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On massive gravitons in 2+1 dimensions

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Abstract. The Fierz-Pauli (FP) free field theory for massive spin-2 particles can be extended, in a spacetime of (1+2) dimensions (3D), to a generally covariant parity-preserving interacting field theory, in at least two ways. One is “new massive gravity” (NMG), with an action that involves curvature-squared terms. Another is 3D “bigravity”, which involves non-linear couplings of the FP tensor field to 3D Einstein-Hilbert gravity. We review the proof of the linearized equivalence of both “massive 3D gravity” theories to FP theory, and we comment on their similarities and differences.

1. Introduction: Fierz-Pauli and NMG

There has been increased interest in recent years in the possibility of massive gravitons. This is motivated in part by the discovery of cosmic acceleration, which might be explainable in terms of an infra-red modification of general relativity that gives the graviton a small mass (e.g. [1]), and in part by the idea that some theory involving massive gravitons could be the low energy limit of a non-critical string-theory underlying QCD (e.g. [2]). As for so many other gravitational physics issues, it may be useful to consider the possibilities for massive gravitons in the simpler context of a three-dimensional (3D) spacetime.

The standard free-field description of massive gravitons, i.e. massive particles of spin-2, is that due to Fierz and Pauli (FP) in terms of a symmetric tensor field, which we will call $f_{\mu\nu}$. The FP action is

$$S_{FP}[f] = S_{EH}^{(2)}[f] - \frac{1}{4}m^2 \int d^3x (f^{\mu\nu} f_{\mu\nu} - f^2), \quad (1)$$

where $f = \eta^{\mu\nu} f_{\mu\nu}$ for Minkowski metric η , which we assume to have “mostly plus” signature, and

$$S_{EH}^{(2)}[f] = -\frac{1}{2} \int d^3x f^{\mu\nu} G_{\mu\nu}^{(lin)}(f), \quad (2)$$

where $G_{\mu\nu}^{(lin)}(f)$ is the linearized Einstein tensor expressed as a self-adjoint differential operator acting on the tensor field f ; the FP action therefore reduces to the quadratic approximation to the Einstein-Hilbert (EH) action in the massless limit. The field equations are equivalent to

$$(\square - m^2) f_{\mu\nu} = 0, \quad \partial^\mu f_{\mu\nu} = 0, \quad f = 0. \quad (3)$$

The differential subsidiary condition is needed to remove ghost modes and the tracefree condition is needed to remove lower spin modes. In 4D the FP equations propagate the 5 helicity states of a massive spin 2 particle. In 3D they propagate the two ± 2 helicity states of a massive spin 2 particle.

The main issue of interest is whether one can generalize the FP free field theory to a consistent interacting theory. One could try to couple $f_{\mu\nu}$ to some linear combination of the stress tensor and its trace but this leads to difficulties (see [3] for a recent review). In 3D there is another option. To see this, we first solve the differential subsidiary condition by writing

$$f_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu}^{\eta\rho}\epsilon_{\nu}^{\tau\sigma}\partial_{\eta}\partial_{\tau}h_{\rho\sigma} \equiv G_{\mu\nu}^{(\text{lin})}(h) \quad (4)$$

for some symmetric tensor potential h . The remaining FP equations become

$$(\square - m^2)G_{\mu\nu}^{(\text{lin})}(h) = 0, \quad R^{(\text{lin})}(h) = 0, \quad (5)$$

where $R^{(\text{lin})}(h)$ is the trace of the linearized Ricci tensor $R_{\mu\nu}^{(\text{lin})}(h)$. These equations are derivable from the quadratic action

$$S_{NMG}^{(2)}[h] = \int d^3x \left\{ \frac{1}{2}h^{\mu\nu}G_{\mu\nu}^{(\text{lin})}(h) + \frac{1}{m^2}G_{(\text{lin})}^{\mu\nu}(h)S_{\mu\nu}^{(\text{lin})}(h) \right\}, \quad (6)$$

where

$$S_{\mu\nu}^{(\text{lin})}(h) = R_{\mu\nu}^{(\text{lin})}(h) - \frac{1}{4}\eta_{\mu\nu}R^{(\text{lin})}. \quad (7)$$

This new quadratic action for symmetric tensor potential h has an obvious extension to a non-linear generally covariant action for metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$; this yields ‘‘new massive gravity’’ (NMG) [4].

It is convenient to give the NMG action in a slightly more general form involving a dimensionless parameter σ ; in appropriate units, this action is [4]

$$S_{NMG}[g] = \int d^3x \sqrt{-\det g} [\sigma R + m^{-2}K], \quad (8)$$

where

$$K := R^{\mu\nu}R_{\mu\nu} - \frac{3}{8}R^2 = G^{\mu\nu}S_{\mu\nu}, \quad S_{\mu\nu} := R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R. \quad (9)$$

The tensor $S_{\mu\nu}$ is the 3D Schouten tensor, which plays an important role in conformal gravity as the gauge potential for conformal boosts¹. To recover the quadratic action (6) we should set $\sigma = -1$, which means that the EH term has the ‘wrong’ sign (relative to the EH action of general relativity).

We have arrived at the action for NMG by a method [5] that guarantees its on-shell equivalence to the FP theory in the linearized limit. This method is a very general one [6] but it does not guarantee *off-shell* equivalence. However, this can be proved very simply for NMG by means of the following alternative action involving an ‘auxiliary’ tensor field f [4]:

$$S_{NMG}[g, f] = \sigma S_{EH}[g] + \int d^3x \sqrt{-\det g} \left[f^{\mu\nu}G_{\mu\nu} - \frac{1}{2}m^2 g^{\mu[\rho}g^{\nu]\sigma} f_{\mu\nu}f_{\rho\sigma} \right]. \quad (10)$$

This action, which is only 2nd order in derivatives, is quadratic in the tensor $f_{\mu\nu}$. The f field equation is $f_{\mu\nu} = (2/m^2)S_{\mu\nu}$; using this equation to eliminate $f_{\mu\nu}$ (or integrating it out by

¹ This is true in any dimension but the expression for the Schouten tensor in terms of the Ricci tensor is dimension dependent. In D spacetime dimensions it is defined as $S_{\mu\nu} = (D-2)^{-1}R_{\mu\nu} - [2(D-1)(D-2)]^{-1}Rg_{\mu\nu}$.

Gaussian functional integration in the path integral) one recovers the action (8) in terms of the metric alone. This is true for any value of σ , so for $\sigma = 0$ the action (10) constitutes a 2nd order version of the “pure- K ” model studied in [7]. Focusing on NMG proper, we set $\sigma = -1$. To linearize about the Minkowski vacuum, we now write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} - f_{\mu\nu} \quad (11)$$

and expand the action in powers of the Minkowski space tensors (h, f) . To quadratic order we find that

$$S_{NMG}^{(2)}[h, f] = -S_{EH}^{(2)}[h] + S_{FP}[f]. \quad (12)$$

Since the quadratic EH action for h propagates no modes, the modes propagated by the linearized NMG theory are the same as those of the FP theory².

Higher-derivative extensions of 4D general relativity have been investigated since Weyl’s 1918 attempt at a unified theory of gravity and electromagnetism. A special feature of curvature-squared invariants is that they contribute to the quadratic approximation in an expansion about a Minkowski vacuum, so they contribute to the linearized field equations, which become fourth order rather than second order. Equations of higher than second order generally propagate ghosts (modes of negative energy) in addition to physical modes, and this is what happens here, in 4D: in addition to the massless graviton, there is also a massive scalar mode and a massive spin-2 ghost [10]. The 3D case is different because the EH action by itself does not propagate any modes, so the massless graviton is now absent. This means that we are free to change the overall sign of the action to arrange for the massive spin-2 modes to be physical, i.e. not ghosts; we can then ensure that they are not tachyons by a choice of relative sign for the EH and curvature-squared terms. However, there is still the massive scalar mode, which is now a ghost. Remarkably, the mass of this scalar ghost mode goes to infinity as the relative coefficient of the two independent curvature-squared terms is tuned to the value that yields the ‘ K -combination’ of (9), so we end up with the unitary massive pure spin-2 model that is NMG. All of this can be read off from the results of Nishino and Rajpoot [11] for the general 3D model with EH and curvature-squared invariants but these authors considered only the ‘right-sign’ EH term and (presumably as a consequence) did not draw attention to the special features of the K -combination.

One motivation for considering the EH action with the ‘wrong’ sign is that this is a feature of “topologically massive gravity” (TMG) [12], which is also “higher-derivative” because it involves the third order, and parity violating, Lorentz Cherns-Simons (LCS) term. TMG propagates only a single spin 2 mode, which is possible in 3D if parity is violated. This possibility can also be realized by a first-order “self-dual” spin-2 theory that can be thought of as the “square-root” of the 3D FP theory [13]. If the differential subsidiary condition of this model is solved, the remaining equations become precisely those of linearized TMG, so TMG can be deduced by the same procedure that we used above to arrive at NMG. Moreover, in this case the on-shell equivalence extends immediately to an off-shell equivalence because with only one propagating mode one can always choose the overall sign of the action so as to ensure that it is physical rather than a ghost. Unitarity for NMG is a more delicate matter because there are two propagating modes. Unitarity follows from the proof of off-shell equivalence to the FP theory reviewed above, but it can also be checked explicitly via a canonical analysis [7]. A nice feature of this method is that one sees explicitly the cancellation of the higher time derivatives; in other words, the action

² In [4] we set $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and used the h equation to eliminate h from the action. This step was criticized in [7] because it involves a back-substitution that is not generally permissible. However, the legitimacy of this back-substitution is manifest in the new basis implicit in (11) because it just eliminates the quadratic EH term, which propagates no modes. This change of basis was first made in [8]; it had occurred independently to us and we used it in our general analysis of unitarity for the ‘cosmological’ extension of NMG [9].

is actually second order in time derivatives despite being fourth order in space derivatives. This is reminiscent of Hořava gravity [14] but without any violation of Lorentz invariance!

There is a natural generalization of NMG that is suggested by TMG: one simply adds a LCS term to the action. This yields what we have called “general massive gravity” (GMG) [4]. There are now two mass parameters, which can be traded for the two masses m_{\pm} of the two spin-2 modes of helicities ± 2 , which we may assume to be such that $m_- \geq m_+ > 0$. As parity flips the sign of helicities, a model with $m_+ \neq m_-$ breaks parity. When $m_+ = m_-$ the parity-violating LCS term is absent and we recover NMG. The limit $m_- \rightarrow \infty$ for fixed m_+ yields TMG, in which the curvature-squared term is absent, so both NMG and TMG are special cases of GMG. Conversely, it is possible to recover GMG by a “soldering” of two TMG models [15], one propagating a helicity +2 mode with mass m_+ and the other propagating a helicity -2 mode with mass m_- .

2. 3D bigravity

In the case of 4D massless gravitons, it is known that any theory of interacting gravitons must reduce to general relativity at low energy [16, 17, 18]. Even if these low energy theorems continue to apply in 3D, they cannot restrict the interactions of massive gravitons because these will decouple at sufficiently low energy. There is therefore no obvious reason why NMG should be the unique interacting extension of the 3D FP theory. Indeed, an alternative parity-preserving generally covariant massive gravity model was recently proposed by Bañados and Theisen [19]; their “bigravity” model for massive 3D gravitons is a 3D version of the 4D “ f - g gravity” of Isham *et al.* [20].

The Bañados-Theisen (BT) model involves a dimensionless parameter, which we shall call α , and two tensor fields; we shall use a different basis for these two fields and we call them (\tilde{g}, f) . Also, we omit here the cosmological terms included in [19]; we will later discuss briefly the cosmological extension. For $\alpha(\alpha + 1) > 0$, the BT action is

$$S_{BT}[\tilde{g}, f] = \alpha S_{EH}[\tilde{g} + \beta f] + S_{EH}[\tilde{g} - \alpha\beta f] - \int d^3x \mathcal{U}(\tilde{g}, f) \quad (13)$$

where

$$\beta = 1/\sqrt{\alpha(\alpha + 1)}. \quad (14)$$

The last term in the action is the potential term, defined by some choice of scalar density \mathcal{U} constructed from the metric tensor and the f -tensor, without derivatives. The choice made in [19] has the form

$$U(\tilde{g}, f) = \frac{1}{2} m^2 \sqrt{-\det \tilde{g}} \tilde{g}^{\mu[\rho} \tilde{g}^{\nu]\sigma} f_{\mu\nu} f_{\rho\sigma} + \mathcal{O}(f^3). \quad (15)$$

A feature of the BT action is that it assumes invertibility of both \tilde{g} and $\tilde{g} + \beta f$. We can interpret the invertibility constraint on, say, \tilde{g} as the geometrical condition that \tilde{g} be a metric tensor but invertibility of $\tilde{g} + \beta f$ then puts a constraint on the tensor field f that is not so obviously geometrical.

The proof that linearized bigravity is equivalent to the FP theory is simple. We write

$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad (16)$$

and expand the action (13) in powers of the Minkowski space tensors (h, f) . The result to quadratic order is

$$S_{BT}^{(2)}[h, f] = (\alpha + 1) S_{EH}^{(2)}[h] + S_{FP}[f]. \quad (17)$$

This is equivalent to the FP theory for any α , because the quadratic EH action for h propagates no modes, although we should recall the constraint $\alpha(\alpha + 1) > 0$.

If β is initially taken as a free parameter then one finds that unitarity is violated if $\alpha(\alpha+1) < 0$ irrespective of the choice of β , and that (14) can be imposed without loss of generality when $\alpha(\alpha+1) > 0$. When $\alpha = 0$ the action (13) is equivalent to the EH action, so we can ignore this case. When $\alpha = -1$, the action (13) is trivial but it is also not equivalent to the action considered in [19]; in this case we should instead consider the action

$$S_{BT}[\tilde{g}, f]|_{\alpha=-1} = -S_{EH}[g] + S_{EH}[g - f] - \int d^3x \mathcal{U}(\tilde{g}, f). \quad (18)$$

Linearization now yields the quadratic action

$$S_{BT}^{(2)}[h, f]|_{\alpha=-1} = \int d^3x \left\{ h^{\mu\nu} G_{\mu\nu}^{(\text{lin})}(f) - \frac{1}{4} m^2 (f^{\mu\nu} f_{\mu\nu} - f^2) \right\}, \quad (19)$$

but this is precisely what one gets from linearization of (10) when $\sigma = 0$. As observed in [9], the h tensor is now a Lagrange multiplier imposing a constraint that has the solution $f_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$. Using this in the f equation, we find that

$$G_{\mu\nu}^{(\text{lin})}(h) = m^2 [\partial_{(\mu} A_{\nu)} - \eta_{\mu\nu} (\partial \cdot A)]. \quad (20)$$

This equation determines the metric fluctuation tensor h in terms of the source, which is itself constrained by the linearized Bianchi identity; this constraint is just the 3D Maxwell equation $\partial^\mu F_{\mu\nu} = 0$, where $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$. It is also true that the action (19) reduces to the 3D Maxwell action upon making the substitution $f_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$. Either way, we see that a single massless mode is propagated, of undefined spin because spin is not defined for massless particles in 3D. This is consistent with the canonical analysis of the ‘‘pure-K’’ model [7].

3. Cosmological NMG and 3D bigravity

We conclude with a few brief observations on the cosmological extension of NMG and 3D bigravity. For NMG one may add the cosmological term

$$S_{\text{cos}}[g] = -2\lambda m^2 \int d^3x \sqrt{-\det g} \quad (21)$$

where λ is a new dimensionless parameter. It is convenient in this context to also allow the parameter σ to be arbitrary, and to allow for $m^2 < 0$. Anti-de Sitter (adS) vacua are possible for $\lambda \neq 0$, and in such cases one may study the nature of the associated 2D conformal field theory (CFT) [21]. It turns out that this has a negative central charge whenever the quadratic approximation to the bulk theory (expanded about the adS vacuum) is unitary, and vice versa [9]. There is one exception, for $\lambda = 3$, in which the massive gravitons are replaced by massive spin 1 modes and the central charge of the CFT is zero; in this case the CFT is of logarithmic type [22].

In the bigravity case we may add the cosmological term

$$S_{\text{cos}}[\tilde{g}] = \frac{2(\alpha+1)}{\ell^2} \int d^3x \sqrt{-\det \tilde{g}}. \quad (22)$$

This allows an adS vacuum of radius ℓ for the metric \tilde{g} , with $f = 0$. Linearization about adS was considered in [19], and also in [23]. ‘Bulk’ unitarity requires either that $\alpha \leq -1$ or that $\alpha \geq 0$. From our understanding of the results of [19], the central charge of the associated 2D CFT is (in our notation) a positive factor times $(\alpha+1)$, which would mean that it is negative for $\alpha < -1$ but positive for $\alpha > 0$. It would appear from this that 3D cosmological bigravity is analogous

to cosmological NMG for $\alpha < -1$ but not for $\alpha > 0$. This can be seen in vestigial form from the Minkowski space actions: when $\alpha < -1$ the quadratic actions (17) are related by a rescaling (which is trivial at $\alpha = -2$) but not when $\alpha > 0$ because of the different sign of the quadratic EH term; although this does not affect the conclusions concerning propagating modes, and may continue to be irrelevant at the non-linear level in a Minkowski vacuum, it makes a significant difference in an adS vacuum.

In conclusion, both NMG and 3D bigravity constitute generally covariant interacting extensions of the free-field 3D Fierz-Pauli model for massive particles of spin 2. Each will likely have its advantages and disadvantages. Whether either will provide insight into the problem of massive 4D gravity remains to be seen.

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References

- [1] Eckhardt D H, Pestana J L G and Fischbach E 2010 Cosmological implications of massive gravitons *New Astron.* **15** 175
- [2] 't Hooft G 2007 Unitarity in the Brout-Englert-Higgs Mechanism for Gravity *Preprint* arXiv:0708.3184
- [3] Goldhaber A S and Nieto M M 2008 Photon and Graviton Mass Limits *Preprint* arXiv:0809.1003
- [4] Bergshoeff E A, Hohm O and Townsend P K 2009 Massive Gravity in Three Dimensions *Phys. Rev. Lett.* **102** 201301
- [5] Andringa R, Bergshoeff E A, de Roo M, Hohm, Sezgin E and Townsend P K 2010 Massive 3D Supergravity *Class. Quantum Grav.* **27** 025010
- [6] Bergshoeff E A, Hohm O and Townsend P K 2009 On Higher Derivatives in 3D Gravity and Higher Spin Gauge Theories *Preprint* arXiv:0911.3061
- [7] Deser S 2009 Ghost-free, finite, fourth order D=3 (alas) gravity *Phys. Rev. Lett.* **103** 101302
- [8] Nakasone M and Oda I 2009 On Unitarity of Massive Gravity in Three Dimensions *Prog. Theor. Phys.* **121** 1389
- [9] Bergshoeff E A, Hohm O and Townsend P K 2009 More on Massive 3D Gravity *Phys. Rev. D* **79** 124042
- [10] Stelle K S 1978 Classical Gravity With Higher Derivatives *Gen. Rel. Grav.* **9** 353
- [11] Nishino H and Rajpoot S 2006 (Curvature)²-terms for supergravity in three dimension *Phys. Lett. B* **639** 110
- [12] Deser S, Jackiw R and Templeton S 1982 Topologically massive gauge theories *Annals Phys.* **140** 372
- [13] Aragone C and Khoudeir A 1986 Selfdual massive gravity *Phys. Lett. B* **173** 141
- [14] Horava P 2009 Quantum Gravity at a Lifshitz Point *Phys. Rev. D* **79** 084008
- [15] Dalmazi D and Mendonça E L 2009 Generalized soldering of ± 2 helicity states in $D = 2 + 1$ *Phys. Rev. D* **80** 025017
- [16] Weinberg S 1965 Photons and gravitons in perturbation theory: Derivation of Maxwell's and Einstein's equations *Phys. Rev.* **138** B988
- [17] Deser S 1970 Self-interaction and gauge invariance *Gen. Rel. Grav.* **1** 9
- [18] Boulware D G and Deser S 1975 Classical General Relativity Derived From Quantum Gravity *Annals Phys.* **89** 193
- [19] Banados M and Theisen S 2009 Three-dimensional massive gravity and the bigravity black hole *J. High Energy Phys.* JHEP11(2009)033
- [20] Isham C J, Salam A and Strathdee J A 1971 F-dominance of gravity *Phys. Rev. D* **3** 867
- [21] Brown J D and Henneaux M 1986 Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity *Commun. Math. Phys.* **104** 207
- [22] Grumiller D and Hohm O 2009 AdS₃/LCFT₂ - Correlators in New Massive Gravity *Preprint* arXiv:0911.4274
- [23] Afshar H R, Alishahiha M and Naseh A 2009 On three dimensional bigravity *Preprint* arXiv:0910.4350