 1, pages 131–136, 2001.

[20] S. Stramigioli. Creating artificial damping by means of damping injection. In K. Danai, editor, *Proceedings of the ASME Dynamic Systems and Control Division*, volume DSC. 58, 1996.

[21] G. E. Swaters. *Introduction to Hamiltonian fluid dynamics and stability theory.* Chapman & Hall / CRC, 2000.

[22] A. J. van der Schaft. *L_2-Gain and Passivity Techniques in Nonlinear Control.* Communication and Control Engineering. Springer Verlag, 2000.

[23] A. J. van der Schaft and B. M. Maschke. Hamiltonian formulation of distributed-parameter systems with boundary energy flow. *Journal of Geometry and Physics*, 2002.