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Essential singularities of rigid analytic functions

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INTRODUCTION

The Picard theorem for a complex analytic function can be formulated as follows:

"Let f be a holomorphic function on $\{z\in\mathbb{C}\mid 0<|z|<1\}$ with values in $\mathbb{C}-\{0,1\}$ then f can be extended as meromorphic function on

$\{z\in\mathbb{C}\mid |z|<1\}$".

A short proof of this statement would be the following: The group

$$\Gamma(2)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in PSl(2,\mathbb{Z})\mid \begin{pmatrix} a & b\\ c & d\end{pmatrix}\equiv\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix} \bmod 2\right\}$$

acts freely as a group of fractional linear transformations on the upper half-space H. The group has 3 parabolic points and the genus of the corresponding algebraic curve is 0. This means that $H/_{\Gamma(2)}\cong\mathbb{C}-\{0,1\}$ and as a consequence $\pi: H\to\mathbb{C}-\{0,1\}$ is the universal covering of $\mathbb{C}-\{0,1\}$.

Let

$$U_1=\{z\in\mathbb{C}\mid 0<|z|<1, \arg(z)=\pi\};$$
$$U_2=\{z\in\mathbb{C}\mid 0<|z|<1, \arg(z)=0\}$$

$U_1\cap U_2=U^+\cup U^-$ where

$$U^+=\{z\in U_1\cap U_2\mid \mathrm{im}(z)>0\}$$

and

$$U^- = \{z \in U_1 \cap U_2 \mid \mathrm{im}(z) < 0\}.$$

There are lifts $f_i : U_i \to H$ of f/U_i (i.e. $\pi \circ f_i = f/U_i$ for $i = 1, 2$) such that $f_1(\frac{1}{2}i) = f_2(\frac{1}{2}i)$. So f_1 coincides with f_2 on U^+. There is a unique $\gamma \in \Gamma(2)$ with $f_1 = \gamma \circ f_2$ on U^-.

We divide H by the action of $\langle\gamma\rangle$, the subgroup of $\Gamma(2)$ generated by γ. The result $H' = H/\langle\gamma\rangle$ is analytically isomorphic to one of the following spaces

(a) $\{z \in \mathbb{C} \mid |z| < 1\}$ if $\gamma = \mathrm{id}$.
(b) $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$ if γ is parabolic
(c) $\{z \in \mathbb{C} \mid r < |z| < 1\}$ for some $r > 0$ if γ is hyperbolic.

Let $\pi' : H' \to \mathbb{C} - \{0, 1\}$ denote the natural map induced by π. From the above it follows that f lifts to a holomorphic map $F : \{z \in \mathbb{C} \mid 0 < |z| < 1\} \to H'$ such that $\pi' \circ F = f$. Since F is bounded, it follows that F (and so also f) extends to $\{z \in \mathbb{C} \mid |z| < 1\}$.

We consider a field K, complete with respect to a non-archimedean valuation. In order to simplify the exposition we suppose that K is algebraically closed. Let $\mathbb{P} = \mathbb{P}^1(K)$ denote the projective line over K. In many situations one has to study holomorphic or meromorphic functions on an open set $\Omega \subset \mathbb{P}$ of the form $\Omega = \mathbb{P} - L$, where L is a compact set. We call L an essential singularity for the meromorphic function f on Ω if f does not extend to a meromorphic function on any $\Omega' = \mathbb{P} - L'$ where L' is a proper closed subset of L.

If L has at least one isolated point then it turns out that $f(\Omega)$ omits at most one value of $\mathbb{P}$. However if L is perfect then $f(\Omega)$ may omit a finite number of values in $\mathbb{P}$ (§ 2, example 1) or $f(\Omega)$ may even omit a compact infinite subset of $\mathbb{P}$ (§ 2, example 2).

The examples are derived from the theory of discontinuous groups over a non-archimedean valued field. In this respect the theory seems quite far from its archimedean analogue. We refer to [1] and [2] for non-archimedean function theory of one variable and for discontinuous groups.

§ 1. POSITIVE RESULTS ON THE VALUES OF HOLOMORPHIC MAPS

A connected affinoid subset X of $\mathbb{P}$ is a subset of the form $X = \mathbb{P} - (B_1 \cup \ldots \cup B_n)$ where $B_1, \ldots, B_n$ are disjoint open disks in $\mathbb{P}$. The $B_1, \ldots, B_n$ are usually called the holes of X; their number is n.

(1.1) PROPOSITION. Let f be a non-constant holomorphic function on a connected affinoid subset X of $\mathbb{P}$. Then $f(X)$ is a connected affinoid subset of $\mathbb{P}$. Moreover the number of holes of $f(X)$ is less than or equal to the number of holes of X.

PROOF. The canonical reduction $\bar{X}$ of X is the maximal ideal space of $\overline{\mathcal{O}(X)}$ ([2] p. 113). According to [2] p. 78, 79 the ring $\overline{\mathcal{O}(X)}$ has the form $\bar{K}[z_1, \ldots, z_n]/I$

where I is an ideal generated by elements $\bar{E}_{i,j}(i \neq j, 1 \leq i, j \leq n)$ of the form

$$\bar{E}_{ij} = z_i z_j + \alpha_{ij} z_i + \beta_{ij} z_j \text{ with } \alpha_{ij}, \beta_{ij} \in \bar{K}.$$

It follows that each component L of $\bar{X}$ is isomorphic to $\mathbb{P}(\bar{K})-V(L)$ where $V(L)$ is a finite non-empty subset of $\mathbb{P}(\bar{K})$. We construct $\hat{\bar{X}}$, the completion of $\bar{X}$, by completing each component L of $\bar{X}$ to a $\mathbb{P}(\bar{K})$. The total number of "missing" points of $\bar{X}$ (i.e. the points of $\hat{\bar{X}}-\bar{X}$) is equal to $\sum \# V(L) = n =$ the number of holes of X.

The set $Y = f(X)$ is according to [2] p. 110, lemma (2,7), the union of an affinoid set and a finite set. Since X is connected it follows that Y is actually a connected affinoid subset of $\mathbb{P}$.

The surjective map $f : X \to Y$ induces a morphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ which is an isometry with respect to the spectral norms $\| \ \|_{sp}$ on X and Y. We obtain an induced, injective $\overline{f^*} : \overline{\mathcal{O}(Y)} \to \overline{\mathcal{O}(X)}$ and a surjective (since [2] p. 114, lemma (2.9.1)) morphism $\bar{f} : \bar{X} \to \bar{Y}$.

The restriction of $\bar{f}$ to any component $L = \mathbb{P}(\bar{K})-V(L)$ of $\bar{X}$ extends uniquely to a morphism of $\mathbb{P}(\bar{K}) \to \hat{\bar{Y}}$. So $\bar{f}$ extends to a morphism $\hat{\bar{f}} : \hat{\bar{X}} \to \hat{\bar{Y}}$. The last map is surjective since $\hat{\bar{f}}(\hat{\bar{X}})$ is complete and contains $\bar{Y}$. Hence the number of missing points of $\bar{Y}$ is $\leq n$. This proves the proposition.

We propose now a second proof of the last statement of the proposition. In [1] § 1, (1.8.9) one has established an exact sequence

$$0 \to A(X) \to \mathcal{O}(X)^* \to \mathbb{Z}^{n-1} \to 0$$

in which $\mathcal{O}(X)^*$ is the group of invertible holomorphic functions on X; n is the number of holes of X; $A(X) = \{\lambda(1+h) \mid \lambda \in K^*, h \in \mathcal{O}(X), \|h\|_{sp} < 1\}$.

Let m be the number of holes of Y. The map f induces $f^* : \mathcal{O}(Y)^* \to \mathcal{O}(X)^*$ such that $(f^*)^{-1}(A(X)) = A(Y)$. So we find an injective map $\mathbb{Z}^{m-1} \to \mathbb{Z}^{n-1}$ and we have shown that $m \leq n$.

(1.2) PROPOSITION. Let L be a compact subset of $\mathbb{P}$ and let $\Omega = \mathbb{P} - L$ denote the analytic subspace of $\mathbb{P}$ defined by the family

$$\{F \mid F \text{ affinoid in } \mathbb{P}; F \cap L = \phi\}.$$

For any non-constant holomorphic map $f : \Omega \to \mathbb{P}$ the set $\mathbb{P} - f(\Omega)$ is compact.

PROOF. We consider the subspace Ω' of $\mathbb{P}$ defined by the family of affinoid sets $\{f(X) \mid X \text{ affinoid}; X \cap L = \phi\}$. If Ω' is not of the form $\mathbb{P} - \{\text{a compact set}\}$ then, according to [2] p. 145, (2.5), there exists a non-constant bounded holomorphic function h on Ω'. The holomorphic function $h \circ f$ on Ω is also bounded and must be constant according to the same result. This implies however that f is constant. So the proposition is proved and we have proved slightly more, namely: every affinoid subset, lying in $f(\Omega)$, is the image of an affinoid subset of Ω under the map f.

(1.3) PROPOSITION. (A version of Picard's theorem). Let f be meromorphic function on $\{z \in K \mid R < |z|\}$ which cannot be extended at ∞. Then f omits at most one value.

PROOF. We note that this result must be known. By lack of reference we include two proofs. Suppose that f omits at least one value, then we may take f to be holomorphic on $\{z \in K \mid R < |z|\}$.

(1) FIRST PROOF. We may express f as a convergent Laurent-series

$$\sum_{n=-\infty}^{\infty} a_n z^n$$

which has infinitely many $a_n \neq 0$ for $n > 0$.

For $\varrho \in |K^*|, R < \varrho < \infty$, we form $\max |a_n| \varrho^n = \alpha(\varrho)$ and we denote the smallest integer n with $|a_n| \varrho^n = \alpha(\varrho)$ by $n(\varrho)$.

Clearly $\lim_{\varrho\to\infty} n(\varrho) = \lim_{\varrho\to\infty} \alpha(\varrho) = \infty$. We will suppose that $\varrho \gg R$ such that $n(\varrho) > 0$. The set $X_\varrho = f(\{z \in K \mid |z| = \varrho\})$ can have the following form:

(a) Suppose that there is only one n with $|a_n| \varrho^n = \alpha(\varrho)$, then

$$X_\varrho = \{z \in K \mid |z| = \alpha(\varrho)\}$$

(b) Suppose that there are more positive integers n with $|a_n| \varrho^n = \alpha(\varrho)$, then $X_\varrho = \{z \in K \mid |z| \leq \alpha(\varrho)\}$.

The above follows from the well-known statement:

$$\sum_{n=-\infty}^{\infty} b_n z^n \in \mathcal{O}(\{z \in K \mid |z| = 1\})$$

has no zeros if and only if there is precisely one m with $|b_m| = \max_n |b_n|$.

Situation (b) occurs for an infinite sequence $\varrho_1, \varrho_2, \ldots$ with $\lim \varrho_i = \infty$. Hence $f(\{z \in K \mid R < |z|\}) = K$.

(2) SECOND PROOF. Suppose that the holomorphic map f omits at least two values in $\mathbb{P}$. Then we may suppose that f omits 0 and ∞. In other words $f \in \mathcal{O}(\{z \in K \mid R < |z|\})^*$. Using [1] § 1, (1.8.9) one sees that f has the form $\lambda z^n(1+h)$ where $\lambda \in K^*$, $n \in \mathbb{Z}$ and h is holomorphic on $\{z \in K \mid R < |z|\}$ such that $|h(z)| < 1$ for all z. But then h can be extended to ∞ and so also f extends at ∞.

§ 2. TWO EXAMPLES

(2.1) *The first example* imitates the proof of Picard's theorem that we have given in the introduction.

Let $k = \mathbb{F}_q((1/t))$ be the Laurent-series field in the variable $1/t$ and with coefficients in the finite field $\mathbb{F}_q$. Let K denote the completion of the algebraic closure of k.

The group $\Gamma(t)$ is the subgroup of $\Gamma(1) = PSl(2, \mathbb{F}_q[t])$ consisting of the matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ modulo}(t).$$

In [2], Chapter 10, it is calculated that: $\Gamma(t)$ has $(q+1)$ inequivalent parabolic points and that the genus of the corresponding algebraic curve is zero.

So the holomorphic map

$$f: \mathbb{P}(K) - \mathbb{P}(k) \to \mathbb{P}(K) - \mathbb{P}(k)/\Gamma(t) \simeq \mathbb{P}(K) - \mathbb{P}(\mathbb{F}_q)$$

omits exactly $q+1$ values. We still have to verify that f has an essential singularity at the compact subset $\mathbb{P}(k)$ of $\mathbb{P}$.

Let L be the smallest compact subset of $\mathbb{P}$ such that f admits an extension as meromorphic function on $\mathbb{P} - L$. One easily sees that L always exists and that L is invariant under $\Gamma(t)$. If $L \neq \phi$ then L turns out to be $\mathbb{P}(k)$ since it is invariant. Further $L = \phi$ would mean that f is a rational function on $\mathbb{P}$. But only a constant rational function can be invariant under $\Gamma(t)$.

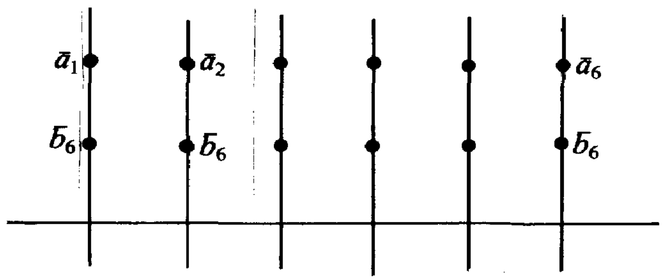
In this example one can clearly vary the finite field $\mathbb{F}_q$ and moreover one can compose f with a rational function on $\mathbb{P}$. This shows the following statement:

"Let the field K have characteristic $\neq 0$ and let $\{a_1, \ldots, a_n\}$ be a subset of $\mathbb{P}(K)$. There exists a perfect compact subset L of $\mathbb{P}(K)$ and a meromorphic function f with an essential singularity at L such that $f(\mathbb{P} - L) = \mathbb{P} - \{a_1, \ldots, a_n\}$".

(2.2) *The second example* works for fields K of any characteristic and residue characteristic. However to simplify matters we assume that the residue field $\bar{K}$ has a characteristic $\neq 2$.

Our construction is a variant of the construction of Whittaker groups done in [2], Chapter 9.

Let the 12 points $a_1, b_1, \ldots, a_6, b_6$ in $\mathbb{P}$ be such that the reduction $\mathbb{P}$ with respect to this set is:



In other terms this means that the position of the 12 points (after an automorphism of $\mathbb{P}$) is such that:

1) all $|a_i| = |b_i| = 1$
2) $|a_i - a_j| = 1$ for $i \neq j$
3) $|b_i - b_j| = 1$ for $i \neq j$
4) $|a_i - b_j| = 1$ for $i \neq j$
5) $|a_i - b_i| < 1$ for all i.

Let s_i $(i = 1, \ldots, 6)$ denote the elliptic element of order 2 with fixed points a_i, b_i. In [2] p. 281 it is shown that the group $\Gamma_0 = \langle s_1, \ldots s_6 \rangle$ generated by the six reflexions is discontinuous and it is shown that the only relations among the generators are $s_1^2 = s_2^2 \ldots = s_6^2 = 1$. Let Ω denote the set of ordinary points of Γ_0.

We introduce now four subgroups Γ_i $(i=1,2,3,4)$ of Γ_0 of finite index. Consider the surjective group homomorphism $\phi : \Gamma_0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2$ given by $\phi(s_i)=(1,0)$ for $i=1,2,3$ and $\phi(s_i)=(0,1)$ for $i=4,5,6$. The kernel Γ_4 of ϕ is a Schottky group on 9 free generators. The generators are

$$s_1s_2,\ s_1s_3,\ s_4s_5,\ s_4s_6,\ s_1s_4s_1s_2s_4s_1,\ s_1s_4s_1s_3s_4s_1,\ s_1s_5s_4s_1,\ s_1s_6s_4s_1,\ s_1s_4s_1s_4$$

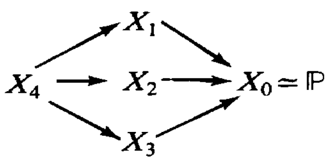
as one easily verifies.

The group Γ_1 is generated by Γ_4 and s_1; the group Γ_2 is generated by Γ_4 and s_4, the group Γ_3 is generated by Γ_4 and s_1s_4. Hence $\Gamma_4 \subset \Gamma_i \subset \Gamma_0$ for $i=1,2,3$ and $[\Gamma_0 : \Gamma_i]=2$ for $i=1,2,3$.

The group Γ_3 turns out to be a free group on 5 generators, namely on $\{s_1s_2, s_1s_3, s_4s_5, s_4s_6, s_1s_4\}$.

The groups Γ_i $(i=0,1,2)$ are not free. One easily calculates that the rank of the abelianized groups $\Gamma_{i/[\Gamma_i,\Gamma_i]}$ is 2 for $i=1,2$.

We write X_i for the algebraic curve Ω/Γ_i $(i=0,\dots,4)$. Although the curve is not always parametrized by a Schottky group (cases $i=0,1,2$) the curve is certainly "locally isomorphic to $\mathbb{P}$" and hence a Mumford curve. (See [2] p. 177). Let g_i denote the genus of X_i, then we have $g_0=0$, $g_1=g_2=2$, $g_3=5$, $g_4=9$ by using [2] p. 250, 251. Moreover we have a diagram of holomorphic maps of degree two between the various curves:



We are especially interested in the morphism $X_4 \to X_1$. The curve X_1 is a Mumford curve of genus 2 and can also be parametrized by a Schottky group Δ with Ω' as set of ordinary points.

The map $f : X_4 \to X_1$ lifts to a holomorphic map $F : \Omega \to \Omega'$ since $\pi : \Omega \to X_4$ and $\pi_1 : \Omega' \to X_1$ are the universal coverings. (Compare [2] p. 149–153). The holomorphic map F omits an infinite compact set since $\mathbb{P}-\Omega'$ is infinite.

Our example is completed with the following lemma.

LEMMA. *F has an essential singularity at the compact perfect set $\mathbb{P}-\Omega$.*

PROOF. Using the Riemann-Hurwitz formula one finds that $f : X_4 \to X_1$ is ramified in 12 points. Let $p \in \Omega$ be a point such that its image in X_4 is one of those 12 points. Since $\pi_4 : \Omega \to X_4$ and $\pi_1 : \Omega' \to X_1$ are locally isomorphisms it follows that also F is ramified (of index 2) at p. The whole orbit $\Gamma_4\ (p)$ consists clearly of ramification points of F. Since p is an ordinary point for Γ_4 the limit points for this orbit are precisely $\mathbb{P}-\Omega$. This implies that F cannot be extended since in any neighbourhood of any $\lambda \in \mathbb{P}-\Omega$ there are infinitely many ramification points of F. So F has an essential singularity at $\mathbb{P}-\Omega$.

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