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Impulse free interconnection of dynamical systems ☆

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A B S T R A C T

In this paper we address the problem that impulses might occur when a to-be-controlled plant is connected to a suitable controller. In the behavioral literature this issue is dealt when studying the so-called 'regular feedback interconnection' (RFI) of the plant and the controller behaviors. We address the question of when a given controlled behavior satisfying desired specifications can be obtained by RFI of the plant and some suitable controller. The paper presents an algorithm to construct a controller that yields the controlled behavior by RFI, if such a controller exists, and then parametrizes all such controllers. We extend the results to the more general case when all the to-be-controlled variables might not be available for interconnection. We show that any disturbances in the plant continue to remain free in the controlled behavior if and only if the interconnection is an RFI. This is also equivalent to the well-posedness of an interconnection. This paper also makes concrete the intuitive relation between absence of inadmissibility of initial conditions and RFI. A door closing mechanism is analysed in relation to the results in this paper.

1. Introduction

Interconnection of two dynamical systems often causes 'impulsive behavior' during interconnection if the systems have not been prepared suitably before interconnection: examples are sparks during

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electrical switchings and jerky behavior during mechanical interconnections. Such situations call for some fine-tuning of system 'states' before the interconnection in order to rule out impulsive behavior. Considering the damage that impulsive behavior can cause to one or more devices in a system, it is of practical importance to design controllers that can be interconnected without requiring any preparation of the to-be-controlled system and the controller, and still guaranteeing absence of any impulsive behavior: we define such controllers as 'RFI controllers' in Section 2.3. This paper deals with this problem for linear time invariant differential systems; we characterize the conditions under which a given desired controlled behavior can be obtained using an RFI controller, and construct such a controller if one exists. This problem has also been dealt in Lomadze [9] using a fairly different approach as elaborated in Section 9.

The occurrence of impulses for certain initial states of an autonomous LTI system is well studied. These issues have been of relevance in the study of singular/descriptor/differential-algebraic-equation systems (see [3,18]). Impulses occurring due to certain initial conditions have also been studied using polynomial matrix representations (see [17,14], for example). Initial states of a dynamical system which lead to impulses are termed *inadmissible*. We establish a relation between the issue of inadmissible initial conditions and RFI: while an RFI controller ensures absence of inadmissible initial conditions, the two are not equivalent, i.e. even when a controller is not an RFI controller, the controlled system could assure absence of any inadmissible initial conditions.

We then consider the general case when all the variables of the plant may not be accessible to the controller for interconnection: the so-called partial interconnection case. We state necessary conditions on the controlled system for existence of an RFI controller for this case. For this general case, we also prove an equivalence between the property of non-impulsive interconnection and unmodelled disturbances remaining free after interconnection with the controller. This brings in the relation of RFI and the familiar concept of well-posedness of interconnections (see Kuijper [8] for related work in this context).

Though we primarily deal with continuous time differential systems, the results we prove are relevant for discrete time difference systems also: RFI controllers are precisely those that allow interconnection at any instant without constraints on the *past* values of the states. Due to space limitations, we deal with just continuous time differential systems; we do not elaborate on this analogy any more in this paper.

The paper is organized as follows: Section 2 states some preliminaries on polynomial matrices and about the behavioral approach. Section 3 relates the regular feedback interconnection problem to the problem of inadmissible initial conditions. Section 4 states this paper's main result, this leads to an algorithm for checking regular feedback implementability and constructing an RFI controller. We also consider a commonly occurring system, viz. a door closing mechanism, to analyse our results by applying the algorithm. We then present a characterization of all controllers which implement the desired behavior by RFI in Section 5. Following this, we extend the results and the algorithm to the partial interconnection case, i.e. when only some of the system variables are available for interconnection. These results have been presented in Section 6. The intuitive notion that a controller should allow any 'unmodelled' disturbances to continue remaining free in the interconnected system also is given a concrete meaning and shown to be equivalent to RFI in Section 7. Finally, we consider the situation of minimal order regular controllers and show using some examples how the straightforward extension of our results cease to be sufficient for this situation (Section 8). We conclude the paper with some remarks in Section 9.

2. Basic definitions and preliminaries

In this section we briefly cover preliminaries and some definitions in the behavioral approach. The notational aspects are also covered here. Table 1 at the end of this paper summarizes the frequently used symbols for easier reading and reference.

We use the symbol $\mathbb{R}$ for the set of real numbers. $\mathbb{R}[\xi]$ is the ring of polynomials in one indeterminate ξ, and $\mathbb{R}(\xi)$ is the field of fractions over this ring. We use the convention that a polynomial with degree less than zero is the zero polynomial. The set of matrices having m rows and n columns with polynomial entries is denoted by $\mathbb{R}^{m\times n}[\xi]$. When one or more dimensions are unambiguous from the context, we use $\bullet$ to denote the dimension, for example $A \in \mathbb{R}^{\bullet\times p}$ is a constant matrix with p columns and a suitable number of rows. At times, we use $\text{col}(A, B)$ as short hand for $\begin{bmatrix} A \\ B \end{bmatrix}$, where A and B have the same number of columns. We denote the number of components of a generic variable with the same letter but in a different font, for example $w(t)$ has $\mathtt{w}$ components, and $w(t) \in \mathbb{R}^{\mathtt{w}}$.

The rest of this section is divided into three parts: preliminaries about polynomial matrices (Section 2.1), definitions about LTI systems and the behavioral approach (Section 2.2), and a review of definitions and results about interconnection of behaviors (Section 2.3).

2.1. Polynomial matrices

The techniques used in this paper rely on various properties of polynomial matrices: this subsection contains relevant definitions and properties. A square polynomial matrix $U \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ is said to be unimodular if its determinant is a nonzero constant.

Let $R \in \mathbb{R}^{p\times\mathtt{w}}[\xi]$ be a polynomial matrix. The *row degree* k_i of its ith row is defined as the degree of the highest degree polynomial in the ith row. The leading row coefficient matrix $R_{\text{lc}} \in \mathbb{R}^{p\times\mathtt{w}}$ of R is defined as the constant matrix whose ith row comprises the coefficients of ξ^{k_i} in $R(\xi)$. A polynomial matrix $R(\xi)$ is said to be *row proper* if its leading row coefficient matrix R_{lc} of $R(\xi)$ has full row rank. Thus row properness of R implies R is full row rank as a polynomial matrix, i.e. its rows are linearly independent over the field $\mathbb{R}(\xi)$. The term *row reduced* is also common in the literature on polynomial matrices and is identical to row proper. A polynomial matrix R is called *column proper* if its transpose R^T is row proper.

The dimension of the state space of a minimal state representation of a dynamical system, the McMillan degree $\mathtt{n}$, plays a key role in this paper. For a full row rank polynomial matrix $R \in \mathbb{R}^{p\times\mathtt{w}}[\xi]$, we use $\mathtt{n}(R)$ to denote the McMillan[1] degree of the corresponding dynamical system: it is the maximal determinantal degree of all the $p \times p$ minors of R. The relation to the dynamical system's state space is elaborated below in Section 2.3; we require the following well-known relation of $\mathtt{n}(R)$ with row properness of R. See Kailath [7, Section 6.3], Wolovich [23, Section 2.5] among various others for a detailed exposition of these concepts.

Proposition 2.1. *Let $R \in \mathbb{R}^{p\times m}[\xi]$ have full row rank. Then, the following statements are true.*

1. *Suppose $k_1, k_2, \ldots, k_p$ are the row degrees of R, then R is row proper if and only if $\sum_{i=1}^{p} k_i = \mathtt{n}(R)$.*
2. *For every unimodular matrix $U \in \mathbb{R}^{p\times p}[\xi]$, $\mathtt{n}(UR) = \mathtt{n}(R)$.*
3. *There exists a unimodular matrix $U \in \mathbb{R}^{p\times p}[\xi]$ such that UR is row proper.*

When R is row proper, and the rows are permuted such that the the row degrees k_i are non-increasing, then the p–tuple $(k_1, k_2, \ldots, k_p)$ is called the Forney invariant indices. It is well-known (see Kailath [7, Lemma 6.3–14], Wedderburn [19]) that if R_1 and R_2 are row proper and related by $R_1 = UR_2$, with U unimodular, then the Forney indices of R_1 and R_2 are equal. As stated above, if R is not row proper but full row rank, then there exists a unimodular matrix U that results in UR being row proper. Such a row reducing unimodular matrix can be written as a product of unimodular matrices of a particularly interesting form: those that have only one row different from the identity matrix. We need such unimodular matrices in this paper. Given a row vector $\beta \in \mathbb{R}^{1\times p}[\xi]$, whose ith entry is a nonzero constant, $U_{\beta,i}$ is the unimodular matrix obtained by replacing the ith row of the identity matrix by β. Thus by definition $U_{\beta,i}$ presupposes that the row vector $\beta \in \mathbb{R}^{1\times p}[\xi]$ has a nonzero constant in its ith entry (and consequently $U_{\beta,i}$ is unimodular). Premultiplication of a matrix R by such a unimodular matrix $U_{\beta,i}$ is nothing but elementary row operations on the ith row of R. The following proposition makes this concrete in the context of row reduction.

[1] Note that this is in deviation with the usual definition of McMillan degree of a polynomial/rational matrix; when the polynomial matrix R is row proper the two are equivalent. See Kailath [7, p. 466] or Vardulakis [17, p. 40].

Proposition 2.2. *Let $R \in \mathbb{R}^{p\times \mathtt{w}}[\xi]$ be a full row rank polynomial matrix. Then, R is not row proper if and only if there exists an $i \in \{1, \ldots, p\}$ such that any one of the following equivalent conditions is true:*

1. *the row degree of the* ith *row, can be decreased by elementary row operations on that row,*
2. *there exists a polynomial row vector $\beta \in \mathbb{R}^{1\times p}[\xi]$, with β_i a nonzero constant, such that the* ith *row of $U_{\beta,i}R$ has a lower row degree than that of the* ith *row of R,*
3. *there exists a* monomial *row vector $\beta \in \mathbb{R}^{1\times p}[\xi]$ with β_i a nonzero constant, such that the* ith *row of $U_{\beta,i}R$ has a lower row degree than that the* ith *row of R.*

The monomial aspect of β in the last statement above plays a crucial role in this paper, hence we explain in detail the construction of the monomial vector $\beta \in \mathbb{R}^{1\times p}[\xi]$.

Construction of a monomial-$\boldsymbol{\beta}$:

1. Let $R_{lc} \in \mathbb{R}^{p\times \mathtt{w}}$ be the leading row coefficient matrix of (non row proper) $R \in \mathbb{R}^{p\times \mathtt{w}}[\xi]$. Find a nonzero $\alpha \in \mathbb{R}^{1\times p}$ such that $\alpha R_{\mathrm{lc}} = 0$; existence of α is guaranteed since R_{lc} is not full row rank. Note that there may be several independent row vectors α; choose any one. Denote the entries of α by $\alpha_1, \ldots, \alpha_p$.
2. Let $N \subseteq \{1, \ldots, p\}$ be the set of indices i such that $\alpha_i \neq 0$, and let $m \in N$ be such that the row degree of the mth row of R is the highest amongst all rows indexed by N. Define d to be the row degree the mth of R. (While d depends only on α, the maximum degree might occur at two or more rows of R indexed by N.)
3. Define $\ell_1, \ldots, \ell_p$ to be the 'slack' integers: $\ell_i := d - \text{rowdegree}_i(R)$ for $i \in \{1, \ldots, p\}$.
4. Finally define $\beta \in \mathbb{R}^{1\times p}[\xi]$ by $\beta = [\alpha_1\xi^{\ell_1}\ \alpha_2\xi^{\ell_2} \cdots \alpha_p\xi^{\ell_p}]$. Notice that by definition of m, $\ell_m = 0$ and $\beta_m = \alpha_m \neq 0$, in spite of the non-uniqueness of m.

The above method is standard: see Wolovich [23, Section 2.5.7], Vardulakis [17, p. 7], Kailath [7, p. 386] or Stefanidis et al [16, p. 49], for example. Due to non-uniqueness in the α above, there could be more than one monomial vectors β, and with different degrees of the monomials within β. This non-uniqueness poses no problems in the way β plays a role in our main results (but can pose difficulties elsewhere: see Remark 8.4 and discussion following that). Construction of β is shown within the door closing example when re-addressed at the end of Section 4 below. The existence of β in statements 3 and 4 above is linked to (the lack of) row properness through the well-known predictable-degree property of row proper matrices. We use this result often and hence we state it below for easy reference. See Forney [4] or Kailath [7, Theorem 6.3–13] for an elaborate exposition and the proof.

Proposition 2.3. *Let $R \in \mathbb{R}^{p\times \mathtt{w}}[\xi]$ be a full row rank polynomial matrix with row degrees k_i. The following are equivalent.*

1. *R is row proper.*
2. *For every $a, b \in \mathbb{R}^{1\times \bullet}[\xi]$ such that $aR = b$, the equality*: degree $b = \max_{i:a_i\neq 0}[k_i + \text{degree } a_i]$ *holds.*
3. *For every $a, b \in \mathbb{R}^{1\times \bullet}[\xi]$ such that $aR = b$, the inequality*: degree $b \geqslant k_i$ *holds for each i satisfying $a_i \neq 0$.*

In the context of inadmissible initial conditions, we need the concept of 'zeroes at infinity' of a polynomial/rational matrix. For a detailed treatment see Kailath [7, Chapter 6].

The Smith–McMillan form and zeroes at infinity: Let $H(s) \in \mathbb{R}^{p\times \mathtt{w}}(s)$ be a full row rank rational matrix such that each entry is proper, i.e. the numerator and denominator are coprime. Write $H(s)$ as $H(s) = N(s)/d(s)$, where $N(s) \in \mathbb{R}^{p\times \mathtt{w}}[s]$ and $d(s)$ is the monic least common multiple of the denominators of the entries of $H(s)$. The polynomial matrix $N(s) = d(s)H(s)$ can be written as $N(s) = U_1(s)[S(s)\ 0_{(\mathtt{w}-p)\times p}]U_2(s)$, where $S(s)$ is the Smith form of $N(s)$, and U_1 and U_2 unimodular matrices of suitable sizes. Further, obtain $\frac{S(s)}{d(s)} = \left[\mathrm{diag}(\frac{\lambda_1(s)}{d(s)}, \ldots, \frac{\lambda_p(s)}{d(s)})\right]$. Suppose on cancelling common factors in each entry of $\frac{\lambda_i(s)}{d(s)}$ we get $\frac{\epsilon_i(s)}{\psi_i(s)}$, $i = 1, \ldots, p$. Then, $H(s) = U_1(s)M(s)U_2(s)$, where

$$M(s) := \left[\mathrm{diag}\left(\frac{\epsilon_1(s)}{\psi_1(s)}, \ldots, \frac{\epsilon_p(s)}{\psi_p(s)}\right) \quad 0_{p\times(\mathtt{w}-p)}\right]$$

is such that M has the divisibility property: ϵ_i divides ϵ_{i+1}, and ψ_{i+1} divides ψ_i for all $i \in \{1, \ldots, p-1\}$. The rational matrix M is called the Smith–McMillan form of $H(s)$. In order to define the structure at ∞ of $H(s)$, it is standard to consider the Smith–McMillan form of $K(\lambda) := H(\frac{1}{\lambda})$ and analyse K at $\lambda = 0$. If any of the numerator polynomials $\epsilon_i(\lambda)$ in $K(\lambda)$ have a root at $\lambda = 0$, then $H(s)$ is said to have zeroes at infinity.

2.2. The behavioral approach

The problem of non-impulsive interconnection is formulated and solved in this paper using the behavioral approach to dynamical systems. For a detailed exposition on the behavioral approach, see Willems [20,22] and Polderman and Willems [11]; we cover only the essential preliminaries here. We consider dynamical systems that are described by a set of linear, constant-coefficient ordinary differential equations. Suppose $R_0 w + R_1 \frac{\mathrm{d}}{\mathrm{d}t} w + \cdots + R_N \frac{d^N}{dt^N} w = 0$, with $R_i \in \mathbb{R}^{p\times\mathtt{w}}$, are a set of p differential equations in the variable w. Constructing the polynomial matrix $R(\xi) := R_0 + R_1\xi + \cdots + R_N\xi^N$, these equations can be written as $R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0$. We assume the trajectories w belong to $\mathfrak{L}^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$: the space of locally integrable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$. We don't assume differentiability of the trajectories; the differential equations are assumed to be satisfied *in the distributional sense*, i.e. in the weak sense.

The set of solutions to the differential equations $R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0$ is called the *behavior* $\mathfrak{B}$ of the system: these are all the trajectories that the system laws allow. More precisely,

$$\mathfrak{B} := \left\{ w \in \mathfrak{L}^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) | R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0 \right\}.$$

The equation $R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0$ is called a *kernel representation* of the behavior, to distinguish it from other ways of describing the set of allowed trajectories. The set of behaviors is denoted by $\mathfrak{L}^{\mathtt{w}}$, the variable $\mathtt{w}$ in the superscript denoting a generic variable in the behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$: this is essential when distinguishing behaviors with different variables.

We say the polynomial matrix $R(\xi) \in \mathbb{R}^{p\times\mathtt{w}}[\xi]$ *induces* a kernel representation for the behavior $\mathfrak{B}$ if $R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0$ is a kernel representation. The matrix R is far from unique: premultiplication of R by any unimodular matrix $U \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}[\xi]$ leaves the set of solutions (to the equations $U\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0$) the same. This allows us to consider R having full row rank without loss of generality. A *minimal* kernel representation $R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0$ is a kernel representation in which $R(\xi)$ has full row rank as a polynomial matrix. The rank of such a matrix is an invariant of the behavior: it does not depend on the particular kernel representation. We call this rank the output cardinality of the behavior and denote it by $\mathtt{p}(\mathfrak{B})$. In addition to a full row rank representation, without loss of generality, we start with a *row proper* kernel representation of a behavior, i.e. $R\left(\frac{\mathrm{d}}{\mathrm{d}t} w\right) = 0$ with R row proper: Proposition 2.1 allows this.

We briefly discussed the McMillan degree $\mathtt{n}(\mathfrak{B})$ of a dynamical system with behavior $\mathfrak{B}$ in Section 2.1. If $R \in \mathbb{R}^{p\times\mathtt{w}}[\xi]$ induces a minimal kernel representation for $\mathfrak{B}$, we denote the McMillan degree of $\mathfrak{B}$ by $\mathtt{n}(R)$, and define it to be the maximal determinantal degree of all the $p \times p$ minors of R. This integer is the dimension of the state space in a minimal state space description of the behavior and it depends only on $\mathfrak{B}$ and not on the R used to define it. See Rapisarda and Willems [13] for a systematic development.

Since the notion of state makes concrete the intuitive idea of non-impulsive concatenability of trajectories within a behavior (see Rapisarda and Willems [13]), the McMillan degree $\mathtt{n}(\mathfrak{B})$ expectedly plays the central role in this paper. We use this in defining Regular Feedback Interconnection in the following subsection.

2.3. Interconnection and control

This subsection contains definitions about interconnection aspects in the behavioral approach. One of the salient features of this approach is that control of a plant is viewed as restriction of the plant behavior $\mathcal{P}$ to a desired sub-behavior $\mathcal{K}$. This restriction is achieved by designing new laws that the system variables have to satisfy in addition to the plant equations. These additional laws themselves constitute a dynamical system which we call the controller. Thus control of a plant $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$ in the behavioral approach is about designing a sub-behavior $\mathcal{K}$ with desired properties and then obtaining the restriction of $\mathcal{P}$ to this controlled behavior $\mathcal{K}$ by a suitable choice of a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ such that $\mathcal{P} \cap \mathcal{C} = \mathcal{K}$, i.e. the trajectories allowed in the controlled system $\mathcal{K}$ are those that satisfy the laws of both $\mathcal{P}$ and $\mathcal{C}$, hence they lie in the intersection of $\mathcal{P}$ and $\mathcal{C}$.

Suppose R and $C \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ induce row proper kernel representations for $\mathcal{P}$ and $\mathcal{C}$ respectively. Then a kernel representation for the controlled behavior $\mathcal{K} := \mathcal{P} \cap \mathcal{C}$ is $\begin{bmatrix} R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ C\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \end{bmatrix} w = 0$. Whether this kernel representation is minimal, and further row proper, is the focus of this section: we define the interconnection of $\mathcal{P}$ and $\mathcal{C}$ to be regular or regular feedback accordingly.

Definition 2.4. Let $\mathcal{P}$ and $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ be two behaviors with minimal kernel representations $R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0$ and $C\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0$ respectively. Their interconnection is said to be regular if rank of $\begin{bmatrix} R(\xi) \\ C(\xi) \end{bmatrix}$ is the sum of the ranks of R and C. In this case, the controlled behavior $\mathcal{K} := \mathcal{P} \cap \mathcal{C}$ is said to be regularly implemented by $\mathcal{C}$, and $\mathcal{C}$ is called a regularly implementing controller. Further, a behavior $\mathcal{K}$ is said to be regularly implementable with respect to $\mathcal{P}$ if there exists $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ such that $\mathcal{K} = \mathcal{P} \cap \mathcal{C}$ and the interconnection of $\mathcal{P}$ and $\mathcal{C}$ is regular.

Regular interconnection means that the output cardinality of the interconnected system is the sum of the output cardinalities of the two behaviors. A more special form of regular interconnection is Regular Feedback Interconnection (RFI): when the McMillan degrees of the two behaviors also sum up to that of the interconnected system.

Definition 2.5. Let $\mathcal{P}$ and $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ be two behaviors. Their interconnection is said to be Regular Feedback Interconnection (RFI) if the interconnection is regular and $\mathtt{n}(\mathcal{P}) + \mathtt{n}(\mathcal{C}) = \mathtt{n}(\mathcal{P} \cap \mathcal{C})$. In other words, suppose $R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0$ and $C\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0$ are row proper kernel representations of $\mathcal{P}$ and $\mathcal{C}$, then their interconnection is said to be RFI if $\begin{bmatrix} R(\xi) \\ C(\xi) \end{bmatrix}$ also is row proper. In this case, the controlled behavior $\mathcal{K} := \mathcal{P} \cap \mathcal{C}$ is said to be regular feedback implemented (RFI) by $\mathcal{C}$, and $\mathcal{C}$ is called a regular feedback implementing (RFI) controller. Further, a behavior $\mathcal{K}$ is said to be regular feedback implementable (RFI) with respect to $\mathcal{P}$ if there exists $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ such that $\mathcal{K} = \mathcal{P} \cap \mathcal{C}$ and the interconnection of $\mathcal{P}$ and $\mathcal{C}$ is an RFI.

The focus of this paper is on regular feedback interconnection and, when unambiguous from the context, we use the abbreviation 'RFI' to mean

- Regular Feedback Interconnection, when speaking about interconnection;
- Regular Feedback Implementing, when referring to a controller that implements a given controlled behavior by regular feedback interconnection; and
- Regular Feedback Implementable, in the context of a controlled behavior that can be obtained by regular feedback interconnection between the plant and some controller.

Note that if an interconnection is a regular feedback interconnection, then it is also regular.[2] Thus we often assume regular interconnection and then formulate questions regarding regular feedback

[2] This is true except for some pathological cases due to equations that are just algebraic and not differential, see Julius [5, p. 94]. We have ignored this aspect in our paper and have defined regular feedback interconnection to be a special kind of a regular interconnection, as done in Kuijper [8] and Willems [22].

interconnection. These notions were introduced in Willems [21], and have been studied further in Kuijper [8], Willems [22,2] and Praagman et al. [12] among various others. In Kuijper [8], the 'degree of singularity' of an interconnection has been defined and zero degree of singularity is called regular feedback interconnection. The door closing mechanism, which we treat in Section 4 below, has been shown to be an interconnection with a degree of singularity equal to two and hence is not a regular feedback interconnection.

The relation of RFI to concatenability of trajectories has been brought out in Willems [21,22], Julius and van der Schaft [6, Section 3.1]. We state this relation as a proposition below for easy reference. It relates concatenability of trajectories to RFI: non-impulsive concatenability of arbitrary trajectories in two behaviors to a suitable common trajectory in the intersection is possible if and only if the interconnection of the behaviors is a regular feedback interconnection. We use the notation $w_{23} := w_2 \underset{\tau}{\wedge} w_3$ for concatenation of w_2 to w_3 at time τ, i.e.

$$w_{23}(t) := w_2 \underset{\tau}{\wedge} w_3(t) = \begin{cases} w_2(t) & \text{for } t \leqslant \tau, \\ w_3(t) & \text{for } t > \tau. \end{cases}$$

Proposition 2.6. *Let $\mathcal{P}$ and $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$. Then the following are equivalent.*

- *The interconnection of $\mathcal{P}$ and $\mathcal{C}$ is RFI.*
- *For every $w_1 \in \mathcal{P}, w_2 \in \mathcal{C}$ and $\tau \in \mathbb{R}$, there exists a $w_3 \in \mathcal{P} \cap \mathcal{C}$ such that $w_{13} := w_1 \underset{\tau}{\wedge} w_3$ and $w_{23} := w_2 \underset{\tau}{\wedge} w_3$ respectively satisfy the equations of $\mathcal{P}$ and $\mathcal{C}$ in a weak sense.*

Note that w_{13} and w_{23} are not $\mathfrak{C}^\infty$ trajectories, but are just $\mathfrak{L}^1_{\text{loc}}$. In this context, satisfying the system's differential equations in a weak sense amounts to no impulse at the time of concatenation τ. According to the above proposition, when the interconnection of $\mathcal{P}$ and $\mathcal{C}$ is an RFI, then during interconnection at any time τ, no matter what the past trajectories w_1 and w_2 of $\mathcal{P}$ and $\mathcal{C}$ were, there exists a suitable non-impulsive transition to $w_3 \in \mathcal{P} \cap \mathcal{C}$. Conversely, when the interconnection is not an RFI, there exist trajectories $w_1 \in \mathcal{P}$ and $w_2 \in \mathcal{C}$ that cannot transition to any trajectory in $\mathcal{P} \cap \mathcal{C}$ without causing an impulse at the time of interconnection. Thus, Regular Feedback Interconnection (RFI) is equivalent to guaranteeing absence of impulsive phenomenon during interconnection.

We finish this section with some notation about row dimension of the polynomial matrices inducing the concerned behaviors' kernel representations. The plant, the controlled system and the controller behaviors are denoted by $\mathcal{P}$, $\mathcal{K}$ and $\mathcal{C}$ respectively. Since we are characterizing conditions about regular *feedback* implementability, we assume the weaker condition of regular implementability whenever relevant. Hence the controllers we begin with are always regular controllers. This allows us to denote/assume the output cardinalities of $\mathcal{P}$, $\mathcal{K}$ and $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ by p, k and $(k-p)$ respectively (see Definition 2.4). Accordingly, minimal kernel representations for $\mathcal{P}$ and $\mathcal{C}$ are, respectively, induced by $R \in \mathbb{R}^{p\times\mathtt{w}}[\xi]$ and $C \in \mathbb{R}^{(k-p)\times\mathtt{w}}[\xi]$.

3. Inadmissible initial conditions

Using the background covered in the previous section, we study the relation between RFI and the occurrence of impulses due to 'inadmissible' initial conditions in an autonomous system. While this relation is intuitively expected, Theorem 3.2 shows that RFI implies absence of any inadmissible initial conditions, but the converse is not true. We illustrate this with a practical example of a door closing mechanism.

We now define an inadmissible initial condition vector for an autonomous system $Q\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w(t) = 0$, with $Q(\xi) \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}[\xi]$ nonsingular; see Verghese et al [18], Dai [3] and Vardulakis [17] for a similar treatment. Let z be equal to the degree of the highest degree entry in $Q(\xi)$. Let $w(0)$, $w^{(1)}(0)$,...,$w^{(z-1)}(0)$ be the values of w, $\frac{\mathrm{d}}{\mathrm{d}t}w$, ..., $\frac{\mathrm{d}^{z-1}}{\mathrm{d}t^{z-1}}w$ at time $t = 0^-$. Define the vector $\bar{w}(0) = (w(0), w^{(1)}(0), \ldots, w^{(z-1)}(0))$. We call $\bar{w}(0) \in \mathbb{R}^{pz}$ an initial condition vector. A vector

$\bar{w}(0) \in \mathbb{R}^{pz}$ is said to be an *inadmissible* initial condition vector if the corresponding solution $w(t)$ contains the Dirac impulse $\delta(t)$ and/or its distributional derivatives.

We use the following result from Vardulakis [17, Theorem 4.32], which gives a necessary and sufficient condition for existence of inadmissible initial conditions for autonomous systems.

Proposition 3.1. *Consider the autonomous system defined by* $Q\left(\frac{d}{dt}\right)w(t) = 0$ *where* $Q \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ *is nonsingular, and suppose z is the degree of the highest degree entry in* $Q(\xi)$*. Then, for every initial condition vector* $\bar{w}(0) = (w(0), w^{(1)}(0), \ldots, w^{(z-1)}(0)) \in \mathbb{R}^{z\mathtt{w}}$*, the corresponding solution* $w(t)$ *has no Dirac impulses nor its distributional derivatives if and only if* $Q(\xi)$ *has no zeroes at infinity. In other words, there exist no inadmissible initial conditions for* $Q\left(\frac{d}{dt}\right)w = 0$ *if and only if Q has no zeros at* ∞.

Using so-called valuations theory (see Kailath [7, Section 6.4] and Vardulakis [17]), a row proper or a column proper matrix matrix does not have any zeroes at infinity. This can also be checked using the definition of zeros at ∞ through the Smith–McMillan form described in the previous section. Use of this relation between row properness and absence of zeroes at ∞ leads to the following theorem: RFI controllers ensure that the controlled system has no inadmissible initial conditions; its proof is straight-forward and follows from the definition of RFI, hence it has been skipped.

Theorem 3.2. *Let* $R\left(\frac{d}{dt}\right)w = 0$ *be a row proper kernel representation of* $\mathcal{P}$*. Suppose the autonomous behavior* $\mathcal{K} \subseteq \mathcal{P}$ *is regular feedback implementable. Then, there exists a controller* $\mathcal{C}$ *with a row proper kernel representation* $C\left(\frac{d}{dt}\right)w = 0$ *(say) such that the system of differential equations* $\begin{bmatrix} R\left(\frac{d}{dt}\right) \\ C\left(\frac{d}{dt}\right) \end{bmatrix} w = 0$ *has no inadmissible initial conditions.*

We now ask the converse question: if the interconnection is not RFI, do there necessarily exist inadmissible initial conditions? We will see in the following example that this is not true. This is because column properness also is sufficient to ensure that there are no zeroes at infinity, but column properness does not imply RFI. We can see this by analysing a very commonplace system, viz. the door closing mechanism.

Let M be the mass/inertia of the door, m the mass/inertia of the door closing mechanism, b the damping coefficient of the damper, and k the spring constant of the spring. Suppose θ is the angle as shown in Fig. 1 and F the force/torque exerted by the door closing mechanism on the door. A kernel representation for this system is given by $K\left(\frac{d}{dt}\right)\begin{bmatrix} \theta \\ F \end{bmatrix} = 0$ where

$$K(\xi) = \begin{bmatrix} M\xi^2 & -1 \\ m\xi^2 + b\xi + k & 1 \end{bmatrix}. \tag{1}$$

Due to K not being row proper, it is seen that this is not RFI; see also Kuijper [8] and Willems [22] for a discussion. However, the matrix in Eq. (1) does not have any zeroes at infinity, since it is column proper, i.e. $K^T(\xi)$ is row proper. This implies that there are no inadmissible initial conditions and, as observed in real life, no impulsive behavior is exhibited for any initial condition of the controlled door system. We will show below in Section 4.1 that the behavior represented in Eq. (1) cannot be implemented by RFI by any controller.

4. Plant equation of reducible degree (PERD) and regular feedback implementability

In this section we study the conditions on a desired control objective $\mathcal{K}$ (and on the plant $\mathcal{P}$) for the existence of a controller that implements $\mathcal{K}$ by regular feedback interconnection (RFI) with the plant $\mathcal{P}$. More precisely, we solve the following problem: given $\mathcal{P}$ and $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$, find necessary and sufficient conditions under which there exists a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ such that $\mathcal{K} = \mathcal{P} \cap \mathcal{C}$ and the interconnection is

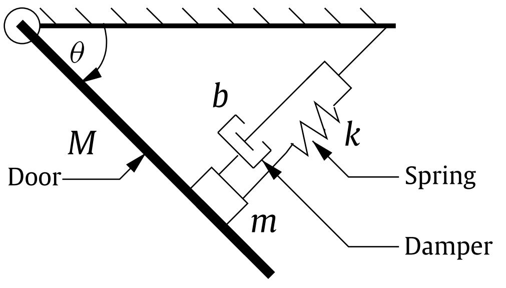

Fig. 1. A door closing mechanism.

an RFI. In order to formulate this condition we define the notion of a 'Plant Equation of Reducible Degree' (PERD). The definition requires use of Proposition 2.2 which states that if a polynomial matrix is not row proper, then there exists a row of this matrix whose row degree can be proper by elementary row operations on that row. Recall that in the text preceding that proposition we defined $U_{\beta,i} \in \mathbb{R}^{k\times k}[\xi]$ as the unimodular matrix constructed by replacing the ith row of the identity matrix by $\beta \in \mathbb{R}^{1\times k}[\xi]$ (assuming the ith entry of β is a nonzero constant). We use this to define a PERD.

Definition 4.1. Consider the interconnection of $\mathcal{P}$ and a regular $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$. Suppose $R \in \mathbb{R}^{p\times \mathtt{w}}[\xi]$ and $C \in \mathbb{R}^{(k-p)\times \mathtt{w}}[\xi]$ induce row proper kernel representations of $\mathcal{P}$ and $\mathcal{C}$, respectively. Define $K := \begin{bmatrix} R \\ C \end{bmatrix}$. Suppose there exists a *monomial* vector $\beta \in \mathbb{R}^{1\times k}[\xi]$ such that

- for some $i \in \{1, \ldots, p\}$, the entry β_i is a nonzero constant and the row degree of the ith row of $U_{\beta,i}K$ is strictly less than that of the ith row of K, and
- none of the last $(k-p)$ entries of β is a nonzero constant.

Then, the ith row of $R(\xi)$ is said to be a Plant Equation of Reducible Degree (PERD).

Note that a PERD is defined using specific kernel representations of $\mathcal{P}$ and $\mathcal{C}$. Further, there could exist more than one PERDs for given row proper kernel representations. In spite of the seeming dependence on the representations used to define PERD, the following theorem shows that existence of a PERD characterizes non-regular-feedback-implementability of $\mathcal{K}$.

Another point about the above definition is noteworthy. The existence of a β clearly implies that the interconnection of $\mathcal{P}$ and *that* controller is not an RFI. However, it is not straightforward to deduce if $\mathcal{K}$ can be obtained by RFI using some other controller, or whether no such RFI controller exists. The condition $\mathtt{n}(\mathcal{K}) \geqslant \mathtt{n}(\mathcal{P})$ is obviously necessary but insufficient for existence of an RFI controller. Suppose we have a regular controller satisfying $\mathtt{n}(\mathcal{C}) > \mathtt{n}(\mathcal{K}) - \mathtt{n}(\mathcal{P})$, then the natural question is: does there exist a lower order controller $\mathcal{C}_{rf}$ that implements $\mathcal{K}$ by RFI (and hence satisfies $\mathtt{n}(\mathcal{C}_{rf}) = \mathtt{n}(\mathcal{K}) - \mathtt{n}(\mathcal{P})$), or is the interconnection of $\mathcal{P}$ and every other regular controller not an RFI? The following theorem puts this matter to rest: we show that regular feedback implementability of $\mathcal{K}$ is equivalent to non-existence of PERD for *every* regular controller.

Theorem 4.2. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$ and let $R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0$ be a row proper kernel representation of $\mathcal{P}$. Suppose $\mathcal{K} \subseteq \mathcal{P}$ is regularly implementable with respect to $\mathcal{P}$. Then, the following are equivalent.*

1. *$\mathcal{K}$ is regular feedback implementable.*
2. *For every controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ that implements $\mathcal{K}$ regularly, there exists no PERD in $\begin{bmatrix} R \\ C \end{bmatrix}$, where $C\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0$ is any row proper kernel representation of $\mathcal{C}$.*

In view of the difficulty posed by a regular controller satisfying $\mathtt{n}(\mathcal{C}) > \mathtt{n}(\mathcal{K}) - \mathtt{n}(\mathcal{P})$ as described above, if $\mathcal{C}$ is not an RFI controller, then the polynomial matrix $\mathrm{col}(R, C)$ is not row proper. Existence of β as pointed in Proposition 2.2 helps in decreasing the row degree of some row. While any modification of the controller equations using the plant equations is justifiable, since this amounts to just choosing another regular controller (see Proposition 4.3), modification of a *plant equation* is not allowed. However, as noted in the text following Proposition 2.2, the elementary row operations and the sequence of operations to obtain row properness is far from unique. Due to this non-uniqueness in the sequence of rows of $\mathrm{col}(R, C)$ whose degrees are decreased successively to obtain row properness, it appears a priori that a PERD's mere existence does not rule out the possibility of a sequence of modifications of just the controller equations to eventually obtain row properness, and hence an RFI controller. The above theorem's significance lies in addressing this concern conclusively.[3]

In order to prove Theorem 4.2, we need the following result from Kuijper [8, Theorem 3.3] (see also Praagman et al. [12, Theorem 11]), which characterizes all regularly implementing controllers $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ for a given controlled behavior $\mathcal{K}$.

Proposition 4.3. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$ and let $R \in \mathbb{R}^{p\times\mathtt{w}}[\xi]$ induce a minimal kernel representation for $\mathcal{P}$. Suppose $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ is a regular controller that implements $\mathcal{K}$ with respect to $\mathcal{P}$, and let $C \in \mathbb{R}^{(k-p)\times\mathtt{w}}[\xi]$ induce a minimal kernel representation for $\mathcal{C}$. Consider a controller $\mathcal{C}' \in \mathfrak{L}^{\mathtt{w}}$ with a minimal kernel representation induced by $C' \in \mathbb{R}^{(k-p)\times\mathtt{w}}[\xi]$. The following are equivalent*:

1. *$\mathcal{C}'$ also regularly implements $\mathcal{K}$.*
2. *There exist a polynomial matrix $P \in \mathbb{R}^{(k-p)\times p}[\xi]$ and a unimodular matrix $V \in \mathbb{R}^{(k-p)\times(k-p)}[\xi]$ such that $C' = PR + VC$.*

We utilize the following easy consequence of the above proposition: any regularly implementing controller $\mathcal{C}'$ admits a minimal kernel representation induced by C' of the form $PR + C$, where P is a suitable polynomial matrix. Moreover, when utilizing the above proposition later to parametrize all RFI controllers, we show that each P above results in a different *controller system* (i.e., a different controller behavior), and not just a different set of equations possibly of the same behavior. (See Lemma 5.2.)

Proof of Theorem 4.2. ($1 \Rightarrow 2$): Suppose $\mathcal{K}$ is RFI with respect to $\mathcal{P}$, and let $\mathcal{C}_r$ be an RFI controller with $C_r \in \mathbb{R}^{(k-p)\times\mathtt{w}}[\xi]$ inducing a row proper kernel representation for $\mathcal{C}_r$. In view of Proposition 4.3, any other controller $\mathcal{C}_2$ with minimal kernel representation $C_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w = 0$ that implements $\mathcal{K}$ regularly can be obtained using a matrix $Q \in \mathbb{R}^{(k-p)\times p}[\xi]$ from R and C_r as $C_2 = QR + C_r$. Suppose $V \in \mathbb{R}^{(k-p)\times(k-p)}[\xi]$ is unimodular such that VC_2 is row proper.

Assume that there exists a PERD for $\begin{bmatrix} R \\ VC_2 \end{bmatrix}$, i.e. there exists a $\beta \in \mathbb{R}^{1\times k}[\xi]$ (β partitioned suitably into $[\beta_p\ \beta_c]$) and $i \in \{1, \ldots, p\}$ such that $\beta_i = 1$ and the ith row of $\left(U_{\beta,i}\begin{bmatrix} R \\ VC_2 \end{bmatrix}\right)$ has a row degree strictly lower than that of the ith row of R. Define b to be the ith row of $\left(U_{\beta,i}\begin{bmatrix} R \\ VC_2 \end{bmatrix}\right)$. Further, there are no nonzero constant entries in $\beta_c \in \mathbb{R}^{1\times(k-p)}[\xi]$. We now show that such a β leads to a contradiction to the Regular Feedback Interconnection of $\mathcal{P}$ and $\mathcal{C}_r$.

Evaluating $[\beta_p\ \ \beta_c]\begin{bmatrix} I & 0 \\ VQ & V \end{bmatrix}$, we get $[\beta_p + \beta_c VQ\ \ \beta_c V] =: a$, say. Since the ith entry of β_p is 1, and since there are no nonzero constant entries in the *monomial* vector β_c, we infer that the ith entry of a is nonzero. Thus we used β to obtain $b \in \mathbb{R}^{1\times\mathtt{w}}[\xi]$ and an $a \in \mathbb{R}^{1\times k}[\xi]$ such that $b = a\begin{bmatrix} R \\ C_r \end{bmatrix}$, and the degree of b is strictly less than the row degree of the ith row of R, with $a_i \neq 0$. By Proposition 2.3, this contradicts the row properness of $\begin{bmatrix} R \\ C_r \end{bmatrix}$. Thus there cannot exist any PERD for any regular controller that implements $\mathcal{K}$ if $\mathcal{K}$ is regular feedback implementable.

[3] On the contrary, in Section 8.2 in the context of minimal order regular controllers, Remark 8.3 and the ensuing text elaborate on the inconclusiveness, despite the absence of any CERD (Controller Equation of Reducible Degree).

$(2 \Rightarrow 1)$: In this part of the proof, we need to assume non-existence of any PERD in $\text{col}(R, C)$ arising out of every regular controller that implements $\mathcal{K}$ (Property 2 of Theorem 4.2), and show the existence of an RFI controller that implements $\mathcal{K}$. We show this as follows: we start with a regular controller $\mathcal{C}_1$ having Property 2 and modify $\mathcal{C}_1$, if it is not already an RFI controller, to obtain $\mathcal{C}_2$ that also implements $\mathcal{K}$ regularly, and so on. In a finite number of steps we show that this method yields an RFI controller due to Property 2. Since this modification is nothing but an algorithm for construction of an RFI controller, if one exists, we write this part of the proof in the form of an algorithm below in the following subsection. The termination of the algorithm in a finite number of steps to yield an RFI controller is also shown there. □

4.1. Construction of an RFI controller

In this subsection we show how a regular controller $\mathcal{C}_1$ that implements $\mathcal{K}$ can be modified to obtain a regular feedback implementing controller $\mathcal{C}_2$ by using the notion of PERD and Theorem 4.2. This construction is chalked out as an algorithm below and it also constitutes the second part of the proof of Theorem 4.2.

Algorithm:
Input: $R \in \mathbb{R}^{p\times \mathtt{w}}[\xi]$ and $C \in \mathbb{R}^{(k-p)\times \mathtt{w}}[\xi]$ respectively inducing row proper kernel representations for $\mathcal{P}$ and a regular controller $\mathcal{C}$.
Output: A controller which implements $\mathcal{K} := \mathcal{P} \cap \mathcal{C}$ by RFI, OR a conclusion that no such controller exists.

1. Check whether $\begin{bmatrix} R \\ C \end{bmatrix}$ is row proper. If yes, then stop: the matrix C induces a kernel representation of the required controller, i.e. an RFI controller. If not, proceed to Step 2.
2. Construct R_{lc}, the leading row coefficient matrix of $\begin{bmatrix} R \\ C \end{bmatrix}$. Find a nonzero vector $\alpha \in \mathbb{R}^{1\times k}$ such that $\alpha R_{\text{lc}} = 0$. Using α, construct the monomial vector $\beta(\xi)$ as explained in the text following Proposition 2.2. Partition $\beta = [\beta_p \;\; \beta_c]$ corresponding to the sizes of R and C. We now have exactly one of the following two situations.
   (a) Situation 1: There is one or more nonzero constant entries *only* in β_p; in that case, the algorithm ends: there exists a PERD and hence $\mathcal{K}$ is not regular feedback implementable.
   (b) Situation 2: β_c has one or more nonzero constant entries; in this case, a lower McMillan degree controller implements $\mathcal{K}$ regularly. This is constructed in the next step.
3. Construct $U_{\beta,i}$ with $i \in \{p+1, \ldots, k\}$ by replacing the ith row of the identity matrix by $\beta(\xi)$ and premultiply $\begin{bmatrix} R \\ C \end{bmatrix}$ by the unimodular matrix $U_{\beta,i}$ so that the degree of the $(i-p)$th row in C is proper by at least one. Call this new controller's representation C_{new} after obtaining row properness, if necessary (see Proposition 2.1, Statement 3). Go to Step 1 with this new controller.

Since the row degree of some row in the controller decreases by at least one in each iteration, the algorithm will terminate in a finite number of steps into one of the following situations: Situation 1 is reached in Step 2, or R_{lc} has full row rank, i.e. the last controller is an RFI controller. Note that the controller is modified each time in Step 3 to obtain a new regular controller with a lower McMillan degree. In the context of the latter part of Theorem 4.2's proof, each new regular controller satisfies Property 2 of Theorem 4.2 by assumption, and this implies that the algorithm never terminates into Situation 1 within Step 2 above, and hence yields an RFI controller in a finite number of steps.

Example. Consider the door closing mechanism mentioned in Eq. (1) and shown in Fig. 1. We have already seen earlier that the system described by Eq. (1) does not have any inadmissible initial conditions. We now analyse what happens when we interconnect the door closing mechanism (construed as the controller) to the door (thought of as the plant). As noted before, the interconnection represented by Eq. (1) is not RFI, see Section 3, Kuijper [8], Willems [22]. However, the interconnection is regular.

Physically this means that the states of the plant and the controller, i.e. θ and $\overset{\bullet}{\theta}$ have to be set to compatible values during interconnection.

Consider the first row (i.e. the equation for the door alone) as the plant. The matrix in Eq. (1) is a minimal kernel representation of $\mathcal{K}$. It can be seen that $\mathcal{K}$ has been regularly implemented by the controller (i.e. the second row in the matrix in Eq. (1)). We now find whether some other controller can implement the same behavior by RFI. We do this by applying the above algorithm.

- Step 1: The given representation is not RFI.
- Step 2: The leading row coefficient matrix is given by $R_{\text{lc}} = \begin{bmatrix} M & 0 \\ m & 0 \end{bmatrix}$. A nonzero vector α satisfying $\alpha R_{\text{lc}} = 0$ is $\alpha = \begin{bmatrix} -\frac{m}{M} & 1 \end{bmatrix}$. Since the row degrees of the matrix $K(\xi)$ are equal, the monomial vector $\beta \in \mathbb{R}^{1\times 2}[\xi]$ in this case is $\beta = \alpha$. This does not have a nonzero constant *only* in $\beta_p(\xi)$, i.e. both the entries in β are nonzero constants.
- Step 3: Constructing $U_{\beta,i}$ by replacing the ith row of the identity matrix $I_{2\times 2}$ by β, we get $U_{\beta,2} = \begin{bmatrix} 1 & 0 \\ -\frac{m}{M} & 1 \end{bmatrix}$. On premultiplying $\begin{bmatrix} R \\ C \end{bmatrix}$ by $U_{\beta,2}$ we get $\begin{bmatrix} M\xi^2 & -1 \\ b\xi + k & 1 + \frac{m}{M} \end{bmatrix}$.
- Back to Step 1: This interconnection is still not RFI.
- Step 2: The new leading row coefficient matrix is $\begin{bmatrix} M & 0 \\ b & 0 \end{bmatrix}$. A vector α for this case is $\alpha = \begin{bmatrix} -\frac{b}{M} & 1 \end{bmatrix}$ and the monomial vector $\beta(\xi) = \begin{bmatrix} -\frac{b}{M} & \xi \end{bmatrix}$. Note that this *does* have a nonzero constant only in $\beta_p(\xi)$, and thus the controlled behavior $\mathcal{K}$ has a regular controller such that there exists a PERD, namely, the first row.

Thus, using Theorem 4.2 we conclude that $\mathcal{K}$ cannot be implemented by RFI using any controller.

5. Characterization of all regular feedback implementing controllers

In the previous section we saw how an RFI controller $\mathcal{C}_{rf}$ that implements $\mathcal{K}$ can be obtained. In this section we show how *all* RFI controllers that implement $\mathcal{K}$ can be obtained from $\mathcal{C}_{rf}$. Section 5.1 provides a way of constructing a minimal kernel representations for all RFI controllers that implement $\mathcal{K}$. Section 5.2 quantifies the size of this set of controllers.

5.1. Parameterization of all RFI controllers

In this subsection, we define a set of polynomial matrices and show in Theorem 5.1 that this set of polynomial matrices parametrizes the set of regular feedback implementing controllers.

Theorem 5.1. *Suppose $R \in \mathbb{R}^{p\times \mathtt{w}}[\xi]$ and $C_{rf} \in \mathbb{R}^{(k-p)\times \mathtt{w}}[\xi]$ induce row proper kernel representations of $\mathcal{P}$ and $\mathcal{C}$, and suppose their row degrees are r_j for $j \in \{1, \ldots, p\}$ and c_i for $i \in \{1, \ldots, k-p\}$ respectively. Assume $\mathcal{C}$ is an RFI controller, and let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ be the resulting controlled behavior. Define the set of polynomial matrices* $\text{DegBnd}(R, C_{rf}) \subseteq \mathbb{R}^{(k-p)\times p}[\xi]$ *with a bound on the degrees of its entries as follows:*

$$\text{DegBnd}(R, C_{rf}) := \{P \in \mathbb{R}^{(k-p)\times p}[\xi] \,|\, \text{degree}(p_{ij}) \leqslant c_i - r_j, \text{ for each } i \in \{1, \ldots, k-p\} \text{ and for each } j \in \{1, \ldots, p\}\}.$$

Then, there is a one to one correspondence between the polynomial matrices $\text{DegBnd}(R, C_{rf})$ *and the set of RFI controllers that implement $\mathcal{K}$, i.e.*

1. *For every $P \in \text{DegBnd}(R, C_{rf})$, the controller $\mathcal{C}' \in \mathfrak{L}^{\mathtt{w}}$ defined by the kernel representation induced by $PR + C_{rf}$ is an RFI controller.*
2. *For every RFI controller $\mathcal{C}' \in \mathfrak{L}^{\mathtt{w}}$ that implements $\mathcal{K}$, there exists a unique $P \in \text{DegBnd}(R, C_{rf})$ such that $PR + C_{rf}$ induces a kernel representation of $\mathcal{C}'$.*

The above theorem is the crux of this section. Before proving it we need a couple of lemmas. First we prove (Lemma 5.2) that when dealing with minimal kernel representations of $\mathcal{P}$ and a regular controller $\mathcal{C}$, manipulation of $\mathcal{C}$ results in, not just a different representation, but a different controller system.

Lemma 5.2. *Let R and C induce minimal kernel representations of $\mathcal{P}$ and $\mathcal{C}$ respectively, and suppose their interconnection is regular. For polynomial matrices $P_1, P_2 \in \mathbb{R}^{\bullet \times p}[\xi]$, define $C_1 := P_1R + C$ and $C_2 := P_2R + C$, and suppose $\mathcal{C}_1$ and $\mathcal{C}_2 \in \mathfrak{L}^{\mathrm{w}}$ are two regular controller behaviors induced by C_1 and C_2 respectively. Then, $P_1 = P_2 \Leftrightarrow \mathcal{C}_1 = \mathcal{C}_2$.*

Proof. The only nontrivial implication to be proved is: $P_1 \neq P_2 \Rightarrow$ there does not exist any unimodular matrix U such that $UC_1 = C_2$; this would prove that the behaviors described by C_1 and C_2 are unequal. Suppose there exists such a U. Then $[(UP_1 - P_2) \quad I - U] \begin{bmatrix} R \\ C \end{bmatrix} = 0$. Since the interconnection is regular $\begin{bmatrix} R \\ C \end{bmatrix}$ has full row rank. This implies $U = I$ and $P_1 = P_2$. This contradiction proves that $\mathcal{C}_1 = \mathcal{C}_2 \Rightarrow P_1 = P_2$. $\square$

As seen from the statement of Theorem 5.1, the parametrization is closely related to the bounds on row degrees of row proper representations of matrices. We now state a lemma which shows that given a block-upper triangular unimodular matrix with certain restrictions on the degrees of the entries, the inverse of this matrix satisfies the *same* restrictions. In order to simplify the presentation for the rest of the section, we assume without loss of generality, that the rows of the matrix which induce a row proper kernel representation are arranged with non-increasing row degrees. The technicalities in the statement of this lemma and its proof are more about the case when one or more rows in the polynomial matrix V, P and Q have the same row degrees. In the following lemma, $\mu_1, \ldots, \mu_\ell$ play a role in the degree constraints, while $\nu_1, \ldots, \nu_\ell$ denote the multiplicities and hence the sizes of the partitioned block matrices. A first reading of the lemma and the proof can be done assuming all row degrees are different, i.e. $\nu_i = 1$ for each i, and $\ell = n$.

Lemma 5.3. *Let $\nu_1, \ldots, \nu_\ell$ be positive integers satisfying $\nu_1 + \cdots + \nu_\ell = n$ and let $\mu_1 > \mu_2 > \cdots > \mu_\ell$ be ℓ non-negative integers. Suppose $V \in \mathbb{R}^{n \times n}[\xi]$ is a unimodular matrix with entries $v_{a,b}$. Partition V into blocks $V_{i,j} \in \mathbb{R}^{\nu_i \times \nu_j}[\xi]$. Assume V satisfies the following:*

A1. V is block-upper-triangular, i.e. $V_{i,j} = 0$ for $i > j$.
A2. $V_{i,i}$ are invertible real matrices.
A3. For every entry $v_{a,b}$ in $V_{i,j}$, with $i < j$, suppose degree $(v_{a,b}) \leqslant \mu_i - \mu_j$.

Define $U := V^{-1}$ and let $U_{i,j} \in \mathbb{R}^{\nu_i \times \nu_j}[\xi]$ be a partition of U like that of V. Suppose $u_{a,b}$ are the entries of U. Then,

C1. U is block-upper-triangular, i.e. $U_{i,j} = 0$ for $i > j$,
C2. $U_{i,i}$ are invertible real matrices and $U_{i,i} = V_{i,i}^{-1}$,
C3. For every entry $u_{a,b}$ in $U_{i,j}$ with $i < j$, we have degree $(u_{a,b}) \leqslant \mu_i - \mu_j$.

Further, let $r_1, r_2, \cdots r_p$ be non-negative integers. Assume $P \in \mathbb{R}^{n \times p}[\xi]$ has entries $p_{a,b}$. Partition P into blocks $P_{i,j} \in \mathbb{R}^{\nu_i \times 1}[\xi]$ with $i \in \{1, \ldots, \ell\}$ and $j \in \{1, \ldots, p\}$.

A4. Suppose every entry $p_{a,b}$ in $P_{i,j}$ satisfies degree $(p_{a,b}) \leqslant \mu_i - r_j$. *Define $Q := UP = V^{-1}P$ with entries $q_{a,b}$ and columns $Q_{i,j} \in \mathbb{R}^{\nu_i \times 1}[\xi]$ with $i \in \{1, \ldots, \ell\}$ and $j \in \{1, \ldots, p\}$.*

Then,

C4. every entry $q_{a,b}$ in $Q_{i,j}$ also satisfies degree $(q_{a,b}) \leqslant \mu_i - r_j$.

Proof. Due to the block-upper triangular assumption on V, it follows that U too is block upper triangular and, consequently, $U_{i,i} = V_{i,i}^{-1}$; this proves claims C1 and C2. We prove claim C3, viz. we show that for $u_{a,b}$ in $U_{i,j}$ with $i < j$, degree $(u_{a,b}) \leqslant \mu_i - \mu_j$. Let $U_{i\bullet}$ denote the ith *block row* of U i.e. $U_{i\bullet} = \begin{bmatrix} U_{i,1} & \cdots & U_{i,l} \end{bmatrix}$ with $U_{i,k} \in \mathbb{R}^{\nu_i \times \nu_k}$. Similarly, let $V_{\bullet j}$ denote the jth *block column* of V. Using $UV = I$ and expanding $U_{i,\bullet}V_{\bullet,i+1} = 0$, we get $U_{i,i}V_{i,i+1} + U_{i,i+1}V_{i+1,i+1} = 0$, and then using $U_{i+1,i+1} = (V_{i+1,i+1})^{-1}$ we obtain $U_{i,i+1} = -U_{i,i}V_{i,i+1}U_{i+1,i+1}$. We noted that for all $v_{a,b}$ in $V_{i,i+1}$, degree $(v_{a,b}) \leqslant \mu_i - \mu_{i+1}$. Since $U_{i,i}$ and $U_{i+1,i+1}$ have only constant entries, the degrees of the entries of $U_{i,i+1}$ are also at most $\mu_i - \mu_{i+1}$. This holds true for all $i \in \{1, \ldots, \ell - 1\}$. This proves claim C3 for the first super(block)-diagonal.

Next, expanding the left hand side of $U_{i,\bullet}V_{\bullet,i+2} = 0$, and using $U_{i+2,i+2} = (V_{i+2,i+2})^{-1}$, we get

$$U_{i,i+2} = -U_{i,i}V_{i,i+2}U_{i+2,i+2} - U_{i,i+1}V_{i+1,i+2}U_{i+2,i+2}$$

Using the same reasoning as for the first super block diagonal, we conclude that the degree of each entry of the first product in the right hand side is at most $\mu_i - \mu_{i+2}$. Consider the second product in the right hand side. For all $u_{a,b}$ in $U_{i,i+1}$ we inferred that degree $(u_{a,b}) \leqslant \mu_i - \mu_{i+1}$ and for all $v_{e,f}$ in $V_{i+1,i+2}$ we assumed degree $(v_{e,f}) \leqslant \mu_{i+1} - \mu_{i+2}$. Therefore, all entries in the product on the right hand side have degrees bounded by $\mu_i - \mu_{i+1} + \mu_{i+1} - \mu_{i+2} = \mu_i - \mu_{i+2}$. This holds true for all $i \in \{1, \ldots, \ell - 2\}$. Thus the degrees of the entries of $U_{i,i+2}$ are at most $\mu_i - \mu_{i+2}$. This proves the degree bound claim (C3) for the second super block diagonal. Continuing in this way, claim C3 of the lemma is proved. We now prove claim C4. Let $P_{\bullet,m}$ denote the mth column of P. Consider the product $U_{i,\bullet}P_{\bullet,m} = U_{i,i}P_{i,m} + U_{i,i+1}P_{i+1,m} + \cdots + U_{i,l}P_{l,m}$. Consider an arbitrary term $U_{i,k}P_{k,m}$, with k in $\{i, \ldots, \ell\}$, of this product. For every entry in this product, the degree is bounded by $(\mu_i - \mu_k) + (\mu_k - d_m) = \mu_i - d_m$; thus proving claim C4 of the lemma. This completes proof of Lemma 5.3. □

With the aid of the above lemmas we now prove Theorem 5.1.

Proof of Theorem 5.1. (**1**): Suppose $P \in \text{DegBnd}(R, C_{rf})$. Consider the controller $\mathcal{C}'$ defined by the kernel representation induced by $C := PR + C_{rf} = \begin{bmatrix} P & I \end{bmatrix}\begin{bmatrix} R \\ C_{rf} \end{bmatrix}$. Let b be the ith row of C and let a be the ith row of $\begin{bmatrix} P & I \end{bmatrix}$. Denote the entries of a by $a_1, \ldots, a_k$. Then $b = a\begin{bmatrix} R \\ C_{rf} \end{bmatrix}$. By the definition of $\text{DegBnd}(R, C_{rf})$, each entry a_j of a for $j \in \{1, \ldots p\}$ has a degree of at most $c_i - r_j$. Also, the $(p + i)$th entry of a is 1. Hence the entries of b have a degree of at most c_i. By the predictable degree property (see Proposition 2.3) we have that degree $(b) = \max_{\{\ell \in 1,\ldots,k : a_\ell \neq 0\}}[k_\ell + \text{degree } a_\ell]$ where k_ℓ are the row degrees of $\begin{bmatrix} R \\ C_{rf} \end{bmatrix}$. Using $k_{p+i} = c_i$, degree $a_{p+i} = 0$ and row-properness of $\begin{bmatrix} R \\ C_{rf} \end{bmatrix}$, we conclude that degree $(b) = c_i$. Thus C and C_{rf} have the same row degrees. Moreover, since they both implement $\mathcal{K}$, $\begin{bmatrix} R \\ C \end{bmatrix}$ induces a minimal kernel representation of $\mathcal{K}$ and is row proper. Hence $\mathtt{n}(C) = \mathtt{n}(C_{rf})$ and $\mathcal{C}'$ too is an RFI controller.

(**2**): Assume $C' \in \mathbb{R}^{(k-p)\times \mathtt{w}}[\xi]$ induces a row proper kernel representation for a regular feedback implementing controller $\mathcal{C}'$. Since $\mathcal{C}$ and $\mathcal{C}'$ are both regular feedback implementing controllers for $\mathcal{K}$, we assume without loss of generality that $\begin{bmatrix} R \\ C' \end{bmatrix}$ is row proper and with the same row degrees as $\begin{bmatrix} R \\ C_{rf} \end{bmatrix}$ (see text following Proposition 2.1 about Forney indices). Note that as a result of this, C' and C_{rf} are both row proper with the same row degrees. Further, since $\mathcal{C}'$ is regularly implementing, due to Proposition 4.3, there exist P and V such that $C' = PR + VC_{rf}$.

Assume without loss of generality that C_{rf} has non-increasing row degrees. Suppose the row degrees are $\mu_1 > \mu_2 > \cdots \mu_\ell \geqslant 0$ with μ_1 the degree of the first ν_1 rows of C_{rf}, μ_2 the degree of the next ν_2 rows, etc. This is essential to handle repeated row degrees. Thus $\nu_1 + \nu_2 + \cdots + \nu_\ell = k - p$ and $\nu_j \geqslant 1$ are the multiplicities of the row degrees. Also let $r_1, \ldots, r_p$ be the row degrees of R.

We will first show that the matrices $P \in \mathbb{R}^{(k-p)\times p}[\xi]$ and $V \in \mathbb{R}^{(k-p)\times(k-p)}[\xi]$ satisfy the degree related hypotheses listed in Lemma 5.3 viz. the degree each entry p_{ij} and v_{ij} in P and V respectively is bounded from above by the Forney indices of R and C_{rf} (claim C4). Having done so, we can then use

Lemma 5.3 to conclude that the matrix $V^{-1}P$ also satisfies the degree bounds mentioned in claim C4 of Lemma 5.3, this would prove that $V^{-1}P \in \text{DegBnd}(R, C_{rf})$.

Let $b \in \mathbb{R}^{1\times \mathtt{w}}[\xi]$ be the ith row of $\begin{bmatrix} R \\ C' \end{bmatrix}$ with $i > p$ and let $a \in \mathbb{R}^{1\times k}[\xi]$ be the $(i-p)$th row of $[P \quad V]$. Hence $b = a\begin{bmatrix} R \\ C_{rf} \end{bmatrix}$, and degree $(b) =$ degree $\left(a\begin{bmatrix} R \\ C_{rf} \end{bmatrix}\right)$. Let a_j with $j \in \{1, \ldots, k\}$, be the entries of a. Using the predictable degree property, viz. Proposition 2.3, we have that degree $(b) = \max_{m\in 1,\ldots,k: a_m \neq 0}[k_m + \text{degree} \ (a_m)]$ where k_m's are the row degrees of $\begin{bmatrix} R \\ C_{rf} \end{bmatrix}$. Therefore, $k_m + \text{degree}(a_m) \leqslant \text{degree}(b)$ for each $m \in \{1, \ldots, k : a_m \neq 0\}$, and consequently, $\text{degree}(a_m) \leqslant \text{degree}(b) - k_m$ for each $m \in \{1, \ldots, k : a_m \neq 0\}$. Thus the degree of a_m, i.e. the $(i-p, m)$th element (note that $i > p$) of $[P \quad V]$, is bounded from above by $\text{rowdegree}_i\left(\begin{bmatrix} R \\ C_{rf} \end{bmatrix}\right) - \text{rowdegree}_m\begin{pmatrix} R \\ C_{rf} \end{pmatrix}$. Note that if $\text{rowdegree}_i\left(\begin{bmatrix} R \\ C_{rf} \end{bmatrix}\right) < \text{rowdegree}_m\left(\begin{bmatrix} R \\ C_{rf} \end{bmatrix}\right)$, then, by our convention that a polynomial of degree less than zero is the zero polynomial, the $(i-p, m)$ entry of $[P \quad V]$ is zero.

Thus we have shown the following:

(1) For $m \leqslant p$, either the $(i-p, m)$ element of $[P \quad V]$ is zero or the degree of the $(i-p, m)$ element of $[P \quad V]$, which is now an element in P, is at most $c_{(i-p)} - r_m$ because $\text{rowdegree}_i\text{col}(R, C) = c_{(i-p)}$ and $m < p \Rightarrow \text{rowdegree}_m\text{col}(R, C) = r_m$.

(2) For $p < m \leqslant k$, we have shown that either the $(i-p, m)$ element of $[P \quad V]$ is zero or the degree of the $(i-p, m)$ element of $[P \quad V]$, which is now an element in V, is at most $c_{(i-p)} - c_{(m-p)}$ because $\text{rowdegree}_m\text{col}(R, C) = c_{(m-p)}$.

Thus the degree related conditions in Lemma 5.3 viz. claims C3 and C4 have been proved. This is utilized in what follows.

Now we shall show that V is block upper-triangular. Consider the ν_2 rows of C_{rf} with degree μ_2 and the set of ν_3 rows of C_{rf} with degree μ_3. From the degree related conditions proved just above (statements (1) and (2)) we see that $v_{i,j} = 0$ for $i \in \{\nu_1 + 1, \ldots, \nu_2\}$ and $j \in \{1, \ldots, \nu_1\}$. The degree bounds established also imply that $v_{i,j} \in \mathbb{R}$ for $i \in \{\nu_1 + 1, \ldots \nu_2\}$ and $j \in \{\nu_1 + 1, \ldots, \nu_2\}$. The same argument applied to other sets of rows of the same degree together imply that V is block upper-triangular and that the blocks on the diagonal are matrices with real (i.e. not polynomial) entries. Also, since V is unimodular and V is block upper-triangular, the blocks on the diagonal must be invertible. Thus the conditions required in Lemma 5.3 are satisfied and this helps conclude that for $q_{a,b} \in Q_{i,j}$ we have $\text{degree}(q_{a,b}) \leqslant \mu_i - r_j$. From this it follows that $V^{-1}P \in \text{DegBnd}(R, C_{rf})$.

Finally, Lemma 5.2 assures that $V^{-1}P$ depends only on the controller behavior $\mathcal{C}'$, thus proving the uniqueness stated in claim 2 of Theorem 5.1. □

5.2. 'Size' of the set of RFI controllers

Using Theorem 5.1 we see that any RFI controller can be constructed from a given RFI controller's kernel representation by various P's with a restriction on the degrees of the entries. Unlike regular interconnection where the set of controller behaviors is infinite dimensional, the set of RFI controller behaviors is finite dimensional for the case of regular feedback interconnection (see Lomadze [9]). Using the above results, in this section we give a precise count of the dimension of this affine space of RFI controllers. Assume $C_{rf} \in \mathbb{R}^{(k-p)\times \mathtt{w}}[\xi]$ is row proper and induces a minimal kernel representation of an RFI controller. Let $\{c_1, \ldots, c_{k-p}\}$ be the $(k-p)$-tuple of row degrees of C_{rf}. Suppose R induces a row proper kernel representation for the plant $\mathcal{P}$ and let $(r_1, \ldots, r_p)$ denote the row degrees of R. Recall the set $\text{DegBnd}(R, C_{rf})$ as defined in Theorem 5.1. The following theorem gives a count of the dimension of the set $\text{DegBnd}(R, C_{rf})$. Its proof is skipped since it follows easily by noting that each entry p_{ij} has an upper bound of $c_i - r_j$ on its degree (see Theorem 5.1), and hence the set of allowed p_{ij}'s form a vector space of dimension $c_i - r_j + 1$. The 'max' operation ensures that when $c_i < r_j$, then no row-operations are allowed, and hence this does not contribute to the count.

Theorem 5.4. *Suppose R and $C_{rf} \in \mathbb{R}^{\bullet\times \mathtt{w}}[\xi]$ are row proper matrices and induce minimal kernel representations of the plant and a regular feedback implementing controller. Let r_j for $j \in \{1, \ldots, p\}$ and c_i for*

$i \in \{1, \ldots, k-p\}$ be the row degrees of R and C_{rf} respectively. Then the set $\mathrm{DegBnd}(R, C_{rf})$ *and hence the set of RFI controllers is a finite dimensional vector space of dimension* $\sum_{j=1}^{p}\sum_{i=1}^{k-p}\max(c_i - r_j + 1, 0)$. *In particular, if $c_i < r_j$ for all i and j, then C_{rf} is the unique*[4] *RFI controller.*

A very similar computation has been done in Rosenthal et al. [15, Section 2]: the stabilizer of the set of row proper matrices of a given McMillan degree is computed in the group of unimodular matrices. In our case the unimodular matrices are of a special form viz. $\begin{bmatrix} I & 0 \\ P & V \end{bmatrix}$ because we are studying the problem of controller design by keeping the structure $\begin{bmatrix} R \\ C \end{bmatrix}$ intact. The computation in Rosenthal et al. [15, Section 2] is about row proper kernel representations (AR systems) with a given set of row degrees, of which our control problem is a special case.

6. Partial interconnection

Until now we had considered the case where all the variables are available for interconnection. In this and the following sections, we look into the more practical case in which only some of the system variables are accessible to the controller for interconnection: we call these variables the control variables c. The variables that we wish to control are termed the to-be-controlled variables w. Thus the *full plant behavior* $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ has typical elements $(w, c) \in \mathcal{P}_{\text{full}}$ (with possibly some components common to w and c). The controller behavior $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ has only the control variables c as its variables. The *full controlled behavior* $\mathcal{K}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ is defined as the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$:

$$\mathcal{K}_{\text{full}} := \{(w, c) \in \mathcal{P}_{\text{full}} | c \in \mathcal{C}\}.$$

Since the interconnection of $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ and $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ is not plain intersection, we use the symbol $\wedge$ to denote the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$, i.e. $\mathcal{K}_{\text{full}} := \mathcal{P}_{\text{full}} \wedge \mathcal{C}$, the behavior $\mathcal{K}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ defined as above. The results in this section require the definition of the behavior obtained by 'eliminating' the to-be-controlled variable w: we define the *control variable plant behavior* $\mathcal{P}_c \in \mathfrak{L}^{\mathtt{c}}$ as

$$\mathcal{P}_c := \{c \in \mathfrak{L}^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{c}}) | \text{there exists a } w \text{such that } (w, c) \in \mathcal{P}_{\text{full}}\}.$$

For the partial interconnection case, $\mathcal{P}_{\text{full}} \wedge \mathcal{C} =: \mathcal{K}_{\text{full}}$ is said to be a regular interconnection if the output cardinalities of the full plant and the controller add up to that of $\mathcal{K}_{\text{full}}$.

Similarly, the interconnection is said to be an RFI if it is regular and $\mathtt{n}(\mathcal{P}_{\text{full}}) + \mathtt{n}(\mathcal{C}) = \mathtt{n}(\mathcal{K}_{\text{full}})$.

The following theorem, one of the main results of this section, shows that the case $\mathcal{P}_c = \mathfrak{L}^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{c}})$ is relatively easy to conclude about regular feedback implementability of $\mathcal{K}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$. The case that $\mathcal{P}_c = \mathfrak{L}^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{c}})$ is equivalent to R_w being full row rank in a minimal kernel representation $R_w\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w + R_c\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) c = 0$ of the full plant $\mathcal{P}_{\text{full}}$: this is standard in behavioral literature, see Polderman and Willems [11, Theorem 6.2.6].

Theorem 6.1. *Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ and suppose $\mathcal{K}_{\text{full}}$ is regularly implementable with respect to $\mathcal{P}_{\text{full}}$ through c. Assume $\mathcal{P}_c = \mathfrak{L}^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{c}})$. Then,*

1. *the controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ that regularly implements $\mathcal{K}_{\text{full}}$ is unique,*
2. *if $\mathcal{K}_{\text{full}}$ is regular feedback implementable, $\mathcal{C}$ is an RFI controller.*

Proof of Theorem 6.1. Claim 1: Suppose

$$\begin{bmatrix} R_w\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) & R_c\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ 0 & C\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \end{bmatrix} \begin{bmatrix} w \\ c \end{bmatrix} = 0$$

[4] We thank an anonymous reviewer for bringing this important special case to our notice.

induces a minimal kernel representation on $\mathcal{K}_{\text{full}}$, where, $[R_w \quad R_c]$ induces a minimal kernel representation on the plant and C induces a minimal kernel representation on the controller. All controllers which implement $\mathcal{K}_{\text{full}}$ by full regular interconnection can be obtained by premultiplying this representation by matrices P and V as in Proposition 4.3. Assume that $\mathcal{K}_{\text{full}}$ can be implemented regularly by another controller through c. Therefore, there exists $P(\xi) \neq 0$ such that $PR_w = 0$. Thus R_w does not have full row rank Hence $\mathcal{P}_c \subsetneq \mathfrak{L}^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^c)$. This is contradictory to the assumption that $\mathcal{P}_c = \mathfrak{L}^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^c)$. Hence claim one is proved. Now we prove claim 2. Since $\mathcal{K}_{\text{full}}$ is regular feedback implementable and since there is a unique regularly implementing controller, this unique controller must also be regular feedback implementing, thus proving the claim. □

Thus if $\mathcal{P}_c$ equals $\mathfrak{L}^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^c)$, then $\mathcal{C}$ is the unique controller behavior that regularly implements $\mathcal{K}_{\text{full}}$ through c. Hence the McMillan degree cannot be decreased using an analogue of the algorithm in Section 4.1. We now state a theorem which is useful when $\mathcal{P}_c$ is a proper subset of $\mathfrak{L}^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^c)$.

This result is a necessary condition for regular feedback implementability of $\mathcal{K}_{\text{full}}$: given $\mathcal{K}_{\text{full}}$ is regular feedback implementable, the *controlled* control variable plant behavior $\mathcal{P}_c \wedge \mathcal{C}$ too is regular feedback implementable.

Theorem 6.2. *Let $\mathcal{K}_{\text{full}} = \mathcal{P}_{\text{full}} \wedge \mathcal{C}$ be a regular interconnection. If $\mathcal{K}_{\text{full}}$ is regular feedback implementable through c, then $\mathcal{P}_c \wedge \mathcal{C}$ is regular feedback implementable.*

Proof of Theorem 6.2. Without loss of generality we assume that $\begin{bmatrix} R_1 & R_2 \\ 0 & R_3 \\ 0 & C \end{bmatrix}$ induces a minimal kernel representation of $\mathcal{K}_{\text{full}}$, ($R_1$ is of full row rank, R_3 is a minimal kernel representation of $\mathcal{P}_c$ and C is a minimal kernel representation of the controller). Suppose $\mathcal{K}_{\text{full}}$ is regular feedback implementable. Then there exists a C' such that $\begin{bmatrix} R_1 & R_2 \\ 0 & R_3 \\ 0 & C' \end{bmatrix}$ is row proper. Thus $\begin{bmatrix} R_3 \\ C' \end{bmatrix}$ too is row proper. Hence we conclude that $\mathcal{P}_c \wedge \mathcal{C}$ is regular feedback implementable. □

Remark 6.3. Regular feedback interconnection of $\mathcal{P}_c$ and $\mathcal{C}$ is not sufficient for that of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ also. A simple example is when $R_w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $R_c = \begin{bmatrix} \xi & 0 \\ 0 & 1 \end{bmatrix}$, and $C = [\xi \ 0]$. The reason is that during the 'elimination' of w to obtain $\mathcal{P}_c$, a notion of 'proper elimination' (see [10]) plays a key role when dealing with the function space $\mathfrak{L}^1_{\text{loc}}$. It appears that proper eliminability of w from $\mathcal{P}_{\text{full}}$ to obtain $\mathcal{P}_c$, together with RFI of $\mathcal{P}_c$ and $\mathcal{C}$, is sufficient to ensure that the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is also an RFI. Due to paucity of space, we do not investigate this matter further.

If $\mathcal{P}_c \subsetneq \mathfrak{L}^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^c)$, then a regularly implementing $\mathcal{C}$ is not unique. We can hence look within the set of regularly implementing controllers to find one that is, in fact, a regular feedback implementing controller. We now state an algorithm which enumerates the steps for this procedure.

RFI controller construction: the partial interconnection case

Input:

1. A minimal kernel representation of the plant in the form mentioned above i.e. $\begin{bmatrix} R_1 & R_2 \\ 0 & R_3 \end{bmatrix}$ with R_3 and $[R_1 \quad R_2]$ row proper.
2. A row proper polynomial matrix C which induces a minimal kernel representation of a controller $\mathcal{C} \in \mathfrak{L}^c$ which implements $\mathcal{K}_{\text{full}}$ by regular interconnection through c.

Output: A controller which implements $\mathcal{K}_{\text{full}}$ by RFI through c if some sufficient conditions are satisfied, OR the conclusion that $\mathcal{K}_{\text{full}}$ is not regular feedback implementable through c if some necessary conditions are not satisfied.

Steps of the algorithm:

1. Consider $\begin{bmatrix} R_3 \\ C \end{bmatrix}$. Use the algorithm mentioned in Section 4.1 to find a controller which implements this interconnection by RFI. If no controller exists, stop; neither $\mathcal{P}_c \cap \mathcal{C}$ nor $\mathcal{K}_{\text{full}}$ can be implemented by RFI. Else,
2. Let C_{rf} be the resulting controller. Check whether $\begin{bmatrix} R_1 & R_2 \\ 0 & R_3 \\ 0 & C_{rf} \end{bmatrix}$ is an RFI. If yes, then the problem is solved. If not, then our results do not conclude/help further.

7. Disturbances, freeness and regular feedback implementability

In most realistic cases of designing a controller, one must account for the presence of external disturbances, which by definition are external and hence are inputs to the systems: both the to-be-controlled and the controlled. The least that is desired is that the controller not restrict the disturbances. In this section we analyse the situation that a given plant behavior $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ is, in fact, obtained from a larger system $\mathcal{P}_{\text{full}}^{\text{ext}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{d}}$ with disturbance variables d also. Assuming that d is unrestricted by the laws describing $\mathcal{P}_{\text{full}}^{\text{ext}}$, we check conditions under which a controller continues to leave the disturbances unrestricted after interconnection also. This is made precise below.

In order to make concrete the notion of 'unrestricted/free', we define the notion of *freeness* of a variable when considering a behavior $\mathfrak{B} \subseteq \mathfrak{L}^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}+\mathtt{d}})$. The variable d is said to be free in $\mathfrak{B}$ if for every $d \in \mathfrak{L}^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{d}})$ there exists a $w \in \mathfrak{L}^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ such that $(w, d) \in \mathfrak{B}$. If $R_w\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) + R_d\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) d = 0$ is a row proper kernel representation of $\mathfrak{B}$, then d being free in $\mathfrak{B}$ is equivalent to $\mathtt{n}([R_w \ R_d]) = \mathtt{n}([R_w])$. See Polderman and Willems [11] for a detailed discussion on this and the relation to properness of the transfer function from d to the output variables amongst w.

Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$. An extension $\mathcal{P}_{\text{full}}^{\text{ext}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{d}}$ (with $\mathtt{d}$ a positive integer) of $\mathcal{P}_{\text{full}}$ is a behavior with variables w,c and d such that

1. d is free in $\mathcal{P}_{\text{full}}^{\text{ext}}$.
2. $\mathcal{P}_{\text{full}} = \{(w, c) | (w, c, 0) \in \mathcal{P}_{\text{full}}^{\text{ext}}\}$.

Similarly, for the controlled behavior $\mathcal{K}_{\text{full}} = \mathcal{P}_{\text{full}} \wedge \mathcal{C}$ with controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$,

$$\mathcal{K}_{\text{full}}^{\text{ext}} = \{(w, c, d) | (w, c, d) \in \mathcal{P}_{\text{full}}^{\text{ext}} \text{ and } c \in \mathcal{C}\}.$$

The theorem below is the main result of this section. Regular feedback interconnection is equivalent to disturbances continuing to be free in the controlled system after interconnection for *every* extension of the plant. When disturbances are restricted to the space of infinitely often differentiable functions $\mathfrak{C}^\infty$, the equivalence to freeness in every extension and *regular interconnection* has been established in [2, Section 7].

Theorem 7.1. *Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ and $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ be a plant and a controller respectively. Then the following are equivalent.*

1. *The interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is an RFI.*
2. *For every extension $\mathcal{P}_{\text{full}}^{\text{ext}}$ of $\mathcal{P}_{\text{full}}$, the disturbance d is free in $\mathcal{K}_{\text{full}}^{\text{ext}}$.*

Proof. $(1 \Rightarrow 2)$: Let $R_w\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w + R_c\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) c = 0$ be a minimal kernel representation of the full plant $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ with McMillan degree $\mathtt{n}(\mathcal{P}_{\text{full}})$, and let $C\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) c = 0$ be a minimal kernel representation of the controller with McMillan degree $\mathtt{n}(\mathcal{C})$. Suppose the interconnection is an RFI. Consider any extension of the plant $\mathcal{P}_{\text{full}}^{\text{ext}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{d}}$ represented minimally by $R_w\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w + R_c\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) c + R_d\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) d = 0$.

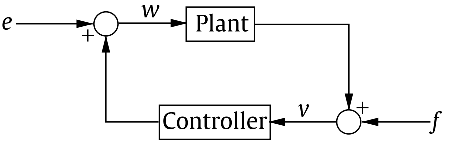

Fig. 2. Well-posedness.

Since d is free in $\mathcal{P}_{\text{full}}^{\text{ext}}$, we noted above that $\mathtt{n}([R_w R_c]) = \mathtt{n}([R_w R_c R_d])$, i.e. a maximal degree minor of $[R_w\ R_c\ R_d]$ is within $[R_w\ R_c]$. Consider the following minimal kernel representation of $\mathcal{K}_{\text{full}}^{\text{ext}} := \mathcal{P}_{\text{full}}^{\text{ext}} \wedge \mathcal{C}$:

$$\begin{bmatrix} R_w\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) & R_c\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) & R_d\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ 0 & C\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) & 0 \end{bmatrix} \begin{bmatrix} w \\ c \\ d \end{bmatrix} = 0. \tag{2}$$

Due to the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ being an RFI, $\mathtt{n}(\mathcal{K}_{\text{full}}) = \mathtt{n}(\mathcal{P}_{\text{full}}) + \mathtt{n}(\mathcal{C})$ and hence a maximal degree minor of $\begin{bmatrix} R_w\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) & R_c\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ 0 & C\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \end{bmatrix}$ continues to be a maximal degree minor of the polynomial matrix inducing the kernel representation of $\mathcal{K}_{\text{full}}^{\text{ext}}$ in Eq. (2) also. Thus d is free in $\mathcal{K}_{\text{full}}^{\text{ext}}$ also.

($2 \Rightarrow 1$): That the interconnection is regular follows from Theorem 12 of [2]. It remains to prove that the interconnection is, in fact, regular feedback. We prove this part as follows. We construct a particular extension $\mathcal{P}_{\text{full}}^{\text{ext}}$ and use the fact that d is free in $\mathcal{K}_{\text{full}}^{\text{ext}}$ to show that the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is an RFI.

Assume $[R_w\ R_c] \in \mathbb{R}^{p\times(\mathtt{w}+\mathtt{c})}[\xi]$ is row proper and induces a kernel representation for $\mathcal{P}_{\text{full}}$. Let $k_1, k_2, \ldots, k_p$ be the row degrees of $[R_w R_c]$. Define $R_d \in \mathbb{R}^{p\times p}[\xi]$ by $R_d(\xi) = \text{diag}(\xi^{k_1}, \xi^{k_2}, \ldots \xi^{k_p})$, and consider the extension $\mathcal{P}_{\text{full}}^{\text{ext}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{d}}$ represented by $R_w\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w + R_c\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)c + R_d\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)d = 0$. Since $\mathtt{n}([R_w\ R_c]) = \mathtt{n}([R_w\ R_c\ R_d])$, the disturbance is free in $\mathcal{P}_{\text{full}}^{\text{ext}}$. Let $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ be a controller behavior with a row proper kernel representation $C\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)c = 0$. We obtain the kernel representation for $\mathcal{K}_{\text{full}}^{\text{ext}}$ as in Eq. (2) above. Due to the assumption that d is free in $\mathcal{K}_{\text{full}}^{\text{ext}}$, we know $\mathtt{n}\left(\begin{bmatrix} R_w & R_c & R_d \\ 0 & C & 0 \end{bmatrix}\right) = \mathtt{n}\left(\begin{bmatrix} R_w & R_c \\ 0 & C \end{bmatrix}\right)$. The LHS of this equality is nothing but $\mathtt{n}(\mathcal{K}_{\text{full}})$, while the RHS is $\mathtt{n}(\mathcal{K}_{\text{full}}^{\text{ext}})$. Due to the particular choice of R_d we made, $\mathtt{n}(\mathcal{P}_{\text{full}}) = \sum_{i=1}^{p} k_i = \mathtt{n}(R_d)$, and further due to the block of zeros below R_d in Eq. (2), $\mathtt{n}(\mathcal{K}_{\text{full}}^{\text{ext}}) = \mathtt{n}(R_d) + \mathtt{n}(\mathcal{C})$. Combining these equalities, we get $\mathtt{n}(\mathcal{K}_{\text{full}}) = \mathtt{n}(\mathcal{P}_{\text{full}}) + \mathtt{n}(\mathcal{C})$, thus proving regular feedback interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$. □

We briefly relate the work in this section with the classical concept of well-posedness of an interconnection. Consider the interconnection of a plant and controller in the feedback configuration shown in Fig. 2. An interconnection is said to be well-posed if the transfer matrix from (e, f) to (w, v) exists and is proper (see Zhou and Doyle [24, Section 5.2]). This is nothing but freeness of external disturbances in the interconnected system. The equivalence of regular feedback interconnection and well-posedness of interconnections has been established in Kuijper [8, Theorem 2.1, (ii)], and we proved the equivalence of RFI to disturbances remaining free after interconnection in *every* extension of the plant. At first sight, it appears that requiring the freeness of the disturbance in *every* extension is rather restrictive, since the traditional notion of well-posedness does not require this[5]. The reason for this is that the plant and controller as operators shown in the figure are assumed to have *proper* transfer function matrices (for the case of LTI systems), and in this case it suffices to define well-posedness in that way. The fact that regular feedback interconnection is related to an input/output feedback interconnection between the plant and the controller with *proper* transfer matrices is well-known: see Kuijper [8] and Willems [22, Theorem 12].

[5] In the traditional notion of well-posedness, the signals e and f are added at the junctions denoted in Fig. 2, and hence $R_d = I$.

8. Minimal McMillan degree systems

In this section we consider some relatively peripheral problems about interconnections pertaining to minimal McMillan degree systems. It is of general interest to obtain a controller with as low McMillan degree as possible. This is important from the point of view of physically implementing a control law: lower the controller order, easier it is to realize it; for instance, micro-controller register requirements are less demanding. Unlike the previous two sections, we deal with the case that *all* variables are available for interconnection, i.e. $\mathcal{P}$ and $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$.

The following remark addresses the question of minimum McMillan degree $\mathcal{K}$ achievable by RFI.

Remark 8.1. For a given $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$, the minimum McMillan degree $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ that can be achieved using an RFI controller is $\mathtt{n}(\mathcal{P})$ itself. (See text preceding Theorem 4.2.) Of course, the controller has to have McMillan degree zero itself for the interconnection to be an RFI. An example of such a controller is a controller which sets all the inputs of the plant to zero. Moreover, as noted in Theorem 5.4, when the degree of every row in the plant equations (from a row proper R) is degree one or more, then such a controller is a unique controller which implements that particular $\mathcal{K}$ by RFI.

The next subsections relax the RFI condition and seek minimum McMillan degree $\mathcal{K}$ (Section 8.1) or controller for a specified $\mathcal{K}$ (Section 8.2).

8.1. Minimum order $\mathcal{K}$ with $\mathcal{K}$ regularly implementable

Consider the following problem: find the McMillan degree of a behavior $\mathcal{K} \subseteq \mathcal{P}$ which satisfies the following properties:

- $\mathcal{K}$ can be obtained by regular interconnection.
- $\mathcal{K}$ has the least McMillan degree possible.

Note that $\mathcal{K}$ need not be regular feedback implementable. We write $R(\xi) = D(\xi)R_{con}(\xi)$, where $D(\xi)$ is nonsingular R_{cont} induces a minimal kernel represent of $\mathcal{P}_{\text{cont}}$, the controllable part[6] of $\mathcal{P}$. Since, R_{cont} is left-prime, we can extend it to a unimodular matrix by augmenting R_{cont} by a suitable polynomial matrix below. Let C be such that $\begin{bmatrix} R_{\text{cont}} \\ C \end{bmatrix}$ is unimodular. The minimum McMillan degree that $\mathcal{K}$ can achieve is the McMillan degree of $D(\xi)$. As is clear, one such interconnection is obtained by using a controller represented by C. In some sense, this interconnection is the ‘maximum’ control possible by regular interconnection. The minimum McMillan degree $\mathcal{K}$ achievable by regular interconnection is thus the McMillan degree of the autonomous part of $\mathcal{P}$. When $\mathcal{P}$ is controllable, then $D(\xi)$ above can be taken to be the identity matrix, and the controller is then exactly the same as that in Kuijper [8, Section 3].

8.2. Minimal order regular controllers

If an RFI controller exists, then it is has the lowest possible McMillan degree amongst all regularly implementing controllers i.e., it is a minimal order controller. However, suppose for a given $\mathcal{K}$ an RFI controller does not exist. We would then have to inevitably ‘prepare’ the states of the plant and the controller before interconnection so that there are no impulses at interconnection. In such a scenario, it is desirable (again for implementation reasons) that the controller have as few states as possible. The precise formulation of the problem follows below. This section discusses how some obvious ways to extend results in this paper to solve this problem do not work, and this problem is left unsolved due to its non-trivial nature.

[6] The decomposition of a behavior into its controllable part and an autonomous part is well-studied in the behavioral literature. See Polderman and Willems [11] for a detailed exposition. $\mathcal{P}_{\text{cont}}$ is the largest controllable behavior within $\mathcal{P}$. Here, R_{cont} is such that $R_{\text{cont}}(\lambda)$ has full row rank for every $\lambda \in \mathbb{C}$. Such an R_{cont} is also called left-prime.

Problem 8.2. Given $\mathcal{P}$ and $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ such that $\mathcal{K}$ is regularly implementable, but not regular feedback implementable, find a minimum McMillan degree regular controller, and parametrize all such controllers.

Unlike other problems in this section, in this problem $\mathcal{K}$ is already specified. Let R be row proper and let C_1 induce a row proper kernel representation of a regular controller. Suppose we apply the algorithm mentioned in Section 4.1 till one is faced with the situation that *every* β results in a PERD, i.e. controller order cannot be decreased further: then one expects that the controller obtained at this step has the least McMillan degree amongst all regular controllers that implement $\mathcal{K}$. Further, having found such a controller, the results of Section 5 suggest that we might be able to find all controllers by restricting P to the set $\mathrm{DegBnd}(R, C)$. We now state this property about minimal order regular controllers precisely (and show that it is *not* true). The first remark is about the possible non-minimality of a regular controller for a $\mathcal{K}$ that is not regular feedback implementable. The following remark says that, even if all equations of reducible degree are plant equations, i.e. there are no 'controller equations of reducible degree', one cannot conclude that the regular controller is of minimal order.

Remark 8.3. Suppose $R \in \mathbb{R}^{p\times\mathtt{w}}[\xi]$ and $C \in \mathbb{R}^{(k-p)\times\mathtt{w}}[\xi]$ induce row proper kernel representations for $\mathcal{P}$ and $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ respectively, and assume the interconnection is regular with $\mathcal{K} := \mathcal{P} \cap \mathcal{C}$. Let K_{lc} be the leading row coefficient matrix of $K := \mathrm{col}(R, C)$. Suppose for every $\alpha \in \mathbb{R}^{1\times k}$ such that $\alpha K_{\mathrm{lc}} = 0$, the following inequality holds for each $i \in \{p+1, \ldots, k\}$ satisfying $\alpha_i \neq 0$:

$$\mathrm{rowdegree}_i(K) < \max_{j\in\{1,\ldots,p\}:\alpha_j\neq 0} (\mathrm{rowdegree}_j(K)). \tag{3}$$

Then, these assumptions do *not* suffice to conclude that $\mathcal{C}$ has the least McMillan degree amongst all regular controllers that implement $\mathcal{K}$.

Of course, the necessity of the assumptions is obvious: if the condition in Eq. (3) is not satisfied then using a 'controller equation of reducible degree' one can decrease the controller McMillan degree. An example to show the lack of sufficiency as claimed in the above remark is as follows. Let $R = \begin{bmatrix}\xi^2 & \xi & 0\end{bmatrix}$, and let $C_1 = \begin{bmatrix}\xi & 2 & 0\\ 0 & -\xi^2 & \xi\end{bmatrix}$. Note that C_1 is row-reduced and the lack of row properness of $\mathrm{col}(R, C)$ is due to only plant equations of reducible degree. However, the controller induced by $C_2 := \begin{bmatrix}\xi & 2 & 0\\ 0 & 0 & \xi\end{bmatrix}$ is of McMillan degree strictly less than that by C_1, and moreover, C_2 can be obtained from R and C_1 using a P and a unimodular V, as discussed in Proposition 4.3.

The next question that naturally arises is as follows: knowing C_2 is minimal, is C_1 within the set of all controller representations that can be got from R and C_2 using a P with a degree bound, like we had shown in Section 5.2? The following remark makes this precise and claims that this too is not true. The remark is followed by a relevant example.

Remark 8.4. Consider $\mathcal{P}$ and $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ and let $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ be a regular controller implementing $\mathcal{K}$ such that $\mathtt{n}(\mathcal{C})$ is minimal amongst all such regular controllers. Let $R \in \mathbb{R}^{p\times\mathtt{w}}[\xi]$ and $C \in \mathbb{R}^{(k-p)\times\mathtt{w}}[\xi]$ induce row proper kernel representations of $\mathcal{P}$ and $\mathcal{C}$ respectively, and let $(r_1, \ldots, r_p)$ and $(c_1, \ldots, c_{k-p})$ be their row degrees. Construct $\mathrm{DegBnd}(R, C) \subseteq \mathbb{R}^{(k-p)\times p}[\xi]$

$$\mathrm{DegBnd}(R, C) := \{P \in \mathbb{R}^{(k-p)\times p}[\xi] | \mathrm{degree}(p_{a,b}) \leqslant c_a - r_b\}. \tag{4}$$

Then, there exist regular controllers $\mathcal{C}'$, with $\mathtt{n}(\mathcal{C}') = \mathtt{n}(\mathcal{C})$, none of whose kernel representations can be obtained from any $P \in \mathrm{DegBnd}(R, C)$ using $PR + C$.

As an example, consider C_2 as defined above just after Remark 8.3. Consider $C_3 := \begin{bmatrix}\xi & 2 & 0\\ 0 & -\xi & \xi\end{bmatrix}$, which is row proper and also regularly implements $\mathcal{K}$, like C_2 does. While they have the same McMillan degree, one can check that there does not exist $P \in \mathrm{DegBnd}(R, C_2)$ nor any unimodular U such that

$C_3 = U(PR + C_2)$. In other words, we require a P with entries which have degree higher than those allowed by the constraints posed within Equation 4. This is more easily evident from the more general controller $C_n := \begin{bmatrix} \xi & 2 & 0 \\ 0 & -\xi^n & \xi \end{bmatrix}$, which: regularly[7] implements $\mathcal{K}$; satisfies row properness; has McMillan degree $n + 1$ (unaffected by premultiplication by unimodular U); and can require P of a degree not bounded by (any function of) row degrees of just R and C_2.

In summary, the problem of finding minimal order regular controllers appears to be non-trivial. What is even more pertinent is that the results of Sections 4 and 5 do not directly extend to the case of minimal order controllers for a $\mathcal{K}$ that is not regular feedback implementable. It is in view of the above examples that Theorem 4.2 is important: the mere existence of a PERD makes the sequence of row-reduction irrelevant, and allows conclusion of non-RFI of $\mathcal{K}$. The characterization of minimal order regular controllers and their parametrization is a topic of further research.

9. Concluding remarks

In this section we summarize the key results in the paper. In Section 3, we showed that regular feedback interconnection ensures that no inadmissible initial conditions exist for the autonomous controlled system, but the converse is not true. While a relation between RFI and absence of inadmissible initial conditions is intuitively expected, we argued (and provided a practical counter example) as to why absence of inadmissible initial conditions is not enough to guarantee absence of impulsive behavior at the time of controller interconnection.

Concerning the question as to when a given desired specification $\mathcal{K}$ can be met using an RFI controller, we stated and proved a necessary and sufficient condition for this: non-existence of PERD (Plant Equation of Reducible Degree) for the controlled system equations arising from any regular controller is equivalent to $\mathcal{K}$ being RFI. This easily verifiable test also allows us to construct an RFI controller, if one exists. Further, we used one such controller and parameterized all the controllers that implement $\mathcal{K}$ by RFI (Theorem 5.1); we also gave a count of the dimension of this affine space (Theorem 5.4).

The regular feedback implementability problem has been addressed in Lomadze [9], though only for the full interconnection case, i.e. when all the variables are available for interconnection with a controller. A necessary and sufficient condition for regular feedback implementability is obtained using observability indices and rational matrix manipulation. Apart from we not requiring the seemingly-unutilized controllability condition assumed[8] in Lomadze [9], our method of characterizing RFI allows systematic modification of a regular controller to obtain an RFI controller, if one exists.

In this paper, as an application of the main results, we checked if the interconnection of a door and a door closing mechanism is an RFI, and investigated the existence of inadmissible initial conditions. Though the controlled system has no inadmissible initial conditions, the interconnection is not RFI, thus implying that before interconnection, some 'preparing'/fine-tuning of the internal states of the controller is inevitable; see Kuijper [8] and Willems [22]. Moreover, we showed that the same controlled behavior cannot be obtained by RFI using any controller, and obtained the unique controller that is of minimal order.

In Section 6 we dealt with the issue of partial interconnection: the case when to-be-controlled variables are possibly different from the control variables. We formulated and proved necessary conditions for a controlled behavior $\mathcal{K}_{\text{full}}$ to be implementable by RFI and used this to get an algorithm for finding controllers that implement $\mathcal{K}_{\text{full}}$ by RFI. Finding necessary and sufficient conditions for this case remains to be investigated.

We also established an equivalence between RFI and the notion that disturbances in a plant ought to remain unrestricted after the controller interconnection also (Theorem 7.1). We noted the similarity between this notion and that of well-posedness; see also Kuijper [8].

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[7] C_n can be obtained from C_2 by the following row operations on row 3 (i.e. r_3) of $\text{col}(R, C_2)$: $r_3 \leftarrow r_3 + \xi^{(n-1)} r_1$ followed by $r_3 \leftarrow r_3 - \xi^n r_2$.

[8] Though controllability is assumed in the paper, and crucially utilized within Example 1 (in Lomadze [9, p. 863]), it appears that the proofs of the main results utilize 'right unimodularity' of only F in Eq. (1) in Lomadze [9, p. 859], and not controllability of the plant behavior $\mathcal{P}$.

Table 1
Notation.

Symbol	Meaning	Symbol	Meaning
$\mathcal{P}$	Plant behavior	PERD	Plant equation of reducible degree
$\mathcal{K}$	Controlled behavior	RFI	Regular feedback interconnection/implementing/implementable
$\mathcal{C}$	Controller behavior	$U_{\beta,i}$	An identity matrix with its ith row replaced by the monomial row vector $\beta(\xi)$
p	Output cardinality of plant $\mathcal{P}$	$\mathcal{B}_1 \wedge \mathcal{B}_2$	Interconnection of two behaviors $\mathcal{B}_1$ and $\mathcal{B}_2$
k	Output cardinality of $\mathcal{K}$	$k-p$	Output cardinality of a regular controller
R_{lc}	Leading coefficient matrix of $R(\xi)$	$\text{DegBnd}(R, C_{rf})$	Set of polynomial matrices constructed using R and C_{rf}
$\mathtt{n}(\bullet)$	McMillan degree	$\mathcal{P}_{\text{full}}^{\text{ext}}$	$\mathcal{P}_{\text{full}}$ with disturbance inputs
$\mathcal{K}_{\text{full}}^{\text{ext}}$	$\mathcal{K}_{\text{full}}$ with disturbance inputs	$\mathcal{P}_{\text{full}}$	Plant behavior with to-be-controlled variables w and control variables c
$\mathcal{P}_c$	$\mathcal{P}_{\text{full}}$ projected on the c variables	$\mathcal{K}_{\text{full}}$	Controlled behavior with to-be-controlled variables w and control variables c
$\mathfrak{L}_{\text{loc}}^1$	Set of locally integrable functions	$\mathfrak{L}^{\mathtt{w}}$	Linear time invariant behavior with $\mathtt{w}$ system variables

Certain auxiliary issues like minimizing the controlled behavior's McMillan degree and/or the controller's McMillan degree were addressed in Section 8. The extension of our results towards finding minimal order controllers when $\mathcal{K}$ is not RFI appears non-trivial. We illustrated this with suitable examples. Some related issues that we did not address are described below. While regular feedback interconnection is closely related to occurrence of impulses when interconnection takes place, an equally important issue is when the plant and controller are *disconnected*: it is well-known that inductors pose a problem when disconnected. Conditions for impulse-free disconnection may or may not be significantly different. Also, when an interconnection is not RFI, often a change of controller is not practically desirable. In such a situation formulating conditions regarding the time instants when interconnecting won't cause impulses can be very useful: the use of electric lamps as synchroscopes to decide when two electric generators can be connected are very popular (see Agrawal [1, p. 542]).

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