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Published in:
Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen. Series A:Mathematical Sciences

DOI:
10.1016/1385-7258(83)90061-6

The formal classification of linear difference operators

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Communicated by Prof. T.A. Springer at the meeting of November 25, 1982

ABSTRACT

A Jordan canonical form for formal difference operators, like the one in [7], is derived in a way inspired by [3], [4]. This yields a classification of meromorphic difference operators in a neighbourhood of infinity, up to formal equivalence.

INTRODUCTION

Let $u(z)$ be an m-dimensional vector function, meromorphic in a full neighbourhood of infinity. T is the operator defined by: $Tu(z)=u(z+1)-A(z)u(z)$, where $A(z)$ is a square $m\times m$ matrix function, meromorphic in the same region. In [7] H.L. Turritin proved that by a formal basis transformation T may be brought into the following form: $Tv(z)=v(z+1)-B(z)u(z)$ where

$$B=\text{ diag }\{B_1,\ldots,B_r\},\ B_i=z^{\lambda_i}\left(b_iI_i+\frac{1}{z}J_i\right),\ \lambda_i\in\frac{1}{m!}\mathbb{Z},$$

with b_i a polynomial of degree $m!$ in $z^{-1/m!}$, $b_i(0)\neq 0$, for $i=1,\ldots,r'$; $b_i=\lambda_i=0$ for $i=r'+1,\ldots,r$; and J_i the matrix:

$$J_i=\begin{pmatrix}0&1&0&.&.&0&0\\0&0&1&.&.&0&0\\.&.&.&.&.&.&.\\0&0&0&.&.&0&1\\0&0&0&.&.&0&0\end{pmatrix}$$

* Research supported by the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

In the same paper he proved an analogous result for differential operators. Recently several authors have proved these last results by entirely different methods: Levelt [3], Malgrange [4] and Robba [5]. Levelt's method is the most complete, since it also yields uniqueness properties. As will be explained in § 7, his method does not work for difference operators. In this paper I shall prove the result mentioned above, by Malgrange's method, and some uniqueness statements in a way inspired by Levelt's.

Just before I finished this paper I received a preprint from Duval [2], in which she proves Turritin's theorem by the method of Robba.

The problem treated in this paper was suggested to me by professor van der Put, whom I would like to thank for all the inspiring discussions we had on the subject.

§ 1. PRELIMINARY REMARKS AND NOTATIONS

For the moment assume $A(z)$ is invertible, and consider $A^{-1}T$ instead of T. Substitute $t=(1/z)$. Then $A^{-1}T$ transforms into an operator defined by:

$$\tilde{u}(t)\mapsto \tilde{A}(t)\tilde{u}\left(\frac{t}{t+1}\right)-\tilde{u}(t).$$

Denote by Φ the operator $\Phi u(t)=A(t)u(t/t+1)$. Then $\Phi(au)(t)=a(t/t+1)u(t)$ for all meromorphic functions a. I shall call Φ a difference operator in the sequel, and my aim is to find a special matrix representation for Φ.

I shall use the following notation:
$\mathcal{O}=\mathbb{C}[[t]]=\{\sum_{i=0}^{\infty} f_i t^i \mid f_i\in\mathbb{C}\}$, the ring of formal power series;
$K=\mathbb{C}((t))$, the quotient field of $\mathcal{O}$;
$\upsilon:\mathcal{O}\to\mathbb{N}\cup\{0\}$ is the additive valuation defined by $\upsilon(\sum_{i=j}^{\infty} f_i t^i)=j$ if $f_j\neq 0$.

υ extends in a unique way to a valuation on K, and even to a valuation on $\bar{K}$, an algebraic closure of K. This valuation will still be denoted by υ. As is well known the field of Puiseux series over $\mathbb{C}$ is an algebraic closure of K. With the usual abuse of notation I shall write $\bar{K}=\bigcup_{q\in\mathbb{N}}\mathbb{C}((t^{1/q}))$. L will be an algebraic extension of K, contained in $\bar{K}$. In general I shall write $L=\mathbb{C}((s))$, with $s^q=t$, and $\mathcal{O}_L=\mathbb{C}[[s]]$, the valuation ring of L. $\varphi:K\to K$ is the $\mathbb{C}$-automorphism defined by $\varphi(t)=(t/t+1)$. Then φ extends to a $\mathbb{C}$-automorphism of $\bar{K}$ by defining for all $q\in\mathbb{N}$:

$$\varphi(t^{1/q})=t^{1/q}\left(\sum_{i=0}^{\infty}\binom{-1/q}{i}t^i\right).$$

V is an m-dimensional linear space over K. We denote by $\Phi: V\to V$ a difference operator, i.e. a $\mathbb{C}$-linear map satisfying $\Phi(av)=\varphi(a)\Phi v$ for all $a\in K$, $v\in V$. If L is an extension of K, then the map $\varphi\otimes\Phi: L\otimes_K V\to L\otimes_K V$ will still be denoted by Φ.

$K[X,\varphi,0]$ is a skew polynomial ring over K. Its elements are polynomials in X over K, which add in the usual way. The multiplication is non-commutative: $Xa=\varphi(a)X$ for all $a\in K$.

Define a left $K[X, \varphi, 0]$-module structure on V by $Xv = \Phi v$ for all $v \in V$. Note that $\bar{K} \otimes_K V$ becomes a $\bar{K}[X, \varphi, 0]$-module in this way.

In general one may define a skew polynomial ring $K[X, \psi, \delta]$ as the set of polynomials in X, with coefficients in K, with the (non-commutative) multiplication $Xa = \psi(a)X + \delta(a)$, for all $a \in K$. Here ψ is a $\mathbb{C}$-automorphism of K, and δ a ψ-derivation, i.e. a $\mathbb{C}$-linear map satisfying $\delta(ab) = \psi(a)\delta(b) + \delta(a)b$, as one may derive from $X(ab) = (Xa)b$. Now if $\theta: V \to V$ is a $\mathbb{C}$-linear map satisfying $\theta(av) = \psi(a)\theta v + \delta(a)v$, then θ defines a $K[X, \psi, \delta]$-module structure on V. In the sequel I shall use the following result: (Cohn [1, p. 67, 299]).

$K[X, \psi, \delta]$ is Euclidean with respect to the degree function, and every finitely generated module M over $K[X, \psi, \delta]$ is the direct sum of cyclic submodules.

Note that this implies that V is the direct sum of subspaces invariant under θ, each having a basis of the form $(v, \theta v, \theta^2 v, \ldots)$, i.e. containing a cyclic vector.

§ 2. THE NEWTON POLYGON ASSOCIATED TO A DIFFERENCE OPERATOR

In this section assume that the difference operator $\Phi: V \to V$ is invertible, and induces the structure of a cyclic $K[X, \varphi, 0]$-module on V. This implies the existence of a (clearly non-unique) $P \in K[X, \varphi, 0]$, say $P = a_m X^m + \ldots + a_0$, with $a_m \neq 0$, $a_0 \neq 0$, such that $V \cong K[X, \varphi, 0]/(P)$. Define the *Newton polygon* of P in the following way (slightly different from [3]): Associate to a_i the half-line in $\mathbb{R}^2 : x = i,\ y \leq v(a_i)$. Then $N(P)$ is the convex hull of the union of the half-lines associated to $a_0, \ldots, a_m$. Number the non-vertical edges from left to right: $\Lambda_1, \ldots, \Lambda_r$ and define λ_i as the slope of Λ_i. Then $-\infty < \lambda_i \ldots < \lambda_r < \infty$. If necessary I shall indicate the dependence on P by writing $\lambda_i(P)$.

In the same way one defines a Newton polygon for elements in an extension $L[X, \varphi, 0]$, denoted by $N_L(P)$ if necessary. The same arguments as used in [3] lead to the following properties:

i) If $P = QR$, then $N(P) = N(Q) + N(R)$, $\{\lambda_i(P)\} = \{\lambda_i(Q)\} \cup \{\lambda_i(R)\}$.

ii) P is in a natural way an element of $\bar{K}[X, \varphi, 0]$, and $N_{\bar{K}}(P) = N_K(P)$.

iii) Substitution of $Y = t^\mu X$, $\mu \in \mathbb{Q}$ yields a polynomial $P_1 \in \bar{K}[Y, \varphi, 0]$. $N(P_1)$ is obtained from $N(P)$ by rotating the lower edge of $N(P)$ by an angle α, with $tg\alpha = -\mu$, around $(0, v(a_0))$. Then $\lambda_i(P_1) = \lambda_i(P) - \mu$.

REMARK. *Note that these properties do not depend on the particular form of φ. In § 9 I shall use this definition and these properties for arbitrary ψ.*

§ 3. A HENSEL LEMMA FOR SKEW POLYNOMIAL RINGS

$\mathcal{O}[X, \psi, \delta]$ is the subset of $K[X, \psi, \delta]$ consisting of all polynomials with coefficients in $\mathcal{O}$. If $P \in \mathcal{O}[X, \psi, \delta]$, P may be written uniquely as $\sum_0^\infty t^j P_j$, $P_j \in \mathbb{C}[X]$. Note that Satz IV of [6] implies that $\psi(\mathcal{O}) \subset \mathcal{O}$, or even $\psi(t) = \psi_0 t + at^2$, with $\psi_0 \in \mathbb{C}^*$, $a \in \mathcal{O}$. In general, however $\delta(\mathcal{O}) \not\subset \mathcal{O}$, so $\mathcal{O}[X, \psi, \delta]$ is a ring only with the additional condition $\delta(t) \in \mathcal{O}$.

LEMMA 1. *Let ψ be as above, and assume $\delta(t)=\delta_0 t+bt^2$, $\delta_0\in\mathbb{C}$, $b\in\mathcal{O}$. Let P be a monic polynomial in $\mathcal{O}[X,\psi,\delta]$. Suppose $P_0=qr$, q and r monic polynomials in $\mathbb{C}[X]$. Further let the following condition be satisfied: If α is any root of r, then $\psi_0^{-k}\alpha-\delta_0\sum_{i=0}^{|k|-1}\psi_0^{i-k}$ is not a root of q for all integers k. Then there exist monic polynomials Q and $R\in\mathcal{O}[X,\psi,\delta]$ such that:*

i) $P=QR$,

ii) $Q_0=q$, $R_0=r$.

Moreover Q and R are unique, and one has an isomorphism of left modules: $K[X,\psi,\delta]/(P)\cong K[X,\psi,\delta]/(Q)\oplus K[X,\psi,\delta]/(R)$.

PROOF. Write $Q=\sum t^jQ_j$ and $R=\sum t^jR_j$ and try to find Q_j and R_j inductively. Define $q_k=(t^{-k}Qt^k)_0$, then $q_k\in\mathbb{C}[X]$, and from $(X-\alpha)t^k=\psi(t^k)X+\delta(t^k)-$ $-\alpha t^k=\psi_0 t^k(X-\alpha)+\delta_0\sum\psi_0^{k-i}+t^{k+1}\hat{Q}$, with $\hat{Q}\in\mathcal{O}[X,\psi,\delta]$, it follows that q_k and r are relatively prime for all integers k. The equation for Q_k and R_k becomes: $Q_kr+q_kR_k=P_k+$ expression in the coefficients of $Q_0,R_0,Q_1,\dots,R_{k-1}$. This equation has a unique solution $Q_k,R_k\in\mathbb{C}[X]$, with degree $Q_k<$ degree q, degree $R_k<$ degree r. In this way one finds by induction on k:

$$\sum_{j=0}^{k} t^jP_j=\Big(\sum_{j=0}^{k} t^jQ_j\Big)\Big(\sum_{j=0}^{k} t^jR_j\Big)\bmod t^{k+1}.$$

Letting $k\to\infty$ one finds a unique solution Q, $R\in\mathcal{O}[X,\psi,\delta]$ such that i) and ii) are satisfied. The proof of the last assertion is identical with the proof of the analogous statement in [3], and will be omitted here.

§ 4. DECOMPOSITION OF V ACCORDING TO THE NEWTON POLYGON

Let Φ, V, P, $N(P)$ be as in § 2. Without loss of generality one may assume that $a_m=1$. Then one may find a decomposition of P into polynomials of lower degree in an extension of $K[X,\varphi,0]$, each having a Newton polygon with one slope; corresponding to this there exists a decomposition of V into subspaces stable under Φ.

THEOREM 2. *There exists a finite extension L of K, say $L=C((s))$, $s^q=t$, such that*

$$P=\prod_{i,j}P_{ij},\ \textit{with}\ P_{ij}=p_{ij}\Big(\prod_{h=1}^{n_{ij}}(t^{\lambda_i-1}-\alpha_{ij}+c_{ijh})+\tilde{P}_{ij}\Big),$$

where

$$p_{ij}\in L,\ \alpha_{ij}=\sum_{k=0}^{q}\alpha_{ijk}s^{-k},\ \alpha_{ijk}\in\mathbb{C},\ \alpha_{ijq}\neq 0,\ c_{ijh}\in\frac{1}{q}\alpha_{ijq}\mathbb{Z},$$

$\tilde{P}_{ij}\in\mathcal{O}_L[t^{\lambda_i-1}X,\varphi,0]$, degree $\tilde{P}_{ij}<n_{ij}$.

Moreover:

i) $L\otimes_K V=\bigoplus_{i,j}V_{ij}$, $V_{ij}\cong L[X,\varphi,0]/(P_{ij})$,

ii) $q\,|\,m!$.

PROOF. By induction on $m=\dim_K V=\text{degr } P$. If $m=1$ the theorem is trivial, so assume that $m>1$, and that the theorem is proved for all $m'<m$.

λ_r, the slope of the last non-vertical edge Λ_r of $N(P)$ is rational, say $\lambda_r=(l_r/q_r)$, with $q_r\in\{1,\dots,m\}$, $gcd(l_r,q_r)=1$. Substitute $Y=t^{\lambda_r}X$, then the resulting polynomial $\tilde{P}=t^{\lambda_r}\varphi(t^{\lambda_r})\ \dots\ \varphi^{m-1}(t^{\lambda_r})P\in L_r[Y,\varphi,0]$, where $[L_r:K]=q_r$. Put $\tilde{P}=Y^m+b_{m-1}Y^{m-1}+\dots+b_0$. Then $N(\tilde{P})$ has slopes $\tilde{\lambda}_i=\lambda_i-\lambda_r\le 0$, so $\tilde{P}\in\mathcal{O}_{L_r}[Y,\varphi,0]$, and since $\tilde{\lambda}_r=0$ we have

$$\tilde{P}_0=Y^m+b_{m-1}(0)Y^{m-1}+\dots+b_n(0)Y^n,\text{ with } b_n(0)\neq 0,\ 0\le n<m.$$

Consider the following argument:
(A) $\tilde{P}_0=\bar{P}_0\cdot Y^n$, where $\bar{P}_0$ is a polynomial in Y^{q_r}. Then $\tilde{P}_0$ splits into q_r+1 factors, which are relatively prime, and hence lemma 1 assures that $\tilde{P}$ splits into q_r+1 factors, one of degree n, with slopes $\tilde{\lambda}_1,\dots,\tilde{\lambda}_{r-1}$, and q_r of degree $(m-n)/q_r$ with slope $\lambda_r=0$.

If $n>0$ or $q_r>1$ then (A) reduces the rank and the induction hypothesis leads to a proof of the theorem. So suppose $q_r=r=1$ (implying $n=0$ and $L_r=K$).

If $\tilde{P}_0$ splits nonetheless, m is reduced again, so assume moreover: $\tilde{P}_0=(Y-a)^m$, $a\in\mathbb{C}^*$. Define $Y_1=(1/t)\ (Y-a)$, then $\tilde{P}(Y)=t\varphi(t)\ \dots\ \varphi^{(m-1)}(t)P_1(Y_1)$, with $P_1\in K[Y_1,\varphi,\delta]$, $\delta=(a/t)(\varphi-1)$. Now consider $N(P_1)$. Let $\lambda_{r'}(P_1)$ have the same meaning for P_1 as $\lambda_r(P)$ for P.
1) If $\lambda_{r'}(P_1)<0$, then $P_1=Y_1^m+t(\dots)$ and hence $P=p((1/t)Y-(a/t))^m+t(\dots)$.
2) If $\lambda_{r'}(P_1)>0$, it is necessarily not an integer, since the construction implies $\lambda_r<1$. Hence one may apply the argument (A) onto P_1.
3. If $\lambda_{r'}(P_1)=0$, then $P_1\in\mathcal{O}[Y_1,\varphi,\delta]$ and $\delta(t)=at+$ higher order terms. If $(P_1)_0$ splits into factors which satisfy the condition of lemma 1, then the rank is reduced, leaving

$$P_1=\prod_{h=1}^{m}(Y_1-\beta+c_h)+t(\dots),\text{ with } c_h\in a\mathbb{Z}.$$

Hence

$$P=p\prod_{h=1}^{m}\left(t^{\lambda_r-1}X-\frac{a}{t}-\beta+c_h\right)+t(\dots),\text{ with } c_h\in a\mathbb{Z}.$$

§ 5. SIMPLIFICATION

Now look at V_{ij} for fixed i and j. For simplicity I shall omit these subscripts in this section: $V\cong L[X,\varphi,0]/(P)$. Choose a basis (e) of V, such that e_1 corresponds to 1, and if e_h corresponds to S, then e_{h+1} corresponds to $(t^{\lambda-1}-\alpha+c_h)S$. Then $t^{\lambda-1}\Phi-\alpha$ is pseudolinear with respect to φ and δ, $\delta=\alpha(\varphi-1)$. And the matrix $\text{Mat}(t^{\lambda-1}\Phi-\alpha,(e))=A_0+sA_1$, where $A_1\in\text{End}\ \mathcal{O}_L^m$, and

$$A_0=\begin{pmatrix} c_1 & 0 & . & . & . & . & 0\\ 1 & c_2 & . & . & . & . & 0\\ . & . & . & . & . & . & .\\ 0 & 0 & . & . & 0 & 1 & c_m\end{pmatrix}$$

LEMMA 3. *There exists a* $B \in Gl_m(L)$*, such that with respect to the basis* (Be) *we have* $\mathrm{Mat}(t^{\lambda-1}\Phi - \alpha, (Be)) = F_0 + sF_1$*, where* F_0 *is a nilpotent matrix with entries in* $\mathbb{C}$*, and* $F_1 \in End\ \mathcal{O}_L^m$.

PROOF. We have $c_h = (1/q)n_h\alpha_q$. Assume for simplicity $0 = n_1 < n_2 < \ldots < n_m$. We may achieve that $c_1 = 0$, by a basis transformation of the type $(e) \mapsto s^k(e)$.

One proves the lemma with induction on n_m. If $n_m = 0$ there is nothing to prove. Assume $n_m > 0$, then there is a constant basis transformation $C \in Gl_m(\mathbb{C})$, such that $CA_0C^{-1} = \mathrm{diag}\ \{F_{11}, F_{22}\}$, where F_{22} has the unique eigenvalue c_m, and F_{11} does not have the eigenvalue c_m. Let (f) be (Ce), the image of (e) under C.

Assume

$$CsA_1C^{-1} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}.$$

If $D = \mathrm{diag}\ \{I, sI\}$, then

$$\mathrm{Mat}(t^{\lambda-1} - \alpha, (Df)) = \begin{pmatrix} F_{11} & 0 \\ (G_{12})_0 & F_{22} - (1/q)\alpha_q \end{pmatrix} + s\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}.$$

The eigenvalues of the leading matrix are the eigenvalues of F_{11} and of $F_{22} - (1/q)\alpha_q$, so that n_m is reduced by one.

The next step is to prove that one may remove all powers of s.

LEMMA 4. *Assume* $\mathrm{Mat}(t^{\lambda-1}\Phi - \alpha), (e)) = F = F_0 + F_1s + F_2s^2 + \ldots$*, with* F_0 *nilpotent. Then there is an* $A \in Gl_m(\mathcal{O}_L)$*, such that* $\mathrm{Mat}(t^{\lambda-1}\Phi - \alpha, (Ae)) = F_0$.

PROOF. Define $f_i = Ae_i$, then $(t^{\lambda-1}\Phi - \alpha)f_i = (t^{\lambda-1}\Phi - \alpha)Ae_i = F\varphi(A)e_i + \delta(A)e_i$. I want to find A such that $(t^{\lambda-1}\Phi - \alpha)f_i = F_0f_i$, so the equation to solve is

$$F\varphi(A) + \delta(A) = AF_0.$$

Try $A = I + \sum_{i=1}^{\infty} A_is^i$. Comparing powers of s, we obtain an equation for A_i:

$$A_iF_0 - \left(F_0 - \frac{i}{q}\,\alpha_qI\right)A_i = \text{an expression in } A_0, \ldots, A_{i-1}, F_0, \ldots, F_i.$$

Since F_0 and $F_0 - (i/q)\alpha_qI$ have no eigenvalues in common, there is a unique solution.

COROLLARY. $V \cong \bigoplus_{k=1}^{r} L[X, \varphi, 0]/(t^{\lambda-1}X - \alpha)^{n_k}$.

§ 6. UNIQUENESS

The corollary at the end of the preceding paragraph shows that statement i) in theorem 2 may be replaced by

i) $\quad L \otimes_K V = \bigoplus V_{ijk},\ V_{ijk} \cong L[X, \varphi, 0]/(t^{\lambda_i-1}X - a_{ij})^{n_{ijk}}$.

This section is dedicated to the study of the uniqueness of this representation. To simplify notations, write for all $n \in \mathbb{N}$, $\lambda \in (1/q)\mathbb{Z}$, $\alpha \in t^{-1}\mathcal{O}_L^*$

$$M_{n,\lambda,\alpha} = L[X,\varphi,0]/(t^{\lambda-1}X-\alpha)^n.$$

Choose $\bar{\alpha} \in \mathbb{C}[s^{-1}]$ such that $\bar{\alpha} = \alpha \bmod s\mathcal{O}_L$.

Note first that lemma 4 implies $M_{n,\lambda,\alpha} \cong M_{n,\lambda,\bar{\alpha}}$.

LEMMA 5. *$M_{n,\lambda,\alpha} \cong M_{n,\lambda,\beta}$ if and only if $\alpha\beta^{-1} \in 1 + (1/q)\mathbb{Z}t + s^{q+1}\mathcal{O}_L$.*

PROOF. Assume there is an isomorphism between the modules. The image of $(X-\alpha)^{n-1}$ has the form $c(X^{n-1}+\dots)$. The image of $(X-\alpha)^n$ then has the form

$$\varphi(c)\left(X - \frac{c}{\varphi(c)}\alpha\right)(X^{n-1}+\dots) = \varphi(c)(X-\beta)^n.$$

Hence

$$\frac{c}{\varphi(c)}\alpha = \beta \text{ or } \alpha\beta^{-1} = \frac{\varphi(c)}{c} \in 1 + \frac{1}{q}\mathbb{Z}t + s^{q+1}\mathcal{O}_L.$$

On the other hand suppose $\alpha = \bar{\alpha}$, $\beta = \bar{\beta}$. This is no restriction in view of the remark preceding the lemma. Then $\alpha\beta^{-1} = 1 + (h/q)t + \dots$ for some $h \in \mathbb{Z}$.

Consider the map from $L[X,\varphi,0] \to L[X,\varphi,0]/((t^{\lambda-1}X-\beta)^m s^h)$ defined by $1 \mapsto s^h$. This map is clearly surjective, its kernel is $(t^{\lambda-1}X-\beta)^m$. Further

$$(t^{\lambda-1}X-\beta)^m s^h = s^h(t^{\lambda-1}X-\alpha)^m + s^{h+1}(\dots).$$

Hence

$$M_{n,\lambda,\beta} \cong L[X,\varphi,0]/((t^{\lambda-1}X-\alpha)^n + s(\dots)) \cong M_{n,\lambda,\alpha}$$

LEMMA 6. *$\bar{K} \otimes_L M_{n,\lambda,\alpha}$ is indecomposable.*

PROOF. Suppose $\bar{K} \otimes_L M_{n,\lambda,\alpha} = M_1 \oplus M_2$, and

$$\mu = 1 + \mu_1(t^{\lambda-1}X-\alpha) + \mu_2(t^{\lambda-1}X-\alpha)^2 + \dots \in M_1.$$

Then

$$(t^{\lambda-1}X-\alpha)\mu = (1+\alpha(\varphi-1)\mu_1)(t^{\lambda-1}X-\alpha) + \nu_2(t^{\lambda-1}X-\alpha)^2 + \dots \in M_1.$$

Now $(1+\alpha(\varphi-1)\mu_1) = 0$ would imply $v((\varphi-1)\mu_1) = 1$. This is impossible, for as one easily calculates the coefficient of t in $(\varphi-1)a$ is zero for all $a \in \bar{K}$. So $(t^{\lambda-1}X-\alpha) + \tilde{\nu}_2(t^{\lambda-1}X-\alpha)^2 + \dots \in M_1$. Proceeding this way we obtain $(t^{\lambda-1}X-\alpha)^{n-1} \in M_1$ and hence $(t^{\lambda-1}X-\alpha)^{n-2}, \dots, 1 \in M_1$. So $M_1 = \bar{K} \otimes_L M_{n,\lambda,\alpha}$, and $M_2 = 0$.

At this stage we may conclude: given the λ_i and the n_{ij}, the α_{ij} are determined up to an equivalence relation. The next lemma assures the uniqueness of the λ_i and the n_{ij}.

LEMMA 7. *Let $T_{\mu,\beta}: M_{n,\lambda,\alpha} \to M_{n,\lambda,\alpha}$ be the map defined by left multiplication by $(t^{\mu-1}X-\beta)$, with $\beta \in (1/t)\mathcal{O}_L^*$, $\mu \in (1/q)\mathbb{Z}$. Then*

$$\dim_{\mathbb{C}} \ker (T^p_{\mu,\beta}; M_{n,\lambda\alpha}) = \min(p,n) \text{ if } \lambda=\mu,\ \alpha\beta^{-1} \in 1+\frac{1}{q}\mathbb{Z}t + s^{q+1}\mathcal{O}_L,$$
$$= 0 \text{ otherwise.}$$

PROOF. First suppose μ or β does not satisfy the stated condition. Let

$$a = \sum_{i=0}^{n-1} a_i(t^{\lambda-1}X-\alpha)^i \in \ker T_{\mu,\beta}.$$

Then $T_{\mu,\beta}\, a = 0$ i.e.

$$\sum_{i=0}^{n-1} \varphi(a_i)\left(t^{\mu-1}X - \frac{\beta a_i}{\varphi(a_i)}\right)(t^{\lambda-1}X-\alpha)^i =$$
$$= \sum_i \varphi(a_i)(t^{\lambda-\mu}(t^{\lambda-1}X-\alpha)^{i+1} + \left(\alpha - \frac{t^{\lambda-\mu}\beta a_i}{\varphi(a_i)}\right)(t^{\lambda-1}X-\alpha)^i).$$

This implies in succession $a_0 = a_1 = \dots = a_{i-1} = 0$, since

$$\alpha - \frac{t^{\lambda-\mu}\beta a_i}{\varphi(a_i)} \neq 0.$$

So $\ker T_{\mu,\beta} = 0$ and a fortiori $\ker T^p_{\mu,\beta} = 0$ for all p.

Since $M_{n,\lambda,\alpha} \cong M_{n,\lambda,\beta}$ *if and only if* $\alpha\beta^{-1} \in 1 + (1/q)\mathbb{Z}t + s^{q+1}\mathcal{O}_L$, it suffices to suppose now $\lambda = \mu$ and $\alpha = \beta$. One proves with induction on p the following stronger statement:

$$\ker(T^p_{\lambda,\alpha}; M_{n,\lambda,\alpha}) = \mathbb{C}(t^{\lambda-1}X-\alpha)^{n-1} +$$
$$+ \mathbb{C}(t^{\lambda-1}X-\alpha)^{n-2} \dots + \mathbb{C}(t^{\lambda-1}X-\alpha)^{n-\min(p,n)}.$$

For $p=0$ there is nothing to prove, so assume that $p>0$, and that the statement is true for all $p'<p$.

Let

$$a = \sum_{i=0}^{n-1} a_i(t^{\lambda-1}X-\alpha)^i \in \ker T^p_{\lambda,\alpha}$$

then $T_{\lambda,\alpha}a \in \ker T^{p-1}_{\lambda,\alpha}$. We have

$$T_{\lambda,\alpha}a = \sum_{i=0}^{n-1} \varphi(a_i)(t^{\lambda-1}X-\alpha)^{i+1} + \alpha(\varphi-1)a_i(t^{\lambda-1}X-\alpha)^i \in \ker T^{p-1}_{\lambda,\alpha}.$$

Hence

$$\varphi(a_{i-1}) + \alpha(\varphi-1)a_i = 0 \text{ for } i = 0,1,\dots,n-\min(p,n)$$
$$\in \mathbb{C} \text{ for } i = n-\min(p,n)+1, \dots n-1.$$

This implies consecutively

$$a_0 = 0, \dots, a_{n-\min(p,n)-1} = 0,\ a_{n-\min(p,n)}, \dots, a_{n-1} \in \mathbb{C},$$

which had to be proved.

§ 7. THE MAIN THEOREM

The results of the preceding paragraphs enable us to reformulate theorem 2.

THEOREM 8. *Assume V is a $K[X,\varphi,0]$ module of dimension m over K. Suppose moreover $Xv=0$ implies $v=0$. Then there exists a finite extension L of K, with $[L:K]=q\,|\,m!$, and a decomposition of $L\otimes_K V$ in cyclic submodules:*

$$L\otimes_K V\cong \bigoplus_{i,j,k} M_{n_{ijk},\lambda_i,\alpha_{ij}}$$

where $n_{ijk}\in\mathbb{N}$, $\sum n_{ijk}=m$, $\lambda_i\in(1/q)\mathbb{Z}$, and $\alpha_{ij}=\sum_{h=0}^{q}\alpha_{ijh}s^{-h}$, $s^q=t$, $\alpha_{ijq}\neq 0$. We have $\mathrm{Re}\,(\alpha_{ij0}/\alpha_{ijq})\in[0,(1/q))$. Moreover the n_{ijk}, λ_i and α_{ij} are unique.

COROLLARY 9. *Let $\Phi: V\to V$ be an invertible difference operator. Then there exists a basis (e) of $L\otimes_K V$ such that*

$$\mathrm{Mat}(\Phi,(e))=\mathrm{diag}\,F_{n_{ijk},\lambda_i,\alpha_{ij}},\ F_{n,\lambda,\alpha}=t^{1-\lambda}\begin{pmatrix}\alpha & 0 & . & . & . & . & 0\\ 1 & \alpha & . & . & . & . & 0\\ . & . & . & . & . & . & .\\ 0 & 0 & . & . & . & 1 & \alpha\end{pmatrix}$$

an $n\times n$ matrix, with n_{ijk}, λ_i and α_{ij} as above. This matrix is unique modulo the order of the blocks.

PROOF. $L\otimes_K V\cong\oplus M_{n_{ijk},\lambda_i,\alpha_{ij}}$. Then

$$(1, t^{\lambda_i-1}X-\alpha_{ij}, (t^{\lambda_i-1}X-\alpha_{ij})^2, (t^{\lambda_i-1}X-\alpha_{ij})^{n_{ijk}-1})$$

is an L-basis of $M_{n_{ijk},\lambda_i,\alpha_{ij}}$. Let (e) represent the images of these elements, then Φ has the above representation.

REMARKS: 1) *The condition $Xv=0$ if and only if $v=0$, implies that V is cyclic* [1, p. 297, 299].

2) *In fact one may dispense with the condition on Φ. For one can find a basis (v) of V such that* $\mathrm{Mat}(\Phi,(v))=\mathrm{diag}\,F_i$, *$F_1$ invertible, and*

$$F_j=\begin{pmatrix}0&1&0&.&.&0&0\\0&0&1&.&.&0&0\\.&.&.&.&.&.&.\\0&0&0&.&.&0&1\\0&0&0&.&.&0&0\end{pmatrix}$$

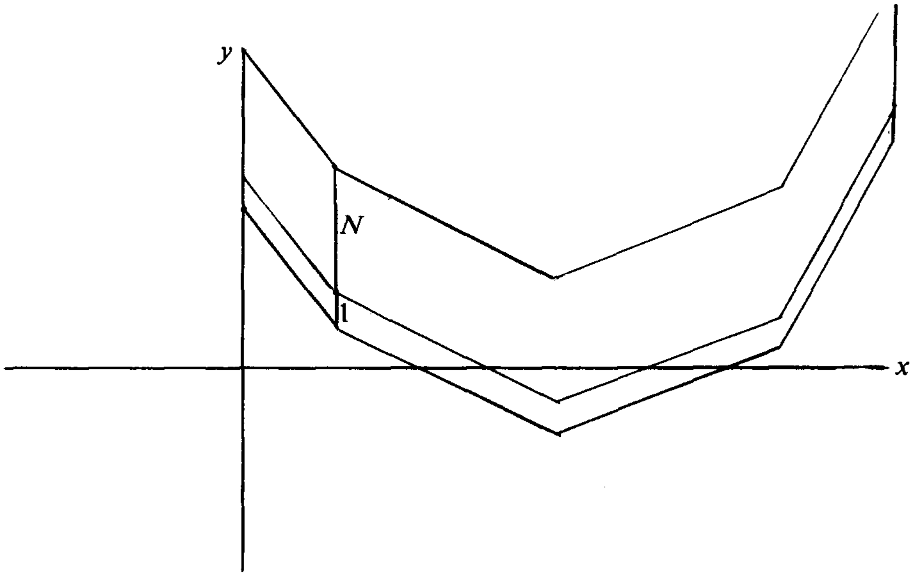
for $j>1$ ([1, p. 297], *together with lemma* 1). *For V_1, the part of V corresponding with F_1, one may apply theorem* 8.

3) *The λ_i in the theorem are the slopes of the Newton polygon associated with V and Φ. The uniqueness of the λ_i and the n_{ijk}'s shows that, although the choice of P in § 4 was not unique, the shape of $N(P)$ is. Note that $n_i=\sum_{j,k}n_{ijk}$ is just the length of the projection of Λ_i on the x-axis.*

4) *The rather artificial condition on* α_{ij0} *is needed to assure the uniqueness of* α_{ij}. *Another way to express this would be: the* α_{ij} *are unique* mod $(\alpha_{ijq}\mathbb{Z})$.

5) *A more careful examination of the splitting in the case* $r = 1$ *(see the proof of theorem 2) leads to the conclusion* $\alpha_{ij} \in \mathbb{C}[t^{-1/q_i}]$, q_i *a divisor of* $n_i!$ *or even of l.c.m.* $(1,2,\ldots,n_i)$. *This also yields a better estimate for* q: q *divides l.c.m.* $(1,2,\ldots,m)$ [3].

6. *One may also verify the following statement: The* α_{ij} *only depend on the monomials of P whose images lie in a strip of width one along the lower edge of N(P). The* n_{ijk} *depend on the monomials whose images lie in a strip of width* $N+1$ *along the lower edge, where* $N = \max_{i,j,l,m} |c_{ijl} - c_{ijm}|$, *the* c_{ijh} *as in theorem* 2.



Let me finish this paragraph with a remark on Levelt's method. In [3] he derives the analogue of theorem 8 for differential operators from the existence of a unique decomposition of the differential operator D in a semisimple differential operator S, and a nilpotent K-linear map N, such that $[S,N] = 0$.

For difference operators, however, such a decomposition does not exist in general, as may be seen from the following example:

Let Φ be given by

$$\mathrm{Mat}(\Phi,(e)) = t^{\nu}\begin{pmatrix} t^{-1} & 0 \\ 1 & t^{-1} \end{pmatrix},$$

and let A be a K-linear map commuting with Φ. An easy calculation shows that

$$\mathrm{Mat}(A,(e)) = \begin{pmatrix} a & 0 \\ b & a \end{pmatrix}, \text{ where } a, b \in \mathbb{C}.$$

However, in that case $\Phi - A$ will not be diagonal, hence not semisimple. For the same reason a decomposition in a product of a semisimple and a unipotent operator does not exist. It is possible to decompose Φ in the sum of a topological semisimple and a topological nilpotent $\mathbb{C}$-linear operator, but in general these maps do not behave well with respect to elements of K.

§ 8. FORMAL CLASSIFICATION OF INVERTIBLE DIFFERENCE OPERATORS

Let (e) be a K-basis of V. What I have done is essentially the following: If $F = \mathrm{Mat}(\Phi, (e))$, then there exists an $A \in Gl_m(L)$ such that

$$\operatorname{diag}(F_{n_{ijk}, \lambda_i, \alpha_{ij}})\varphi(A) = AF.$$

Now let g be a generator of Gal $(L|K)$. If $\lambda_i = (l_i/q)$, then the action of g yields

$$\operatorname{diag}(\xi^{-l_i} F_{n_{ijk}, \lambda_i, g(\alpha_{ij})})\varphi(g(A)) = g(A)F,$$

where ξ is a primitive q-th root of unity, since $\varphi g = g\varphi$.

By a constant transformation $B \in Gl_m(\mathbb{C})$ we find:

$$\operatorname{diag}(F_{n_{ijk}, \lambda_i, \xi^{-l_i} g(\alpha_{ij})})\varphi(g(BA)) = (g(BA))F.$$

But as a consequence of the uniqueness of the representation by $F_{n,\lambda,\alpha}$'s we have $g(t^{1-\lambda_i}\alpha_{ij}) = t^{1-\lambda_i}\alpha_{ih}, n_{ijk} = n_{ihk}$.

DEFINITION. *Let $\Phi_1, \Phi_2 : V \to V$ be two invertible difference operators. Φ is formally equivalent to Φ_2, $\Phi_1 \sim \Phi_2$ if there exists an $A \in Gl_m(\bar{K})$ such that $\Phi_1 A = A\Phi_2$.*

It is clear from the preceding paragraph that every equivalence class is determined completely by the finite set of pairs (β_i, n_i), $\beta_i \in \bar{K}^*/(1 + Qt + \bar{\mathfrak{m}}t)$ where $\bar{\mathfrak{m}}$ is the maximal ideal of $\bar{\mathcal{O}}$, and $n_i \in \mathbb{N}$ $(\beta_i = t^{1-\lambda_i}\alpha_i)$. Another question we should consider is, given a finite set $\{(\beta_i, n_i)_i\}$ as above, does there exist a linear space V over K, and a $\Phi : V \to V$ such that Φ belongs to the equivalence class defined by $\{(\beta_i, n_i)_i\}$? Obviously a necessary condition is that if (β_i, n_i) occurs, then also $(g(\beta_i), n_i)$, for all $g \in \mathrm{Gal}(\bar{K}|K)$. I shall show that this condition is sufficient.

It is possible to reindex the set $\{(\beta_i, n_i)\}$ as follows: $\{(g_i^k(\beta_i), n_{ij})\}$, where g_i is a generator of Gal $(K(\beta_i)|K)$ and $k = 1, \ldots, q_i$ with $q_i = [K(\beta_i) : K]$. Choose originals of β_i in K^* in the following way:

$$\beta_{ij} = \sum_{h=0}^{q_i} c_{ijh} t^{v(\beta_i) + h/q_i}, \text{ such that } c_{ijq_i} - c_{ikq_i} \notin \frac{1}{q_i}\mathbb{Z} \text{ if } j \neq k.$$

Define $R_{ij} \in \bar{K}[T]$ by

$$R_{ij} = \prod_{k=1}^{q_i} (T - g_i^k(\beta_{ij}))^{n_{ij}} = \sum_{h=0}^{n_{ij}q_i} r_{ijh} T^h,$$

then R_{ij} lies in $K[T]$, so $r_{ijh} \in K$. Define $P_{ij} \in K[X, \varphi, 0]$ by

$$P_{ij} = \sum_{h=0}^{n_{ij}q_i} r_{ijh} X^h, \text{ and } P = \prod_{i,j} P_{ij}.$$

Put $V = K[X, \varphi, 0]/(P)$, and define $\Phi : V \to V$ by $\Phi v = Xv$. Now Φ is an element of the equivalence class defined by $\{(\beta_i, n_i)\}$ as one may verify by applying the procedure described in sections 4, 5 and 6, or as follows immediately from remark 6 of § 7. So I conclude this paragraph with the following theorem:

THEOREM 10. *The set of equivalence classes of invertible difference operators on V is represented by $\{(\beta_i, n_i)\}$, $\sum n_i = m$, $\beta_i \in \bar{K}^*/(1+Qt+\bar{\mathfrak{m}}t)$. The class represented by $\{(\beta_i, n_i)\}$ is nonempty if and only if for all $g \in \mathrm{Gal}\,(\bar{K}|K)$ there is a j such that $g(\beta_i) = \beta_j$, $n_i = n_j$.*

§ 9. A GENERALIZATION

The special form of φ did not play a decisive rôle in the preceding paragraphs. In fact the method described here may be generalized to arbitrary automorphisms, or even to arbitrary pseudolinear operators. Let $\theta: V \to V$ be a pseudolinear operator, i.e. $\mathbb{C}$-linear and satisfying $\theta(av) = \psi(a)\theta v + \delta(a)v$ all $a \in K$, $v \in V$, with ψ a $\mathbb{C}$-automorphism of K, δ a ψ-derivation. Distinguish the following cases:

1) $\psi = \mathrm{id}$, $\delta = 0$. Then θ is a K-linear operator, and the result is well-known: The formal equivalence classes are represented by the eigenvalues and their multiplicities.

2) $\psi =$ identity and δ a derivation. This case is treated by Malgrange [3], and Levelt [4]. The analogue of theorem 10 would be: the formal equivalence classes of differential operators are represented by $\{(\beta_i, n_i)\}$ where $\beta_i \in \mathbb{C}/\mathbb{Q} \oplus \bar{K}/\bar{\mathcal{O}}$, satisfying the condition: for all $g \in \mathrm{Gal}\,(\bar{K}|K)$, there is a j such that $g(\beta_i) = \beta_j$, $n_i = n_j$.

3) $\psi(t) = t + at^{1+f} + h.o$, $a \in \mathbb{C}^*$, $f \geq 0$; if $f = 0$, $1+a$ not a root of unity. Then δ has to be of the form $\gamma(\psi - 1)$, $\gamma \in K$. (Cohn [1, p. 295]) Let $\theta' = \theta - \gamma$, then this reduces to the case $\delta = 0$. So assume $\delta = 0$, V is a $K[X, \psi, 0]$ module, and recall (Cohn [1, p. 297, 299]) that V is a direct sum of cyclic submodules V_k, such that $V_k = K[X, \psi, 0]/(X^{r_k})$ if $k \geq 2$. If $V_1 \cong K[X, \psi, 0]/(PX^r)$, r maximal then

$$V_1 \cong K[X, \psi, 0]/(P) \oplus K[X, \psi, 0]/(X^r),$$

as one may prove in the same way as lemma 1.

So it is no restriction to assume θ is invertible, and V is cyclic (as I did in § 4). Proceeding now as I did for Φ, with some modifications, induced by f, one may formulate theorem 8 in the same way with the following alterations:

$$M_{n,\lambda,\alpha} = L[X, \psi, 0]/(t^{\lambda-f}X - \alpha)^n,$$

$v(\alpha) = -f$, α unique modulo multiplication by $1 + (a\mathbb{Z}/q)t^f + \mathfrak{m}_L t^f$, if $f \geq 1$, α unique modulo multiplication by $(1 + a/q) + \mathfrak{m}_L$ if $f = 0$.

4) The final case $\psi(t) = at + t^2(\ldots)$, $a^p = 1$ is more complicated. Recall first, that by choosing a suitable uniformizer t, ψ has the following form $\psi(t) = at + \tau$, $v(\tau) = pd + 1$, $\tau \in \mathbb{C}[[t^p]]$. Next specify extensions of ψ in the following way:

If $[L:K] = q$, and g.c.d. $(q,p) = p_1$, then $\psi(s) = \alpha s + \ldots$, α a primitive p_1p-th root of unity, such that $\alpha^q = a$. Now follow the procedure described in § 4 and 5. Difficulties arise in the proof of lemma 5. Instead one proves the existence of a basis (e) such that $\mathrm{Mat}(t^{\lambda}\theta-\alpha,(e))$ has entries in $\mathbb{C}[[t^p]]$. This yields a cyclic vector and $p_1=Y^m+p_{m-1}Y^{m-1}+\ldots+p_0$, $Y=t^{\lambda}X$, $p_i\in C[[t^p]]$. Now $\psi(t^p)=$ $=t^p+h.o$ in t^p, so if $(p_1)_0$ splits there exist Q and R with coefficients in $\mathbb{C}[[t^p]]$ as in lemma 1. Repeating this procedure until one may use the fact that $\psi(t^p)=t^p+ct^{pd+p}+\ldots$ to prove an analogon of lemma 5, one arrives at a theorem analogous to theorem 8: $M_{n,\lambda,\alpha}=L[X,\psi,0]/(t^{\lambda-pd}X-\alpha)^n$. The uniqueness part is very complicated, since one has to determine the set $\{\psi(c)/c\}$.

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