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# Nonlinear Control of a Class of Underactuated Systems

Mahmut Reyhanoglu<sup>1</sup> Arjan van der Schaft<sup>2</sup>

Department of Applied Mathematics  
 University of Twente  
 Enschede, The Netherlands

<sup>1</sup>m.reyhanoglu@math.utwente.nl

<sup>2</sup>twarjan@math.utwente.nl

N. Harris McClamroch<sup>3</sup> Ilya Kolmanovsky<sup>4</sup>

Department of Aerospace Engineering  
 University of Michigan  
 Ann Arbor, Michigan

<sup>3</sup>nhm@engin.umich.edu

<sup>4</sup>ilya@engin.umich.edu

**Abstract**— A theoretical framework is established for the dynamics and control of underactuated systems, defined as systems which have fewer inputs than degrees of freedom. Control system formulation of underactuated systems is addressed and the class of second-order nonholonomic systems is identified. Controllability and stabilizability results are derived for this class of underactuated systems. Examples are included to illustrate the results.

## 1. Introduction

In the past few years, there has been a considerable amount of interest in the control of nonholonomic systems. These studies were primarily limited to first-order nonholonomic systems, in particular, systems satisfying classical nonholonomic velocity relations (see e.g. [2],[6] and references therein). In this paper the ideas in [2] are extended to second-order nonholonomic systems, i.e. systems that satisfy nonintegrable relations involving not only generalized coordinates and velocities but also the generalized accelerations.

Second-order nonholonomic systems can arise by imposition of certain design conditions on the allowable motions of redundant manipulators. Such systems can also arise as models of underactuated systems, defined as systems which have fewer inputs than degrees of freedom. While many interesting techniques and results have been presented for underactuated systems, the control of these systems still remains an open problem. Important issues are: how can nonlinear control models be formulated for such systems; what are their controllability and stabilizability properties; how can open-loop and closed-loop control problems be solved. The first two issues are addressed in this paper.

The organization of this paper is as follows. In Section 2, formulation of the problem is given. Section 3 derives certain fundamental controllability and stabilizability properties. Examples are considered in Section 4. Finally, Section 5 contains a summary of the paper.

## 2. Models of Underactuated Systems

Consider first a dynamic system with configuration manifold  $\mathbf{Q}$ . Let  $(q, \dot{q}) = (q^1, \dots, q^n, \dot{q}^1, \dots, \dot{q}^n)$  denote local coordinates on  $\mathbf{M} = \mathbf{TQ}$ . We refer to  $q, \dot{q}$ , and  $\ddot{q}$

as the vectors of generalized coordinates, generalized velocities and generalized accelerations, respectively. Assume that the system is under the action of  $m < n$  independent control forces and/or torques, i.e. there are fewer control inputs than degrees of freedom. Also let  $u \in \mathbf{R}^m$  denote the vector of control variables. We partition the set of generalized coordinates  $q = (q^1, \dots, q^n)$  as  $q = (q_1, q_2)$ ,  $q_1 \in \mathbf{R}^m$ ,  $q_2 \in \mathbf{R}^{n-m}$ . Without loss of generality, we assume that the actuated degrees of freedom are represented by the elements of  $q_1$  and the unactuated degrees of freedom are represented by the elements of  $q_2$ . Lagrange's equations can then be written as

$$M_{11}(q)\ddot{q}_1 + M_{12}(q)\ddot{q}_2 + F_1(q, \dot{q}) = B(q)u, \quad (1)$$

$$M_{21}(q)\ddot{q}_1 + M_{22}(q)\ddot{q}_2 + F_2(q, \dot{q}) = 0, \quad (2)$$

where  $B(q) \in \mathbf{R}^{m \times m}$  is invertible for all  $q \in \mathbf{Q}$ ,  $F_1(q, \dot{q}) \in \mathbf{R}^m$ ,  $F_2(q, \dot{q}) \in \mathbf{R}^{n-m}$ , and  $M_{ij}(q)$ ,  $i, j = 1, 2$ , represent components of an  $n \times n$  inertia matrix which is symmetric and positive definite for all  $q \in \mathbf{Q}$ . Throughout this paper all functions are assumed to be smooth ( $C^\infty$ ) functions defined on  $\mathbf{M}$ .

Following Spong [15], we may solve for  $\ddot{q}_2$  as

$$\ddot{q}_2 = -M_{22}^{-1}(q)[M_{21}(q)\ddot{q}_1 + F_2(q, \dot{q})]$$

and substitute into (1) to obtain

$$\bar{M}(q)\ddot{q}_1 + \bar{F}(q, \dot{q}) = B(q)u,$$

where

$$\bar{M}(q) = M_{11}(q) - M_{12}(q)M_{22}^{-1}(q)M_{21}(q),$$

$$\bar{F}(q, \dot{q}) = F_1(q, \dot{q}) - M_{12}(q)M_{22}^{-1}(q)F_2(q, \dot{q}).$$

Consequently, using the partial feedback linearizing controller

$$u = B^{-1}(q)[\bar{M}(q)v + \bar{F}(q, \dot{q})]$$

the equations (1)-(2) can be rewritten as

$$\ddot{q}_1 = v. \quad (3)$$

$$\ddot{q}_2 = J(q)\ddot{q}_1 + R(q, \dot{q}), \quad (4)$$

where

$$J(q) = -M_{22}^{-1}(q)M_{21}(q) ,$$

$$R(q, \dot{q}) = -M_{22}^{-1}(q)F_2(q, \dot{q}) .$$

Equations (3)-(4) have a special triangular or cascade form that appropriately captures the important attributes of underactuated mechanical systems. Equation (4) defines  $n-m$  relations involving generalized coordinates as well as their first and second time derivatives. If these relations do not admit any nontrivial integral, i.e. any smooth function  $h(q, \dot{q}, t)$  such that  $\frac{dh}{dt} = 0$  along the solutions, then these relations may be interpreted as  $n-m$  second-order nonholonomic constraints [13]. As will be seen in the subsequent development, controllability and stabilizability properties of underactuated mechanical systems are closely related to this property. Hence, it is crucial to identify underactuated mechanical systems where the relations defined by equation (4) are second-order nonholonomic.

Let  $I_k$  denote the set  $\{1, \dots, k\}$ . Define the  $n-m$  covector fields

$$\omega^i = \sum_{j=1}^m J_{ij}(q)d\dot{q}_{1,j} - d\dot{q}_{2,i} + R_i(q, \dot{q})dt , \quad i \in I_{n-m} ,$$

on  $\mathbf{M} \times \mathbf{R}$  so that the  $n-m$  relations given by the equation (4) can be rewritten as  $\omega^i = 0$ ,  $i \in I_{n-m}$ . Augment the above covector fields with

$$\tilde{\omega}^j = dq_{1,j} - \dot{q}_{1,j}dt , \quad j \in I_m ,$$

$$\tilde{\omega}^{m+l} = dq_{2,l} - \dot{q}_{2,l}dt , \quad l \in I_{n-m} ,$$

and let  $\Omega \subset \mathbf{T}^*(\mathbf{M} \times \mathbf{R})$  denote the codistribution

$$\Omega = \text{span}\{\omega^i, \tilde{\omega}^j, i \in I_{n-m}, j \in I_n\} .$$

The annihilator of  $\Omega$ , denoted  $\Omega^\perp$ , is spanned by  $m+1$  linearly independent smooth vector fields

$$\tau_0 = \sum_{j=1}^m \dot{q}_{1,j} \frac{\partial}{\partial q_{1,j}} + \sum_{k=1}^{n-m} (\dot{q}_{2,k} \frac{\partial}{\partial q_{2,k}} + R_k(q, \dot{q}) \frac{\partial}{\partial \dot{q}_{2,k}}) + \frac{\partial}{\partial t} ,$$

$$\tau_j = \frac{\partial}{\partial \dot{q}_{1,j}} + \sum_{i=1}^{n-m} J_{ij}(q) \frac{\partial}{\partial \dot{q}_{2,i}} , \quad j \in I_m .$$

We present the following definition.

**Definition 1:** Consider the distribution  $\Omega^\perp$  and let  $\tilde{\mathcal{C}}$  denote its accessibility algebra, i.e. the smallest subalgebra of  $\mathbf{V}^\infty(\mathbf{M} \times \mathbf{R})$  that contains  $\tau_0, \tau_1, \dots, \tau_m$ . Let  $\tilde{\mathcal{C}}$  denote the accessibility distribution generated by the accessibility algebra  $\tilde{\mathcal{C}}$ . The nonlinear control system (3)-(4) is said to be completely second-order nonholonomic if  $\dim \tilde{\mathcal{C}}(x, t) = 2n + 1$ ,  $\forall (x, t) \in \mathbf{M} \times \mathbf{R}$ .

Previous, less general, definitions of second-order nonholonomic systems have been given in [11], [18]. Examples of underactuated systems that satisfy Definition 1 include underactuated robot manipulators ([8],[11]), underactuated marine vehicles ([12],[18]), the planar

vertical takeoff and landing aircraft ([5],[9]), the rotational translational actuator system ([4],[7]), and the acrobot system [14].

In this paper it is assumed throughout that the constraints (4) are (completely) second-order nonholonomic. In contrast to the first-order nonholonomic case, second-order nonholonomic relations do not reduce the dimension of the state space.

A particularly important class of solutions are the equilibrium solutions of (3)-(4) with  $v(t) = 0$ ,  $\forall t \geq 0$ . A solution is an equilibrium solution if it is a constant solution; note that if  $(q, \dot{q}) = (q^e, 0)$  is an equilibrium solution we refer to  $q^e$  as an equilibrium configuration. Clearly, the set of equilibrium configurations of the system (3)-(4) is given by

$$\{q \in Q \mid R(q, 0) = 0\} .$$

Equations (3)-(4) can be expressed in the usual nonlinear control system form by defining the following state variables

$$x_1 = q_1, \quad x_2 = q_2, \quad x_3 = \dot{q}_1, \quad x_4 = \dot{q}_2 .$$

Then the state equations are given by

$$\dot{x}_1 = x_3 , \tag{5}$$

$$\dot{x}_2 = x_4 , \tag{6}$$

$$\dot{x}_3 = v , \tag{7}$$

$$\dot{x}_4 = J(x_1, x_2)v + R(x_1, x_2, x_3, x_4) . \tag{8}$$

Clearly, these state equations define a drift vector field  $f(x) = (x_3, x_4, 0, R(x_1, x_2, x_3, x_4))$  and control vector fields  $g_i(x) = (0, 0, e_i, J_i(x_1, x_2))$ , where  $e_i$  denotes the  $i$ 'th standard basis vector in  $\mathbf{R}^m$  and  $J_i(x_1, x_2)$  denotes the  $i$ 'th column of the matrix function  $J(x_1, x_2)$ , for all  $i \in I_{n-m}$ , according to the standard control system form

$$\dot{x} = f(x) + \sum_{i=1}^m g_i(x)v_i . \tag{9}$$

Note that an equilibrium solution  $x^e$ , corresponding to  $v = 0$ , of equation (9) has the form  $x_1^e \in \mathbf{R}^m$ ,  $x_2^e \in \mathbf{R}^{n-m}$ , where  $R(x_1^e, x_2^e, 0, 0) = 0$ , and  $x_3^e = x_4^e = 0$ , i.e. an equilibrium solution corresponds to a motion of the system for which all the configuration variables remain constant. The controllability and stabilizability properties of the system (3)-(4) near an equilibrium configuration  $q^e$  can be obtained by studying local properties of the system (5)-(8) near the corresponding equilibrium solution  $(x_1^e, x_2^e, 0, 0)$ .

### 3. Controllability and Stabilizability

This section develops controllability and stabilizability results for second-order nonholonomic systems. The reader is referred to [10] and [16] for the controllability

concepts developed in the nonlinear control literature.

We first demonstrate that the second-order nonholonomic system, defined by equations (3)-(4), does satisfy certain nonlinear controllability properties. In particular, we show that the system is strongly accessible.

**Theorem 1:** *Let  $n-m \geq 1$ . The second-order nonholonomic system, defined by equations (3)-(4), is strongly accessible.*

**Proof:** Since we have assumed that the system (3)-(4) is completely second-order nonholonomic the distribution  $\Omega^\perp$  spanned by  $\tau_0, \tau_1, \dots, \tau_m$  satisfies the accessibility Lie algebra rank condition at any  $(q, \dot{q}, t) \in \mathbf{M} \times \mathbf{R}$ , i.e.

$$\dim \tilde{C}(x, t) = 2n + 1, \quad \forall (x, t) \in \mathbf{M} \times \mathbf{R} .$$

Let  $\pi_M : \mathbf{M} \times \mathbf{R} \rightarrow \mathbf{M}$  denote the projection onto  $\mathbf{M}$ . Then, clearly  $\pi_{M^*} \tau_0 = f$  and  $\pi_{M^*} \tau_i = g_i, i \in I_m$ . Let  $C_0$  denote the strong accessibility algebra associated with  $f, g_1, \dots, g_m$ , i.e. the smallest subalgebra which contains  $g_1, \dots, g_m$  and satisfies  $[f, X] \in C_0, \forall X \in C_0$ , and let  $\tilde{C}_0$  denote the strong accessibility distribution generated by the strong accessibility algebra  $C_0$ . Since we have

$$\dim \tilde{C}(x, t) = \dim C_0(x) + 1 ,$$

it follows that

$$\dim C_0(x) = 2n, \quad \forall x \in \mathbf{M} .$$

Hence, the system (5)-(8) is strongly accessible. Consequently, the second-order nonholonomic system, defined by equations (3)-(4), is strongly accessible.

The following result illustrates the fact that in certain cases a given equilibrium configuration cannot be asymptotically stabilized using time-invariant continuous (static or dynamic) state feedback.

**Theorem 2.** *Assume that  $R_i(q, 0) = 0, \forall q \in \mathbf{Q}$ , for some  $i \in I_{n-m}$ . Let  $n-m \geq 1$  and let  $(q^e, 0)$  denote an equilibrium solution. Then the second-order nonholonomic system, defined by equations (3)-(4), is not asymptotically stabilizable to  $(q^e, 0)$  using time-invariant continuous (static or dynamic) state feedback law.*

**Proof:** A necessary condition for the existence of a time-invariant continuous ( $C^0$ ) asymptotically stabilizing state feedback law for system (5)-(8) is that the image of the mapping

$$(x, v) \mapsto (x_3, x_4, v, J(x_1, x_2)v + R(x_1, x_2, x_3, x_4))$$

contain some neighborhood of zero (see Brockett [3]). No points of the form

$$\begin{pmatrix} 0 & 0 & 0 & \epsilon \end{pmatrix}, \quad \epsilon \neq 0 ,$$

are in its image: it follows that the necessary condition is not satisfied. Hence system (5)-(8) cannot be asymptotically stabilized to  $(x_1^e, x_2^e, 0, 0)$  by a time-invariant continuous (static or dynamic) state feedback

law. Consequently, the second-order nonholonomic system, defined by equations (3)-(4), is not asymptotically stabilizable to  $(q^e, 0)$  using a time-invariant continuous (static or dynamic) state feedback.

There are examples of second-order nonholonomic systems for which the assumption of Theorem 2 is not satisfied; in such cases an equilibrium solution may be smoothly (even linearly) stabilizable. The planar V/STOL problem is viewed from this perspective in [9].

It is well-known that strong accessibility is far from being sufficient for the existence of a feedback control which asymptotically stabilizes the system at an equilibrium solution. In certain cases it is possible to prove a stronger controllability property such as small time local controllability (STLC) which guarantees the existence of a piecewise analytic feedback control for asymptotic stabilization in the real analytic case [17]. Examples are studied in Section 4 to illustrate this point.

## 4. Examples

### Control of a Manipulator with an Unactuated Joint

Consider the planar 3-DOF redundant manipulator (Figure 1), moving in a horizontal plane so that gravity can be ignored [1]. The base body can translate and rotate freely in the plane. A massless arm is attached to the base body by an unactuated revolute joint, with joint angle  $\phi$ . Let  $(x, y)$  denote the end-effector position of the manipulator and let  $\theta$  be the orientation angle of the base body. Also, let the base body have mass  $M$  and rotational inertia  $I$ , the end-effector and payload combination have mass  $m$ , and let  $l$  be the length of the massless arm.

Assume that initially  $\phi(0) = \dot{\phi}(0) = 0$ . The control problem is to move the manipulator between any given initial configuration  $(x^0, y^0, \theta^0)$  and final configuration  $(x^f, y^f, \theta^f)$  such that there is no change in the unactuated joint angle, i.e.  $\phi(t) = 0, \forall t \geq 0$ .

We use the ideas introduced previously to formulate the above problem as a nonlinear control problem. Let  $u = (u_x, u_y, u_\theta) \in \mathbf{R}^3$  denote the vector of control inputs applied to the base body; where  $(u_x, u_y)$  are the force inputs in the  $x$  and  $y$  direction, respectively, and  $u_\theta$  is the torque input. Setting  $\phi \equiv 0$ , we obtain

$$(M + m)\ddot{x} + Ml\ddot{\theta} \sin \theta + Ml\dot{\theta}^2 \cos \theta = u_x , \quad (10)$$

$$(M + m)\ddot{y} - Ml\dot{\theta} \cos \theta + Ml\dot{\theta}^2 \sin \theta = u_y , \quad (11)$$

$$I\ddot{\theta} = u_\theta , \quad (12)$$

$$\ddot{x} \sin \theta - \ddot{y} \cos \theta = 0 . \quad (13)$$

Note that equation (13) represents a second-order nonholonomic relation which implies that there is no net torque on the unactuated joint. This condition can be

viewed as a design constraint.

In order to satisfy the above equations, it is required to select

$$u_\theta = \frac{I}{Ml}(u_x \sin \theta - u_y \cos \theta). \quad (14)$$

It is then straightforward to show that the above equations can be equivalently written as

$$\ddot{x} = u_1, \quad (15)$$

$$\ddot{\theta} = u_2, \quad (16)$$

$$\ddot{y} = u_1 \tan \theta, \quad (17)$$

where

$$u_1 = \frac{1}{M+m}(u_x \cos \theta + u_y \sin \theta - Ml\dot{\theta}^2) \cos \theta, \quad (18)$$

$$u_2 = \frac{1}{Ml}(u_x \sin \theta - u_y \cos \theta). \quad (19)$$

It is easy to check that equations (15)-(17) satisfy Definition 1 and hence define a second-order nonholonomic system. Note that now the control problem is reduced to designing controls  $u_1$  and  $u_2$  for the system (15)-(17). Once these controls are designed one can use relations (18)-(19) to determine the controls  $u_x$  and  $u_y$ . Finally,  $u_\theta$  can be determined from (14).

Define the variables

$$x_1 = x, x_2 = \theta, x_3 = y, x_4 = \dot{x}, x_5 = \dot{\theta}, x_6 = \dot{y},$$

so that the state equations are given by

$$\dot{x}_1 = x_4, \quad (20)$$

$$\dot{x}_2 = x_5, \quad (21)$$

$$\dot{x}_3 = x_6, \quad (22)$$

$$\dot{x}_4 = u_1, \quad (23)$$

$$\dot{x}_5 = u_2, \quad (24)$$

$$\dot{x}_6 = u_1 \tan x_2. \quad (25)$$

Note that the state space is  $\mathbf{M} = \mathbf{R} \times (-\pi/2, \pi/2) \times \mathbf{R} \times \mathbf{R}^3$ . The drift and control vector fields on  $\mathbf{M}$  are given by

$$f = x_4 \frac{\partial}{\partial x_1} + x_5 \frac{\partial}{\partial x_2} + x_6 \frac{\partial}{\partial x_3},$$

$$g_1 = \frac{\partial}{\partial x_4} + \tan x_2 \frac{\partial}{\partial x_6}, \quad g_2 = \frac{\partial}{\partial x_5}.$$

Note also that the set of equilibrium solutions corresponding to  $u = 0$  is given by

$$\mathbf{M}_e = \{x \in \mathbf{M} \mid x_4 = x_5 = x_6 = 0\}.$$

The following results characterize the controllability and stabilizability properties of the constrained manipulator dynamics.

**Proposition 1.** *Let  $\mathbf{M}_e$  denote the equilibrium manifold and let  $x^e \in \mathbf{M}_e$  denote an equilibrium solution. The following hold for the manipulator dynamics described by equations (20)-(25).*

1. *The system is strongly accessible since the space spanned by the vectors*

$$g_1, g_2, [f, g_1], [f, g_2], [g_2, [f, g_1]], [f, [g_2, [f, g_1]]]$$

*has dimension 6 at any  $x \in \mathbf{M}$ .*

2. *The system is small time locally controllable at  $x^e$  since the brackets satisfy sufficient conditions for small time local controllability.*

3. *There exist time-invariant piecewise analytic feedback laws which asymptotically stabilize  $x^e$ .*

4. *There is no time-invariant continuous feedback law which asymptotically stabilizes the closed loop to  $x^e$ .*

Clearly, the controllability properties given in Proposition 1 guarantee the existence of solutions to the problem of controlling the manipulator while not exciting the unactuated joint. As indicated in [1], if there is a torsional spring at the unactuated joint, the above control problem is equivalent to controlling the manipulator without energy storage in the torsional spring.

#### Control of an Underactuated Surface Vessel

Consider the problem of controlling the Cartesian position and orientation of a surface vessel (marine vehicle) with two independent propellers as shown in Figure 2, where  $(x, y)$  denotes the position of the center of mass of the vehicle and  $\psi$  denotes the orientation of the vehicle [18]. Let  $F_x$  and  $T_z$  denote the external force (along the body-fixed  $x$ -direction) and the external torque (about the body-fixed  $z$ -direction), respectively, which are generated by the two propellers. For simplicity, assume that the vehicle is neutrally buoyant. Also assume that the hydrodynamic damping is not coupled and that the damping terms of order higher than one are negligible. Then, the dynamic equations of motion can be written as

$$m(\ddot{x} \cos \psi + \ddot{y} \sin \psi) + c_x(\dot{x} \cos \psi + \dot{y} \sin \psi) = F_x, \quad (26)$$

$$I\ddot{\psi} + c_z\dot{\psi} = T_z, \quad (27)$$

$$m(-\ddot{x} \sin \psi + \ddot{y} \cos \psi) + c_y(-\dot{x} \sin \psi + \dot{y} \cos \psi) = 0, \quad (28)$$

where  $m$  and  $I$  denote the mass and the rotational inertia (around the body-fixed  $z$ -axis passing through the center of mass) of the vehicle, respectively, and  $c_x$ ,  $c_y$  and  $c_z$  are positive constants representing the hydrodynamic damping coefficients.

Clearly, the above equations can be equivalently written as

$$\ddot{x} = u_1, \quad (29)$$

$$\ddot{\psi} = u_2, \quad (30)$$

$$\ddot{y} = u_1 \tan \psi + \frac{c_y}{m}(\dot{x} \tan \psi - \dot{y}), \quad (31)$$

where

$$u_1 = \frac{1}{m}(F_x \cos \psi - \dot{x}(c_y \sin^2 \psi + c_x \cos^2 \psi) + (c_y - c_x)\dot{y} \sin \psi \cos \psi), \quad (32)$$

$$u_2 = \frac{1}{I}(T_z - c_z \dot{\psi}). \quad (33)$$

It is easy to check that equations (29)-(31) satisfy Definition 1 and hence define a second-order nonholonomic system.

Define the variables

$$x_1 = x, \quad x_2 = \psi, \quad x_3 = y, \quad x_4 = \dot{x}, \quad x_5 = \dot{\psi}, \quad x_6 = \dot{y}$$

so that the state equations are given by

$$\dot{x}_1 = x_4, \quad (34)$$

$$\dot{x}_2 = x_5, \quad (35)$$

$$\dot{x}_3 = x_6, \quad (36)$$

$$\dot{x}_4 = u_1, \quad (37)$$

$$\dot{x}_5 = u_2, \quad (38)$$

$$\dot{x}_6 = u_1 \tan x_2 + \frac{c_y}{m}(x_4 \tan x_2 - x_6). \quad (39)$$

Note that the state space is  $\mathbf{M} = \mathbf{R} \times (-\pi/2, \pi/2) \times \mathbf{R} \times \mathbf{R}^3$ . The drift and control vector fields on  $\mathbf{M}$  are given by

$$f = x_4 \frac{\partial}{\partial x_1} + x_5 \frac{\partial}{\partial x_2} + x_6 \frac{\partial}{\partial x_3} + \frac{c_y}{m}(x_4 \tan x_2 - x_6) \frac{\partial}{\partial x_6},$$

$$g_1 = \frac{\partial}{\partial x_4} + \tan x_2 \frac{\partial}{\partial x_6}, \quad g_2 = \frac{\partial}{\partial x_5}.$$

Note also that the set of equilibrium solutions corresponding to  $u = 0$  is given by

$$\mathbf{M}_e = \{x \in \mathbf{M} \mid x_4 = x_5 = x_6 = 0\}.$$

The following results characterize the controllability and stabilizability properties of the underactuated vehicle dynamics.

**Proposition 2.** *Let  $\mathbf{M}_e$  denote the equilibrium manifold and let  $x^e \in \mathbf{M}_e$  denote an equilibrium solution. The following hold for the vehicle dynamics described by equations (34)-(39).*

1. *The system is strongly accessible since the space spanned by the vectors*

$$g_1, g_2, [f, g_1], [f, g_2], [g_2, [f, g_1]], [f, [g_2, [f, g_1]]]$$

*has dimension 6 at any  $x \in \mathbf{M}$ .*

2. *The system is small time locally controllable at  $x^e$  since the brackets satisfy sufficient conditions for small time local controllability.*

3. *There exist time-invariant piecewise analytic feedback laws which asymptotically stabilize  $x^e$ .*

4. *There is no time-invariant continuous feedback law which asymptotically stabilizes the closed loop to  $x^e$ .*

Time-invariant discontinuous feedback control laws have been developed for this problem in [12], based on the above theoretical results.

## 5. Conclusions

A theoretical framework has been presented for the dynamics and control of second-order nonholonomic systems. In particular, a nonlinear control system formulation has been introduced and certain controllability and stabilizability properties have been analyzed. These fundamental properties should provide a foundation for further research in this area.

We argue that identification of specific nonlinear control systems as second-order nonholonomic systems is a useful categorization. Second-order nonholonomic systems arise as models for a large class of underactuated mechanical systems, and study of this class of nonlinear control systems will necessarily provide insight into the challenging problem of controlling underactuated mechanical systems. The controllability and stabilizability properties developed in this paper are readily applicable to such control problems, as shown by the examples of underactuated systems considered previously in the paper.

In addition, we believe that motion planning algorithms and feedback stabilization schemes can be developed for the class of second order nonholonomic systems, just as such developments have been made for first-order nonholonomic systems [6]. Specific feedback stabilization schemes have recently been developed for the control of an underactuated surface vessel [12] and for hover control of an V/STOL aircraft [9]. In the former problem, a time-invariant discontinuous feedback law is developed based on a nonsmooth state transformation. In the latter problem, a time-invariant discontinuous feedback law is developed based on introduction of a piecewise constant switching signal. These particular feedback stabilization approaches, and other approaches that have been introduced for first-order nonholonomic systems, can perhaps be extended to the class of second-order nonholonomic systems. These extensions are not direct, but the results in ([9], [12]) are encouraging.

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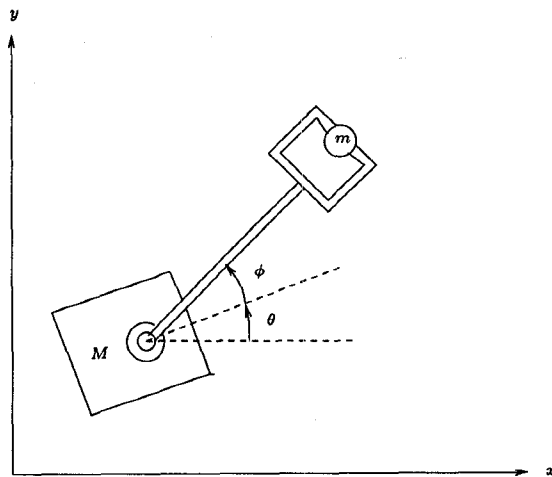


Figure 1: Model of a Manipulator.

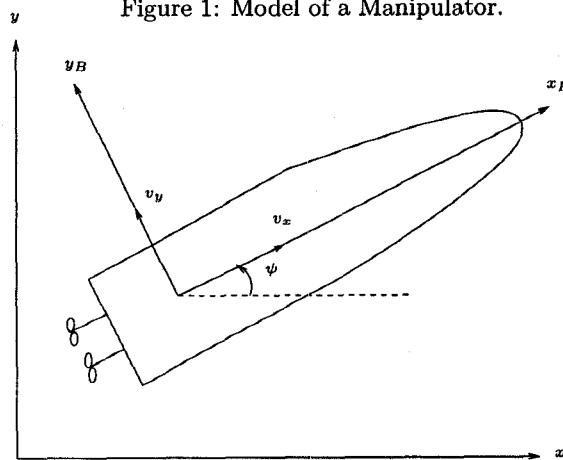


Figure 2: Model of a Surface Vessel.