Moduli of singularities of vector fields

Takens, Floris

[Link to publication in University of Groningen/UMCG research database](#)

MODULI OF SINGULARITIES OF VECTOR FIELDS

Floris Takens

RECENTLY a number of situations were discovered where ratios between eigenvalues, of singularities of vectorfields, or of fixed points of diffeomorphisms, appeared as topological invariants. These phenomena were related with non-transverse intersections of stable and unstable manifolds. See [2, 3]. In this note I give an example where such moduli appear at isolated singularities of vectorfields (i.e. independent of non-transverse intersection of stable and unstable manifolds). Singularities with such moduli can in general not be avoided in generic 6-parameter families of vectorfields.

§1. INTRODUCTION: DEFINITIONS AND RESULTS

The vector fields I consider in this paper will be assumed to be C^∞ (though C^k $k \geq 14$ is enough for the arguments we present here). If X is a vector field on $\mathbb{R}^n$ (or on some manifold) we say that p is a singularity of X if $X(p) = 0$. If X and X' are vector fields on $\mathbb{R}^n$ with singularities in p, respectively p', we say that these singularities are *topologically equivalent* if there are neighbourhoods U and U' of p and p' in $\mathbb{R}^n$ are a homeomorphism $h: (U, p) - (U', p')$ which maps integral curves of $X|U$ direction preserving on integral curves of $X'|U'$.

Without loss of generality we may restrict to vector fields on $\mathbb{R}^n$ which have a singularity in 0. Two such vector fields X and X' are said to have the same k-*jet* if all partial derivatives up to (and including) order k of X and X' are equal in the origin. The set of such k-jets is denoted by $J^{k,n}$. So roughly speaking an element $[X] \in J^{k,n}$ "is" a Taylor's expansion, up to order k, of a vector field X in $0 \in \mathbb{R}^n$, with $X(0) = 0$. $J^{k,n}$ can be identified with a vector space, e.g. by using the coefficients of the corresponding Taylor's expansion.

There does not seem to be any generally accepted definition of modulus, so also here we choose one which is suitable for our purpose.

Definition. A *modulus* for k-jets of singularities of vector fields on $\mathbb{R}^n$ is a pair, consisting of a smooth submanifold $M \subset J^{k,n}$ and a function $\sigma: M \to \mathbb{R}$ with $d\sigma \neq 0$ such that for $[X]$, $[X'] \in M$ with $\sigma([X]) \neq \sigma([X'])$, any two representatives X, X' of $[X]$, $[X']$ are topologically non-equivalent. We shall call the co-dimension of M in $J^{k,n}$ the *co-dimension* of the modulus.

Note that if (M, σ) is such a modulus with co-dimension l, then one cannot avoid the occurence of singularities whose k-jet belongs to M in generic l-parameter families of vector fields on $\mathbb{R}^n$. Such families are unstable since a nearby family can have a singularity whose k-jet is also in M, but where the value of σ is different.

We shall prove

THEOREM. *There exists a modulus for 5-jets of singularities of vector fields on $\mathbb{R}^4$ with co-dimension 6*

I don't know whether such modulus exists for singularities of vector fields on $\mathbb{R}^3$; for vector fields on $\mathbb{R}^2$ no such moduli exists, this follows from [1]. It seems that in dimensions greater than 4 also such moduli exist, but I was not yet able to prove this.

§2. NORMAL FORMS

The material in this section is not new but is mainly included to make the paper self contained. Proofs can be found, e.g. in [4].

Let $X = \Sigma_{i=1}^4 X_i(\partial/\partial x_i)$ be a vector field on $\mathbb{R}^4$ with a singularity in the origin 0. We assume that, possibly after a linear change of coordinates, the linear part of X has the form

$$\lambda \cdot \left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}\right) + \mu \cdot \left(x_3 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_3}\right)$$

with $m \cdot \lambda + n \cdot \mu \neq 0$ whenever $m, n \in \mathbb{Z}$ and $1 \leq |m| + |n| \leq k + 1$ (the integer k will be specified later). Note that for any k, the set of such jets is a submanifold of co-dimension two (in the space of all jets of singularities of vector fields on $\mathbb{R}^4$).

In this situation there is a non-linear change of coordinates which puts X in the following form:

$$X = \lambda \cdot \left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}\right) + \mu \cdot \left(x_3 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_3}\right) + f_1(x_1^2 + x_2^2, x_3^2 + x_4^2) \cdot \left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}\right)$$

$$+ f_2(x_1^2 + x_2^2, x_3^2 + x_4^2) \cdot \left(x_3 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_3}\right) + g_1(x_1^2 + x_2^2, x_3^2 + x_4^2) \cdot \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\right)$$

$$+ g_2(x_1^2 + x_2^2, x_3^2 + x_4^2) \cdot \left(x_3 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4}\right) + R_k,$$

where R_k is a vector field with zero k-jet and where $f_1, f_2, g_1, g_2 \colon \mathbb{R}^2 \to \mathbb{R}$ are zero in the origin. The proof can be found in [4]; the geometric meaning can be stated as "up to order k, the singularity can be made symmetric with respect to the rotations in the one-jet". We call

$$\tilde{X} = g_1(y_1^2, y_2^2)y_1 \frac{\partial}{\partial y_1} + g_2(y_1^2, y_2^2)y_2 \frac{\partial}{\partial y_2}$$

the reduced vector field; since it is only defined modulo terms of order $\geq k + 1$, we should speak about a k-jet instead of a vector field.

We now restrict our attention to those singularities for which the 1-jets of the corresponding functions g_1 and g_2 are zero. This is equivalent with four more conditions so we are now dealing with singularities whose jets form a co-dimension 6 submanifold in the jet space of singularities.

Within this collection of singularities we restrict to those singularities for which the reduced vector field $\tilde{X}$ has the following phase portrait (Fig. 1) and such that the 5-jet of $\tilde{X}$ determines its topological type. To be more specific, we require that the vector field, obtained from $\tilde{X}$ by blowing up (or writing it in polar coordinates (r, φ)) and deviding by r^4 is hyperbolic and has the following phase portrait (Fig. 2) near the circle $\{r = 0\}$. This condition on the 5-jet of X is certainly open; in order to see that it does not define the empty set, one can take

$$\tilde{X} = \frac{\partial H}{\partial y_1} \frac{\partial}{\partial y_2} - \frac{\partial H}{\partial y_2} \frac{\partial}{\partial y_1}$$

with

$$H(y_1, y_2) = y_1 \cdot y_2 \cdot (2y_1^4 + 2y_2^4 - 5y_1^2 y_2^2) = -y_1 \cdot y_2 \cdot (y_1^2 - 2y_2^2)(y_2^2 - 2y_1^2).$$
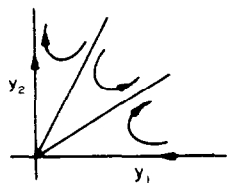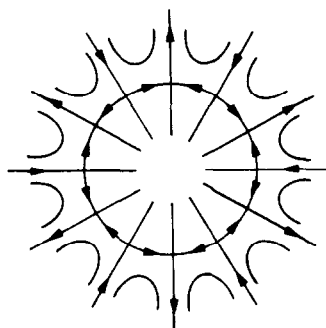
Fig. 1.   Fig. 2.

We now want that the corresponding vector field X has invariant cones corresponding to the separatrices of $\tilde{X}$ in the positive quadrant. This follows from the invariant manifold theorem in [4] provided in the condition on the eigenvalues λ, μ we take $k \geq 14$ (this estimate is not supposed to be harp). It also follows from this invariant manifold theorem that, in a small neighbourhood of the origin, the only points whose X orbit have the origin as α- or ω-limit, are in these invariant cones (and manifolds), which are the following:

2-manifolds W^s and W^u, corresponding to the separatrices of $\tilde{X}$ which coincide with the y_1-, respectively y_2-axis;
cones C^s and C^u on tori, corresponding to the stable, respectively unstable, separatix of $\tilde{X}$ in the interior of the positive quadrant.

§3. CONSTRUCTION OF THE MODULUS

We take the co-dimension 6 submanifold $M \subset J^{5,4}$ consisting of those 5 jets of vector fields on $\mathbb{R}^4$ as considered in the previous section, namely with two pairs of purely imaginary eigenvalues $\pm i\lambda$ and $\pm i\mu$ satisfying nonresonance conditions with $k = 14$ and such that the reduced vectorfield has a trivial 3-jet and a 5-jet as specified in the previous section. The function $\sigma: M \to \mathbb{R}$ is defined as

$$\sigma([X]) = \frac{\lambda([X])}{\mu([X])},$$

where the eigenvalues of the linear part of $[X]$ are $\pm i\lambda$, $\pm i\mu$ with $\lambda > 0$ (because of the nonresonance conditions, both are non-zero).

In order to show that (M, σ) is indeed a modulus, we have to show that if X and X' are vector fields on $\mathbb{R}^4$ with a singularity in 0 such that their 5-jets in 0 are in M and such that their singularities (in the origin) are topologically equivalent, then $\sigma([X]) = \sigma([X'])$.

Let U, U' be neighbourhoods of the origin in $\mathbb{R}^4$ and $h: (U, 0) \to (U', 0)$ a homeomorphism mapping orbits of $X|U$ direction preserving to orbits of $X'|U'$. The eigenvalues of X, X' are denoted by $\pm i\lambda$, $\pm i\mu$, respectively $\pm i\lambda'$, $\pm i\mu'$; the stable and unstable manifolds and cones of X, X', are, as in the previous section, denoted by W^s, W^u, C^s, C^u, respectively. W'^s, W'^u, C'^s, W'^u. Clearly h maps these manifolds and cones (belonging) to X), intersected with U, to the corresponding intersection of U' with the invariant manifolds and cones of X'.

We assume that the pairs $(U, W^s \cap U)$ and $(U, W^u \cap U)$ are homeomorphic with the pair consisting of the open unit disc in $\mathbb{R}^4$ and the intersection of this disc with a plane through the origin; if this were not the case, we can reach this by selecting a smaller U.

Let D^s, D^u be fundamental domains in $C^s \cap U$ and $C^u \cap U$. We choose a sequence of integral curves γ_i: $[0, T_i] \to U$ of X such that $\gamma_i(0)$ approaches D^s and $\gamma_i(T_i)$ approaches D^u. Since D^s and D^u are compact, namely tori, we may assume that $\gamma_i(0)$ and $\gamma_i(T_i)$ converge to points, say to $p \in D^s \subset C^s$ and $q \in D^u \subset C^u$. Let now V_p and V_q be small neighbourhoods of p and q in U and let $\tilde{\gamma}$ be a curve in U from p to q not intersecting W^s or W^u. For the above sequence of integral curves γ_i, it is clear that for i sufficiently big, say for $i \geq i_0$, $\gamma_i(0) \in V_p$ and $\gamma_i(T_i) \in V_q$. For each of these integral curves γ_i, $i \geq i_0$ we make a closed curve Γ_i:

Γ_i consists of a curve from p to $\gamma_i(0)$ in V_p, followed by γ_i, followed by a curve in V_q from $\gamma_i(T_i)$ to q followed by $\tilde{\gamma}$. For sufficiently small V_p and V_q the winding numbers of Γ_i around $W^s \cap U$ and around $W^u \cap U$ are independent of the choice of the curves in V_p, V_q. The winding number $w(\Gamma_i, W^s)$ of Γ_i around $W^s \cap U$ is the non-negative integer such that the fundamental class of Γ_i is $w(\Gamma_i, W^s)$·times a generator of $H_1(U - W^s \cap U)$. In the following we assume that V_p and V_q are sufficiently small in this sense. We observe that

$$\lim_{i \to \infty} \frac{w(\Gamma_i, W^s)}{w(\Gamma_i, W^u)} = \frac{\mu}{\lambda}.$$

This follows from the fact that, in a very small neighbourhood of $0 \in \mathbb{R}^4$, integral curves rotate around W^s with angular velocity μ and around W^u with angular velocity λ, and the fact that, for $i \to \infty$, $\gamma_i(t)$ is for most t in a very small neighbourhood of 0.

Finally, since h is a homoeomorphism (and hence preserves topological notions as winding numbers) and maps integral curves to integral curves, it follows that

$$\lim_{i \to \infty} \frac{w(\Gamma_i, W^s)}{w(\Gamma_i, W^u)} = \lim_{i \to \infty} \frac{w(h(\Gamma_i, W'^s)}{w(h(\Gamma_i), W'^u)}$$

and hence it follows that $\sigma([X]) = \sigma([X'])$.

§4. CONCLUDING REMARK

While writing this paper, F. Dumortier showed me that no moduli of this sort can appear for co-dimension two singularities.

REFERENCES

1. F. DUMORTIER: Singularities of vector fields on the plane. *J. Diff. Equ.* **23** (1977), 53–106.
2. J. PALIS: Moduli of stability of bifurcation theory. *Proc. Int. Congres of Mathematics*, Helsinki (1978).
3. S. VAN STRIEN: One parameter families of vector fields: bifurcations near saddle-connections, to be submitted as Ph.D. thesis at the state University of Utrecht.
4. F. TAKENS: Singularities of vector fields. *Publ. I.H.E.S.* **43** (1974), 47–100.

Mathematisch Instituut
Rijksuniversiteit Groningen
Hoogbouw WSN, Nettelbosje 2
Postbus 800, 9700 *Av Groningen*
The Netherlands