Model Reduction of port-Hamiltonian Systems as Structured Systems

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Rostyslav V. Polyuga and Arjan van der Schaft

***Abstract*— The goal of this work is to demonstrate that a specific projection-based model reduction method, which provides an $\mathcal{H}_2$ error bound, turns out to be applicable to port-Hamiltonian systems, preserving the *port-Hamiltonian structure* for the reduced order model, and, as a consequence, *passivity*.**

I. INTRODUCTION

The port-Hamiltonian approach to modeling and control of complex physical systems has arisen as a systematic and unifying framework during the last twenty years, see [20], [13], [21] and the references therein. The port-Hamiltonian modeling captures the physical properties of the considered system including the energy dissipation, stability and passivity properties as well as the presence of conservation laws. Another important issue the port-Hamiltonian approach deals with is the interconnection of the physical system with other physical systems creating the so-called physical network. In real applications the dimensions of such interconnected port-Hamiltonian state-space systems rapidly grow both for lumped- and (spatially discretized) distributed-parameter models. Therefore an important issue concerns (structure preserving) *model reduction* of these high-dimensional models for further analysis and control.

The goal of this work is to demonstrate that a specific projection-based model reduction method, which provides an $\mathcal{H}_2$ error bound, turns out to be applicable to port-Hamiltonian systems, preserving the *port-Hamiltonian structure* for the reduced order model, and, as a consequence, *passivity*. Preservation of port-Hamiltonian structure was studied in [10], [16], [9], [21] and the references therein, along with the preservation of moments in [11], [15]. Recent work [14] presents a summary of latest structure preserving model reduction methods for port-Hamiltonian systems. For an overview of the general model reduction theory we refer to [1], [18].

In this paper we are looking at port-Hamiltonian systems as first order systems which are a subclass of so-called structured systems. Structured systems, studied in [19], are defined using notion of differential operator. The projection of such systems onto a dominant eigenspace of the appropriate controllability Gramian results in the reduced order model which inherits the underlying structure of the full order model. In fact, the frequency domain representation of the controllability Gramian leads in this case to the error bound in the $\mathcal{H}_2$ norm [19]. The preservation of the first order structure can be further shown to preserve the port-Hamiltonian structure for the reduced order model, implying passivity and stability properties.

In Section II we provide a description of the method used. The application of this method to port-Hamiltonian systems is considered in Section III.

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II. DESCRIPTION OF THE METHOD

In the systems and control literature the most usual representation of physical and engineering systems is the first order representation, possibly with a feed-through term D

$$\left\{\begin{array}{rcl}\dot{x} &=& Ax+Bu,\\ y &=& Cx+Du,\end{array}\right. \tag{1}$$

with $x(t)\in\mathbb{R}^n$, $u(t)\in\mathbb{R}^m$, $y(t)\in\mathbb{R}^p$, and $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $C\in\mathbb{R}^{p\times n}$, $D\in\mathbb{R}^{p\times m}$ constant matrices. At the same time in many applications higher order structures naturally arise. One important class of structured systems is the class of so-called second order systems described by the system of equations

$$\left\{\begin{array}{rcl}M\ddot{x}+D\dot{x}+Kx &=& Bu,\\ y &=& Cx,\end{array}\right. \tag{2}$$

with $x(t)\in\mathbb{R}^{n/2}$, $u(t)\in\mathbb{R}^m$, $y(t)\in\mathbb{R}^p$, $M,D,K\in\mathbb{R}^{n/2\times n/2}$, $B\in\mathbb{R}^{n/2\times m}, C\in\mathbb{R}^{p\times n/2}$. For mechanical applications the matrices M,D and K represent, respectively, the *mass* (or *inertia*), *damping* and *stiffness* matrices, with M invertible. Of course, the matrix D and the vector x in (2) are different from those in (1). The system (2) can be easily represented in the form (1).

In general model reduction methods applied to (1) produce reduced order models of the form

$$\left\{\begin{array}{rcl}\dot{x}_r &=& A_rx_r+B_ru,\\ y_r &=& C_rx_r+D_ru,\end{array}\right. \tag{3}$$

with $r\ll n$, $x_r(t)\in\mathbb{R}^r$, $u(t)\in\mathbb{R}^m$, $y_r(t)\in\mathbb{R}^p$, $A_r\in\mathbb{R}^{r\times r}$, $B_r\in\mathbb{R}^{r\times m}$, $C_r\in\mathbb{R}^{p\times r}$, $D_r\in\mathbb{R}^{p\times m}$. The second (higher) order structure (2) for the reduced order models quite often fails to be extracted from (3). Therefore special structure preserving methods are required.

Model reduction of second order systems was studied in [6], [12], [5], [4], along with the use of the Krylov methods in [2], [17], [8], [3]. In this work we are using the method of [19] which provides an $\mathcal{H}_2$ error bound and turns out to be applicable to port-Hamiltonian systems, preserving the port-

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Hamiltonian structure for the reduced order model, and, as a consequence, passivity.

A. System representation using differential operators

In order to proceed we need the following notation. Let $K(s), P(s)$ be polynomial matrices in s:

$$K(s) = \sum_{j=0}^{l} K_j s^j,\ K_j \in \mathbb{R}^{n\times n},$$
$$P(s) = \sum_{j=0}^{l} P_j s^j,\ P_j \in \mathbb{R}^{n\times m},$$

where K is invertible, $K^{-1}P$ is a strictly proper rational matrix and l is the order of the system ($l = 1$ for (1) and $l = 2$ for (2)). Then $K(\frac{d}{dt}), P(\frac{d}{dt})$ denote the differential operators

$$K(\tfrac{d}{dt}) = \sum_{j=0}^{l} K_j \frac{d^j}{dt^j}, \quad P(\tfrac{d}{dt}) = \sum_{j=0}^{l} P_j \frac{d^j}{dt^j}.$$

The systems (without a feed-through term) can be now defined by the following set of equations:

$$\Sigma : \left\{ \begin{array}{rcl} K(\frac{d}{dt})x &=& P(\frac{d}{dt})u, \\ y &=& Cx, \end{array} \right. \tag{4}$$

where $C \in \mathbb{R}^{p\times n}$.

This is a more general representation of (1), (2), which allows for derivatives of the input u.

B. Reachability Gramian

Recall from [1] that for the first order stable system (1) the corresponding (infinite) reachability Gramian is defined as

$$W := \int_0^\infty e^{A\tau} B B^T e^{A^T\tau} d\tau. \tag{5}$$

This Gramian is one of the central objects in the mathematical systems theory. It is a symmetric positive semidefinite matrix which satisfies the following Lyapunov equation

$$AW + WA^T + BB^T = 0. \tag{6}$$

The eigenvalues of the Gramian W are measures of the reachability of the system (1).

The Gramian (5) can be rewritten as

$$W := \int_0^\infty x(t)x(t)^T dt \tag{7}$$

for $x(t)$ being the state of the corresponding (first order) system when the input u is the δ-distribution. Indeed, the solution of $\dot{x} = Ax + Bu,\ x(0) = 0$, to the input $u(t) = I\delta(t)$ is given as $x(t) = e^{At}B$.

In a similar way as in (7) the reachability Gramians of higher order systems can be defined. In particular, the reachability Gramian of the second order system (2) can be shown to be the left upper block of the reachability Gramian of the corresponding first order system (1).

Using Parseval's theorem, the Gramian (7) can be considered in the frequency domain:

$$W = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(i\omega)x(i\omega)^* dt, \tag{8}$$

where the star denotes the conjugate transpose and $x(i\omega)$ is the Laplace transform of the time signal $x(t)$ (for simplicity of notation, quantities in the time and frequency domains are denoted by the same symbol x).

The transfer function of (4) in the frequency domain is given as

$$G(s) = CK(s)^{-1}P(s),$$

while the input-to-state and the input-to-output maps are

$$x(s) = K(s)^{-1}P(s)u(s), \quad y(s) = G(s)u(s).$$

For the input being the unit impulse $u(t) = \delta(t)I$ it follows that $u(s) = I$ and the about expressions read

$$x(s) = K(s)^{-1}P(s), \quad y(s) = G(s).$$

In the time domain we have

$$\begin{array}{rcl} \mathrm{trace}\{\int_0^\infty y(t)y(t)^T dt\} &=& \mathrm{trace}\{\int_0^\infty Cx(t)x(t)^T C^T dt\} \\ &=& \mathrm{trace}\{CWC^T\}. \end{array}$$

Using the notation

$$F(s) := K(s)^{-1}P(s)$$

and the Parceval's theorem we obtain for the frequency domain

$$\mathrm{trace}\{\int_0^\infty y(t)y(t)^T dt\} =$$
$$\mathrm{trace}\{C(\tfrac{1}{2\pi}\int_{-\infty}^{\infty} F(i\omega)F(i\omega)^* d\omega)C^T\}.$$

This reasoning results in the conclusion that the reachability Gramian of a system with the corresponding order is given in the frequency domain as

$$W \quad = \quad \tfrac{1}{2\pi}\int_{-\infty}^{\infty} F(i\omega)F(i\omega)^* d\omega. \tag{9}$$

C. Model reduction procedure

Model reduction of the systems (4), as explained in [19], is based on the projection of (4) on the dominant eigenspace of a Gramian W of the state x.

The eigenvalue decomposition of the corresponding Gramian W gives

$$W = V\Lambda V^T,\ \Lambda = diag(\Lambda_1,\ \Lambda_2). \tag{10}$$

where $\Lambda \in \mathbb{R}^{n\times n}$ is a diagonal matrix containing the real

eigenvalues of the Gramian W in decreasing order, and $V \in \mathbb{R}^{n\times n}$ is an orthogonal matrix.

Choosing the dimension of the reduced order model r leads to the partitioning

$$\Lambda = diag(\Lambda_1,\ \Lambda_2),\ V = [V_1,\ V_2], \tag{11}$$

where $\Lambda_1 \in \mathbb{R}^{r\times r},\ \Lambda_2, \in \mathbb{R}^{(n-r)\times(n-r)},\ V_1 \in \mathbb{R}^{n\times r},\ V_2 \in \mathbb{R}^{n\times(n-r)}$.

An orthogonal basis for the dominant eigenspace of dimension r is used to construct a reduced order model:

$$\hat{\Sigma}: \left\{ \begin{array}{rcl} \hat{K}_{11}(\frac{d}{dt})\hat{x} &=& \hat{P}(\frac{d}{dt})u, \\ \hat{y} &=& \hat{C}_1\hat{x}, \end{array} \right. \tag{12}$$

where

$$\begin{aligned} &\hat{x} \in \mathbb{R}^r, && \hat{C}_1 = CV_1 \in \mathbb{R}^{p\times r}, \\ &\hat{K}_{11}(\tfrac{d}{dt}) = \sum_{j=0}^{l} \hat{K}_j \frac{d^j}{dt^j}, && \hat{K}_j = V_1^T K_j V_1 \in \mathbb{R}^{r\times r}, \\ &\hat{P}(\tfrac{d}{dt}) = \sum_{j=0}^{l} \hat{P}_j \frac{d^j}{dt^j}, && \hat{P}_j = V_1^T P_j \in \mathbb{R}^{r\times m}. \end{aligned}$$

This model reduction method by construction preserves the second or higher order structure of the full order model Σ in (4) for the reduced order model in (12)

Suppose the polynomial matrix $\hat{K}(s)$ has the following splitting corresponding to the dimension of the reduced order model

$$\begin{aligned} \hat{K}(s) &= V^T K(s) V = \begin{bmatrix} V_1^T K(s) V_1 & V_1^T K(s) V_2 \\ V_2^T K(s) V_1 & V_2^T K(s) V_2 \end{bmatrix} \\ &=: \begin{bmatrix} \hat{K}_{11}(s) & \hat{K}_{12}(s) \\ \hat{K}_{21}(s) & \hat{K}_{22}(s) \end{bmatrix}. \end{aligned}$$

Let $L(s)$ be the polynomial matrix

$$L(s) := (\hat{K}_{11}(s))^{-1}\hat{K}_{12}(s). \tag{13}$$

If the reduced order system has no poles on the imaginary axis, $\sup_{\omega} \|L(iw)\|_2$ is finite. Then the model reduction method results in the following $\mathcal{H}_2$ error bound.

Theorem 1: [19] Consider the full order structured system Σ in (4) and the reduced order structured system $\hat{\Sigma}$ in (12). Then the error system

$$\mathcal{E} = \Sigma - \hat{\Sigma}$$

satisfies the following $\mathcal{H}_2$ error bound

$$\|\mathcal{E}\|_{\mathcal{H}_2}^2 \leqslant \mathrm{trace}\{\hat{C}_2\Lambda_2\hat{C}_2^T\} + \kappa\ \mathrm{trace}\{\Lambda_2\}, \tag{14}$$

where κ is a constant depending on $\Sigma,\ \hat{\Sigma}$, and the diagonal elements of Λ_2 are the neglected smallest eigenvalues of W:

$$\begin{aligned} \kappa &= \sup_{\omega} \|(\hat{C}_1 L(i\omega))^*(\hat{C}_1 L(i\omega) - 2\hat{C}_2)\|_2, \\ \hat{C}_2 &= CV_2. \end{aligned}$$

The frequency domain representation of the Gramian (9) results in the following expressions [19] in the coordinates, where the Gramian is diagonal $W = \Lambda$:

$$\begin{aligned} \Lambda &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(i\omega)\hat{F}(i\omega)^* d\omega, \\ \Lambda_1 &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}_1(i\omega)\hat{F}_1(i\omega)^* d\omega, \\ \Lambda_2 &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}_2(i\omega)\hat{F}_2(i\omega)^* d\omega, \\ 0 &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}_2(i\omega)\hat{F}_1(i\omega)^* d\omega, \end{aligned} \tag{15}$$

where $\hat{F}_1(s),\ \hat{F}_2(s)$ come from the splitting according to the dimension of the reduced order model of $\hat{F}(s)$, which is nothing but the defined before matrix $F(s)$ in the new coordinates:

$$\hat{F}(s) = \begin{bmatrix} \hat{F}_1(s) \\ \hat{F}_2(s) \end{bmatrix} = V^T F(s) = V^T K(s)^{-1} P(s). \tag{16}$$

The expressions (15) are of the direct use in the proof of the error bound in Theorem 1. The proof of Theorem 1 can be found in [19] and is also sketched in [14].

III. APPLICATION OF THE METHOD TO PORT-HAMILTONIAN SYSTEMS

Consider linear port-Hamiltonian systems [20], [7]

$$\Sigma_{PHS}: \left\{ \begin{array}{rcl} \dot{x} &=& (J-R)Qx + Bu, \\ y &=& B^T Q x. \end{array} \right. \tag{17}$$

As discussed in [15], [14], there exists a coordinate transformation $S,\ x = Sx_I$, such that in the new coordinates

$$Q_I = S^T Q S = I. \tag{18}$$

By defining the transformed system matrices as $J_I = S^{-1}JS^{-T},\ R_I = S^{-1}RS^{-T}, B_I = S^{-1}B$, we obtain the transformed port-Hamiltonian system

$$\left\{ \begin{array}{rcl} \dot{x}_I &=& (J_I - R_I)x_I + B_I u, \\ y &=& B_I^T x_I, \end{array} \right. \tag{19}$$

with energy $H(x_I) = \frac{1}{2}\|x_I\|^2$. System (19) can be rewritten as

$$\left\{ \begin{array}{rcl} I\dot{x}_I - (J_I - R_I)x_I &=& B_I u, \\ y &=& B_I^T x_I, \end{array} \right. \tag{20}$$

which is of the form (4) with

$$\begin{aligned} K(\tfrac{d}{dt}) &= I\tfrac{d}{dt} - (J_I - R_I), \\ P(\tfrac{d}{dt}) &= B_I, \\ C &= B_I^T. \end{aligned}$$

The Gramian of the transformed port-Hamiltonian system (20)

$$W := \int_0^{\infty} x_I(t)x_I(t)^T dt \tag{21}$$

can be decomposed using the eigenvalue decomposition as shown in (10) with the splitting as in (11) according to the chosen dimension r of the reduced order model .

This leads to the main result.

Theorem 2: Consider a full order port-Hamiltonian system (17) and construct V_1 as in (11) using the eigenvalue decomposition of the Gramian (21) of the transformed port-Hamiltonian system (20). Then the r^{th} order reduced system

$$\hat{\Sigma}_{PHS}: \left\{ \begin{array}{rcl} \dot{\hat{x}}_I &=& (\hat{J}_I - \hat{R}_I)\hat{x}_I + \hat{B}_I u, \\ \hat{y} &=& \hat{C}_I \hat{x}_I, \end{array} \right. \tag{22}$$

with the interconnection matrices $\hat{J}_I, \hat{B}_I$, energy matrix $\hat{Q}_I$, dissipation matrices $\hat{R}_I$ and output matrix $\hat{C}_I$ given as

$$\begin{array}{rclcrcl} \hat{J}_I &=& V_1^T J_I V_1, & \hat{R}_I &=& V_1^T R_I V_1, & \hat{Q}_I = I, \\ \hat{B}_I &=& V_1^T B_I, & \hat{C}_I &=& B_I^T V_1, \end{array}$$

is a port-Hamiltonian system as well as the first order system. Furthermore the error system

$$\mathcal{E} = \Sigma_{PHS} - \hat{\Sigma}_{PHS}$$

satisfies the following $\mathcal{H}_2$ error bound

$$\|\mathcal{E}\|^2_{\mathcal{H}_2} \leqslant B_I^T V_2 \Lambda_2 V_2^T B_I + \kappa \ \mathrm{trace}\{\Lambda_2\}, \tag{23}$$

where κ is a constant depending on Σ_{PHS}, $\hat{\Sigma}_{PHS}$ and the diagonal elements of Λ_2 are the neglected smallest eigenvalues of W:

$$\begin{array}{rcl} \kappa &=& \sup\limits_{\omega} \|(B_I^T V_1 L(i\omega))^* (B_I^T V_1 L(i\omega) - 2B_I^T V_2)\|_2, \\ L(s) &=& (V_1^T (J_I - R_I) V_1 - Is)^{-1} V_1^T (J_I - R_I) V_2. \end{array}$$

Proof: Projection of the transformed port-Hamiltonian system (20) leads to the reduced order system

$$\left\{ \begin{array}{rcl} I\dot{\hat{x}}_I - (\hat{J}_I - \hat{R}_I)\hat{x}_I &=& \hat{B}_I u, \\ \hat{y} &=& \hat{C}_I \hat{x}_I, \end{array} \right.$$

which is of the form (12), preserving the first order structure of (20), as well as (17). This further results in the reduced order model (22) where $\hat{J}_I$ is clearly skew-symmetric and $\hat{R}_I$ is symmetric and positive semi-definite. Moreover $\hat{C}_I = \hat{B}_I^T \hat{Q}_I$. Therefore the reduced order system (22) is port-Hamiltonian. The error bound (23) follows directly from Theorem 1. ■

Note that the reduced order system (22) is automatically passive because of the preservation of the port-Hamiltonian structure. See also [20], [7].

IV. CONCLUSIONS

In this paper we considered a representation of port-Hamiltonian systems using a notion of a differential operator. The projection of such (first order) systems onto the dominant eigenspace of the corresponding reachability Gramian results in the reduced order model which is shown to preserve the port-Hamiltonian structure, and therefore passivity and stability. General error bound derived in [19] is adopted to port-Hamiltonian systems.

An extension of the method when the full order system is projected on the dominant eigenspace of the product of the observability and reachability Gramians with the relation to Lyapunov balancing as well as the applications of other methods preserving higher order structure to port-Hamiltonian systems are left for future research.

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