Full-order observer design for a class of port Hamiltonian systems

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A. Venkatraman[*], *A. J. van der Schaft*[†]

Abstract We consider a special class of port-Hamiltonian systems and propose a design methodology for constructing globally exponentially stable full-order observers for them by a passivity based approach. The essential idea is to make the augmented system consisting of the plant and the observer dynamics to become strictly passive with respect to an invariant manifold defined on the extended state space, on which the state estimation error is zero. We first introduce the concept of passivity of a system with respect to a manifold by defining a new input and output on the extended state space and then perform a partial state feedback passivation which leads to the construction of the observer. We finally illustrate this procedure for three well known physical systems, modeled in port-Hamiltonian form.

1   Introduction

The observer problem for a plant is to construct a dynamical system that asymptotically estimates the states of the plant by using the measurements of the inputs and outputs. It is well known that for linear systems, the so-called Luenberger observer fulfills the task of state estimation whereas there does not exist a general observer design framework for nonlinear systems. However, observers have been proposed for various special classes of nonlinear systems.

The first attempts towards nonlinear observer design were to identify necessary and sufficient conditions on a nonlinear system for converting it into a simpler form (like linear or bilinear up to a nonlinear output injection term) for which an observer can be easily constructed [1]-[8]. Reference [25, Chapter 14] proposes a

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different observer design approach where the state dependent nonlinearities are not cancelled in the error dynamics but are dominated by using high gain linear terms and therefore caters to a larger class of nonlinear systems.

Another class of nonlinear systems that was studied, consisted of those in which the state-dependent nonlinearities satisfied certain conditions, like being globally Lipschitz as in [9]-[12], are a monotonic function of a linear combination of the states as in [13], [14] or have a bounded slope [15]. Observer design for such systems was performed by employing quadratic Lyapunov functions. Quite recently, the observer design was studied as a problem of rendering a selected manifold in the extended state-space of the plant and observer as positively invariant and globally attractive [16], [17]. Reference [17] in particular allows for non-monotonic nonlinearities to appear in the unmeasured state dynamics and proposes a reduced-order observer design for such systems.

Reference [18] proposes a different approach to observer design as compared to the above works by invoking passivity based concepts. The underlying idea is to make the augmented system consisting of the plant and the observer dynamics to become strictly passive with respect to an invariant set in which the state estimation error becomes zero. In order to establish passivity, a new input and output is defined on the extended state space and further, under some assumptions on the plant and the observer, it is proved that passivation can be done. It has also been shown that the proposed observer, on account of its passivity property, admits a redesign which makes it robust to measurement disturbances.

In this paper, we consider port-Hamiltonian systems [26, Chapter4] and propose to construct globally exponentially stable full-order observers for them by following a similar approach as stated in [18]. Our main contribution is to identify a special class of port-Hamiltonian systems that admit such a passivity based observer and give a methodology for designing the observer. We allow the observer gain matrices to depend on the observer states unlike in [18] (where they are assumed constant) and thus also enlarge the admissible class of nonlinear systems. Interestingly, as a part of our full-order observer construction, we obtain an exponentially stable reduced-order observer. We finally illustrate the proposed observer design for some well known physical systems.

2 Passivity based observer design for port-Hamiltonian systems

2.1 Introduction to the class of port-Hamiltonian systems

We consider the following class of port-Hamiltonian systems whose dynamics can be described by the model:

$$\dot{x} = [J(x_1, u_1) - R(x_1)]\frac{\partial H}{\partial x}(x) + g(y)u_2, \tag{1}$$

$$J = \begin{bmatrix} J_1(x_1, u_1) & T(x_1, u_1) \\ -T^{\mathrm{T}}(x_1, u_1) & J_2(x_1, u_1) \end{bmatrix}, R = \begin{bmatrix} R_1(x_1) & 0 \\ 0 & R_2(x_1) \end{bmatrix}, g = \begin{bmatrix} g_1(y) \\ g_2(y) \end{bmatrix} \tag{2}$$

where $x = (x_1, x_2)$, $x \in \mathbb{R}^n$ ($x_1 \in \mathbb{R}^p$, $x_2 \in \mathbb{R}^{n\text{-}p}$) is the state, $u_1 \in U \subset \mathbb{R}^m$, $u_2 \in \mathbb{R}^m$ are the inputs where U is a compact set. The matrices $J_1 \in \mathbb{R}^{p \times p}$, $J_2 \in \mathbb{R}^{n\text{-}p \times n\text{-}p}$ are skew-symmetric, $R_1 \in \mathbb{R}^{p \times p}$, $R_2 \in \mathbb{R}^{n\text{-}p \times n\text{-}p}$ are symmetric positive semi-definite and further $T \in \mathbb{R}^{p \times n\text{-}p}$, $g_1 \in \mathbb{R}^{p \times m}$, $g_2 \in \mathbb{R}^{n\text{-}p \times m}$. We assume each of the matrices $J_1, J_2, R_1, R_2, T, g_1, g_2$ to be smooth in their arguments. The Hamiltonian $H :$ $\mathbb{R}^p \times \mathbb{R}^{n\text{-}p} \to \mathbb{R}$ assumes the form

$$H(x_1, x_2) = x_2^{\mathrm{T}} Q x_2 + K(x_1), \tag{3}$$

where $Q^{\mathrm{T}} = Q > 0$ is a constant matrix and K is a smooth nonlinear function of x_1. We consider only x_1 to be measurable, that is the system output is $y = x_1$ (which may or may not equal the port-Hamiltonian output, $y_{\mathrm{p}} = g^{\top}(y)\frac{\partial H}{\partial x}(x)$).

We now proceed to design under some assumptions a globally exponentially stable full-order observer for the port-Hamiltonian system (1). We start by defining a passivity based observer and the notion of strict passivity with respect to a manifold for (1).

2.2 Problem Formulation

Definition 1. *We call the dynamical system represented as*

$$\dot{\hat{x}} = [J(\hat{x}_1, u_1) - R(\hat{x}_1)]\frac{\partial H}{\partial \hat{x}}(\hat{x}) + g(y)u_2 + L(\hat{x}_1)v, \ \ L(\hat{x}_1) = \begin{bmatrix} L_1(\hat{x}_1) \\ L_2(\hat{x}_1) \end{bmatrix} \tag{4}$$

where $\hat{x} = (\hat{x}_1, \hat{x}_2)$, $\hat{x} \in \mathbb{R}^n$ ($\hat{x}_1 \in \mathbb{R}^p$, $\hat{x}_2 \in \mathbb{R}^{n-p}$), $v \in \mathbb{R}^p$, a passivity based observer for the system (1) if there exists a smooth globally invertible matrix $L_1 \in \mathbb{R}^{p \times p}$, a smooth matrix $L_2 \in \mathbb{R}^{n-p \times p}$, a constant matrix $X(\in \mathbb{R}^{p \times p}) = X^{\top} > 0$ and a continuous scalar function $k : \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ such that the feedback law $v = L_1^{-1}(\hat{x}_1)X^{-1}\{k(y, \hat{x}, u_1)y_d + v_d\}$ makes the augmented system[1] composed of (1) and (4) to become strictly passive with respect to the manifold $\mathcal{M} = \{(x, \hat{x}) : x = \hat{x}\}$, from the new input v_d to the new output $y_d = \hat{x}_1 - x_1$.

Definition 2. *The system (1), (4) is strictly passive with respect to the manifold $\mathcal{M}$, uniformly for all $u_1 \in U$ and $u_2 \in \mathbb{R}^m$ if there exists a storage function $S(x, \hat{x}) > 0$ for every $x \neq \hat{x}$, $S(x, \hat{x}) = 0$ on $\mathcal{M}$ and the time derivative of S along the system trajectory satisfies:*

$$\frac{\partial^{\top} S}{\partial x}(x, \hat{x})[J(x_1, u_1) - R(x_1)]\frac{\partial H}{\partial x}(x) + \frac{\partial^{\top} S}{\partial \hat{x}}(x, \hat{x})[J(\hat{x}_1, u_1) - R(\hat{x}_1)]\frac{\partial H}{\partial \hat{x}}(\hat{x})$$

$$+ k(y, \hat{x}, u_1)\frac{\partial^{\top} S}{\partial \hat{x}}(x, \hat{x})L(\hat{x}_1)L_1^{-1}(\hat{x}_1)X^{-1}y_d \leq -\alpha(\|x - \hat{x}\|), \tag{5}$$

$$\{\frac{\partial^{\top} S}{\partial x}(x, \hat{x}) - \frac{\partial^{\top} S}{\partial \hat{x}}(x, \hat{x})\}g(y) \equiv 0, \ \ \frac{\partial^{\top} S}{\partial \hat{x}}(x, \hat{x})L(\hat{x}_1)L_1^{-1}(\hat{x}_1) = y_d^{\top} \tag{6}$$

where α is a positive definite function.

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[1] In the sequel we shall always use the term augmented system to refer to the system composed of (1) and (4).

If the augmented system becomes strictly passive with respect to $\mathcal{M}$ for some functions L_1, L_2 and k, then upon letting $v_d = 0$, the manifold $\mathcal{M}$ becomes positively invariant[2] and globally attractive[3]. The observer design problem then follows by noting that the state estimation error is zero on $\mathcal{M}$.

2.3 Observer Design

The notion of strict passivity is usually associated with respect to a point in the state space rather than a manifold. It has been established in [20] that any affine control system can be rendered strictly passive by a smooth static state feedback iff the system has a vector relative degree $\{1, \ldots\ldots, 1\}$ and is globally minimum phase. In situations where some of the states are not measurable, additional sufficiency conditions have been proposed in [21] which ensure feedback passivation by a static output feedback while reference [18] gives sufficient conditions for rendering a system strictly passive with respect to a set, by a partial state feedback. Our situation is similar to [18] and [21] as we need to achieve strict passivity of the augmented system with respect to $\mathcal{M}$ by using a feedback law which is independent of x_2.

We next state two key assumptions on (1) - (4) and use them to prove that:
1. There exist matrices $L_1(\hat{x}_1)$ and $L_2(\hat{x}_1)$ such that the augmented system satisfies a vector relative degree and global minimum phase condition with respect to $\mathcal{M}$ which are analogous to the conditions needed for static state feedback passivation.

2. The augmented system satisfies an additional nonlinear growth inequality which is sufficient to make it strictly passive with respect to $\mathcal{M}$ by a partial state feedback law $v = L_1^{-1}(\hat{x}_1)X^{-1}\{k(y, \hat{x}, u_1)y_d + v_d\}$, which is independent of x_2.

Assumption 1. *There exists a smooth globally invertible matrix $L_1(x_1) \in \mathbb{R}^{p \times p}$ and a smooth matrix $L_2(x_1) \in \mathbb{R}^{n\text{-}p \times p}$ such that*

$$A^\top(x_1, u_1) + A(x_1, u_1) > \epsilon I_{p \times p}, \ \epsilon > 0 \tag{7}$$

holds for all x_1, uniformly for all $u_1 \in U$ where

$$A(x_1, u_1) = \{L_2(x_1)L_1^{-1}(x_1)T(x_1, u_1) + R_2(x_1)\}Q.$$

Assumption 2. *There exists a smooth function $\beta : \mathbb{R}^p \to \mathbb{R}^{n\text{-}p}$ such that*

$$L_2(x_1)L_1^{-1}(x_1) = \frac{\partial \beta}{\partial x_1}(x_1) \tag{8}$$

holds for all $x_1 \in \mathbb{R}^p$.

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[2]The manifold $\mathcal{M}$ is positively invariant if $(x(0), \hat{x}(0)) \in \mathcal{M} \Rightarrow (x(t), \hat{x}(t)) \in \mathcal{M}$ for all $t \geq 0$ and for every initial condition $(x(0), \hat{x}(0))$.

[3]The manifold $\mathcal{M}$ is globally attractive if, for every initial condition $(x(0), \hat{x}(0))$, the distance of the augmented state vector to $\mathcal{M}$ globally asymptotically goes to zero, i.e., $\lim_{t \to \infty} \text{dist}\{(x(t), \hat{x}(t)), \mathcal{M}\} = 0$.

We next state the following theorem.

Theorem 1. *Under Assumption 1,*

1.   The augmented system has a vector relative degree $\{1, ..., 1\}$ with respect to the input v and the output $y_d = \hat{x}_1 - x_1$.

2.   The zero dynamics of the augmented system with respect to the output y_d renders the manifold $\mathcal{P} = \{(x_1, x_2, \hat{x}_2) : \hat{x}_2 = x_2\}$, positively invariant and globally exponentially attractive.

Proof. We compute the derivative of y_d and see that the input v appears in it pre-multiplied by the matrix L_1. From Assumption 1, since L_1 is invertible for all x_1, we conclude that the augmented system has a vector relative degree $\{1, ..., 1\}$ with respect to the input v and the output y_d

We next see that the zero dynamics of the augmented system with respect to output y_d essentially consists of (1) and the equations

$$0 = T(x_1, u_1)Q\{\hat{x}_2 - x_2\} + L_1(x_1)v, \tag{9}$$

$$\dot{\hat{x}}_2 = \{J_2(x_1, u_1) - R_2(x_1)\}Q\hat{x}_2 + g_2(y)u_2 - T^\mathrm{T}(x_1, u_1)\frac{\partial K}{\partial x_1}(x_1) + L_2(x_1)v, \tag{10}$$

where we make use of (3). We now consider the manifold $\mathcal{P}$ and denote its off-the-manifold coordinate as $z = \hat{x}_2 - x_2$. Computing the derivative of z along (1), (10) and eliminating v by using (9) yields

$$\dot{z} = \{J_2(x_1, u_1) - A(x_1, u_1)\}Qz. \tag{11}$$

We can clearly see from (11) that the manifold $\mathcal{P}$ is positively invariant and further if we consider the Lyapunov function $V = \frac{1}{2}z^\mathrm{T}Qz$, then Assumption 1 verifies $\dot{V} \leq -\frac{\epsilon\lambda_\mathrm{m}^2(Q)}{\lambda_\mathrm{M}(Q)}V$ with λ_m, λ_M denoting the minimum and maximum eigenvalue. Thus, V exponentially decays to zero with convergence rate $\frac{\epsilon\lambda_\mathrm{m}^2(Q)}{\lambda_\mathrm{M}(Q)}$. □

An interesting corollary that follows from Theorem 1 is,

Corollary 3. *Under Assumptions 1 and 2, the dynamical system*

$$\dot{\eta} = -\frac{\partial\beta}{\partial x_1}(x_1)\{(J_1(x_1, u_1) - R_1(x_1))\frac{\partial K}{\partial x_1}(x_1) + g_1(y)u_2\} - T^T(x_1, u_1)\frac{\partial K}{\partial x_1}(x_1)$$

$$+ \{J_2(x_1, u_1) - R_2(x_1) - \frac{\partial\beta}{\partial x_1}(x_1)T(x_1, u_1)\}Q\{\eta + \beta(x_1)\} + g_2(y)u_2,$$

where $\eta \in \mathbb{R}^{n\text{-}p}$, is a reduced order observer for x_2.

Proof. We consider the manifold $\mathcal{N} = \{(x_1, x_2, \eta) : \eta = x_2 - \beta(x_1)\}$ and differentiate its off-the-manifold coordinate, $z = \eta - x_2 + \beta(x_1)$ along the system dynamics to obtain

$$\dot{z} = \{J_2(x_1, u_1) - R_2(x_1) - \frac{\partial\beta}{\partial x_1}(x_1)T(x_1, u_1)\}Qz. \tag{12}$$

Using Assumptions 1, 2 and employing the Lyapunov function $V = \frac{1}{2}z^{\mathrm{T}}Qz$, we can show that $\mathcal{N}$ is positively invariant and globally exponentially attractive. Hence, η is a reduced order observer[4] for x_2 whose asymptotic estimate is $\eta + \beta(x_1)$. $\square$

Remark 1. *Assumptions 1 and 2 involve finding matrices $L_1(x_1)$, $L_2(x_1)$ such that $A(x_1, u_1)$ is strictly positive definite and $L_2(x_1)L_1^{-1}(x_1)$ is integrable respectively. Designing such state dependent matrices that satisfy (7) and (8) would involve solving a set of algebraic and partial differential equations which is usually a difficult task. Reference [18] studies the observer design problem by restricting L_1, L_2 to be constant matrices which trivializes Assumption 2 (as $\beta(x_1) = L_2 L_1^{-1} x_1$) and hence narrows the applicable class of nonlinear systems. Indeed, as we show later in our examples, whenever T is a constant matrix, letting L_1, L_2 to be constant matrices would suffice for the observer design, whereas in situations where T depends on x_1, it is natural to allow L_1, L_2 to depend on x_1 in order to satisfy Assumption 1.*

Remark 2. *Another interesting situation is when the damping matrix $R(x_1) > 0$, in which case the Assumption 1 gets satisfied with $L_2 = 0$ and hence the resulting reduced order observer for x_2 exactly emulates the x_2 dynamics. The convergence rate of z in (12) would then solely depend on the natural damping of the system which could be negligible or could be subject to high uncertainities, in which case such a reduced-order observer is not generally preferred.*

We next prove under Assumptions 1 and 2 that the system (1) - (4) admits a partial state feedback v (independent of x_2) that renders it strictly passive with respect to the manifold $\mathcal{M}$ and also leads to the construction of the full-order observer.

Theorem 2. *Under Assumptions 1 and 2,*
1. The system (1), (4) expressed in the coordinates $(x_1, x_2, \zeta_1, \zeta_2)$ where

$$\zeta_1 = \hat{x}_1 - x_1, \ \ \zeta_2 = \hat{x}_2 - x_2 - \{\beta(\hat{x}_1) - \beta(x_1)\} \tag{13}$$

assumes the global normal form[5] with respect to the input v and the output y_d.
2. Under the additional assumption that $g_1 \equiv 0$ in (1), there exists non-negative scalar functions $f_1(\zeta_1, \hat{x}_1, \hat{x}_2, u_1)$, $f_2(\zeta_1, \hat{x}_1, \hat{x}_2, u_1)$ such that the feedback law

$$v = L_1^{-1}(\hat{x}_1)X^{-1}\{-[\delta + f_1 + f_2^2]\zeta_1 + v_d\}, \tag{16}$$

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[4]The approach to observer design as a problem of rendering an invariant manifold in the extended state-space of the plant and observer as attractive has been detailed in [17] and [16]- see also the references in there.

[5]A dynamical system with input u and output y, both of the same dimension, is said to be expressed in a global normal form if its dynamics can be written down in some suitable coordinates (z, y) as

$$\dot{z} = f_{11}(z) + f_{12}(z, y)y, \tag{14}$$
$$\dot{y} = f_{21}(z, y) + f_{22}(z, y)u, \tag{15}$$

where the square matrix $f_{22}(z, y)$ is invertible for every (z, y).

where $X(\in \mathbb{R}^{p \times p}) = X^\top > 0$, $\delta(\in \mathbb{R}) > 0$, makes the system strictly passive with respect to the manifold $\mathcal{M}$, uniformly for all $u_1 \in U$, $u_2 \in \mathbb{R}^m$, from the input v_d to the output ζ_1, with the storage function being given by $W(\zeta) = \frac{1}{2}\zeta_2^T Q \zeta_2 + \frac{1}{2}\zeta_1^T X \zeta_1$.

Proof. We begin by defining the functions $F_i(\zeta_1, \zeta_2, x_1, x_2, u_1)$, $i = 1, 2, 3$, with $F_1 \in \mathbb{R}^p$, $F_2 \in \mathbb{R}^{n-p}$, $F_3 \in \mathbb{R}^{n-p}$ as:

$$\begin{bmatrix} F_1(\zeta_1, \zeta_2, x_1, x_2, u_1) \\ F_2(\zeta_1, \zeta_2, x_1, x_2, u_1) \end{bmatrix} = [J(\hat{x}_1, u_1) - R(\hat{x}_1)]\frac{\partial H}{\partial \hat{x}}(\hat{x}) - [J(x_1, u_1) - R(x_1)]\frac{\partial H}{\partial x}(x),$$

where the matrices J and R are as defined in (2) and

$$F_3(\zeta_1, \zeta_2, x_1, x_2, u_1) = \frac{\partial \beta}{\partial \hat{x}_1}(\hat{x}_1)\{[J_1(\hat{x}_1, u_1) - R_1(\hat{x}_1)]\frac{\partial H}{\partial \hat{x}_1}(\hat{x}) + T(\hat{x}_1, u_1)\frac{\partial H}{\partial \hat{x}_2}(\hat{x})\}$$
$$- \frac{\partial \beta}{\partial x_1}(x_1)\{[J_1(x_1, u_1) - R_1(x_1)]\frac{\partial H}{\partial x_1}(x) + T(x_1, u_1)\frac{\partial H}{\partial x_2}(x)\}.$$

We next compute the dynamics

$$\dot{\zeta}_1 = F_1(\zeta_1, \zeta_2, x_1, x_2, u_1) + L_1(\hat{x}_1, u_1)v, \quad \dot{\zeta}_2 = \{F_2 - F_3\}(\zeta_1, \zeta_2, x_1, x_2, u_1), \quad (17)$$

where we have used the fact $g_1 \equiv 0$, and see that the system (1), (17) is in its global normal form with respect to input v and output $y_d(= \zeta_1)$.

We note that for each, $i = 1, 2, 3$, $F_i(\zeta_1, \zeta_2, x_1, x_2, u_1) = F_i(0, \zeta_2, x_1, x_2, u_1) + F_i(\zeta_1, \zeta_2, x_1, x_2 + \zeta_2, u_1)$ and further $F_i(\zeta_1, \zeta_2, x_1, x_2 + \zeta_2, u_1) = 0$ whenever $\zeta_1 = 0$. So, there exists continuous matrix functions $A_1(\zeta_1, x_1, x_2 + \zeta_2, u_1) \in \mathbb{R}^{p \times p}$, $A_i(\zeta_1, x_1, x_2 + \zeta_2, u_1) \in \mathbb{R}^{n-p \times p}$, $i = 2, 3$, such that $F_i(\zeta_1, \zeta_2, x_1, x_2 + \zeta_2, u_1) = A_i(\zeta_1, x_1, x_2 + \zeta_2, u_1)\zeta_1$, $i = 1, 2, 3$. Now, it is always possible to find non-negative continuous scalar functions $\psi_i(\zeta_1, x_1, x_2 + \zeta_2, u_1)$, $i = 1, 2, 3$ such that,

$$\|A_i(\zeta_1, x_2 + \zeta_2, x_1, u_1)\| \leq \psi_i(\zeta_1, x_1, x_2 + \zeta_2, u_1), \tag{18}$$

holds for all $\zeta_1, x_1, x_2 + \zeta_2, u_1$, where $\|\cdot\|$ is the induced norm of any general matrix. From the form of J, R in (1) and using (3) we obtain the inequality,

$$\|F_1(0, \zeta_2, x_1, x_2, u_1)\| \leq \|T(x_1, u_1)\|\|Q\zeta_2\|. \tag{19}$$

We next make use of the matrix norm property to obtain, $\|Q\zeta_2\| \leq \alpha\sqrt{\epsilon}\|\zeta_2\|$ where $\alpha = \lambda_M(Q)/\sqrt{\epsilon}$ and ϵ as in (7). We then obtain the inequalities

$$\|\zeta_2^T Q\{F_2 - F_3\}(\zeta_1, \zeta_2, x_1, x_2 + \zeta_2, u_1)\| \leq \alpha\{\psi_2 + \psi_3\}(\zeta_1, x_1, x_2 + \zeta_2, u_1)\sqrt{\epsilon}\|\zeta_2\|\|\zeta_1\|, \tag{20}$$

$$\|\zeta_1^T X L_1^{-1}(x_1 + \zeta_1)F_1(\zeta_1, \zeta_2, x_1, x_2 + \zeta_2, u_1)\| \leq \psi_1(\zeta_1, \zeta_2, x_1, x_2 + \zeta_2, u_1)\lambda_M(X)\lambda_M(L_1^{-1})\|\zeta_1\|^2, \tag{21}$$

$$\|\zeta_1^T X L_1^{-1}(x_1 + \zeta_1)F_1(0, \zeta_2, x_1, x_2, u_1)\| \leq \alpha\|T(x_1, u_1)\|\lambda_M(X)\lambda_M(L_1^{-1})\sqrt{\epsilon}\|\zeta_2\|\|\zeta_1\|. \tag{22}$$

We now consider the observer feedback law (16) with $f_1 = \psi_1\lambda_M(X)\lambda_M(L_1^{-1})$ and $f_2 = \alpha\{\psi_2 + \psi_3 + \|T\|\lambda_M(X)\lambda_M(L_1^{-1})\}$. We differentiate the storage function

$W(\zeta_1, \zeta_2) = \frac{1}{2}\zeta_2^{\mathrm{T}}Q\zeta_2 + \frac{1}{2}\zeta_1^{\mathrm{T}}X\zeta_1$ along (1), (17) and use (20), (21), (22) to finally obtain

$$\dot{W} \leq -\delta\|\zeta_1\|^2 + \zeta_1^{\mathrm{T}}v_{\mathrm{d}} - \frac{3}{4}\epsilon\|\zeta_2\|^2 - \{\frac{1}{2}\sqrt{\epsilon}\|\zeta_2\| - \|\zeta_1\|f_2\}^2.$$

Thus, the system is strictly passive with respect to the manifold $\mathcal{M}$, from input v_{d} to the output $y_{\mathrm{d}}(=\zeta_1)$ with the storage function being $W(\zeta_1, \zeta_2)$. Further, upon letting $v_{\mathrm{d}} = 0$ and performing some simple computations, we get that $\dot{W} \leq -\frac{1}{c}W$ where $c = \max(\frac{\lambda_{\mathrm{M}}(X)}{2\delta}, \frac{2\lambda_{\mathrm{M}}(Q)}{3\epsilon})$ and hence the Lyapunov function $W(\zeta_1, \zeta_2)$ exponentially decays to zero with convergence rate $\frac{1}{c}$. □

Remark 3. *The Assumption $g_1 \equiv 0$ ensures that the input u_2 is decoupled from the dynamics of (ζ_1, ζ_2). This would be the case in mechanical systems where the input is the external force applied and appears in the dynamics of the (unmeasured) generalized momenta. In case of constant L_1, L_2 matrices, that is, when β is a linear function, this assumption can be relaxed.*

In the next section, we illustrate our proposed observer design by considering three physical examples which come under the class of (1).

3 Physical examples

3.1 Permanent Magnet Synchronous Motor

We consider the permanent magnet synchronous motor example [26, Ch. 4],

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{bmatrix} 0 & 0 & \Phi_{q^0} \\ 0 & -R_{\mathrm{s}} & L_0 x_1 \\ -\Phi_{q^0} & -L_0 x_1 & -R_{\mathrm{s}} \end{bmatrix} \begin{pmatrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial x_2} \\ \frac{\partial H}{\partial x_3} \end{pmatrix} + \begin{bmatrix} -\frac{1}{n_{\mathrm{p}}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u, \quad (23)$$

with state $x = (\frac{j}{n_{\mathrm{p}}}\omega, L_{\mathrm{d}}i_{\mathrm{d}}, L_{\mathrm{q}}i_{\mathrm{q}})$, where ω is the angular velocity, $i_{\mathrm{d}}, i_{\mathrm{q}}$ are the currents and $j, n_{\mathrm{p}}, L_{\mathrm{d}}, L_{\mathrm{q}}, R_{\mathrm{s}}, \Phi_{q^0}$ are physical constants with $L_0 = L_{\mathrm{d}}n_{\mathrm{p}}/j$. The three inputs are the the stator voltage $(v_{\mathrm{d}}, v_{\mathrm{q}})$ and the constant load torque. The Hamiltonian $H(x)$ is given as

$$H(x_1, x_2, x_3) = \frac{1}{2}\frac{n_{\mathrm{p}}}{j}x_1^2 + \frac{1}{2L_{\mathrm{d}}}x_2^2 + \frac{1}{2L_{\mathrm{q}}}x_3^2. \quad (24)$$

We assume that only ω is measurable i.e $y = \omega$. Hence, (23) fits in the framework of (1). We let $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ be the state estimates, define their dynamics as in (4) and let $(e_1, e_2, e_3) = (\hat{x}_1 - x_1, \hat{x}_2 - x_2, \hat{x}_3 - x_3)$ be the estimation errors. We see that the damping matrix $\begin{bmatrix} R_{\mathrm{s}} & 0 \\ 0 & R_{\mathrm{s}} \end{bmatrix}$ is positive definite and hence as stated in Remark 2, we choose $L_2 = 0$. We next differentiate the Lyapunov function $V(e_2, e_3) = \frac{1}{2L_{\mathrm{d}}}e_2^2 + \frac{1}{2L_{\mathrm{q}}}e_3^2$ along the augmented system dynamics, subject to $e_1 \equiv 0$, to obtain, $\dot{V} \leq -\epsilon\{e_2^2 + e_3^2\}$, where $\epsilon = \frac{R_{\mathrm{s}}}{(L_{\mathrm{d}})^2+(L_{\mathrm{q}})^2}$. We choose $L_1 = \frac{j}{n_{\mathrm{p}}}$ and

the total storage function for the plant and observer dynamics as $W(e_1, e_2, e_3) = H(e_1, e_2, e_3)$. We next compute the inequalities (20), (21), (22) and obtain the functions f_1, f_2 as $f_1 = 0$, $f_2 = \frac{1}{\sqrt{\epsilon}} |\frac{L_0}{L_q L_d}| \{|\hat{x}_2| + |\hat{x}_3|\}$. We finally choose $v = -[\delta + \frac{1}{\epsilon} |\frac{L_0}{L_q L_d}|^2 \{|\hat{x}_2| + |\hat{x}_3|\}^2] e_1 + v_d$ and complete the observer construction.

3.2 Magnetic Levitation System

We consider the magnetic levitation system [16], consisting of an iron ball in a vertical magnetic field created by a single electromagnet, described by the model

$$
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{bmatrix} -R_2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{pmatrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial x_2} \\ \frac{\partial H}{\partial x_3} \end{pmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u
\tag{25}
$$

where x_1, x_2, x_3 correspond to the flux, position and momentum respectively and the system's energy is given as $H(x_1, x_2, x_3) = \frac{1}{2m} x_3^2 + mgx_2 + \frac{1}{2k} x_1^2 \{1 - x_2\}$ with $m > 0$, $k > 0$ being system constants. We assume only the flux and position to be measurable. Thus, (25) fits in the framework of (1). We let $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ be the state estimates, define their dynamics as in (4) and let $(e_1, e_2, e_3) = (\hat{x}_1 - x_1, \hat{x}_2 - x_2, \hat{x}_3 - x_3)$ denote the error. Upon choosing $L_1 = \begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix}$, $L_2 = [0 \ \ m]$ in our observer construction, we obtain the zero dynamics of $(x, \hat{x})$ with respect to the outputs (e_1, e_2) as $\dot{e}_3 = -\frac{e_3}{m}$. Computing the derivative of the Lyapunov function $V(e_3) = \frac{1}{2m} e_3^2$ along the zero dynamics yields $\dot{V} = -\{\frac{e_3}{m}\}^2$. We introduce the change of coordinates $\zeta = (\zeta_1, \zeta_2, \zeta_3) = (e_1, e_2, e_3 - e_2)$ to obtain the dynamics in the global normal form. We choose the total storage function as $W(\zeta) = \frac{1}{2} \zeta_1^2 + \frac{1}{2m} \zeta_2^2 + \frac{1}{2m} \zeta_3^2$ and compute the inequalities (20), (21), (22) to get $f_1 = \frac{1}{m^2} + \frac{R_2}{k} \{|1 - \hat{x}_2| + |x_1|\}$ and $f_2 = \frac{1}{2k} |x_1 + \hat{x}_1|$. We accordingly choose the observer feedback v as,

$$
\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -[\delta + \frac{1}{m^2} + \frac{R_2}{k} \{|1 - \hat{x}_2| + |x_1|\} + \frac{1}{4k^2} |x_1 + \hat{x}_1|^2] \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} + \begin{bmatrix} v_{d1} \\ v_{d2} \end{bmatrix}, \tag{26}
$$

and complete the observer construction.

3.3 Inverted Pendulum on a Cart

We consider the inverted pendulum on a cart example [22] which is a two-degree of freedom underactuated mechanical system whose inertia and input matrices are given as

$$
M = \begin{bmatrix} 1 & b\cos q_1 \\ b\cos q_1 & m_3 \end{bmatrix}, G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\tag{27}
$$

and its potential energy is $V(q) = a \cos q_1$. We introduce the coordinates, $(q, \bar{p}) = (q, T^\top(q)p)$ where q is the position vector, p is the momentum vector and the matrix $T(q)$ satisfies $T(q)T^\top(q) = M^{-1}(q)$ and is given as,

$$
T(q) = \begin{bmatrix} \frac{\sqrt{m_3}}{\sqrt{m_3 - b^2 \cos^2 q_1}} & 0 \\ \frac{-b\cos q_1}{\sqrt{m_3}\sqrt{m_3 - b^2 \cos^2 q_1}} & \frac{1}{\sqrt{m_3}} \end{bmatrix}.
\tag{28}
$$

Upon expressing the system in the coordinates $(q, \bar{p})$, we obtain the system dynamics

$$\begin{pmatrix} \dot{q} \\ \dot{\bar{p}} \end{pmatrix} = \begin{bmatrix} 0 & T(q) \\ -T^\top(q) & 0 \end{bmatrix} \begin{pmatrix} \frac{\partial V}{\partial q} \\ \bar{p} \end{pmatrix} + \begin{bmatrix} 0 \\ T^\top(q)G \end{bmatrix} u. \tag{29}$$

We assume that only q is measurable while p (and hence $\bar{p}$) is not measurable. We let $(\hat{q}, \hat{p})$ be the state estimates and define their dynamics as in (4). Upon choosing $L_1(q) = I_{2\times 2}$, $L_2(q) = \begin{bmatrix} 1 & 0 \\ \frac{b\cos q_1}{m_3} & 1 \end{bmatrix}$, it can be verified that Assumptions 1, 2 hold with $\epsilon = 2\min(1, \frac{1}{\sqrt{m_3}})$ and $\beta(q) = \begin{bmatrix} q_1 \\ \frac{b\sin q_1}{m_3} + q_2 \end{bmatrix}$. For constructing the full order observer we introduce the coordinates $\zeta_1 = \hat{q} - q$, $\zeta_2 = \hat{p} - \bar{p} - \{\beta(\hat{q}) - \beta(q)\}$. We then use the total storage function $W(\zeta_1, \zeta_2) = \frac{1}{2}\{\zeta_1^\top \zeta_1 + \zeta_2^\top \zeta_2\}$ and compute the inequalities (20), (21), (22) to get $f_1 = \sqrt{M_1 M_5} + \|\hat{p}\|\sqrt{M_3}$, $f_2 = \frac{1}{\sqrt{\epsilon}}\{\sqrt{M_5} + M_2 + \|\hat{p}\|M_4 + M_6\sqrt{M_1}\}$, where $M_1, M_2, M_3, M_4, M_5, M_6$ are constants which were computed from the inequalities. We accordingly design the observer feedback law given by (16) to complete the problem.

Remark 4. *Mechanical systems which can be expressed in the form (29) by a suitable change of the momentum coordinates $(p \rightsquigarrow \bar{p}(q, p))$ clearly belong to the class (1). It is known from the literature on differential geometry that if the Riemannian metric corresponding to the inertia matrix M of a mechanical system has zero curvature then there exists a factorization $T(q)T^\top(q) = M^{-1}(q)$ such that the dynamics expressed in the coordinates $(q, \bar{p}) = (q, T^\top(q)p)$ is affine in $\bar{p}$ and is of the form (29). The inverted pendulum on the cart satisfies this property. We refer the reader to the references [27], [23], [24] which give a more detailed explanation of this property in the context of linearization.*

4 Conclusion

We have proposed a passivity based full-order observer for a class of port-Hamiltonian systems. The idea is to render the augmented system (composed of the plant and observer dynamics) strictly passive with respect to an invariant manifold defined on the extended state space on which the state estimation error is zero. The observer construction is done in two steps: (1) Compute the observer gain matrices $L_1(\hat{x}_1)$ and $L_2(\hat{x}_1)$ such that Assumptions 1 and 2 get satisfied, (2) Compute the partial state feedback law $v(y, \hat{x}, u_1)$ by the procedure outlined in the proof of Theorem 2.

We have not given an algorithmic procedure for finding $L_1(\hat{x}_1)$ and $L_2(\hat{x}_1)$, that involves solving a set of algebraic and partial differential equations, which is usually a difficult task. However, we have illustrated the observer design procedure on three physical systems: a permanent magnet synchronous motor and a magnetic levitation system, where we use constant L_1, L_2 matrices, and the inverted pendulum on the cart system where L_2 is state dependent. We also identified a class of mechanical systems (as stated in Remark 4) which admit this passivity based observer. However, the task of computing the matrices $L_1(\hat{x}_1)$ and $L_2(\hat{x}_1)$ for such systems still remains to be done.

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