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## UNITARY COLLIGATIONS IN $\Pi_\kappa$ -SPACES, CHARACTERISTIC FUNCTIONS AND ŠTRAUS EXTENSIONS

A. DIJKSMA, H. LANGER AND H. S. V. DE SNOO

*Dedicated to Earl A. Coddington on the occasion of his 65th birthday.*

**The main result of this paper is the description of certain linear manifolds  $T(\lambda)$ , associated with a symmetric operator, in terms of certain boundary values of the characteristic function of a unitary colligation.**

**1. Introduction.** Let  $\mathfrak{H}$  be a Hilbert space and let  $\tilde{\mathfrak{H}}$  be a  $\pi_\kappa$ -space, i.e., a Pontryagin space with  $\kappa$  negative squares, such that  $\tilde{\mathfrak{H}}$  contains  $\mathfrak{H}$  and the indefinite inner product on  $\tilde{\mathfrak{H}}$  restricted to  $\mathfrak{H}$  coincides with the Hilbert inner product on  $\mathfrak{H}$ ; we denote this situation by  $\mathfrak{H} \subset_s \tilde{\mathfrak{H}}$ . Let  $A$  be a selfadjoint subspace in  $\tilde{\mathfrak{H}}^2$  with nonempty resolvent set  $\rho(A)$ . With  $A$  we associate as in [8] a family  $\{T(l) \mid l \in \mathbf{C} \cup \{\infty\}\}$  of linear manifolds  $T(l) \subset \mathfrak{H}^2$  defined by

$$(1.1) \quad \begin{cases} T(l) = \{ \{ Pf, Pg \} \mid \{ f, g \} \in A, g - lf \in \mathfrak{H} \}, & l \in \mathbf{C}, \\ T(\infty) = \{ \{ f, Pg \} \mid \{ f, g \} \in A, f \in \mathfrak{H} \}. \end{cases}$$

Here  $P$  denotes the orthogonal projection from  $\tilde{\mathfrak{H}}$  onto  $\mathfrak{H}$ . We note that  $A \cap \mathfrak{H}^2$  is a symmetric subspace in  $\mathfrak{H}^2$ , with adjoint  $(P^{(2)}A)^c$ , i.e., the closure in  $\mathfrak{H}^2$  of the set

$$P^{(2)}A = \{ \{ Pf, Pg \} \mid \{ f, g \} \in A \}.$$

The following inclusions are obvious:

$$A \cap \mathfrak{H}^2 \subset T(l) \subset P^{(2)}A, \quad l \in \mathbf{C} \cup \{\infty\},$$

and also

$$T(\bar{l}) \subset T(l)^*, \quad l \in \mathbf{C} \cup \{\infty\},$$

with equality when  $l \in \rho(A)$ .

Now let  $S$  be a symmetric subspace in  $\mathfrak{H}^2$ . We consider the selfadjoint extensions  $A \subset \tilde{\mathfrak{H}}^2$  of  $S$ , with nonempty resolvent set  $\rho(A)$ , where  $\mathfrak{H} \subset_s \tilde{\mathfrak{H}}$ . The corresponding families  $\{T(l) \mid l \in \mathbf{C} \cup \{\infty\}\}$ , form the class of Štraus extensions of  $S$  and  $T(l)$  for  $l \in \mathbf{C} \setminus \mathbf{R}$  was characterized in [8], to which

we refer for notations and definitions. An important tool in this note is the characteristic function of a unitary colligation of the form  $(\mathfrak{H}, \mathfrak{F}, \mathfrak{G}; U)$ , where the inner space  $\mathfrak{H}$  is a  $\pi_\kappa$ -space, the outer spaces are fixed and given by  $\mathfrak{F} = \nu(S^* - \bar{\mu})$  and  $\mathfrak{G} = \nu(S^* - \mu)$ , the corresponding defect spaces of  $S$ , and where  $U$  is the restriction of  $C_\mu(A)$ , the Cayley transform of  $A$ , where  $\mu \in \rho(A) \setminus \mathbf{R}$ . The main result of this paper is the description of  $T(\lambda)$  for real  $\lambda$ , i.e.,  $\lambda \in \mathbf{R} \cup \{\infty\}$ , in terms of certain boundary values of this characteristic function. Štraus [19], [21], [22] investigated the case where  $\kappa = 0$  and the manifolds involved are single-valued, i.e., (graphs of) linear operators. His method we could not easily extend to the case where  $\kappa > 0$  and the manifolds are multivalued. However, by generalizing the theory of unitary colligations from the case where the inner space is a Hilbert space (see Brodskii [4]) to the case where it is a  $\pi_\kappa$ -space, we obtain a method which is simpler than the one used by Štraus (for instance, the two cases  $\lambda \in \mathbf{R}$  and  $\lambda = \infty$  need not be treated separately) and, at the same time, works just as well in the more general situation.

We outline the contents of this paper. In §2 we consider unitary colligations and their characteristic functions and state results to be used in the rest of this paper. The proofs will appear in [9]. Such characteristic functions associated with  $\pi_\kappa$ -spaces were also considered by Krein and Langer [12], [13], [14]. The new ingredient in our treatment is the systematic use of unitary colligations. Closely related results are announced by Arov and Grossman [2], Azizov [3] and Filimonov [10]. In §3 we give the above mentioned characterization of  $T(\lambda)$ ,  $\lambda \in \mathbf{R} \cup \{\infty\}$ . We give a sufficient condition for  $T(\lambda)$ ,  $\lambda \in \mathbf{R} \cup \{\infty\}$  to be selfadjoint. This includes a result of Stenger [17]. Also we will characterize the symmetric linear manifold  $A \cap \mathfrak{S}^2$  as the intersection of a finite number of manifolds  $T(I)$ , thereby sharpening and generalizing a result of Brown [5]. In [20] Štraus presented a characterization of the subspaces  $\{\{f, g\} \in S^* \mid g = \lambda f\}$ ,  $\lambda \in \mathbf{R}$ , for the case of a densely defined symmetric operator. In order to apply his previous theorems from [19], [21], he had to introduce special Štraus extensions in an auxiliary Hilbert space. In §4 we show that such results when  $S$  is a symmetric subspace follow directly by making use of the theory of unitary colligations.

We dedicate this paper to Earl A. Coddington, whose work in the theory of subspaces and its applications to ordinary differential equations has been very stimulating for us. The first and last author wish to express their gratitude to Prof. Coddington for many inspiring ideas and many years of cooperation.

**2. Unitary colligations.** In this section we will collect some statements about unitary colligations in  $\pi_\kappa$ -spaces, which will be proved in [9]. Let  $\mathfrak{F}$  and  $\mathfrak{G}$  be arbitrary Hilbert spaces and let  $\hat{\mathfrak{H}}$  be a  $\pi_\kappa$ -space. We shall use  $[ \cdot, \cdot ]$  as the notation for the scalar or inner product for these and other spaces; it should be clear from the context to which space it refers. By  $S_\kappa(\mathfrak{F}, \mathfrak{G})$  we denote the class of all functions  $\Theta$  with the following two properties:

(a)  $\Theta$  is defined and meromorphic on  $\mathbf{D} = \{z \in \mathbf{C} \mid |z| < 1\}$ , with  $0 \in \mathfrak{D}_\Theta$ , the domain of holomorphy of  $\Theta$  in  $\mathbf{D}$ , and has values in  $[\mathfrak{F}, \mathfrak{G}]$ .

(b) The kernel

$$S_\Theta(z, \zeta) = \frac{I - \Theta^*(\zeta)\Theta(z)}{1 - z\bar{\zeta}}, \quad z, \zeta \in \mathfrak{D}_\Theta,$$

has  $\kappa$  negative squares, i.e., for arbitrary choices of  $n \in \mathbf{N}$ ,  $z_i \in \mathfrak{D}_\Theta$  and  $f_i \in \mathfrak{F}$ ,  $i = 1, \dots, n$ , the  $n \times n$  hermitian matrix

$$\left( [S_\Theta(z_i, z_j)f_i, f_j] \right)_{i, j=1, \dots, n}$$

has at most  $\kappa$  and for at least one such choice exactly  $\kappa$  negative eigenvalues.

Let  $\Delta$  be a unitary colligation, i.e., a quadruple of the form  $\Delta = (\hat{\mathfrak{H}}, \mathfrak{F}, \mathfrak{G}; U)$ , where

$$(2.1) \quad U = \begin{pmatrix} T & F \\ G & H \end{pmatrix} : \begin{pmatrix} \hat{\mathfrak{H}} \\ \mathfrak{F} \end{pmatrix} \rightarrow \begin{pmatrix} \hat{\mathfrak{H}} \\ \mathfrak{G} \end{pmatrix}$$

is unitary, i.e., isometric and surjective. Here  $\begin{pmatrix} \hat{\mathfrak{H}} \\ \mathfrak{G} \end{pmatrix}$  ( $\begin{pmatrix} \hat{\mathfrak{H}} \\ \mathfrak{F} \end{pmatrix}$ ) is the orthogonal direct sum of  $\hat{\mathfrak{H}}$  and  $\mathfrak{F}$  ( $\mathfrak{G}$ , respectively),  $T \in [\hat{\mathfrak{H}}, \hat{\mathfrak{H}}]$ ,  $F \in [\mathfrak{F}, \hat{\mathfrak{H}}]$ ,  $G \in [\hat{\mathfrak{H}}, \mathfrak{G}]$  and  $H \in [\mathfrak{F}, \mathfrak{G}]$ . According to M. G. Krein the characteristic function  $\Theta = \Theta_\Delta$  of  $\Delta$  is defined by

$$(2.2) \quad \Theta(z) = H + zG(I - zT)^{-1}F, \quad z^{-1} \in \rho(T),$$

see Brodskii [4]. As  $U$  is unitary, it is easy to see that  $T$  is a contraction and hence the spectrum  $\sigma(T)$  of  $T$  consists of points from  $\mathbf{D}^c$ , the closure of  $\mathbf{D}$  in  $\mathbf{C}$ , and of at most  $\kappa$  points from  $\mathbf{C} \setminus \mathbf{D}^c$ , which are normal eigenvalues of  $T$ , cf. [11]. Thus  $\Theta(z)$  is defined for  $z \in \mathbf{D}$ , with the exception of at most  $\kappa$  points.

The unitary colligation  $\Delta$  is called closely connected if the linear span of all elements of the form  $T^m Ff$  or  $(T^*)^n G^*g$  with  $m, n \in \mathbf{N} \cup \{0\}$ ,  $f \in \mathfrak{F}$  and  $g \in \mathfrak{G}$ , is dense in  $\hat{\mathfrak{H}}$ , or, equivalently, if there exists no nontrivial subspace  $\mathfrak{R} \subset \hat{\mathfrak{H}}$  with  $T(\mathfrak{R}) = \mathfrak{R}$  and  $T|_{\mathfrak{R}}$  is isometric. In particular, if  $\Delta$  is closely connected,  $T$  has no eigenvalues on the unit circle  $\partial\mathbf{D}$ . Finally, two

unitary colligations  $\Delta = (\hat{\mathfrak{H}}, \mathfrak{F}, \mathfrak{G}; U)$  and  $\Delta' = (\hat{\mathfrak{H}}', \mathfrak{F}, \mathfrak{G}; U')$  are called unitarily equivalent if there exists a unitary operator  $Z \in [\hat{\mathfrak{H}}, \hat{\mathfrak{H}}']$  such that

$$U' = \begin{pmatrix} Z & 0 \\ 0 & I_{\mathfrak{G}} \end{pmatrix} \begin{pmatrix} T & F \\ G & H \end{pmatrix} \begin{pmatrix} Z^{-1} & 0 \\ 0 & I_{\mathfrak{F}} \end{pmatrix}.$$

**THEOREM 2.1.** (i) *Let  $\Delta = (\hat{\mathfrak{H}}, \mathfrak{F}, \mathfrak{G}; U)$  be a unitary colligation where  $\hat{\mathfrak{H}}$  is a  $\pi_\kappa$ -space. Then  $\Theta_\Delta \in S_{\kappa'}(\mathfrak{F}, \mathfrak{G})$  for some  $\kappa'$  with  $0 \leq \kappa' \leq \kappa$ . If  $\Delta$  is closely connected then  $\kappa' = \kappa$ .*

(ii) *If  $\Theta \in S_\kappa(\mathfrak{F}, \mathfrak{G})$  then there exists a unitary colligation  $\Delta = (\hat{\mathfrak{H}}, \mathfrak{F}, \mathfrak{G}; U)$ , where  $\hat{\mathfrak{H}}$  is a  $\pi_\kappa$ -space, such that  $\Theta = \Theta_\Delta$ . The colligation  $\Delta$  can be chosen such that it is closely connected also, in which case it is uniquely determined up to unitary equivalence, and  $\mathfrak{D}_\Theta = \{z \in \mathbb{C} \mid z^{-1} \notin \sigma_p(T)\}$ .*

It follows from the construction, that if the colligation  $\Delta$  is closely connected, the dimension of the inner space  $\hat{\mathfrak{H}}$  is equal to the sum of the numbers of negative and positive squares of the kernel  $S_{\Theta_\Delta}$ .

The next theorem deals with the behaviour of  $\Theta \in S_\kappa(\mathfrak{F}, \mathfrak{G})$  on the boundary  $\partial\mathbf{D}$  of  $\mathbf{D}$ . By  $\lim_{z \rightarrow \zeta}$  we denote the nontangential limit as  $z \in \mathbf{D}$  tends to  $\zeta \in \partial\mathbf{D}$ .

**THEOREM 2.2.** *Let  $\Theta \in S_\kappa(\mathfrak{F}, \mathfrak{G})$ ,  $\zeta \in \partial\mathbf{D}$ ,  $\psi \in \mathfrak{F}$ . Then the following statements are equivalent:*

(i) *there exists an element  $\varphi \in \mathfrak{G}$ , such that  $\|\varphi\| = \|\psi\|$  and*

$$\lim_{z \rightarrow \zeta} \frac{[\psi, \psi] - [\Theta(z)\psi, \varphi]}{1 - \bar{\zeta}z} \text{ exists,}$$

(ii)

$$\lim_{z \rightarrow \zeta} \frac{\|\psi\| - \|\Theta(z)\psi\|}{1 - |z|} \text{ exists,}$$

(iii) *there exists a sequence  $(z_n)$  in  $\mathfrak{D}_\Theta$  with  $z_n \rightarrow \zeta$  such that*

$$\sup_n \left| \frac{\|\psi\| - \|\Theta(z_n)\psi\|}{1 - |z_n|} \right| < \infty.$$

*If one of these statements is valid, then  $\varphi$  in (i) is uniquely determined and  $\varphi = \lim_{z \rightarrow \zeta} \Theta(z)\psi$ .*

*If  $\Theta = \Theta_\Delta$  where  $\Delta$  is a closely connected unitary colligation, then equivalent to (i), (ii) or (iii) is*

(iv)

$$F\psi \in \Re(I - \zeta T),$$

and then (iv) implies

$$\varphi = (H + \zeta G(I - \zeta T)^{-1} F)\psi.$$

Let  $\Theta \in S_\kappa(\mathfrak{F}, \mathfrak{G})$  and let  $\Theta = \Theta_\Delta$  where  $\Delta$  is as in Theorem 2.2. As  $\zeta \in \partial\mathbf{D}$  is not an eigenvalue of  $T$  we may extend the definition of  $\Theta$  to  $\zeta$  by putting

$$\mathfrak{D}_0(\Theta(\zeta)) = \{\psi \in \mathfrak{F} \mid F\psi \in \mathfrak{R}(I - \zeta T)\},$$

$$\Theta(\zeta) = H + \zeta G(I - \zeta T)^{-1} F \quad \text{on } \mathfrak{D}_0(\Theta(\zeta)).$$

In general,  $\mathfrak{D}_0(\Theta(\zeta))$  need not be closed, but, of course, if we also have that  $\zeta^{-1} \in \rho(T)$ , then  $\mathfrak{D}_0(\Theta(\zeta)) = \mathfrak{F}$  and the definition of  $\Theta(\zeta)$  coincides with the one in (2.2). Theorem 2.2 shows that this extension of  $\Theta$  to  $\mathbf{D}^c$  can be described without making use of the fact that it is a characteristic function of a colligation: for  $\zeta \in \partial\mathbf{D}$  we have

$$\mathfrak{D}_0(\Theta(\zeta)) = \left\{ \psi \in \mathfrak{F} \mid \lim_{z \rightarrow \zeta} \frac{\|\psi\| - \|\Theta(z)\psi\|}{1 - |z|} \text{ exists} \right\}$$

and

$$\Theta(\zeta)\psi = \lim_{z \rightarrow \zeta} \Theta(z)\psi, \quad \text{strongly,} \quad \psi \in \mathfrak{D}_0(\Theta(\zeta)).$$

We note that for  $\zeta \in \partial\mathbf{D}$ ,  $\Theta(\zeta)$  is an isometry on  $\mathfrak{D}_0(\Theta(\zeta))$ . We remark that  $\mathfrak{D}_0(\Theta(\zeta))$  is in general contained in the set

$$\left\{ \psi \in \mathfrak{F} \mid \lim_{z \rightarrow \zeta} \Theta(z)\psi \text{ exists strongly} \right\},$$

cf. [19]. Also we note that if  $\zeta \in \partial\mathbf{D}$  and  $1/\zeta \in \rho(T)$ , then  $\Theta(\zeta)$  is a unitary mapping from  $\mathfrak{F}$  onto  $\mathfrak{G}$ . In particular, this happens when  $\dim \hat{\mathfrak{S}} < \infty$ , for then  $\partial\mathbf{D} \subset \rho(T)$ .

**THEOREM 2.3 (maximum principle).** *Let  $\Theta \in S_\kappa(\mathfrak{F}, \mathfrak{G})$ ,  $\psi \in \mathfrak{F}$ ,  $\varphi \in \mathfrak{G}$  and assume that the relation  $\varphi = \Theta(z)\psi$  holds for more than  $\kappa$  points  $z \in \mathfrak{D}_\Theta$ . Then we have the inequality  $\|\varphi\| \leq \|\psi\|$ . If we have  $\|\varphi\| = \|\psi\|$ , then  $\varphi = \Theta(z)\psi$  for all  $z \in \mathfrak{D}_\Theta$ .*

If  $\Delta = (\hat{\mathfrak{S}}, \mathfrak{F}, \mathfrak{G}; U)$  is a unitary colligation, where  $U$  is of the form (2.1), then it is clear that  $H$  maps the nullspace  $\nu(F)$  isometrically onto the nullspace  $\nu(G^*)$  with inverse  $H^*$ . In terms of the characteristic function  $\Theta = \Theta_\Delta$ , we have that

$$\Theta(z) \mid_{\nu(F)} = H \mid_{\nu(F)}, \quad z \in \mathfrak{D}_\Theta,$$

and hence with  $\psi \in \nu(F)$  and  $\varphi = H\psi$ , that

$$(2.3) \quad \varphi = \Theta(z)\psi, \quad \|\varphi\| = \|\psi\|, \quad z \in \mathfrak{D}_\Theta.$$

Conversely, if we have (2.3) and  $\Delta$  is closely connected if  $\kappa > 0$ , then  $\psi \in \nu(F)$ ,  $\varphi \in \nu(G^*)$  and  $\varphi = H\psi$ .

**3. Štraus extensions.** Let  $\mathfrak{H}$  be a Hilbert space and let  $S \subset \mathfrak{H}^2$  be a symmetric subspace. It is well-known that  $S^*$  can be written as

$$S^* = S \dot{+} M_l \dot{+} M_l, \quad \text{direct sum in } \mathfrak{H}^2,$$

where  $l \in \mathbf{C} \setminus \mathbf{R}$  and  $M_l = \{ \{f, g\} \in S^* \mid g = lf \}$ , is the defect subspace of  $S$  at  $l \in \mathbf{C}$ . We fix  $\mu \in \mathbf{C} \setminus \mathbf{R}$  and consider a selfadjoint extension  $A \subset \mathfrak{H}^2$  of  $S$  with  $\mu \in \rho(A)$  where  $\mathfrak{H}$  is a Pontryagin space with  $\kappa$  negative squares such that  $\mathfrak{H} \supset_s \mathfrak{H}$ . The condition  $\mu \in \rho(A)$  is a restriction only if  $\kappa > 0$ . For, if  $\kappa = 0$   $\mathfrak{H}$  is a Hilbert space and then  $\mathbf{C} \setminus \mathbf{R} \subset \rho(A)$ . But if  $\kappa > 0$  then either  $\rho(A) = \emptyset$  or  $\mathbf{C} \setminus \mathbf{R} \subset \rho(A)$  with the exception of at most  $2\kappa$  points, which are normal eigenvalues of  $A$ , and one of these could coincide with  $\mu$ . We denote the Cayley transform and its inverse at  $l \in \mathbf{C} \setminus \mathbf{R}$  by  $C_l$  and  $F_l$  respectively. Then

$$C_\mu(A) = C_\mu(S) \dot{+} U, \quad C_{\bar{\mu}}(A) = C_{\bar{\mu}}(S) \dot{+} U^*, \quad \text{direct sums in } \mathfrak{H}^2,$$

and

$$(3.1) \quad A = S \dot{+} F_\mu(U) = S \dot{+} F_{\bar{\mu}}(U^*), \quad \text{direct sums in } \mathfrak{H}^2,$$

where  $U$  is (the graph of) a unitary operator with a matrix representation of the form (2.1) in which  $\mathfrak{G} = \mathfrak{H} \ominus \mathfrak{H}$  is a Pontryagin space with  $\kappa$  negative squares,

$$\mathfrak{F} = \mathfrak{D}(M_{\bar{\mu}}) = \nu(S^* - \bar{\mu}) \quad \text{and} \quad \mathfrak{G} = \mathfrak{D}(M_\mu) = \nu(S^* - \mu).$$

Writing the equalities in (3.1) in full detail, we obtain:

$$(3.2) \quad \left\{ \begin{aligned} A &= S \dot{+} \left\{ \left( \begin{array}{l} \hat{\psi} - (T\hat{\psi} + F\psi) \\ \psi - (G\hat{\psi} + H\psi) \end{array} \right), \left( \begin{array}{l} \bar{\mu}\hat{\psi} - \mu(T\hat{\psi} + F\psi) \\ \bar{\mu}\psi - \mu(G\hat{\psi} + H\psi) \end{array} \right) \right\} \Big| \psi \in \mathfrak{F}, \hat{\psi} \in \mathfrak{H} \Big\} \\ &= S \dot{+} \left\{ \left( \begin{array}{l} \hat{\varphi} - (T^*\hat{\varphi} + G^*\varphi) \\ \varphi - (F^*\hat{\varphi} + H^*\varphi) \end{array} \right), \left( \begin{array}{l} \mu\hat{\varphi} - \bar{\mu}(T^*\hat{\varphi} + G^*\varphi) \\ \mu\varphi - \bar{\mu}(F^*\hat{\varphi} + H^*\varphi) \end{array} \right) \right\} \Big| \varphi \in G, \hat{\varphi} \in \mathfrak{H} \Big\}, \\ &\quad \text{direct sums in } \mathfrak{H}^2. \end{aligned} \right.$$

A direct consequence of these formulas is a description of  $A \cap \mathfrak{H}^2$ :

$$\begin{aligned} A \cap \mathfrak{H}^2 &= S \dot{+} \{ \{ \psi - H\psi, \bar{\mu}\psi - \mu H\psi \} \mid \psi \in \nu(F) \} \\ &= S \dot{+} \{ \{ \varphi - H^*\varphi, \mu\varphi - \bar{\mu}H^*\varphi \} \mid \varphi \in \nu(G^*) \}, \quad \text{direct sums in } \mathfrak{H}^2. \end{aligned}$$

Using definition (1.1) we obtain from (3.2)

$$(3.3) \quad \left\{ \begin{aligned} T(l) &= S \dot{+} \{ \{ \psi - (G\hat{\psi} + H\psi), \bar{\mu}\psi - \mu(G\hat{\psi} + H\psi) \} | \\ &\quad \psi \in \mathfrak{F}, \hat{\psi} \in \mathfrak{H}, (l - \bar{\mu})\hat{\psi} = (l - \mu)(T\hat{\psi} + F\psi) \} \\ &= S \dot{+} \{ \{ \varphi - (F^*\hat{\varphi} + H^*\varphi), \mu\varphi - \bar{\mu}(F^*\hat{\varphi} + H^*\varphi) \} | \\ &\quad \varphi \in \mathfrak{G}, \hat{\varphi} \in \mathfrak{H}, (l - \mu)\hat{\varphi} = (l - \bar{\mu})(T^*\hat{\varphi} + G^*\varphi) \}, \quad l \in \mathbf{C}, \\ &\quad \text{direct sums in } \mathfrak{H}^2, \\ T(\infty) &= S \dot{+} \{ \{ \psi - (G\hat{\psi} + H\psi), \bar{\mu}\psi - \mu(G\hat{\psi} + H\psi) \} | \\ &\quad \psi \in \mathfrak{F}, \hat{\psi} \in \mathfrak{H}, \hat{\psi} = T\hat{\psi} + F\psi \} \\ &= S \dot{+} \{ \{ \varphi - (F^*\hat{\varphi} + H^*\varphi), \mu\varphi - \bar{\mu}(F^*\hat{\varphi} + H^*\varphi) \} | \\ &\quad \varphi \in G, \hat{\varphi} \in \mathfrak{H}, \hat{\varphi} = F^*\hat{\varphi} + G^*\varphi \}, \quad \text{direct sums in } \mathfrak{H}^2. \end{aligned} \right.$$

Let  $C_\mu = \{l \in \mathbf{C} \mid \text{Im } l \text{Im } \mu > 0\}$ , let  $z: C_\mu \rightarrow \mathbf{D}$  be the fractional linear transformation  $z(l) = (l - \mu)/(l - \bar{\mu})$  and put  $z(\infty) = 1$ . Then (3.3) implies that the Štraus extension of  $S$  associated with  $A$  via formula (1.1) can be written in the following way: for all  $l \in C_\mu$  with  $z(l) \in \mathfrak{D}_\Theta$

$$(3.4) \quad T(l) = S \dot{+} \{ \{ \psi - \Theta(z)\psi, \bar{\mu}\psi - \mu\Theta(z)\psi \} \mid \psi \in \nu(S^* - \bar{\mu}) \},$$

$$z = z(l),$$

and

$$(3.5) \quad T(\bar{l}) = S \dot{+} \{ \{ \varphi - \Theta(z)^*\varphi, \mu\varphi - \bar{\mu}\Theta(z)^*\varphi \} \mid \varphi \in \nu(S^* - \mu) \},$$

$$z = z(l),$$

direct sums in  $\mathfrak{H}^2$ , where  $\Theta = \Theta_\Delta$  is the characteristic function of the unitary colligation  $\Delta = (\mathfrak{H}, \mathfrak{F}, \mathfrak{G}; U)$  and belongs to class  $S_{\kappa'}(\mathfrak{F}, \mathfrak{G})$  for some  $\kappa'$  with  $0 \leq \kappa' \leq \kappa$ . We note that  $\Delta$  is closely connected if and only if  $A$  is closely connected, i.e., the linear span of all elements of the form  $(A - l)^{-1}h$ ,  $h \in \mathfrak{H}$ ,  $l \in \rho(A)$ , together with the elements of  $\mathfrak{H}$  is dense in  $\mathfrak{H}$ . It follows from Theorem 2.1 that if  $A$  is closely connected, then  $\kappa' = \kappa$ . Theorem 2.1 and (3.3) imply the following result

**THEOREM 3.1.** *Let  $S$  be a symmetric subspace in  $\mathfrak{H}^2$  and  $\mu \in \mathbf{C} \setminus \mathbf{R}$ .*

(i) *Let  $\{T(l) \mid l \in \mathbf{C} \cup \{\infty\}\}$  be a Štraus extension of  $S$  associated with a selfadjoint extension  $A$  of  $S$  in  $\mathfrak{H}^2$  with  $\mu \in \rho(A)$ , where  $\mathfrak{H} \supset_s \mathfrak{H}$  is a  $\pi_\kappa$ -space. Then there exist uniquely a  $\kappa'$  with  $0 \leq \kappa' \leq \kappa$  and a function  $\Theta \in S_{\kappa'}(\nu(S^* - \bar{\mu}), \nu(S^* - \mu))$  such that, for all  $l \in C_\mu$  with  $z(l) \in \mathfrak{D}_\Theta$ ,  $T(l)$  is given by (3.4). Furthermore, for these values of  $l$  (3.5) is valid. If  $A$  is closely connected then  $\kappa' = \kappa$ .*

(ii) *If for some  $\Theta \in S_{\kappa'}(\nu(S^* - \bar{\mu}), \nu(S^* - \mu))$  and all  $l \in C_\mu$  with  $z(l) \in \mathfrak{D}_\Theta$ ,  $T(l) \subset \mathfrak{H}^2$  is given by (3.4), then  $T(l)$  can be extended to all  $l \in \mathbf{C} \cup \{\infty\}$  such that  $\{T(l) \mid l \in \mathbf{C} \cup \{\infty\}\}$  is a Štraus extension of  $S$*



associated with a selfadjoint extension  $A$  of  $S$  in  $\tilde{\mathfrak{H}}^2$  with  $\mu \in \rho(A)$ , where  $\tilde{\mathfrak{H}} \supset_s \mathfrak{H}$  is a  $\pi_\kappa$ -space and then  $T(l)$ ,  $z(l) \in \mathfrak{D}_\Theta$ , is given by (3.5).  $A$  and  $\tilde{\mathfrak{H}}$  can be chosen such that  $A$  is closely connected, in which case they are uniquely determined up to isomorphisms which, when restricted to  $\mathfrak{H}$ , are equal to the identity on  $\mathfrak{H}$ .

This theorem is actually another formulation of Theorem 5.1 of [8] and shows that the description of Štraus extensions given there is one in terms of characteristic functions of unitary colligations. In [18] Štraus identified the mapping  $\Theta$  in (3.4), restricted to the operator case with  $\kappa = 0$ , with a characteristic function (in his sense) of some operator in  $\tilde{\mathfrak{H}}$ . It can be shown that this notion is equivalent to that of a characteristic function of a colligation associated with this operator.

For  $\lambda \in \mathbf{R} \cup \{\infty\}$  we reformulate (3.3) as follows

$$(3.6) \quad T(\lambda) = S \dot{+} \left\{ \left\{ \psi - (H + \zeta G(I - \zeta T)^{-1} F)\psi, \right. \right. \\ \left. \left. \bar{\mu}\psi - \mu(H + \zeta G(I - \zeta T)^{-1} F)\psi \right\} \mid \right. \\ \left. \psi \in \mathfrak{F}, F\psi \in \mathfrak{R}(I - \zeta T) \right\}, \quad \text{direct sum in } \mathfrak{H}^2,$$

where  $\zeta = (\lambda - \mu)/(\lambda - \bar{\mu})$  if  $\lambda \in \mathbf{R}$  and  $\zeta = 1$  if  $\lambda = \infty$ . Theorem 2.2 now implies that  $T(\lambda)$  for these values of  $\lambda$  can be characterized as a boundary value of  $T(l)$ ,  $l \in \mathbf{C}_\mu$ .

**THEOREM 3.2.** *Let  $S$  be a symmetric subspace in  $\mathfrak{H}^2$ ,  $\mu \in \mathbf{C} \setminus \mathbf{R}$  and let  $T(l) \subset \mathfrak{H}^2$  be given by (3.4),  $l \in \mathbf{C}_\mu$ ,  $z(l) \in \mathfrak{D}_\Theta$ , where  $\Theta \in S_\kappa(\nu(S^* - \bar{\mu}), \nu(S - \mu))$ . Then with  $\zeta = (\lambda - \mu)/(\lambda - \bar{\mu})$  if  $\lambda \in \mathbf{R}$  and  $\zeta = 1$  if  $\lambda = \infty$  we have*

$$T(\lambda) = S \dot{+} \left\{ \left\{ \psi - \Theta(\zeta)\psi, \bar{\mu}\psi - \mu\Theta(\zeta)\psi \right\} \mid \psi \in \mathfrak{D}_0(\Theta(\zeta)) \right\},$$

where

$$\mathfrak{D}_0(\Theta(\zeta)) = \left\{ \psi \in \nu(S^* - \bar{\mu}) \mid \lim_{z \rightarrow \zeta} \frac{\|\psi\| - \|\Theta(z)\psi\|}{1 - |z|} \text{ exists} \right\},$$

and

$$\Theta(\zeta)\psi = \lim_{z \rightarrow \zeta} \Theta(z)\psi, \quad \psi \in \mathfrak{D}_0(\Theta(\zeta)).$$

Theorem 3.1 restricted to the operator case with  $\kappa = 0$  coincides with the main results of Štraus in [19], [21] and [22].

As  $T(l)^* = T(\bar{l})$ ,  $l \in \rho(A)$ ,  $T(l)$  is a subspace for  $l \in \rho(A)$ . However, for those  $\lambda \in \mathbf{R} \cup \{\infty\}$  which do not belong to  $\rho(A)$   $T(\lambda)$  need not be closed in general. Sufficient conditions for  $T(\lambda)$  to be closed for all  $\lambda \in \mathbf{R} \cup \{\infty\}$  are that  $\dim \nu(S^* - \mu) < \infty$  and  $\dim \nu(S^* - \bar{\mu}) < \infty$ . Another sufficient condition is that  $\dim \hat{\mathfrak{S}} < \infty$ , as follows from the following theorem.

**THEOREM 3.3.** *Let  $A$  be a selfadjoint subspace in  $\tilde{\mathfrak{S}}^2$  with  $\rho(A) \neq \emptyset$ , where  $\tilde{\mathfrak{S}}$  is a  $\pi_\kappa$ -space, and let  $\mathfrak{S} \subset_s \tilde{\mathfrak{S}}$  be a Hilbert space such that  $\dim \tilde{\mathfrak{S}} \ominus \mathfrak{S} < \infty$ . Let  $P$  be the orthogonal projection from  $\tilde{\mathfrak{S}}$  onto  $\mathfrak{S}$ . Then*

$$T(\lambda) = \{ \{ Pf, Pg \} \mid \{ f, g \} \in A, g - \lambda f \in \mathfrak{S} \}, \quad \lambda \in \mathbf{R},$$

and

$$T(\infty) = \{ \{ f, Pg \} \mid \{ f, g \} \in A, f \in \mathfrak{S} \} = PA|_{\mathfrak{S}},$$

are selfadjoint in  $\mathfrak{S}^2$ .

*Proof.* Let  $\mu \in \rho(A) \cap (\mathbf{C} \setminus \mathbf{R})$  and let  $\Delta = (\hat{\mathfrak{S}}, \mathfrak{S}, \mathfrak{S}; U)$  where  $\hat{\mathfrak{S}} = \tilde{\mathfrak{S}} \ominus \mathfrak{S}$  and  $U = C_\mu(A)$ . Then  $\Delta$  is a unitary colligation and without loss of generality we may assume that it is closely connected. Write

$$U = \begin{pmatrix} T & F \\ G & H \end{pmatrix} : \begin{pmatrix} \hat{\mathfrak{S}} \\ \mathfrak{S} \end{pmatrix} \rightarrow \begin{pmatrix} \hat{\mathfrak{S}} \\ \mathfrak{S} \end{pmatrix}$$

and let  $\Theta = \Theta_\Delta$ . Then, as  $\dim \hat{\mathfrak{S}} < \infty$ , for all  $\zeta \in \partial \mathbf{D}$  we have that  $1/\zeta \in \rho(T)$  and therefore

$$\Theta(\zeta) = H + \zeta G(I - \zeta T)^{-1} F$$

is unitary for all  $\zeta \in \partial \mathbf{D}$ . Formula (3.6) with  $S = \{ \{0, 0\} \} \subset \mathfrak{S}^2$  and  $\mathfrak{F} = \mathfrak{S}$  now implies that  $T(\lambda)$  is selfadjoint for all  $\lambda \in \mathbf{R} \cup \{\infty\}$ .

The statement in Theorem 3.3 about  $T(\infty)$  coincides with a result of Stenger [17] in case  $\kappa = 0$  and  $A$  is a selfadjoint operator. The theorem is still valid when  $\mathfrak{S}$  is a  $\pi_{\kappa'}$ -subspace of  $\tilde{\mathfrak{S}}$ ,  $0 \leq \kappa' \leq \kappa$ .

**THEOREM 3.4.** *Let  $\mathfrak{S}$  be a Hilbert space and  $\tilde{\mathfrak{S}}$  be a  $\pi_\kappa$ -space with  $\mathfrak{S} \subset_s \tilde{\mathfrak{S}}$ . Let  $A$  be a selfadjoint subspace in  $\tilde{\mathfrak{S}}^2$  with  $\rho(A) \neq \emptyset$ , which is closely connected if  $\kappa > 0$ . We assume  $\mu \in \rho(A) \setminus \mathbf{R}$ , so that the corresponding Štraus family is represented for all  $l \in C_\mu$  with  $z(l) \in \mathfrak{D}_\Theta$  by*

$$T(l) = \{ \{ \psi - \Theta(z(l))\psi, \bar{\mu}\psi - \mu\Theta(z(l))\psi \} \mid \psi \in \mathfrak{S} \}$$

for some  $\Theta \in S_\kappa(\mathfrak{S}, \mathfrak{S})$ . If  $S_1 \subset \mathfrak{S}^2$  is a symmetric subspace in  $\mathfrak{S}^2$ , such that

$$S_1 \subset T(l),$$

for more than  $\kappa$  points  $l \in \mathbf{C}_\mu$  with  $z(l) \in \mathfrak{D}_\Theta$ , then we have

$$S_1 \subset A \cap \mathfrak{S}^2.$$

*Proof.* We assume  $S_1 \subset T(l)$ , for  $l = l_1, \dots, l_{\kappa+1}$  in  $\mathbf{C}_\mu$  with  $z(l_i) \in \mathfrak{D}_\Theta$ ,  $i = 1, \dots, \kappa + 1$ . Let  $\{\alpha, \beta\} \in S_1$ , then

$$\{\beta - \mu\alpha, \beta - \bar{\mu}\alpha\} = \{\psi, \Theta(z(l_i))\psi\}$$

for some  $\psi \in \mathfrak{S}$ . The symmetry of  $S_1$  implies  $\|\beta - \mu\alpha\| = \|\beta - \bar{\mu}\alpha\|$ , so that we have

$$\varphi = \Theta(z(l_i))\psi, \quad \|\varphi\| = \|\psi\|,$$

for more than  $\kappa$  points  $l_i$  such that  $z(l_i) \in \mathfrak{D}_\Theta$ . Using Theorem 2.3 and the formula following (3.2) with  $S = \{\{0, 0\}\}$ , we obtain the desired result.

This result, stated in a slightly different way, can be found in [8]. The present proof is similar to the one given by McKelvey [16] who showed this result for the case of a Hilbert space  $\tilde{\mathfrak{S}}$  and operators  $S$  and  $A$ . A direct consequence of Theorem 3.4 is

$$A \cap \mathfrak{S}^2 = \left( \bigcap_{i=1}^{\kappa+1} T(l_i) \right) \cap T(\bar{l}_j), \quad 1 \leq j \leq \kappa + 1,$$

where  $l_i$ ,  $i = 1, \dots, \kappa + 1$ , are distinct points in  $\mathbf{C}_\mu$  with  $z(l_i) \in \mathfrak{D}_\Theta$ . This can be seen by checking that the set on the right-hand side is symmetric. A more general result is contained in the following corollary.

**COROLLARY.** *Let  $\mathfrak{S}$  be a Hilbert space and  $\tilde{\mathfrak{S}}$  be a  $\pi_\kappa$ -space with  $\mathfrak{S} \subset_s \tilde{\mathfrak{S}}$ . Let  $A$  be a selfadjoint subspace in  $\tilde{\mathfrak{S}}^2$  with  $\rho(A) \neq \emptyset$ , which is closely connected if  $\kappa > 0$ . We assume  $\mu \in \rho(A) \setminus \mathbf{R}$ , so that the corresponding Štraus family is represented by  $\Theta \in \mathcal{S}_\kappa(\mathfrak{S}, \mathfrak{S})$ . Let  $l_1, \dots, l_{\kappa+1} \in \mathbf{C}_\mu$  be  $\kappa + 1$  distinct points such that  $z(l_i) \in \mathfrak{D}_\Theta$ ,  $i = 1, \dots, \kappa + 1$ , and let  $m_1, \dots, m_{\kappa+1} \in \mathbf{C}_\mu$  be  $\kappa + 1$  distinct points such that  $z(m_i) \in \mathfrak{D}_\Theta$ ,  $i = 1, \dots, \kappa + 1$ . Then*

$$A \cap \mathfrak{S}^2 = \bigcap_{i=1}^{\kappa+1} (T(l_i) \cap T(\bar{m}_i)).$$

*Proof.* It is sufficient to show that the manifold on the right-hand side is symmetric, because then we may apply Theorem 3.4. So we assume

$$\{\alpha, \beta\} \in \bigcap_{i=1}^{\kappa+1} (T(l_i) \cap T(\bar{m}_i)),$$

which implies by (3.4) and (3.5) that

$$\{\beta - \mu\alpha, \beta - \bar{\mu}\alpha\} = \{\psi, \Theta(z(l_i))\psi\} = \{\Theta(z(m_i))^* \varphi, \varphi\},$$

for  $\psi, \varphi \in \mathfrak{H}$ ,  $i = 1, \dots, \kappa + 1$ . This representation yields  $\|\psi\| = \|\varphi\|$ , see Theorem 2.3, or, equivalently,  $\text{Im}(\alpha, \beta) = 0$ , and the proof is complete.

This corollary contains a result of Brown [5]. He considered a densely defined symmetric operator  $S$  in  $\mathfrak{H}$ , and assumed that  $A$  is a selfadjoint operator extension of  $S$  in the Hilbert space  $\tilde{\mathfrak{H}}$ . In that case he proved

$$A \cap \mathfrak{H}^2 = \bigcap_{l \in \mathbb{C} \setminus \mathbb{R}} T(l),$$

or, strictly speaking, the equivalent result

$$\{f \in \mathfrak{D}(A) \cap \mathfrak{H} \mid Af \in \mathfrak{H}\} = \bigcap_{l \in \mathbb{C} \setminus \mathbb{R}} \mathfrak{D}(T(l)).$$

**4. A special extension of a symmetric subspace.** Let  $\mathfrak{H}$  be a Hilbert space and let  $S \subset \mathfrak{H}^2$  be a symmetric subspace. We define the linear manifolds  $S(l)$ ,  $l \in \mathbb{C} \cup \{\infty\}$ , by

$$(4.1) \quad \begin{cases} S(l) = S \dot{+} M_l, & l \in \mathbb{C}, \\ S(\infty) = S \dot{+} \{0, g\} \in S^*. \end{cases}$$

In this section we will study the boundary behaviour of  $S(l)$  as  $l$  tends to  $\lambda \in \mathbb{R} \cup \{\infty\}$ , analogous to the results in Theorem 3.2. First we note some obvious consequences of the definition. We have  $S \subset S(l) \subset S^*$ ,  $l \in \mathbb{C} \cup \{\infty\}$ ;  $S(l)$  is maximal dissipative for  $l \in \mathbb{C}^+$ ,  $-S(l)$  is maximal dissipative for  $l \in \mathbb{C}^-$ ,  $S(l)^* = S(\bar{l})$  for  $l \in \mathbb{C} \setminus \mathbb{R}$  and  $S(\lambda)$  is a (not necessarily closed) symmetric linear manifold for  $\lambda \in \mathbb{R} \cup \{\infty\}$ . Note that for  $\lambda \in \mathbb{R}$   $S(\lambda)$  is selfadjoint if and only if

$$\Re(S - \lambda) = \Re(S^* - \lambda) \cap (\Re(S - \lambda))^c,$$

while (see [7])  $S(\infty)$  is selfadjoint if and only if

$$\mathfrak{D}(S) = \mathfrak{D}(S^*) \cap (\mathfrak{D}(S))^c.$$

The manifold  $S(\infty)$  plays a role in determining whether an extension of  $S$  is an operator or not, cf. [6] and [22].

**THEOREM 4.1.** *Let  $S$  be a symmetric subspace in  $\mathfrak{H}^2$  and let  $\mu \in \mathbb{C} \setminus \mathbb{R}$ ,  $l \in \mathbb{C}_\mu$ . Then we have*

$$C_\mu(S(l))|_{\nu(S^* - \bar{\mu})} = \Theta\left(\frac{l - \bar{\mu}}{l - \mu}\right),$$

where  $\Theta \in S_0(\nu(S^* - \bar{\mu}), \nu(S^* - \mu))$  is the characteristic function of the unitary operator colligation

$$\begin{pmatrix} C_{\bar{\mu}}(S(\mu)) & I|_{\nu(S^* - \bar{\mu})} \\ P_{\mu} & 0 \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \nu(S^* - \bar{\mu}) \end{pmatrix} \rightarrow \begin{pmatrix} \mathfrak{H} \\ \nu(S^* - \mu) \end{pmatrix}.$$

where  $P_{\mu}$  denotes the orthogonal projection from  $\mathfrak{H}$  onto  $\nu(S^* - \mu)$ .

*Proof.* For a fixed  $\mu \in \mathbb{C} \setminus \mathbb{R}$  we have for  $l \in \mathbb{C}_{\bar{\mu}}$

$$\mathfrak{H} = \mathfrak{R}(S - l) \dot{+} \nu(S^* - \mu), \quad \text{direct sum.}$$

This decomposition defines a projection of  $\mathfrak{H}$  onto  $\nu(S^* - \mu)$ , parallel to  $\mathfrak{R}(S - l)$ , which we denote by  $P_{l,\mu}$ . Completely analogous to Štraus [20] we obtain

$$P_{l,\mu} = \frac{l - \mu}{\bar{\mu} - \mu} P_{\mu} (I + (l - \bar{\mu})(S(\mu) - l)^{-1}),$$

where  $P_{\mu}$  denotes the orthogonal projection from  $\mathfrak{H}$  onto  $\nu(S^* - \mu)$ . Using the identity

$$\frac{l - \mu}{l - \bar{\mu}} (I + (l - \bar{\mu})(S(\mu) - l)^{-1}) = \left( I - \frac{l - \bar{\mu}}{l - \mu} C_{\bar{\mu}}(S(\mu)) \right)^{-1},$$

$l \in \mathbb{C}_{\bar{\mu}}, l \neq \bar{\mu},$

we obtain

$$P_{l,\mu} = P_{\mu} \left( I - \frac{l - \bar{\mu}}{l - \mu} C_{\bar{\mu}}(S(\mu)) \right)^{-1}, \quad l \in \mathbb{C}_{\bar{\mu}}.$$

Note that the identity

$$C_{\bar{\mu}}(S(\mu)) = C_{\bar{\mu}}(S)|_{\mathfrak{R}(S - \bar{\mu})} \oplus 0|_{\nu(S^* - \mu)},$$

shows that  $C_{\bar{\mu}}(S(\mu))$  is a partial isometry on  $\mathfrak{H}$ . We have for all  $\psi \in \nu(S^* - \bar{\mu})$  that  $C_{\mu}(S(l))\psi \in \nu(S^* - \mu)$  and

$$(l - \bar{\mu})\psi - (l - \mu)C_{\mu}(S(l))\psi \in \mathfrak{R}(S - l).$$

Using the notion of parallel projection we obtain the desired result.

Applying Schwarz' lemma to this characteristic function  $\Theta$ , note  $\Theta(0) = 0$ , we obtain the following corollary.

**COROLLARY.** For  $\mu \in \mathbb{C} \setminus \mathbb{R}$  and  $l \in \mathbb{C}_{\bar{\mu}}$  we have

$$\|C_{\mu}(S(l))|_{\nu(S^* - \bar{\mu})}\| \leq \left| \frac{l - \bar{\mu}}{l - \mu} \right|.$$

If for some  $\psi \in \nu(S^* - \bar{\mu})$  and some  $l \in \mathbf{C}_{\bar{\mu}}$

$$\|C_\mu(S(l))\psi\| = \left| \frac{l - \bar{\mu}}{l - \mu} \right| \|\psi\|,$$

then  $\psi \in \nu(S^*) \cap S^*(0)$  and, consequently,

$$C_\mu(S(l))\psi = \frac{l - \bar{\mu}}{l - \mu} \psi, \quad l \in \mathbf{C}_{\bar{\mu}}.$$

Let  $\lambda \in \mathbf{R} \cup \{\infty\}$ , then it is not difficult to show that  $C_\mu(S(\lambda))|_{\nu(S^* - \bar{\mu})}$  is the boundary value of  $C_\mu(S(l))|_{\nu(S^* - \bar{\mu})}$  as  $l \rightarrow \lambda$ ,  $l \in \mathbf{C}_{\bar{\mu}}$ . In order to apply Theorem 2.2 we remark that the symmetric subspace  $S \subset \mathfrak{S}^2$  can be written as an orthogonal sum  $S_1 \oplus H$ , where  $S_1$  is a simple closed symmetric operator in  $\mathfrak{S}_1^2$ ,  $H$  is a selfadjoint subspace in  $\mathfrak{S}_2^2$  and  $\mathfrak{S}_j$ ,  $j = 1, 2$ , are subspaces of  $\mathfrak{S}$  with  $\mathfrak{S} = \mathfrak{S}_1 \oplus \mathfrak{S}_2$ , cf. [15]. This shows that for  $l \in \mathbf{C} \cup \{\infty\}$

$$S(l) = S_1(l) \oplus H(l) = S_1(l) \oplus H,$$

where

$$S_1(l) = S_1 \dot{+} \{ \{f, g\} \in S_1^* \mid g = lf \}, \quad l \in \mathbf{C},$$

$$S_1(\infty) = S_1 \dot{+} \{ \{0, g\} \in S_1^* \}.$$

In terms of the colligation the above splitting implies

$$\mathbf{C}_{\bar{\mu}}(S(\mu)) = \mathbf{C}_{\bar{\mu}}(S_1(\mu)) \oplus \mathbf{C}_{\bar{\mu}}(H),$$

i.e., a splitting in a unitary operator and a partial isometry, which does not have a non-trivial unitary part. Hence without loss of generality we assume  $S$  to be simple, which is equivalent to the corresponding unitary colligation being closely connected.

**THEOREM 4.2.** *Let  $S$  be a simple symmetric closed operator in  $\mathfrak{S}^2$  and let  $S(\lambda)$ ,  $\lambda \in \mathbf{R} \cup \{\infty\}$ , be given by (4.1). Let  $\mu \in \mathbf{C} \setminus \mathbf{R}$  and let  $\Theta \in S_0(\nu(S^* - \bar{\mu}), \nu(S^* - \mu))$  be as given in Theorem 4.1. Then with  $\zeta = (\lambda - \bar{\mu})/(\lambda - \mu)$  if  $\lambda \in \mathbf{R}$  and  $\zeta = 1$  if  $\lambda = \infty$  we have*

$$S(\lambda) = S \dot{+} \{ \{ \psi - \Theta(\zeta)\psi, \bar{\mu}\psi - \mu\Theta(\zeta)\psi \} \mid \psi \in \mathfrak{D}_0(\Theta(\zeta)) \},$$

where

$$\mathfrak{D}_0(\Theta(\zeta)) = \left\{ \psi \in \nu(S^* - \bar{\mu}) \mid \lim_{z \rightarrow \zeta} \frac{\|\psi\| - \|\Theta(z)\psi\|}{1 - |z|} \text{ exists} \right\},$$

and

$$\Theta(\zeta)\psi = \lim_{z \rightarrow \zeta} \Theta(z)\psi, \quad \psi \in \mathfrak{D}_0(\Theta(\zeta)).$$

Štraus [19], [21] obtained this result by extending the operator to a selfadjoint operator in a larger Hilbert space and then used his previous results about Štraus extensions [18]. We prove this result by directly relying on Theorem 2.2. In [20] Štraus gives necessary and sufficient conditions for the operator  $S$  in Theorem 4.2 to be densely defined. A more general version is given in the following theorem.

**THEOREM 4.3.** *Let  $\mathfrak{F}$  and  $\mathfrak{G}$  be Hilbert spaces and let  $\Theta \in S_0(\mathfrak{F}, \mathfrak{G})$  with  $\Theta(0) = 0$ . Then there exists a Hilbert space  $\mathfrak{H}$ , a simple symmetric closed operator  $S \subset \mathfrak{H}^2$ ,  $\mu \in \mathbf{C} \setminus \mathbf{R}$ , an isometry  $F$  from  $\mathfrak{F}$  onto  $\nu(S^* - \bar{\mu})$  and an isometry  $G$  from  $\nu(S^* - \bar{\mu})$  onto  $\mathfrak{G}$ , such that*

$$\Theta\left(\frac{l - \bar{\mu}}{l - \mu}\right) = G\left[C_\mu(S(l))|_{\nu(S^* - \bar{\mu})}\right]F, \quad l \in \mathbf{C}_\mu.$$

The operator  $S$  is densely defined if and only if for all  $\psi \in \mathfrak{F}$

$$\lim_{z \rightarrow 1} \frac{\|\psi\| - \|\Theta(z)\psi\|}{1 - |z|} = \infty,$$

or, equivalently, if for all  $\psi \in \mathfrak{F}$  and for all  $\varphi \in \mathfrak{G}$  with  $\|\psi\| = \|\varphi\|$

$$\lim_{z \rightarrow 1} \left| \frac{[\psi, \psi] - [\Theta(z)\psi, \varphi]}{1 - z} \right| = \infty.$$

*Proof.* According to Theorem 2.1 there exists a closely connected unitary colligation  $\Delta = (\hat{\mathfrak{H}}, \mathfrak{F}, \mathfrak{G}; U)$ , where  $\hat{\mathfrak{H}}$  is a Hilbert space and  $U$  has the form

$$\begin{pmatrix} T & F \\ G & 0 \end{pmatrix}: \begin{pmatrix} \hat{\mathfrak{H}} \\ \mathfrak{F} \end{pmatrix} \rightarrow \begin{pmatrix} \hat{\mathfrak{H}} \\ \mathfrak{G} \end{pmatrix},$$

where  $T$  is a completely non-unitary partial isometry. If we denote the isometric part of  $T$  by  $V$ , we have

$$\nu(T) = G^*\mathfrak{G} = \mathfrak{D}(V)^\perp, \quad \nu(T^*) = F\mathfrak{F} = \mathfrak{R}(V)^\perp,$$

and we have

$$\Theta(z) = G\left[zP_{\mathfrak{D}(V)^\perp}(I - zT)^{-1}I|_{\mathfrak{R}(V)^\perp}\right]F, \quad z \in \mathbf{D},$$

where  $F$  and  $G$  are isometries from  $\mathfrak{F}$  onto  $\mathfrak{R}(V)^\perp$ , and from  $\mathfrak{D}(V)^\perp$  onto  $\mathfrak{G}$  respectively. For  $\mu \in \mathbf{C} \setminus \mathbf{R}$  we define  $S = F_\mu(V)$ , so that  $S$  is a simple symmetric closed operator in  $\mathfrak{H}$ . Note that

$$\mathfrak{D}(V) = \mathfrak{R}(S - \bar{\mu}), \quad \mathfrak{R}(V) = \mathfrak{R}(S - \mu),$$

so that

$$\mathfrak{D}(V)^\perp = \nu(S^* - \mu), \quad \mathfrak{R}(V)^\perp = \nu(S^* - \bar{\mu}).$$

Hence we obtain

$$\begin{pmatrix} T & I|_{\mathfrak{R}(V)^\perp} \\ P_{\mathfrak{D}(V)^\perp} & 0 \end{pmatrix} = \begin{pmatrix} C_{\bar{\mu}}(S(\mu)) & I|_{\nu(S^* - \bar{\mu})} \\ P_\mu & 0 \end{pmatrix}$$

and hence

$$\Theta\left(\frac{l - \bar{\mu}}{l - \mu}\right) = G[C_\mu(S(l))|_{\nu(S^* - \bar{\mu})}]F, \quad l \in \mathbf{C}_{\bar{\mu}}.$$

Next we observe that  $S$  is densely defined if and only if  $S^*(0) = \{0\}$ . If  $S^*(0) = \{0\}$ , then  $S(\infty) = S$ , which by Theorem 4.2 implies that  $\mathfrak{D}_0(\Theta(1)) = \{0\}$ . Conversely, if  $\mathfrak{D}_0(\Theta(1)) = \{0\}$ , then we have by Theorem 4.2 that  $S(\infty) = S$ , which implies  $S^*(0) = \{0\}$ . The final statement in our theorem follows from Theorem 2.2.

If we have  $\mathfrak{G} = \mathfrak{F}$ , then we may write  $G = F^*W$ , where  $W$  is a unitary extension of  $V$  in  $\mathfrak{F}$ , see [12]. Our result resembles a similar statement in [15]. The special case  $\mathfrak{F} = \mathfrak{G} = \mathbf{C}$  goes back to Livšic, and can be found in [1].

#### REFERENCES

- [1] N. I. Achieser and I. M. Glasmann, *Theorie der linearen Operatoren im Hilbert Raum*, 8th edition, Akademie Verlag, Berlin, 1981.
- [2] D. Z. Arov and L. Z. Grossman, *Scattering matrices in the theory of dilations of isometric operators*, Dokl. Akad. Nauk SSSR, **270** (1983), 17–20 (English transl.: Sov. Math. Dokl., **27** (1983), 518–522).
- [3] T. Ya. Azizov, *Extensions of  $J$ -isometric and  $J$ -symmetric operators*, Funkcional. Anal.: Priložen, **18** (1984), 57–58 (English transl.: Functional Anal. Appl., **18** (1984), 46–48).
- [4] M. S. Brodskii, *Unitary operator colligations and their characteristic functions*, Uspekhi Mat. Nauk, **33**: 4 (1978), 141–168 (English transl.: Russian Math. Surveys, **33**: 4 (1978), 159–191).
- [5] K. J. Brown, *A symmetric operator maximal with respect to a generalised resolution of the identity*, J. London Math. Soc., (2) **7** (1974), 609–616.
- [6] E. A. Coddington, *Extension theory of formally normal and symmetric subspaces*, Mem. Amer. Math. Soc., **134** (1973).
- [7] E. A. Coddington and H. S. V. de Snoo, *Positive selfadjoint extensions of positive symmetric subspaces*, Math. Z., **159** (1978), 203–214.
- [8] A. Dijksma, H. Langer and H. S. V. de Snoo, *Selfadjoint  $\pi_\kappa$ -extensions of symmetric subspaces: an abstract approach to boundary problems with spectral parameter in the boundary conditions*, Integral equations and operator theory, **7** (1984), 459–515.
- [9] \_\_\_\_\_, *Characteristic functions of unitary operator colligations in  $\pi_\kappa$ -spaces*, to be published.



- [10] O. P. Filimonov, *On the inverse problem of the theory of characteristic operator functions of metric nodes in the space  $\Pi_{\kappa}$* , Dokl. Akad. Nauk Ukrain. SSR, Ser. A, no. 9 (1977), 798–802.
- [11] I. S. Iohvidov, M. G. Krein and H. Langer, *Introduction to the Spectral Theory of Operators in Spaces with an indefinite Metric*, Akademie Verlag, Berlin 1982.
- [12] M. G. Krein and H. Langer, *Über die verallgemeinerten Resolventen und die charakteristische Funktion eines isometrischen Operators in Raume  $\Pi_{\kappa}$* , Hilbert Space Operators and Operator Algebras (Proc. Int. Conf., Tihany, 1970) Colloquia Math. Soc. János Bolyai, no. 5, North-Holland, Amsterdam (1972), 353–399.
- [13] ———, *Über einige Fortsetzungsprobleme, die eng mit der Theorie hermitescher Operatoren im Raume  $\Pi_{\kappa}$  zusammenhängen*, Teil I: Einige Funktionenklassen und ihre Darstellungen, Math. Nachr., **77** (1977), 187–236.
- [14] ———, *Some propositions on analytic matrix functions related to the theory of operators in the space  $\Pi_{\kappa}$* , Acta Sci. Math. (Szeged), **43** (1981), 181–205.
- [15] H. Langer and B. Textorius, *On generalized resolvents and  $Q$ -functions of symmetric linear relations (subspaces) in Hilbert space*, Pacific J. Math., **72** (1977), 135–165.
- [16] R. McKelvey, *Spectral measures, generalized resolvents and functions of positive type*, J. Math. Anal. Appl., **11** (1965), 447–477.
- [17] W. Stenger, *On the projection of a selfadjoint operator*, Bull. Amer. Math. Soc., **74** (1968), 369–372.
- [18] A. V. Štraus, *Characteristic functions of linear operators*, Izv. Akad. Nauk SSSR Ser. Mat., **24** (1960), 43–74 (English transl.: Amer. Math. Soc. Transl., (2) **40** (1964), 1–37).
- [19] ———, *On the extensions of symmetric operators depending on a parameter*, Izv. Akad. Nauk SSSR Ser. Mat., **29** (1965), 1389–1416 (English transl.: Amer. Math. Soc. Transl., (2) **61** (1967), 113–141).
- [20] ———, *On the extensions and the characteristic function of a symmetric operator*, Izv. Akad. Nauk SSSR Ser. Mat., **32** (1968), 186–207 (English transl.: Math. USSR Izv., **2** (1968), 181–204).
- [21] ———, *On one-parameter families of extensions of a symmetric operator*, Izv. Akad. Nauk SSSR Ser. Mat., **30** (1966), 1325–1352 (English transl.: Amer. Math. Soc. Transl., (2) **90** (1970), 135–164).
- [22] ———, *Extensions and generalized resolvents of a symmetric operator which is not densely defined*, Izv. Akad. Nauk SSSR Ser. Mat., **34** (1970), 175–202 (English transl.: Math. USSR Izv., **4** (1970), 179–208).

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