Self-Adjoint Subspaces and Eigenfunction Expansions for Ordinary Differential Subspaces

Coddington, Earl A.; Dijksma, Aalt

Self-Adjoint Subspaces and Eigenfunction Expansions for Ordinary Differential Subspaces

EARL A. CODDINGTON* AND AALT DIJKSMA

Mathematisch Instituut, Rijksuniversiteit Groningen, Nederland

Received October 20, 1974

1. INTRODUCTION

Let L be an ordinary differential expression of order n on an open real interval $\iota = (a, b)$,

$$L = \sum_{k=0}^{n} p_k D^k, \qquad D = d/dx,$$

where $p_k \in C^k(\iota)$, and $p_n(x) \neq 0$, $x \in \iota$. Its Lagrange adjoint is L^+, where

$$L^+ = \sum_{k=0}^{n} (-1)^k D^k \bar{p}_k .$$

Naturally associated with L in the Hilbert space $\mathfrak{H} = \mathfrak{L}^2(\iota)$ are two closed operators, the minimal operator T_0 and the maximal operator T. We shall identify operators with their graphs in the Hilbert space $\mathfrak{H}^2 = \mathfrak{H} \oplus \mathfrak{H}$. Thus T_0 can be described as the closure in $\mathfrak{H}^2$ of the set of all $\{f, Lf\}$ with $f \in C_0^n(\iota)$, the functions of class $C^n(\iota)$ with compact support. If we denote the minimal and maximal operators for L^+ by T_0^+, T^+, then we have the relations $T_0 \subset T$, $T_0^+ \subset T^+$, and $T_0^* = T^+$, $(T_0^+)^* = T$, where T_0^*, $(T_0^+)^*$ are the adjoints of T_0, T_0^+, respectively. In order to be more specific, let us suppose we are in the regular case where a, b are finite, $p_k \in C^k(\bar{\iota})$, and $p_n(x) \neq 0$, $x \in \bar{\iota}$, where $\bar{\iota}$ is the closure of ι. Then T is the set of all $\{f, Lf\}$ such that $f \in C^{n-1}(\bar{\iota})$, $f^{(n-1)}$ is absolutely continuous on $\bar{\iota}$, and $Lf \in \mathfrak{H}$. The minimal operator T_0 is given by

$$T_0 = \{\{f, Lf\} \in T \mid \hat{f}(a) = \hat{f}(b) = 0\},$$

where $\hat{f}(x)$ is the $n \times 1$ matrix with rows $f(x), f'(x), ..., f^{(n-1)}(x)$. A typical boundary value problem associated with L in $\mathfrak{H}$ is one of finding solutions

473

f of the equation $Lf = h$, where h is given in $\mathfrak{H}$, and f is required to satisfy a finite set of boundary conditions:

$$Lf = h, \quad b_j(f) = \sum_{k=1}^{n} m_{jk} f^{(k-1)}(a) + n_{jk} f^{(k-1)}(b) = 0, \quad j = 1,\dots, p, \quad (1.1)$$

where m_{jk}, n_{jk} are given complex constants. Similarly, a typical eigenvalue problem for L is given by:

$$Lf = \lambda f, \quad b_j(f) = 0, \quad j = 1,\dots, p. \quad (1.2)$$

Associated with these two problems is the operator A_1 defined by

$$A_1 = \{\{f, Tf\} \mid f \in \mathfrak{D}(T), b_j(f) = 0, j = 1,\dots, p\},$$

where $\mathfrak{D}(T)$ is the domain of T. It clearly satisfies $T_0 \subset A_1 \subset T$. The problem (1.1) is just the problem of computing $A_1^{-1}(h)$, and (1.2) is the problem of determining the eigenvalues and eigenfunctions of the operator A_1.

The boundary functionals b_j are examples of continuous linear functionals on T, considered as a subspace of $\mathfrak{H}^2$. Therefore there exist elements $\{\sigma_j, \tau_j\}, j = 1,\dots, p$, in $\mathfrak{H}^2$ such that

$$b_j(f) = (\{f, Tf\}, \{\sigma_j, \tau_j\}) = (f, \sigma_j) + (Tf, \tau_j), f \in \mathfrak{D}(T).$$

If B is the subspace in $\mathfrak{H}^2$ spanned by $\{\sigma_1, \tau_1\},\dots, \{\sigma_p, \tau_p\}$, then we see that

$$A_1 = T \cap B^\perp = \{\{f, Tf\} \mid f \in \mathfrak{D}(T), (f, \sigma) + (Tf, \tau) = 0, \text{ all } \{\sigma, \tau\} \in B\}.$$

The boundary functionals b_j are not the only type of continuous linear functionals on T. For example, if $\mu_1,\dots, \mu_n$ are functions of bounded variation on $\bar{\imath}$, then the linear functional b given by

$$b(f) = \sum_{k=1}^{n} \int_a^b f^{(k-1)}(x) \, d\mu_k(x)$$

is continuous on T. Of course, the most general finite set of continuous linear functionals on T is prescribed by a finite-dimensional subspace B of $\mathfrak{H}^2$, and it is natural to consider the operator $A_1 = T \cap B^\perp$ which is associated with T and such a subspace B. Clearly $T_0 \cap B^\perp \subset A_1 \subset T$. Now, since A_1 need not be densely defined, its adjoint,

$$A_1{}^* = \{\{h, k\} \in \mathfrak{H}^2 \mid (g, h) = (f, k), \text{ all } \{f, g\} \in A_1\},$$

need not be (the graph of) an operator (single-valued function), and, even if A_1 is densely defined, A_1^* need no longer be a differential operator. From Section 2 it follows that A_1^* is an algebraic sum:

$$A_1^* = T_0^+ \dotplus (-B^{-1}) = \{\{f + \tau, \, T_0^+f - \sigma\} \mid f \in \mathfrak{D}(T_0^+), \, \{\sigma, \tau\} \in B\}.$$

More generally, for the given finite-dimensional B, we can study subspaces (closed linear manifolds) $A \subset \mathfrak{H}^2$ satisfying

$$T_0 \cap B^{\perp} \subset A \subset (T_0^+ \cap B^{\perp})^* = T \dotplus (-B^{-1}).$$

Such subspaces A can be described as restrictions of $T \dotplus (-B^{-1})$, namely as the intersection of the null spaces of a finite number of continuous linear functionals on $T \dotplus (-B^{-1})$ which vanish on $T_0 \cap B^{\perp}$. These functionals, which might be called generalized boundary values, involve not only the boundary values at a and b, but integral terms as well. The results of Section 2 imply that

$$T_0^+ \cap B^{\perp} \subset A^* \subset (T_0 \cap B^{\perp})^* = T^+ \dotplus (-B^{-1}).$$

The case when L is formally symmetric, $L = L^+$, is important, and we concentrate our attention on this case. Then we write S_0, S_0^* instead of $T_0 = T_0^+$, $T = T^+$, where S_0 is now a symmetric operator, $S_0 \subset S_0^*$. We consider the general (not necessarily regular) case of an arbitrary open interval ι, and study the possible self-adjoint subspace extensions $H = H^*$ of $S = S_0 \cap B^{\perp}$. When such H exist in $\mathfrak{H}^2$ they can be characterized by corresponding generalized boundary values. Self-adjoint extensions H always exist in an appropriate larger Hilbert space $\mathfrak{K}^2$, where $\mathfrak{H} \subset \mathfrak{K}$. We show how each such extension H gives rise to an eigenfunction expansion result.

We briefly summarize the contents of the subsequent sections. In Section 2 we consider a general subspace T_0 in the sum $X \oplus Y$ of two Banach spaces X, Y, and a finite-dimensional subspace B of the dual space $X^* \oplus Y^*$, and compute the adjoint of $T_0 \cap {}^{\perp}B$, where

$$^{\perp}B = \{\{f, g\} \in X \oplus Y \mid (\sigma, f) + (\tau, g) = 0, \text{ all } \{\sigma, \tau\} \in B\}.$$

We then specialize to the case when $X = Y = \mathfrak{H}$, a Hilbert space, $^{\perp}B = B^{\perp} = \mathfrak{H}^2 \ominus B$, and $T_0 = S_0$ is a densely defined symmetric operator in $\mathfrak{H}$. The adjoint of the symmetric operator $S = S_0 \cap B^{\perp}$ is then just $S^* = S_0^* \dotplus (-B^{-1})$. In [4] was considered the special case where B has the form $B = \mathfrak{H}_0 \oplus \{0\}$, with $\mathfrak{H}_0$ being a finite-dimensional subspace of $\mathfrak{H}$. In Theorem 2.3 we indicate how S may be represented as $S = S_1 \cap (\mathfrak{H}_0 \oplus \{0\})^{\perp}$ for an appropriate densely defined symmetric operator

S_1 and a subspace $\mathfrak{H}_0$ of $\mathfrak{H}$. Thus, in a certain sense, the general case for S is reduced to the special case. We show in Section 3 that the symmetric operator $S = S_0 \cap B^\perp$ has self-adjoint extensions in $\mathfrak{H}^2$ if and only if S_0 does, that is, if and only if S_0 has equal deficiency indices. Then, assuming S_0 does have self-adjoint extensions in $\mathfrak{H}^2$, all self-adjoint extensions of S in $\mathfrak{H}^2$ are characterized in Theorem 3.3. In Section 4 it is shown how this result applies to the case of an S_0 which is the minimal operator for a formally symmetric ordinary differential expression L in $\mathfrak{H} = \mathfrak{L}^2(\iota)$. The regular case is considered in detail in Theorem 4.1. We show that problems involving multipoint boundary conditions, and, more generally, problems involving measures (Stieltjes boundary conditions), can be considered as special cases of Theorem 4.1. Moreover, certain singular problems involving measures can be considered as special cases of the general result Theorem 3.3. Facts about self-adjoint extensions of $S = S_0 \cap B^\perp$ in larger spaces $\mathfrak{K}^2$, $\mathfrak{H} \subset \mathfrak{K}$, are summarized in Section 5. For an S_0 which is the minimal operator for a formally symmetric L in $\mathfrak{H} = \mathfrak{L}^2(\iota)$, and each self-adjoint subspace extension of S in $\mathfrak{K}^2$, $\mathfrak{H} \subset \mathfrak{K}$, we give an eigenfunction expansion result. Two proofs are presented. One, in Section 6, follows the general scheme in [5], where the special case $B = \mathfrak{H}_0 \oplus \{0\}$ was treated. It depends upon an analysis of the generalized resolvent corresponding to H. The other proof is given in Section 7; it follows the ideas in [6], and makes use of the Riesz representation theorem. In deducing the eigenfunction expansion we obtain a map V of $\mathfrak{H}$ into a transform space $\hat{\mathfrak{H}}$ which is in general a contraction. It is an isometry on a certain subspace of $\mathfrak{H}$ which has finite codimension. We show that this isometry is surjective if and only if the generalized spectral family for S corresponding to H is the spectral family for a self-adjoint subspace extension of S in $\mathfrak{H}^2$ itself.

Our results carry over to the case of systems of ordinary differential operators. In fact, only a minor reinterpretation of the symbols is required in order to obtain the results for a system of n first-order operators.

The real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively, and we let $\mathbb{C}^\pm = \{l \in \mathbb{C} \mid \mathrm{Im}\, l \gtrless 0\}$, $\mathbb{C}_0 = \mathbb{C}^+ \cup \mathbb{C}^-$. For any interval $\iota \subset \mathbb{R}$ we denote by $C^n(\iota)$ the set of all complex-valued functions on ι having n continuous derivatives there, and $C_0^n(\iota)$ is the set of all those $f \in C^n(\iota)$ with compact support. Although we denote by $\bar{\iota}$ the closure of an interval ι, for other sets $\mathfrak{D}$ in a Hilbert space $\mathfrak{H}$ the closure is denoted by $\mathfrak{D}^c$. The identity operator is denoted by I. The $j \times k$ zero matrix is designated by 0_j^k, and the $n \times n$ identity matrix is represented by I_n. The transpose of the matrix A is denoted by A^t, and the conjugate transpose of A is A^*. If A, B are two matrices with the same number of rows, then $(A : B)$ denotes the matrix obtained by placing the columns of B next to those of A in the order indicated. If f is a one-rowed matrix-valued function whose elements

have $n-1$ derivatives, then $\tilde{f}(x)$ is the matrix with rows $f(x), f'(x),...,f^{(n-1)}(x)$. If $F = (F_{kj})$, $G = (G_{kj})$, are matrices, with elements in a Hilbert space $\mathfrak{H}$ over $\mathbb{C}$, and which have the same number of rows, we define the *matrix inner product* (F, G) to be the matrix whose (i, j)th element is

$$(F, G)_{ij} = \sum_k (F_{kj}, G_{ki}).$$

For example, if the elements of F, G are in $\mathfrak{H} = \mathbb{C}$, then $(F, G) = G^*F$, and if the elements of F, G are in $\mathfrak{H} = \mathfrak{L}^2(\iota)$, $\iota = (a, b)$, then $(F, G) = \int_a^b G^*F$. This matrix inner product has the properties:

$$(F, F) \geqslant 0, \quad \text{and} \quad (F, F) = 0 \quad \text{if and only if} \quad F = 0,$$
$$(G, F) = (F, G)^*,$$
$$(F_1 + F_2, G) = (F_1, G) + (F_2, G),$$
$$(FC, G) = (F, G)C, \quad (F, GD) = D^*(F, G),$$

where C, D are matrices with elements in $\mathbb{C}$. A true inner product is given by $F \cdot G = \text{trace}(F, G)$, and hence a norm is given via $\|F\|^2 = \text{trace}(F, F)$.

2. THE ADJOINT OF A SUBSPACE

We extend some of the definitions given in [3] to Banach spaces. Let X and Y be Banach spaces over the complex field $\mathbb{C}$. We denote by $X \oplus Y$ the Banach space of all pairs $\{f, g\}$, $f \in X$ and $g \in Y$, with a linear structure defined component-wise and with the norm defined by

$$\|\{f, g\}\| = (\|f\|_X^2 + \|g\|_Y^2)^{1/2},$$

where $\| \|_X$ and $\| \|_Y$ are the norms of the spaces X and Y. A *subspace* T in $X \oplus Y$ is a closed linear manifold T in $X \oplus Y$. We treat such a subspace T as a linear relation and define the domain $\mathfrak{D}(T)$ and the range $\mathfrak{R}(T)$ of T by

$$\mathfrak{D}(T) = \{f \in X \mid \{f, g\} \in T \text{ for some } g \in Y\},$$
$$\mathfrak{R}(T) = \{g \in Y \mid \{f, g\} \in T \text{ for some } f \in X\}.$$

Let T and S be subspaces in $X \oplus Y$. We define the sets αT ($\alpha \in \mathbb{C}$), $T + S$ in $X \oplus Y$ and T^{-1} in $Y \oplus X$ by

$$\alpha T = \{\{f, \alpha g\} \mid \{f, g\} \in T\},$$
$$T + S = \{\{f, g + k\} \mid \{f, g\} \in T, \{f, k\} \in S\},$$
$$T^{-1} = \{\{g, f\} \mid \{f, g\} \in T\}.$$

For $f \in \mathfrak{D}(T)$ we let

$$T(f) = \{g \in Y \mid \{f, g\} \in T\}.$$

If T is a subspace in $X \oplus Y$ satisfying $T(0) = \{0\}$, then T is the graph of a closed operator, that is, a closed linear function, from X into Y. We shall frequently identify this operator with its graph, denote it by T and replace $T(f)$ by the usual $Tf, f \in \mathfrak{D}(T)$. Conversely, if T is a closed operator from X into Y we shall often identify it with its graph which is a subspace in $X \oplus Y$. The null space of the subspace T in $X \oplus Y$ is the set

$$\nu(T) = \{f \in X \mid \{f, 0\} \in T\} = T^{-1}(0).$$

The *algebraic sum* $T \dotplus S$ in $X \oplus Y$ of the subspaces T, S in $X \oplus Y$ is the linear manifold

$$T \dotplus S = \{\{f + h, g + k\} \mid \{f, g\} \in T, \{h, k\} \in S\}.$$

It is called a *direct algebraic sum* if $T \cap S = \{\{0, 0\}\}$. If $T \cap S = \{\{0, 0\}\}$ then each $\{u, v\} \in T \dotplus S$ has a unique decomposition

$$\{u, v\} = \{f, g\} + \{h, k\}, \qquad \{f, g\} \in T, \qquad \{h, k\} \in S.$$

The dual space Z^* of a Banach space Z is the Banach space of all continuous conjugate linear functionals on Z. If $h \in Z^*$ then its value at $g \in Z$ will be denoted by (h, g). Let X, Y be Banach spaces. Then the dual of $X \oplus Y$, $(X \oplus Y)^*$, is isometrically isomorphic to the Banach spaces $X^* \oplus Y^*$ and $Y^* \oplus X^*$. With the subspaces T in $X \oplus Y$ and S in $X^* \oplus Y^*$ we associate the subspaces $T^\perp$ in $X^* \oplus Y^*$, $^\perp S$ in $X \oplus Y$ and T^* in $Y^* \oplus X^*$ given by

$$T^\perp = \{\{h, k\} \in X^* \oplus Y^* \mid (h, f) + (k, g) = 0 \text{ for all } \{f, g\} \in T\},$$
$$^\perp S = \{\{f, g\} \in X \oplus Y \mid (h, f) + (k, g) = 0 \text{ for all } \{h, k\} \in S\},$$
$$T^* = \{\{h, k\} \in Y^* \oplus X^* \mid (h, g) - (k, f) = 0 \text{ for all } \{f, g\} \in T\}.$$

T^* is called the *adjoint* of T, and, clearly, $T^* = (-T^{-1})^\perp$.

Let T_0 be a subspace in $X \oplus Y$ and let B be a finite dimensional subspace of $X^* \oplus Y^*$. Let $T = T_0 \cap {}^\perp B$. Then T is a subspace in $X \oplus Y$ and its adjoint is given by the following theorem.

THEOREM 2.1. *Let* $T_0 \subset X \oplus Y$ *and* $B \subset X^* \oplus Y^*$ *be subspaces with* $\dim B < \infty$, *and let* $T = T_0 \cap {}^\perp B$. *Then* $T^* = T_0^* \dotplus -B^{-1}$, *and the algebraic sum is direct if and only if* $T_0^\perp \cap B = \{\{0, 0\}\}$.

Proof. If M and N are linear manifolds in a Banach space such that $M^\perp + N^\perp$ is closed in the dual space then $M^\perp + N^\perp = (M \cap N)^\perp$ (cf. [9, p. 221]). Set $M = -T_0^{-1}$ and $N = -(^\perp B)^{-1}$. Then M and N are subspaces in $Y \oplus X$, $M^\perp = T_0^*$ and, since dim $B < \infty$, $N^\perp = -((^\perp B)^\perp)^{-1} = -B^{-1}$ (cf. [12, p. 227, Problem 2]). Again since B is finite dimensional, $M^\perp + N^\perp$ is closed in $Y^* \oplus X^*$ (cf. [9, p. 130]). Hence,

$$T_0^* \dotplus -B^{-1} = M^\perp + N^\perp = (M \cap N)^\perp = (-(T_0 \cap {}^\perp B)^{-1})^\perp$$
$$= (-T^{-1})^\perp = T^*.$$

The equality $T^* \cap -B^{-1} = -(T_0^\perp \cap B)^{-1}$ shows that $T^* \dotplus -B^{-1}$ is a direct sum if and only if $T_0^\perp \cap B = \{\{0, 0\}\}$.

We now set $X = Y = \mathfrak{H}$, where $\mathfrak{H}$ is a Hilbert space. Then $\mathfrak{H}^2 = \mathfrak{H} \oplus \mathfrak{H}$ is a Hilbert space also, with inner product

$$(\{f, g\}, \{h, k\}) = (f, h) + (g, k), \qquad \{f, g\}, \{h, k\} \in \mathfrak{H}^2.$$

We identify $\mathfrak{H}$ with its dual in the usual manner. Then all the above definitions coincide with the ones in [3]. In particular, if T is a subspace in $\mathfrak{H}^2$ then $T^\perp$ and $^\perp T$ coincide and are equal to the orthogonal complement of T in $\mathfrak{H}^2$, which we denote by $T^\perp$. If S is a subspace in $\mathfrak{H}^2$ which is orthogonal to T then $T \dotplus S$ is a direct algebraic sum which is denoted by $T \oplus S$ and called the *orthogonal sum*. If $S \subset T$ then the orthogonal complement of S in T will be denoted by $T \ominus S$.

For any subspace T in $\mathfrak{H}^2$ let T_∞ be the set of all elements of the form $\{0, g\}$ in T. Then $T_\infty = \{0\} \oplus T(0)$. Let $T_s = T \ominus T_\infty$. Then T_s is an operator in $\mathfrak{H}$, called the *operator part* of T, with $\mathfrak{D}(T_s) = \mathfrak{D}(T)$ dense in $(T^*(0))^\perp$ and $\mathfrak{R}(T_s) \subset (T(0))^\perp$.

A *symmetric subspace* S in $\mathfrak{H}^2$ is one satisfying $S \subset S^*$, and a *self-adjoint subspace* H in $\mathfrak{H}^2$ is one for which $H = H^*$. If $H = H_s \oplus H_\infty$ is a self-adjoint subspace in $\mathfrak{H}^2$, then H_s is a densely defined self-adjoint operator in the Hilbert space $(H(0))^\perp$.

Let S_0 be a symmetric subspace in $\mathfrak{H}^2$ and let B be a subspace in $\mathfrak{H}^2$ with dim $B = p < \infty$ and $S_0^\perp \cap B = \{\{0, 0\}\}$. Let $S = S_0 \cap B^\perp$. The following result is an immediate consequence of Theorem 2.1.

COROLLARY 2.2. *S is a symmetric subspace in $\mathfrak{H}^2$ and $S^* = S_0^* \dotplus -B^{-1}$, where the algebraic sum is direct.*

We remark here that $S_0^\perp \cap B = \{\{0, 0\}\}$ is not a real restriction. For, without this condition, $S = S_0 \cap B^\perp = S_0 \cap [B \ominus (S_0^\perp \cap B)]^\perp$ and $S_0^\perp \cap [B \ominus (S_0^\perp \cap B)] = \{\{0, 0\}\}$.

From now on we shall assume that $\mathfrak{D}(S_0)$ is dense in $\mathfrak{H}$. Then S_0 is a *densely defined symmetric operator* in $\mathfrak{H}$, for $S_0(0) = (\mathfrak{D}(S_0{}^*))^\perp = \{0\}$. We decompose B into two subspaces B_1 and B_2, where

$$B_2 = \{\{\sigma, \tau\} \in B \mid \tau \in \mathfrak{D}(S_0{}^*)\},$$
$$B_1 = B \ominus B_2.$$

Thus $B = B_1 \oplus B_2$, and, since B_2 contains all elements of B of the form $\{\sigma, 0\}$ we see that B_1^{-1} is an operator. Let $m = \dim B_2$, and, consequently, $\dim B_1 = p - m$. We define

$$\mathfrak{H}_0 = \{\varphi \in \mathfrak{H} \mid \varphi = S_0{}^*\tau + \sigma, \{\sigma, \tau\} \in B_2\},$$

and $S_1 = S_0 \cap B_1{}^\perp$. In the following theorem we list some of the properties of the subspaces defined above.

THEOREM 2.3. (i) S_1 *is a densely defined symmetric operator in* $\mathfrak{H}$ *and* $S_1{}^* = S_0{}^* \dotplus -B_1^{-1}$, *where the algebraic sum is direct,*

(ii) $\dim \mathfrak{H}_0 = m$,

(iii) $(S^*)_\infty = \{\{0, \varphi\} \mid \varphi \in \mathfrak{H}_0\} = \{0\} \oplus \mathfrak{H}_0$,

(iv) $S = S_1 \cap B_2{}^\perp = S_1 \cap (\mathfrak{H}_0 \oplus \{0\})^\perp$,

(v) $S^* = S_1{}^* \dotplus -B_2^{-1} = S_1{}^* \dotplus (\{0\} \oplus \mathfrak{H}_0) = S_0{}^* \dotplus -B_1^{-1} \dotplus (S^*)_\infty$,

where the algebraic sums are direct.

Proof. (i) Since $S_1 \subset S_0$, S_1 is an operator. Since $S_0{}^\perp \cap B_1 = \{\{0, 0\}\}$, Corollary 2.2 with S, B replaced by S_1, B_1, implies that S_1 is a symmetric subspace in $\mathfrak{H}^2$ and that $S_1{}^* = S_0{}^* \dotplus -B_1^{-1}$, where the algebraic sum is direct. To show that S_1 is densely defined it suffices to prove that $S_1{}^*(0) = \{0\}$, for $\mathfrak{D}(S_1)$ is dense in $(S_1{}^*(0))^\perp$. Let $g \in S_1{}^*(0)$. Then $\{0, g\} \in S_1{}^* = S_0{}^* \dotplus -B_1^{-1}$ and there exists a unique decomposition

$$\{0, g\} = \{h, k\} + \{\tau, -\sigma\}, \quad \{h, k\} \in S_0{}^*, \quad \{\sigma, \tau\} \in B_1.$$

It follows that $\tau = -h \in \mathfrak{D}(S_0{}^*)$. Hence $\{\sigma, \tau\} \in B_1 \cap B_2 = \{\{0, 0\}\}$. Therefore $h = 0$ and, since $S_0{}^*$ is an operator, $g = k = S_0{}^*h = 0$, that is $S_1{}^*(0) = \{0\}$.

(ii) Let $\kappa: B_2 \to \mathfrak{H}_0$ be defined by $\kappa(\{\sigma, \tau\}) = S_0{}^*\tau + \sigma$. Clearly κ is linear and surjective. We show that κ is also injective. If $S_0{}^*\tau + \sigma = 0$, $\{\sigma, \tau\} \in B_2$, then $\sigma = -S_0{}^*\tau$ and by Corollary 2.2

$$\{\tau, -\sigma\} = \{\tau, S_0{}^*\tau\} \in S_0{}^* \cap -B_2^{-1} \subset S_0{}^* \cap -B^{-1} = \{\{0, 0\}\}.$$

Hence $\{\sigma, \tau\} = \{0, 0\}$. Consequently $\dim \mathfrak{H}_0 = \dim B_2 = m$.

(iii)   Let $\{0, \varphi\} \in (S^*)_\infty \subset S_0^* \dotplus -B^{-1}$. Then $\{0, \varphi\}$ can be written as

$$\{0, \varphi\} = \{h, k\} + \{\tau, -\sigma\}, \qquad \{h, k\} \in S_0^*, \qquad \{\sigma, \tau\} \in B.$$

As in (i) this implies that $\{\sigma, \tau\} \in B_2$ and $\tau = -h$. Hence $\varphi = k - \sigma = S_0^*h - \sigma = -(S_0^*\tau + \sigma) \in \mathfrak{H}_0$. Thus $(S^*)_\infty \subset \{0\} \oplus \mathfrak{H}_0$. Conversely if $\varphi \in \mathfrak{H}_0$, then $\varphi = S_0^*\tau + \sigma$ for some $\{\sigma, \tau\} \in B_2$. It follows that

$$\{0, \varphi\} = \{\tau, S_0^*\tau\} + \{-\tau, \sigma\} \in S_0^* \dotplus -B^{-1} = S^*,$$

and so $\{0, \varphi\} \in (S^*)_\infty$. This proves (iii).

(iv)   Since $B^\perp = B_1^\perp \cap B_2^\perp$, $S = S_0 \cap B^\perp = S_1 \cap B_2^\perp$. To prove the second equality, let $\{f, g\} \in S_1 \cap B_2^\perp = S$ and let $\{\varphi, 0\} \in \mathfrak{H}_0 \oplus \{0\}$. Then by (iii) $\{0, \varphi\} \in S^*$ and hence $(f, \varphi) = 0$. Thus $\{f, g\} \in S_1 \cap (\mathfrak{H}_0 \oplus \{0\})^\perp$. Conversely, let $\{f, g\} \in S_1 \cap (\mathfrak{H}_0 \oplus \{0\})^\perp$ and let $\{\sigma, \tau\} \in B_2$. Then $g = S_0f$, $\varphi = S_0^*\tau + \sigma \in \mathfrak{H}_0$, and

$$(f, \sigma) + (g, \tau) = (f, \sigma) + (f, S_0^*\tau) = (f, \varphi) = 0.$$

Hence $\{f, g\} \in S_1 \cap B_2^\perp$.

(v)   Let $\{\sigma, \tau\} \in S_1^\perp \cap B_2$. Then for all $\{f, S_0f\} \in S_1 = S_0 \cap B_1^\perp$ we have

$$0 = (\{f, S_0f\}, \{\sigma, \tau\}) = (f, \sigma) + (S_0f, \tau) = (f, \sigma + S_0^*\tau).$$

Since $\mathfrak{D}(S_1)$ is dense in $\mathfrak{H}$, this implies $\sigma = -S_0^*\tau$ and consequently

$$\{\tau, -\sigma\} = \{\tau, S_0^*\tau\} \in S_0^* \cap -B^{-1} = \{\{0, 0\}\}.$$

Thus $S_1^\perp \cap B_2 = \{\{0, 0\}\}$, and similarly $S_1^\perp \cap (\mathfrak{H}_0 \oplus \{0\}) = \{\{0, 0\}\}$. The equalities in (v) now follow from Corollary 2.2(i), (iii), and (iv).

3. SELF-ADJOINT SUBSPACE EXTENSIONS IN $\mathfrak{H}^2$

For any subspace S in $\mathfrak{H}^2$ and $l \in \mathbb{C}$ we define

$$M_S(l) = \{\{f, g\} \in S^* \mid g = lf\}.$$

If S is symmetric and $l \in \mathbb{C}^+$, then

$$S^* = S \dotplus M_S(l) \dotplus M_S(\bar{l}),$$

where the algebraic sums are direct. A symmetric subspace S in $\mathfrak{H}^2$ always has self-adjoint extensions in suitably chosen Hilbert spaces $\mathfrak{K}^2$, $\mathfrak{H} \subset \mathfrak{K}$,

but there exist self-adjoint extensions of S in $\mathfrak{H}^2$ if and only if for some $l \in \mathbb{C}^+$ (and hence for all $l \in \mathbb{C}^+$) dim $M_S(l) =$ dim $M_S(\bar{l})$.

Let S_0, B and $S = S_0 \cap B^\perp$ be as in Section 2. We shall write $M_0(l)$, $M(l)$ instead of $M_{S_0}(l)$, $M_S(l)$.

THEOREM 3.1.   *For* $l \in \mathbb{C}_0$

$$\dim M(l) = \dim M_0(l) + \dim B.$$

Proof.   Let $l \in \mathbb{C}_0$. Each $\{f, lf\} \in M(l) \subset S^* = S_0^* \dotplus -B^{-1}$ can be uniquely decomposed into

$$\{f, lf\} = \{u, v\} + \{\tau, -\sigma\}, \qquad \{u, v\} \in S_0^*, \qquad \{\sigma, \tau\} \in B.$$

We define the linear map $\kappa: M(l) \to B$ by $\kappa(\{f, lf\}) = \{\sigma, \tau\}$. Let $\{\sigma, \tau\} \in B$. Since $\mathfrak{R}(S_0^* - lI) = \mathfrak{H}$, there exists a $u \in \mathfrak{D}(S_0^*)$ such that $(S_0^* - lI)u = \sigma + l\tau$. If $f = u + \tau$, then

$$\{f, lf\} = \{u, S_0^*u\} + \{\tau, -\sigma\} \in S^*.$$

Thus $\{f, lf\} \in M(l)$ and $\kappa(\{f, lf\}) = \{\sigma, \tau\}$, that is, κ is surjective. It is easy to see that $M_0(l)$ is the null space of κ. It follows that κ restricted to $M(l) \ominus M_0(l)$ is a linear bijection onto B. Hence $\dim(M(l) \ominus M_0(l)) = \dim B$, which proves the theorem.

COROLLARY 3.2.   *S has self-adjoint extensions in $\mathfrak{H}^2$ if and only if S_0 has self-adjoint extensions in $\mathfrak{H}^2$.*

In the remainder of Section 3 we assume that dim $M_0(l) =$ dim $M_0(\bar{l}) < \infty$ for $l \in \mathbb{C}_0$, and we put $\omega =$ dim $M_0(l)$, $q = \omega + p =$ dim $M_0(l) +$ dim $B =$ dim $M(l)$. Then S_0 has self-adjoint extensions in $\mathfrak{H}^2$, and so do S_1 and S. By Theorem 3.1

$$\dim M_{S_1}(l) = \dim M_0(l) + \dim B_1 = \omega + p - m,$$

and

$$\dim M(l) = \dim M_{S_1}(l) + \dim \mathfrak{H}_0 = \omega + p - m + m = q.$$

We can now apply [4, Theorem 3] and describe all self-adjoint extensions H of $S = S_1 \cap (\mathfrak{H}_0 \oplus \{0\})^\perp$ in $\mathfrak{H}^2$. We shall use the following notation. For $h, f \in \mathfrak{D}(S_0^*)$, $\{\sigma, \tau\}$, $\{\varphi, \psi\} \in B_1$,

$$\begin{aligned}
\langle h + \tau, f + \psi \rangle_1 &= (S_1^*(h + \tau), f + \psi) - (h + \tau, S_1^*(f + \psi)) \\
&= (S_0^*h - \sigma, f + \psi) - (h + \tau, S_0^*f - \varphi) \\
&= \langle h, f \rangle + (\{h, S_0^*h\}, \{\varphi, \psi\}) \\
&\quad - (\{\sigma, \tau\}, \{f, S_0^*f\}) - \langle \tau, \psi \rangle_{B_1}, \qquad (3.1)
\end{aligned}$$

where

$$\langle h, f \rangle = (S_0{}^*h, f) - (h, S_0{}^*f),$$

$$\langle \tau, \psi \rangle_{B_1} = (B_1^{-1}\tau, \psi) - (\tau, B_1^{-1}\psi) = (\sigma, \psi) - (\tau, \varphi).$$

THEOREM 3.3. *Let H be a self-adjoint subspace extension of S in $\mathfrak{H}^2$ with* dim $H(0) = s$. *Let $\varphi_1, ..., \varphi_s$ be an orthonormal basis for $H(0)$, and $\varphi_1, ..., \varphi_s$, $\varphi_{s+1}, ..., \varphi_m$ an orthonormal basis for $\mathfrak{H}_0$. Then there exist $\gamma_{s+1}, ..., \gamma_m$, $\delta_{m+1}, ..., \delta_q$ in $\mathfrak{D}(S_0{}^*)$, $\tau_{s+1}, ..., \tau_q \in \mathfrak{R}(B_1)$ and $E_{rk} \in \mathbb{C}$, $r, k = s+1, ..., m$, such that*

$$\delta_{m+1} + \tau_{m+1}, ..., \delta_q + \tau_q \text{ are linearly independent} \bmod \mathfrak{D}(S_1), \quad (3.2)$$

$$\langle \delta_l + \tau_l, \delta_j + \tau_j \rangle_1 = 0, \qquad j, l = m+1, ..., q, \quad (3.3)$$

$$E_{rk} = \bar{E}_{kr}, \qquad r, k = s+1, ..., m, \quad (3.4)$$

and if

$$\psi_k = \sum_{r=s+1}^{m} [E_{rk} - \tfrac{1}{2}\langle \gamma_k + \tau_k, \gamma_r + \tau_r \rangle_1] \varphi_r, \qquad k = s+1, ..., m, \quad (3.5)$$

$$\zeta_l = - \sum_{r=s+1}^{m} \langle \delta_l + \tau_l, \gamma_r + \tau_r \rangle_1 \varphi_r, \qquad l = m+1, ..., q, \quad$$

then

H is the set of all $\{h + \tau, S_0{}^*h - \sigma + \varphi\}$, $h \in \mathfrak{D}(S_0{}^*)$, $\{\sigma, \tau\} \in B_1$, $\varphi \in \mathfrak{H}_0$, (3.6)

such that

$$(h + \tau, \varphi_j) = 0, \qquad j = 1, ..., s,$$

$$\langle h + \tau, \delta_l + \tau_l \rangle_1 - (h + \tau, \zeta_l) = 0, \qquad l = m+1, ..., q,$$

$$\varphi = c_1 \varphi_1 + \cdots + c_s \varphi_s$$

$$+ \sum_{k=s+1}^{m} [(h + \tau, \psi_k) - \langle h + \tau, \gamma_k + \tau_k \rangle_1] \varphi_k, \qquad c_j \in \mathbb{C},$$

and the operator part H_s of H is given by

$$H_s(h + \tau) = S_0{}^*h - \sigma - \sum_{j=1}^{s} (S_0{}^*h - \sigma, \varphi_j) \varphi_j$$

$$+ \sum_{k=s+1}^{m} [(h + \tau, \psi_k) - \langle h + \tau, \gamma_k + \tau_k \rangle_1] \varphi_k. \quad (3.7)$$

Conversely, if $\varphi_1, ..., \varphi_s, \varphi_{s+1}, ..., \varphi_m$ *is an orthonormal basis for* $\mathfrak{H}_0$, γ_k, $\delta_k \in \mathfrak{D}(S_0^*)$, $\tau_j \in \mathfrak{R}(B_1)$ *and* $E_{kr} \in \mathbb{C}$ *exist satisfying* (3.2), (3.3) *and* (3.4), *and* ψ_k, ζ_l *are defined by* (3.5), *then* H *defined by* (3.6) *is a self-adjoint extension of* S *such that* $H(0) = \mathrm{span}\{\varphi_1, ..., \varphi_s\}$ *and* H_s *is given by* (3.7).

We observe that if $B_1 = \{\{0, 0\}\}$ then Theorem 3.3 coincides with [4, Theorem 3]. We refer to [4] for comments about other special cases of Theorem 3.3.

4. PROBLEMS INVOLVING ORDINARY DIFFERENTIAL OPERATORS

Let L be a formally symmetric ordinary differential operator of order n,

$$L = \sum_{k=0}^{n} p_k D^k = \sum_{k=0}^{n} (-1)^k D^k \bar{p}_k, \qquad D = d/dx,$$

where $p_k \in C^k(\iota)$, $\iota = (a, b) \subset \mathbb{R}$ and $p_n(x) \neq 0$ for all $x \in \iota$. We consider the Hilbert space $\mathfrak{H} = \mathfrak{L}^2(\iota)$, and define S_0 to be the closure in $\mathfrak{H}^2$ of the set of all $\{f, Lf\}$ with $f \in C_0^n(\iota)$. Then S_0 is a closed densely defined symmetric operator in $\mathfrak{H}$, called the *minimal operator* for L in $\mathfrak{H}$. Its adjoint S_0^* is the set of all $\{f, Lf\}$ where $f \in C^{n-1}(\iota) \cap \mathfrak{H}$, $f^{(n-1)}$ is locally absolutely continuous on ι and $Lf \in \mathfrak{H}$. S_0^* is called the *maximal operator* for L in $\mathfrak{H}$. The operator S_0 satisfies the conditions set in the previous sections and we define B, S, B_1, B_2 and $\mathfrak{H}_0$ as in those sections. Let $u, v \in \mathfrak{D}(S_0^*)$. Then, as is well known, the limits of

$$[uv](x) = \sum_{m=1}^{n} \sum_{j+k=m-1} (-1)^j u^{(k)}(x)(p_m \bar{v})^{(j)}(x)$$

exist as x tends to a or b and

$$\langle u, v \rangle = [uv](b) - [uv](a).$$

Thus $\langle u, v \rangle$ represents boundary terms, and in Theorem 3.3 we see that the domain $\mathfrak{D}(H)$ is prescribed by certain boundary-integral conditions, cf. (3.6) and (3.1), and H_s involves the differential operator L as well as boundary-integral terms.

Regular problems. We shall consider in more detail the case when L is regular. In this case a and b are finite, $p_k \in C^k(\bar{\iota})$ and $p_n(x) \neq 0$ for all $x \in \bar{\iota}$, the closure of ι. The operator S_0^* is the set of all $\{f, Lf\}$ where $f \in C^{n-1}(\bar{\iota})$, $f^{(n-1)}$ is absolutely continuous on $\bar{\iota}$ and $Lf \in \mathfrak{H}$, and S_0 is the set

of all $\{f, Lf\} \in S_0^*$ for which $\tilde{f}(a) = \tilde{f}(b) = 0$. Since for each $l \in \mathbb{C}$ all solutions of $(L - l)u = 0$ belong to $C^n(\tilde{\imath})$,

$$\omega = \dim M_0(l) = \dim \nu(S_0^* - lI) = n.$$

Thus S_0, and hence S, has self-adjoint extensions H in $\mathfrak{H}^2$ and $q = \dim M(l) = p + n$.

Using the vector notation described in the Introduction we shall write down the various conditions of Theorem 3.4. We put

$$\{\delta, S_0^*\delta\} = (\{\delta_{m+1}, S_0^*\delta_{m+1}\},..., \{\delta_q, S_0^*\delta_q\}),$$
$$\{\gamma, S_0^*\gamma\} = (\{\gamma_{s+1}, S_0^*\gamma_{s+1}\},..., \{\gamma_m, S_0^*\gamma_m\}),$$

and

$$\delta = (\delta_{m+1},..., \delta_q), \qquad \gamma = (\gamma_{s+1},..., \gamma_m), \qquad \text{etc.}$$

Let $\{\sigma^1, \tau^1\}$ denote the $1 \times (p - m)$ matrix whose entries form a basis for B_1. Then the elements $\tau_{s+1},..., \tau_q \in \mathfrak{R}(B_1)$ given in Theorem 3.3 uniquely determine elements $\{\sigma_{s+1}, \tau_{s+1}\},..., \{\sigma_q, \tau_q\} \in B_1$ and they may be expressed in terms of $\{\sigma^1, \tau^1\}$. Thus there exist matrices A_1 and A_2 of complex constants of order $(p - m) \times (q - m)$ and $(p - m) \times (m - s)$, respectively, such that

$$(\{\sigma_{m+1}, \tau_{m+1}\},..., \{\sigma_q, \tau_q\}) = \{\sigma^1, \tau^1\} A_1,$$
$$(\{\sigma_{s+1}, \tau_{s+1}\},..., \{\sigma_m, \tau_m\}) = \{\sigma^1, \tau^1\} A_2. \tag{4.1}$$

Using the notion of a matrix inner product described in the Introduction, and the above notation, we see that condition (3.3) reads

$$\langle \delta, \delta \rangle + A_1^*F - F^*A_1 - A_1^*T_1A_1 = 0, \tag{4.2}$$

where

$$\langle \delta, \delta \rangle = (S_0^*\delta, \delta) - (\delta, S_0^*\delta),$$
$$F = (\{\delta, S_0^*\delta\}, \{\sigma^1, \tau^1\}) = (\delta, \sigma^1) + (S_0^*\delta, \tau^1),$$
$$T_1 = \langle \tau^1, \tau^1 \rangle_{B_1} = (\sigma^1, \tau^1) - (\tau^1, \sigma^1).$$

The form $[uv](x)$ may be written as

$$[uv](x) = \tilde{v}^*(x) B(x) \tilde{u}(x), \qquad u, v \in \mathfrak{D}(S_0^*),$$

where B is a continuous, invertible, skew-hermitian, $n \times n$ matrix-valued function on $\tilde{\imath}$, and then we have

$$\langle u, v \rangle = \tilde{v}^*(b) B(b) \tilde{u}(b) - \tilde{v}^*(a) B(a) \tilde{u}(a).$$

We remark that this relation remains valid if u, v are one-rowed matrices whose elements are in $\mathfrak{D}(S_0{}^*)$. If $M = \tilde{\delta}^*(a)\, B(a)$, $N = -\tilde{\delta}^*(b)\, B(b)$, then

$$\langle \delta, \delta \rangle = MB^{-1}(a)\, M^* - NB^{-1}(b)\, N^*, \qquad (4.3)$$

and (4.3) combined with (4.2) shows that condition (3.3) is equivalent to

$$MB^{-1}(a)\, M^* - NB^{-1}(b)\, N^* + A_1{}^*F - F^*A_1 - A_1{}^*T_1A_1 = 0.$$

We now consider (3.5). Let

$$C = \tilde{\gamma}^*(a)\, B(a), \qquad D = -\tilde{\gamma}^*(b)\, B(b),$$

$$G = (\{\gamma, S_0{}^*\gamma\}, \{\sigma^1, \tau^1\}) = (\gamma, \sigma^1) + (S_0{}^*\gamma, \tau^1),$$

$$\Phi_0 = (\varphi_1, ..., \varphi_s), \qquad \Phi_1 = (\varphi_{s+1}, ..., \varphi_m),$$

$$\Psi = (\psi_{s+1}, ..., \psi_m), \qquad Z = (\zeta_{m+1}, ..., \zeta_q), \qquad E = (E_{rk}).$$

Then (3.5) can be replaced by

$$\Psi = \Phi_1\{E + \tfrac{1}{2}[DB^{-1}(b)\, D^* - CB^{-1}(a)\, C^* + G^*A_2 - A_2{}^*G + A_2{}^*T_1A_2]\},$$

$$Z = \Phi_1\{DB^{-1}(b)\, N^* - CB^{-1}(a)\, M^* + G^*A_1 - A_2{}^*F + A_2{}^*T_1A_1\}.$$

We now turn to condition (3.2). Using the above notations (3.2) says that $\{\delta, S_0{}^*\delta\} + \{\tau^1, -\sigma^1\}A_1$ is a $1 \times (q - m)$ matrix whose components are linearly independent mod S_1. Suppose that these components are linearly dependent mod S_1. Then there exists a $(q - m) \times 1$ matrix d of complex constants, not all equal to 0, such that

$$[\{\delta, S_0{}^*\delta\} + \{\tau^1, -\sigma^1\}A_1]\, d \in S_1 = S_0 \cap B_1{}^\perp.$$

It follows that $\delta d + \tau^1 A_1 d \in \mathfrak{D}(S_0) \subset \mathfrak{D}(S_0{}^*)$, and since $\delta d \in \mathfrak{D}(S_0{}^*)$ we have $\tau^1 A_1 d \in \mathfrak{D}(S_0{}^*)$. Hence

$$\{\sigma^1, \tau^1\}\, A_1 d \in B_1 \cap B_2 = \{\{0, 0\}\}.$$

Since the components of $\{\sigma^1, \tau^1\}$ form a basis for B_1, this implies that $A_1 d = 0$, or $d^*A_1{}^* = 0$, and $\{\delta, S_0{}^*\delta\}d \in S_0 \cap B_1{}^\perp$. The fact that $\{\delta, S_0{}^*\delta\}d \in B_1{}^\perp$ implies that

$$d^*F^* = (\{\sigma^1, \tau^1\}, \{\delta, S_0{}^*\delta\}d) = 0.$$

The fact that $\{\delta, S_0{}^*\delta\}d \in S_0$ implies that $\tilde{\delta}(a)d = \tilde{\delta}(b)d = 0$ and hence that $d^*M = d^*N = 0$. Now let $(M : N : A_1{}^* : F^*)$ be the $(q - m) \times 2(q - m)$ matrix formed by setting the columns of N, $A_1{}^*$, F^* next to those of M

in the order indicated. Then we have just shown that if the components of $\{\delta, S_0{}^*\delta\} + \{\tau^1, -\sigma^1\}A_1$ are linearly dependent mod S_1,

$$\operatorname{rank}(M : N : A_1{}^* : F^*) < q - m.$$

The above argument can be traced in reverse, to show that the converse also holds. Hence condition (3.2) is equivalent to

$$\operatorname{rank}(M : N : A_1{}^* : F^*) = q - m.$$

THEOREM 4.1. *In the regular case of an nth order formally symmetric differential operator L as given above, let H be a self-adjoint extension of S in $\mathfrak{H}^2$ with* dim $H(0) = s$. *Let $\varphi_1 ,..., \varphi_s$ be an orthonormal basis for $H(0)$ and $\varphi_1 ,..., \varphi_s , \varphi_{s+1} ,..., \varphi_m$ be an orthonormal basis for $\mathfrak{H}_0$, where $m = \dim B_2$. Let $\{\sigma^1, \tau^1\}$ be a $1 \times (p - m)$ matrix whose entries form a basis for B_1 , $\dim B_1 = \dim(B \ominus B_2) = p - m$, and put $T_1 = \langle \tau^1, \tau^1 \rangle_{B_1} = (\sigma^1, \tau^1) - (\tau^1, \sigma^1)$. Let $q = p + n$ and put*

$$\Phi_0 = (\varphi_1 ,..., \varphi_s), \qquad \Phi_1 = (\varphi_{s+1} ,..., \varphi_m).$$

Then there exist matrices of complex constants M, N, C, D, F, G, A_1, A_2, E of order $(q - m) \times n$, $(q - m) \times n$, $(m - s) \times n$, $(m - s) \times n$, $(p - m) \times (q - m)$, $(p - m) \times (m - s)$, $(p - m) \times (q - m)$, $(p - m) \times (m - s)$, $(m - s) \times (m - s)$, respectively, such that

$$\operatorname{rank}(M : N : A_1{}^* : F^*) = q - m, \tag{4.4}$$

$$MB^{-1}(a)\, M^* - NB^{-1}(b)\, N^* + A_1{}^*F - F^*A_1 - A_1{}^*T_1A_1 = 0, \tag{4.5}$$

$$E = E^*, \tag{4.6}$$

and if

$$\Psi = \Phi_1\{E + \tfrac{1}{2}[DB^{-1}(b)\, D^* - CB^{-1}(a)\, C^* + G^*A_2 - A_2{}^*G + A_2{}^*T_1A_2]\},$$

$$Z = \Phi_1\{DB^{-1}(b)\, N^* - CB^{-1}(a)\, M^* + G^*A_1 - A_2{}^*F + A_2{}^*T_1A_1\}, \tag{4.7}$$

then

> *H is the set of all $\{h + \tau^1 c_1 , S_0{}^*h - \sigma^1 c_1 + \varphi\}$, where*
> *$h \in \mathfrak{D}(S_0{}^*)$, $\varphi \in \mathfrak{H}_0$, and c_1 is a $(p - m) \times 1$ matrix of*
> *complex constants such that* (4.8)

$$(h + \tau^1 c_1 , \Phi_0) = 0,$$

$$(M : N : -A_1{}^* : F^* + A_1{}^*T_1)\, h^1 + (h + \tau^1 c_1 , Z) = 0,$$

$$\varphi = \Phi_0 c + \Phi_1[(C : D : -A_2{}^* : G^* + A_2{}^*T_1)\, h^1 + (h + \tau^1 c_1 , \Psi)],$$

where c is an arbitrary $s \times 1$ matrix of complex constants,

$$h^1 = \begin{pmatrix} \tilde{h}(a) \\ \tilde{h}(b) \\ (\{h, S_0{}^*h\}, \{\sigma^1, \tau^1\}) \\ c_1 \end{pmatrix},$$

and

$$H_s(h + \tau^1 c_1) = Lh - \sigma^1 c_1 - \Phi_0(Lh - \sigma^1 c_1, \Phi_0)$$
$$+ \Phi_1[(C : D : -A_2{}^* : G^* + A_2{}^* T_1) h^1 + (h + \tau^1 c_1, \Psi)].$$
$$\tag{4.9}$$

Conversely, if $\varphi_1, \ldots, \varphi_s$, $\varphi_{s+1}, \ldots, \varphi_m$ is an orthonormal basis for $\mathfrak{H}_0$, the entries of $\{\sigma^1, \tau^1\}$ form a basis for B_1, and $M, N, C, D, F, G, A_1, A_2, E$ exist satisfying (4.4), (4.5) and (4.6), and Ψ, Z are defined by (4.7), then H defined by (4.8) is a self-adjoint extension of S such that $H(0) = \mathrm{span}\{\varphi_1, \ldots, \varphi_s\}$ and the operator part H_s of H is given by (4.9).

Proof. If H is a self-adjoint extension of S with $H(0) = \mathrm{span}\{\varphi_1, \ldots, \varphi_s\}$, then, as we have seen, (3.2)–(3.5) are equivalent to (4.4)–(4.7). One can readily verify that the descriptions of $\mathfrak{D}(H)$ and H_s in (3.6) and (3.7) coincide with the descriptions of $\mathfrak{D}(H)$ and H_s in (4.8) and (4.9).

To prove the converse, all we need to show is that for given matrices M, N, C, D, F, G, A_1 and A_2 there exist

$$\gamma_{s+1}, \ldots, \gamma_m, \delta_{m+1}, \ldots, \delta_q \in \mathfrak{D}(S_0{}^*), \qquad \tau_{s+1}, \ldots, \tau_q \in \mathfrak{R}(B_1),$$

such that (4.1) holds, and such that

$$\tilde{\delta}(a) = -B^{-1}(a) M^*, \qquad \tilde{\delta}(b) = B^{-1}(b) N^*,$$
$$(\{\delta, S_0{}^*\delta\}, \{\sigma^1, \tau^1\}) = F,$$
$$\tilde{\gamma}(a) = -B^{-1}(a) C^*, \qquad \tilde{\gamma}(b) = B^{-1}(b) D^*,$$
$$(\{\gamma, S_0{}^*\gamma\}, \{\sigma^1, \tau^1\}) = G.$$
$$\tag{4.10}$$

We let (4.1) define $\tau_{s+1}, \ldots, \tau_q \in \mathfrak{R}(B_1)$. The existence of $\gamma_{s+1}, \ldots, \gamma_m$, $\delta_{m+1}, \ldots, \delta_q \in \mathfrak{D}(S_0{}^*)$ satisfying (4.10) is established once it is shown that the linear mapping $\kappa: \mathfrak{D}(S_0{}^*) \to \mathbb{C}^{2n+p-m}$, defined by

$$\kappa(g) = \begin{pmatrix} \tilde{g}(a) \\ \tilde{g}(b) \\ (\{g, S_0{}^*g\}, \{\sigma^1, \tau^1\}) \end{pmatrix}, \qquad g \in \mathfrak{D}(S_0{}^*),$$

is surjective. To prove this let $d = (d_a : d_b : d_1) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^{p-m}$ be a $1 \times (2n + p - m)$ matrix such that $d\kappa(g) = 0$ for all $g \in \mathfrak{D}(S_0{}^*)$. Let $g \in \mathfrak{D}(S_0)$. Then $\tilde{g}(a) = \tilde{g}(b) = 0$, and hence

$$(g, \sigma) + (S_0{}^*g, \tau) = 0,$$

where $\{\sigma, \tau\} = \{\sigma^1, \tau^1\}d_1{}^*$. Since $g \in \mathfrak{D}(S_0)$ is arbitrary it follows that $\{\tau, -\sigma\} \in S_0{}^* \cap -B_1^{-1} = \{\{0, 0\}\}$. Hence $\{\sigma^1, \tau^1\}d_1{}^* = 0$, and thus $d_1 = 0$. So $d_a\tilde{g}(a) + d_b\tilde{g}(b) = 0$ for all $g \in \mathfrak{D}(S_0{}^*)$. For the given d_a, d_b one can find a $g \in \mathfrak{D}(S_0{}^*)$ such that $\tilde{g}(a) = d_a{}^*$, $\tilde{g}(b) = d_b{}^*$; see, e.g., [5, Proof of Theorem 1]. Consequently, $d_a = d_b = 0$, and thus $d = 0$, showing that κ is surjective.

If $B_1 = \{\{0, 0\}\}$ Theorem 4.1 reduces to [5, Theorem 1].

Remark. Theorem 4.1 holds almost verbatim if L is not an nth-order differential operator but a system of n first-order differential operators and $\mathfrak{H} = \mathfrak{L}_n{}^2(\iota)$, the Hilbert space of $n \times 1$ matrix-valued functions on ι, whose magnitudes are square integrable (see [5] for more details). The only change is that $\{\sigma^1, \tau^1\}$ now is an $n \times (p - m)$ matrix whose $(p - m)$ columns form a basis for B_1. Observe that in this case Φ_0, Φ_1 are $n \times s$, $n \times (m - s)$ matrices whose columns are given by $(\varphi_1, ..., \varphi_s)$ and $(\varphi_{s+1}, ..., \varphi_m)$, respectively. Self-adjoint *operator* extensions of such systems have been studied in a number of papers; see [10], for instance. Zimmerberg [14] deals with genuine subspaces (multivalued operators) associated with systems of first-order differential operators. His Theorem 3.1 with $\lambda = 0$ coincides with [5, Theorem 9], which is the system analog of Theorem 4.1 above, in the case $B_1 = \{\{0, 0\}\}$. The parameter mentioned in the title of [14] is the matrix c in [5, Theorem 9].

Problems with multipoint boundary conditions. Let L be as in Theorem 4.1, and let $c \in (a, b)$. We define $\{\sigma, \tau\} = (\{\sigma_1, \tau_1\}, ..., \{\sigma_n, \tau_n\})$ on $[a, b]$ as follows: $\tau = \sigma = O_1{}^n$ on $[a, c)$, $\tau_j \in C^n[c, b]$, $\sigma_j = -L\tau_j$ on $[c, b], j = 1, ..., n$, $\tilde{\tau}(c) = -B^{-1}(c)$, $\tilde{\tau}(b) = O_n{}^n$. Let B be the space spanned by the components of $\{\sigma, \tau\}$. Then $B_2 = \{\{0, 0\}\}$, $p = \dim B = \dim B_1 = n$, $T_1 = -B^{-1}(c)$, $S_0{}^\perp \cap B = \{\{0, 0\}\}$, and $(\{h, S_0{}^*h\}, \{\sigma, \tau\}) = -\tilde{h}(c)$. Let H be a self-adjoint extension of $S = S_0 \cap B^\perp$. Then H is necessarily an operator and can be described as follows: $\mathfrak{D}(H)$ is the set of all $v \in C^{n-1}(\iota \backslash \{c\})$, such that $v^{(n-1)}$ is absolutely continuous on each compact subset of the components of $\iota \backslash \{c\}$, $Lv \in \mathfrak{H}$, and

$$M\tilde{v}(a) + N\tilde{v}(b) + C\tilde{v}(c + 0) - D\tilde{v}(c - 0) = 0,$$

where the matrices $M, N, C (= -F^*B(c) + A_1{}^*), D = (-F^*B(c))$ satisfy

$$\text{rank}[M : N : C : D] = 2n,$$
$$MB^{-1}(a) M^* - NB^{-1}(b) N^* + CB^{-1}(c) C^* - DB^{-1}(c) D^* = 0,$$

and $Hv = Lv$ on $\iota\backslash\{c\}$. This example can easily be extended to cover the case of finitely many points $c_1, \ldots, c_k$ in (a, b). A recent paper on this subject is by Locker [11].

Problems involving measures. More generally, Theorems 3.3 and 4.1 can be applied to certain problems where the side conditions involve measures, which need not be concentrated at a finite number of points. For example, let us consider a formally symmetric ordinary differential operator L in the regular case, which is the situation obtaining in Theorem 4.1. Let S_0 be the minimal operator for L in $\mathfrak{H} = \mathfrak{L}^2(\iota)$, and define $S \subset S_0$ by

$$\mathfrak{D}(S) = \left\{ f \in \mathfrak{D}(S_0) \,\middle|\, \int_a^b f^{(j-1)} \, d\bar{\mu}_{ij} = 0, \, i = 1,\ldots,p_j, \, j = 1,\ldots,n \right\},$$

$$(4.11)$$

where the $\mu_{ij} \in BV(\iota)$, the set of all functions of bounded variation on ι. This S is clearly symmetric in $\mathfrak{H}$, and it may be described as $S = S_0 \cap B^\perp$ for an appropriate subspace $B \subset \mathfrak{H}^2$, as the following shows.

THEOREM 4.2.   *There exists a $1 \times p$ vector $\tau = (\tau_1, \ldots, \tau_p)$, $\tau_j \in \mathfrak{H}$, such that*

$$\mathfrak{D}(S) = \{ f \in \mathfrak{D}(S_0) \mid (S_0 f, \tau) = 0 \}. \tag{4.12}$$

Thus

$$S = S_0 \cap B^\perp, \qquad B = \text{span}\{\{0, \tau_1\}, \ldots, \{0, \tau_p\}\}.$$

Proof.   It is sufficient to show that, if $\mu \in BV(\iota)$ and j is fixed, $j = 1,\ldots,n$, then there exists a $\tau_j \in \mathfrak{H}$ such that

$$\left\{ f \in \mathfrak{D}(S_0) \,\middle|\, \int_a^b f^{(j-1)} \, d\bar{\mu} = 0 \right\} = \{ f \in \mathfrak{D}(S_0) \mid (S_0 f, \tau_j) = 0 \}.$$

Since $f \in \mathfrak{D}(S_0)$ implies that $f^{(j-1)} \in C(\iota) \cap BV(\iota)$, integration by parts yields

$$\int_a^b f^{(j-1)} \, d\bar{\mu} = f^{(j-1)}(b) \, \bar{\mu}(b) - f^{(j-1)}(a) \, \bar{\mu}(a) - \int_a^b f^{(j)} \bar{\mu} = -(f^{(j)}, \mu),$$

for $\tilde{f}(a) = \tilde{f}(b) = 0$. We show that for each $j = 1,\ldots, n$ there exists a $\tau_j \in \mathfrak{H}$ such that

$$(f^{(j)}, \mu) = (S_0 f, \tau_j), \qquad f \in \mathfrak{D}(S_0). \tag{4.13}$$

To do this we use the well-known right inverse R_0 of $S_0{}^*$, which is an integral operator

$$R_0 h(x) = \int_a^b k_0(x, y) \, h(y) \, dy = \int_a^x k_0(x, y) \, h(y) \, dy, \qquad h \in \mathfrak{H},$$

with a kernel given explicitly by

$$k_0(x, y) = s(x)[ss]^{-1} s^*(y), \quad a \leqslant y \leqslant x \leqslant b,$$
$$= 0, \quad a \leqslant x < y \leqslant b.$$

Here $s = (s_1, \ldots, s_n)$ is a basis for the solutions of $Lu = 0$, and $[ss] = [ss](x) = \tilde{s}^*(x) B(x) \tilde{s}(x)$ is independent of x and invertible. If $D^j = (d/dx)^j$, then

$$D^j R_0 h(x) = \int_a^b k_j(x, y) h(y) \, dy, \quad j = 0, 1, \ldots, n - 1,$$

$$D^n R_0 h(x) = \int_a^b k_n(x, y) h(y) \, dy + (h(x)/p_n(x)),$$

where

$$k_j(x, y) = D^j s(x)[ss]^{-1} s^*(y), \quad y \leqslant x,$$
$$= 0, \quad y > x.$$

We have $S_0^* R_0 h = h$ for all $h \in \mathfrak{H}$. Since $\nu(S_0) = \{0\}$ we see that S_0^{-1} exists as an operator. In fact, S_0^{-1} is R_0 restricted to $\mathfrak{R}(S_0)$, that is,

$$S_0 R_0 h = h, \quad h \in \mathfrak{R}(S_0), \tag{4.14}$$

$$R_0 S_0 f = f, \quad f \in \mathfrak{D}(S_0). \tag{4.15}$$

As to (4.14), note that if $f = R_0 h$ then $\tilde{f}(a) = 0$ and $\tilde{f}(b) = \tilde{s}(b)[ss]^{-1} (h, s)$. Since $\tilde{s}(b)$, $[ss]^{-1}$ are nonsingular, we see that $f \in \mathfrak{D}(S_0)$ if and only if $(h, s) = 0$, or $h \in [\nu(S_0^*)]^\perp$. Hence $\mathfrak{R}(S_0) = [\nu(S_0^*)]^\perp$ and (4.14) is true. If $f \in \mathfrak{D}(S_0)$ then $h = S_0 f$ is such that $g = R_0 h \in \mathfrak{D}(S_0)$ and $S_0 g = S_0 R_0 S_0 f = S_0 f$, from (4.14). But $\nu(S_0) = \{0\}$ implies $f = g$, or (4.15). The operators $R_j = D^j R_0$, $j = 0, 1, \ldots, n$, are defined on $\mathfrak{H}$ as bounded operators there, and so their adjoints R_j^* are bounded on $\mathfrak{H}$.

We return to the proof of (4.13). From (4.15) we have that

$$f^{(j)} = D^j f = D^j R_0 S_0 f = R_j S_0 f, \quad f \in \mathfrak{D}(S_0),$$

and thus

$$(f^{(j)}, \mu) = (R_j S_0 f, \mu) = (S_0 f, R_j^* \mu), \quad f \in \mathfrak{D}(S_0).$$

This is just (4.13) with $\tau_j = R_j^* \mu$, and so the proof of Theorem 4.2 is complete.

In order to apply Theorem 4.1 to the S described above it is necessary to identify the subspace B_2 of B, that is, identify those $\tau \in \mathrm{span}\{\tau_1, \ldots, \tau_p\}$ which are in $\mathfrak{D}(S_0^*)$, and to make sure of the nontriviality condition $S_0^\perp \cap B = \{\{0, 0\}\}$. In order to illustrate these ideas we present a simple

example. Let $L = iD$, and let S_0 be the minimal operator for L on $\mathfrak{H} = \mathfrak{L}^2[0, 1]$. Let $\tau \in BV[0, 1]$, and suppose τ is not a constant function. We define $S \subset S_0$ via

$$\mathfrak{D}(S) = \left\{ f \in \mathfrak{D}(S_0) \,\middle|\, \int_0^1 f \, d\bar{\tau} = 0 \right\} = \{ f \in \mathfrak{D}(S_0) \mid (S_0 f, \tau) = 0 \}.$$

Thus if $B = \{0\} \oplus \{\tau\}$, where $\{\tau\}$ is the subspace in $\mathfrak{H}$ spanned by τ, then $S = S_0 \cap B^\perp$. Also $B \cap S_0^\perp = \{\{0, 0\}\}$ since τ is not a constant. We have

$$S^* = S_0^* \dotplus (-B^{-1}) = \{\{h, ih'\} + \{c\tau, 0\} \mid h \in \mathfrak{D}(S_0^*), c \in \mathbb{C}\},$$

and for $l \in \mathbb{C}^+$ we have dim $M_0(l) = 1$, dim $B = 1$, and thus dim $M(l) = 2$. There are two cases according as (1) $\tau \in \mathfrak{D}(S_0^*)$, or (2) $\tau \notin \mathfrak{D}(S_0^*)$.

In case (1), $B = B_2$, $\mathfrak{H}_0 = S^*(0) = \{S_0^*\tau\} = \{i\tau'\}$. There are two subcases: (1i) $H(0) = \{0\}$, and H is an operator, or (1ii) $H(0) = \{i\tau'\}$, where H is a self-adjoint extension of S given by Theorem 4.1. In case (1i) $H = \{\{h, ih' + \alpha i\tau'\}\}$ where $h \in \mathfrak{D}(S_0^*)$ and

$$mh(0) + nh(1) + i(\bar{d}n - \bar{c}m)(h, i\tau') = 0,$$
$$\alpha = ch(0) + dh(1) + [e + (i/2)(|\,d\,|^2 - |\,c\,|^2)](h, i\tau'),$$
$$|\,m\,| = |\,n\,| \neq 0, \qquad e \in \mathbb{R}, \quad c, d \in \mathbb{C} \text{ arbitrary.}$$

In case (1ii) we have $H = \{\{h, ih' + \alpha i\tau'\}\}$ where $h \in \mathfrak{D}(S_0^*)$, $\alpha \in \mathbb{C}$ is arbitrary, and

$$(h, i\tau') = 0, \qquad mh(0) + nh(1) = 0, \qquad |\,m\,| = |\,n\,| \neq 0.$$

In case (2), $B = B_1$, $\mathfrak{H}_0 = S^*(0) = \{0\}$, and so S^* is an operator. All self-adjoint extensions H of S are operators, and have the form $H = \{\{h + \alpha\tau, ih'\}\}$, where $h \in \mathfrak{D}(S_0^*)$, $\alpha \in \mathbb{C}$, satisfy

$$m_1 h(0) + n_1 h(1) + a_1 (ih', \tau) + f_1 \alpha = 0,$$
$$m_2 h(0) + n_2 h(1) + a_2 (ih', \tau) + f_2 \alpha = 0,$$

with $m_j, n_j, a_j, f_j \in \mathbb{C}$ such that

$$\operatorname{rank} \begin{pmatrix} m_1 & n_1 & a_1 & f_1 \\ m_2 & n_2 & a_2 & f_2 \end{pmatrix} = 2,$$
$$m_j \bar{m}_k - n_j \bar{n}_k = i(a_j \bar{f}_k - f_j \bar{a}_k), \qquad j, k = 1, 2.$$

As we remarked just after the proof of Theorem 4.1, an analog of Theorem 4.1 is valid for systems. A number of authors have considered first-order systems together with Stieltjes boundary conditions; see e.g. [10, 13]. For

example, Krall [10] considered in $\mathfrak{H} = \mathfrak{L}_n^2[0, 1]$ the set $\mathfrak{D}$ of all $f \in \mathfrak{H}$ satisfying:

(a) For each f there is an $s \times 1$ matrix of constants ψ such that $f + H[Cf(0) + Df(1)] + H_1\psi$ is absolutely continuous;

(b) $Tf = -i(f + H[Cf(0) + Df(1)] + H_1\psi)' + Qf$ exists almost everywhere and is in $\mathfrak{H}$;

(c) $Af(0) + Bf(1) + \int_0^1 dK(t) f(t) = 0$,

$$\int_0^1 dK_1(t) \, f(t) = 0.$$

Here H, H_1 are $n \times (2n - m)$ and $n \times s$ matrix-valued functions of bounded variation; A, B are $m \times n$ matrices of constants $(m \leqslant 2n)$ with rank$(A : B) = m$; C, D are $(2n - m) \times n$ matrices such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is nonsingular; K, K_1 are matrix-valued functions of bounded variation of order $m \times n$ and $r \times n$, respectively; and Q is a continuous $n \times n$ matrix-valued function on $[0, 1]$. The map $f \in \mathfrak{D} \to Tf$ defines an operator T in $\mathfrak{H}$. If we let S_0 be the minimal operator for $L = -iD + Q$ on $\mathfrak{H} = \mathfrak{L}_n^2[0, 1]$, and we define $S \subset S_0$ via

$$\mathfrak{D}(S) = \left\{ f \in \mathfrak{D}(S_0) \,\middle|\, \int_0^1 (dK)f = \int_0^1 (dK_1)f = 0 \right\},$$

then clearly $S \subset T$, and we must have $T^* \subset S^*$. If we define the $n \times (m + r)$ matrix-valued function μ by $\mu = (K^* : K_1^*)$, then we have

$$\mathfrak{D}(S) = \left\{ f \in \mathfrak{D}(S_0) \,\middle|\, \int_0^1 (d\mu^*)f = 0 \right\}.$$

An integration by parts yields

$$\int_0^1 (d\mu^*)f = -\int_0^1 \mu^*f' = -(f', \mu) = -i(S_0f - Qf, \mu).$$

Therefore,

$$\mathfrak{D}(S) = \{ f \in \mathfrak{D}(S_0) \mid (f, \sigma) + (S_0f, \tau) = 0 \},$$

where $\sigma = -Q^*\mu$, $\tau = \mu$. If $\mu_1, \ldots, \mu_{m+r}$ are the columns of μ, and we define B as the span of $\{-Q^*\mu_1, \mu_1\}, \ldots, \{-Q^*\mu_{m+r}, \mu_{m+r}\}$, then clearly

$S = S_0 \cap B^\perp$, and we have the situation to which the system analog of Theorem 4.1 applies. If T is self-adjoint it must be among those operator extensions H given in Theorem 4.1.

The singular case. There are some problems involving measures in the general singular case of an open interval $\iota = (a, b)$ (possibly infinite) which can be dealt with in the same manner as in Theorem 4.2. Suppose S_0 is the minimal operator for an nth-order differential operator L in $\mathfrak{H} = \mathfrak{L}^2(\iota)$, as indicated at the beginning of Section 4. Let now D_j, $j = 1,..., n$, denote the maximal operator for $D^j = (d/dx)^j$ in $\mathfrak{H}$. Thus $\mathfrak{D}(D_j)$ is the set of all $f \in C^{j-1}(\iota) \cap \mathfrak{H}$, such that $f^{(j-1)}$ is locally absolutely continuous on ι, $f^{(j)} \in \mathfrak{H}$, and $D_j f = f^{(j)}$ for $f \in \mathfrak{D}(D_j)$. Suppose $\mathfrak{D}(S_0^*) \subset \mathfrak{D}(D_n)$ and let $S \subset S_0$ be defined by

$$\mathfrak{D}(S) = \left\{ f \in \mathfrak{D}(S_0) \,\middle|\, \int_a^b f^{(j-1)} \, d\bar{\mu}_{ij} = 0, \ i = 1,..., p_j, \ j = 1,..., n \right\}, \quad (4.16)$$

where now $\mu_{ij} \in BV(\iota) \cap \mathfrak{H}$. Then the symmetric operator S may be described as $S = S_0 \cap B^\perp$ for an appropriate $B \subset \mathfrak{H}^2$. This will be indicated in Theorem 4.3 below.

We remark that a sufficient condition for the inclusion $\mathfrak{D}(S_0^*) \subset \mathfrak{D}(D_n)$ is that there exist constants $c, d > 0$ such that $|p_n(x)| \geqslant c$ and $|p_j(x)| \leqslant d$, $j = 0, 1,..., n - 1$, for $x \in \iota$. In fact, in [8, Lemma 2.1] it is shown that under these conditions $\mathfrak{D}(S_0^*) \subset \mathfrak{D}(D_j)$ for $j = 1,..., n$. In particular, $\mathfrak{D}(D_n) \subset \mathfrak{D}(D_j)$ for $j = 1,..., n - 1$, and $\mathfrak{D}(S_0^*) \subset \mathfrak{D}(D_n)$ implies $\mathfrak{D}(S_0^*) \subset \mathfrak{D}(D_j)$ for $j = 1,..., n$. The map $\{f, S_0^* f\} \to \{f, D_j f\}$ of S_0^* into D_j is clearly closed, and thus the closed graph theorem implies that it is bounded. Therefore there exist constants $c_j > 0$ such that

$$\|f\|^2 + \|D_j f\|^2 \leqslant c_j(\|f\|^2 + \|S_0^* f\|^2), \quad f \in \mathfrak{D}(S_0^*), \quad j = 1,..., n. \quad (4.17)$$

In particular these inequalities are valid for $f \in \mathfrak{D}(S_0)$. Under the assumptions

$$\mathfrak{D}(S_0^*) \subset \mathfrak{D}(D_n), \quad \mu_{ij} \in BV(\iota) \cap \mathfrak{H}, \quad (4.18)$$

we see that the integrals involved in (4.16) can be given a meaning as follows. For $f \in C_0^n(\iota)$ we have

$$\int_a^b f^{(j-1)} \, d\bar{\mu}_{ij} = -(f^{(j)}, \mu_{ij}),$$

and for an arbitrary $f \in \mathfrak{D}(S_0)$ there exists a sequence $f_k \in C_0^n(\iota)$ such that $\{f_k, S_0 f_k\} \to \{f, S_0 f\}$, and from (4.17) we see that $\{f_k, D_j f_k\} \to \{f, D_j f\}$. Hence,

$$\int_a^b f_k^{(j-1)} \, d\bar{\mu}_{ij} \to -(f^{(j)}, \mu_{ij}),$$

and, since the limit is independent of the sequence chosen, we can define $\int_a^b f^{(j-1)} \, d\bar{\mu}_{ij}$ as $-(f^{(j)}, \mu_{ij})$.

THEOREM 4.3. Let S_0, S be as above with (4.18) assumed. Then there exists a $1 \times p$ vector $\{\sigma, \tau\} = \{\{\sigma_1, \tau_1\}, ..., \{\sigma_p, \tau_p\}\}$, $\{\sigma_j, \tau_j\} \in \mathfrak{H}^2$, such that

$$\mathfrak{D}(S) = \{f \in \mathfrak{D}(S_0) \mid (\{f, S_0 f\}, \{\sigma, \tau\}) = 0\},$$

and hence

$$S = S_0 \cap B^\perp, \qquad B = \mathrm{span}\{\{\sigma_1, \tau_1\}, ..., \{\sigma_p, \tau_p\}\}.$$

Proof. Here we use the existence of a right inverse $G(l)$ of $S_0^* - lI$, $l \in \mathbb{C}_0$; see [1]. It has the properties:

$$(S_0^* - lI) \, G(l)h = h, \quad h \in \mathfrak{H}; \quad \| \, G(l)\| \leqslant 1/|\, \mathrm{Im} \, l\,|; \quad (G(l))^* = G(\bar{l}).$$

Now S_0 being symmetric implies that $(S_0 - lI)^{-1}$ exists as an operator defined on $\mathfrak{R}(S_0 - lI) = [\nu(S_0^* - \bar{l}I)]^\perp$, and it is easy to see that $(S_0 - lI)^{-1}$ is just $G(l)$ restricted to $\mathfrak{R}(S_0 - lI)$, that is,

$$(S_0 - lI) \, G(l)h = h, \qquad h \in \mathfrak{R}(S_0 - lI), \tag{4.19}$$

$$G(l)(S_0 - lI)f = f, \qquad f \in \mathfrak{D}(S_0). \tag{4.20}$$

As to (4.20), since $(S_0^* - lI) \, G(l)(S_0 - lI)f = (S_0 - lI)f = (S_0^* - lI)f$, we have $G(l)(S_0 - lI)f = f + \chi(l)$, where $\chi(l) \in \nu(S_0^* - lI)$. But then

$$\begin{aligned}
(f, \chi(l)) + \| \, \chi(l)\|^2 &= (G(l)(S_0 - lI)f, \chi(l)) \\
&= (f, (S_0^* - \bar{l}I) \, G(\bar{l}) \chi(l)) \\
&= (f, \chi(l))
\end{aligned}$$

implies that $\chi(l) = 0$, and thus (4.20) is true. This shows that $G(l)$ restricted to $\mathfrak{R}(S_0 - lI)$ has a range in $\mathfrak{D}(S_0)$, and hence (4.19) follows.

For any $h \in \mathfrak{H}$, $G(l)h \in \mathfrak{D}(S_0^*) \subset \mathfrak{D}(D_j)$ for $j = 1, ..., n$, and so we define $R_j(l)$ by

$$R_j(l) = D_j G(l), \quad j = 1, ..., n, \qquad R_0(l) = G(l).$$

Now $R_j(l)$ is defined on all of $\mathfrak{H}$, and, since it is closed, it follows from the closed graph theorem that each $R_j(l)$ is bounded. Thus the adjoint operator $R_j^*(l)$ is defined on $\mathfrak{H}$ as a bounded operator. For $f \in \mathfrak{D}(S_0)$ we have by (4.20)

$$f^{(j)} = D_j G(l)(S_0 - lI)f = R_j(l)(S_0 - lI)f,$$

and hence for $\mu \in BV(\iota) \cap \mathfrak{H}$ we see that

$$
\begin{aligned}
\int_a^b f^{(j-1)} \, d\bar{\mu} &= -(f^{(j)}, \mu) = -(R_j(l)(S_0 - lI)f, \mu) \\
&= -((S_0 - lI)f, R_j{}^*(l)\mu) \\
&= (f, \sigma) + (S_0 f, \tau),
\end{aligned}
$$

where $\sigma = \bar{l} R_j{}^*(l)\mu$, $\tau = -R_j{}^*(l)\mu$. This implies the statement of the theorem with $p = \sum_{j=1}^{n} p_j$.

Clearly, Theorem 3.3 can now be applied to the S of Theorem 4.3.

5. Self-Adjoint Extensions in Larger Spaces

We now return to the general situation considered in Section 2 and at the beginning of Section 3. Thus S_0 is a densely defined symmetric operator in a Hilbert space $\mathfrak{H}$, B is a subspace in $\mathfrak{H}^2$ with $\dim B = p < \infty$, $S = S_0 \cap B^\perp$, $S_0{}^\perp \cap B = \{\{0, 0\}\}$, $S^* = S_0{}^* \dotplus -B^{-1}$ (a direct algebraic sum), and

$$
\dim M(l) = \dim M_0(l) + \dim B, \qquad l \in \mathbb{C}_0.
$$

If $\dim M_0(l) = \omega^\pm$, $\dim M(l) = q^\pm$, $l \in \mathbb{C}^\pm$, then

$$
q^\pm = \omega^\pm + p, \qquad l \in \mathbb{C}^\pm.
$$

We do not assume that $\omega^+ = \omega^-$, and so S need not have any self-adjoint extensions in $\mathfrak{H}^2$. However, S always has self-adjoint subspace extensions in some larger space $\mathfrak{K}^2$, $\mathfrak{H} \subset \mathfrak{K}$. Let $H = H_s \oplus H_\infty$ be a self-adjoint subspace in $\mathfrak{K}$ satisfying $S \subset H$. Then H_s is a self-adjoint operator in $H(0)^\perp = \mathfrak{K} \ominus H(0)$ with a spectral resolution

$$
H_s = \int_{-\infty}^{\infty} \lambda \, dE_s(\lambda),
$$

where $E_s = \{E_s(\lambda) \mid \lambda \in \mathbb{R}\}$ is the unique suitably normalized *spectral family* of projections in $H(0)^\perp$ for H_s. The *resolvent* R_H of H is an operator-valued function defined for $l \in \mathbb{C}_0$ by $R_H(l) = (H - lI)^{-1}$. The operator $R_H(l)$ is defined on all of $\mathfrak{K}$ and satisfies:

$$
\begin{aligned}
\| R_H(l) \| &\leqslant 1/|\operatorname{Im} l|, \\
(R_H(l))^* &= R_H(\bar{l}), \\
R_H(l) - R_H(m) &= (l - m)\, R_H(l)\, R_H(m).
\end{aligned}
$$

Moreover, R_H is analytic in the uniform topology, and

$$R_H(l) = \int_{-\infty}^{\infty} \frac{dE(\lambda)}{\lambda - l}, \qquad l \in \mathbb{C}_0, \tag{5.1}$$

where

$$\begin{aligned} E(\lambda)f &= E_s(\lambda)f, \qquad f \in H(0)^{\perp}, \\ &= 0, \qquad\quad f \in H(0). \end{aligned}$$

Thus $E(\lambda) = E_s(\lambda) \oplus O_0$, where O_0 is the zero operator on $H(0)$. The family $E = \{E(\lambda) \mid \lambda \in \mathbb{R}\}$ is called the *spectral family* of projections in $\mathfrak{K}$ for the subspace H.

Let P be the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{H}$, and put

$$R(l)f = PR_H(l)f, \qquad f \in \mathfrak{H}, \quad l \in \mathbb{C}_0.$$

Then R is called a *generalized resolvent* of S corresponding to H. The operator $R(l)$ is defined on all of $\mathfrak{H}$ and satisfies:

    (i) $\| R(l) \| \leqslant 1/|\operatorname{Im} l|$,

    (ii) $(R(l))^* = R(\bar{l})$,

    (iii) $\operatorname{Im}(R(l)f, f)/\operatorname{Im} l \geqslant \| R(l)f \|^2$,   (5.2)

    (iv) $S \subset T(l) \subset S^*$,   where $T(l) = \{\{R(l)f, lR(l)f + f\} \mid f \in \mathfrak{H}\}$,

and R is analytic in the uniform topology. For $f \in \mathfrak{H}$ the relation (5.1) implies that

$$(R(l)f, f) = (R_H(l)f, f) = \int_{-\infty}^{\infty} \frac{d(F(\lambda)f, f)}{\lambda - l}, \tag{5.3}$$

where

$$F(\lambda)f = PE(\lambda)f, \qquad f \in \mathfrak{H}.$$

The family $F = \{F(\lambda) \mid \lambda \in \mathbb{R}\}$ is a *generalized spectral family* for S corresponding to H. An inversion of (5.3) yields

$$(F(\Delta)f, f) = \lim_{\epsilon \to +0} \frac{1}{\pi} \int_{\Delta} \operatorname{Im}(R(\nu + i\epsilon)f, f) \, d\nu, \quad f \in \mathfrak{H}, \tag{5.4}$$

where

$$\Delta = \{\nu \mid \mu < \nu \leqslant \lambda\}, \qquad F(\Delta) = F(\lambda) - F(\mu),$$

and λ, μ are continuity points of F. See [5, Section 4] and [7] for more details concerning R_H, R, E, F. Note that in [5] a generalized spectral family was defined on the smaller space $\mathfrak{H} \cap H(0)^{\perp} = \mathfrak{H} \ominus PH(0)$. Clearly, for $f \in \mathfrak{K}$, $E(\lambda)f \to E(\infty)f$, as $\lambda \to +\infty$, where $E(\infty) = P_s$, the orthogonal projection

of $\mathfrak{H}$ onto $H(0)^{\perp}$. This implies that for $f \in \mathfrak{H}$, $F(\lambda)f \to F(\infty)f$, where $F(\infty) = PP_s$.

For any given symmetric S in $\mathfrak{H}^2$ and self-adjoint extension H in $\mathfrak{R}^2$, $\mathfrak{H} \subset \mathfrak{R}$, the fundamental problem is to compute the $F(\lambda)$. We show how this can be done for the case when S_0 is an ordinary differential operator. This leads directly to an eigenfunction expansion result. Two proofs are presented in the following two sections. One, following the general scheme in [5], depends upon an analysis of the nature of the generalized resolvent, and the use of (5.4). The other approach follows ideas in [6] and makes use of the Riesz representation theorem to yield the $F(\lambda)$.

6. Eigenfunction Expansions

In this section we consider the general case of a formally symmetric ordinary differential operator L of order n on an arbitrary open interval $\iota = (a, b)$, as described at the beginning of Section 4. Thus S_0, $S_0{}^*$ are the minimal and maximal operators for L in $\mathfrak{H} = \mathfrak{L}^2(\iota)$, B, S are as given in Section 5, and H is a self-adjoint extension of S in $\mathfrak{R}^2$, $\mathfrak{H} \subset \mathfrak{R}$, with generalized resolvent R and generalized spectral family F. We first show that $R(l)$ is an integral operator of Carleman type, and determine the smoothness properties of its kernel. To do this we use the existence of a right inverse $G(l)$ of $S_0{}^* - lI$ which was described in [5, Section 5]. It is an integral operator of Carleman type

$$G(l) f(x) = \int_a^b G(x, y, l) f(y)\, dy, \qquad l \in \mathbb{C}_0, \quad f \in \mathfrak{H},$$

with the properties:

$$\| G(l) \| \leqslant 1/|\operatorname{Im} l|,$$
$$(G(l))^* = G(\bar{l}),$$
$$(S_0{}^* - lI)\, G(l)f = f,$$

and G is analytic in the uniform topology.

Since for all $f \in \mathfrak{H}$,

$$\{R(l)f,\, lR(l)f + f\} \in S^*, \qquad \{G(l)f,\, lG(l)f + f\} \in S_0{}^* \subset S^*,$$

we see that if $A(l) = R(l) - G(l)$, then $\{A(l)f,\, lA(l)f\} \in M(l)$, and $A(l)f \in \nu(S^* - lI) = \mathfrak{D}(M(l))$. Let

$$\alpha(l) = (\alpha_1(l), ..., \alpha_{q\pm}(l)), \qquad l \in \mathbb{C}^{\pm},$$

where the components of $\alpha(l)$ form an orthonormal basis for $\nu(S^* - lI)$, $l \in \mathbb{C}^{\pm}$. Then,

$$A(l)f = \alpha(l)(A(l)f, \alpha(l)) = \alpha(l)(f, A(\bar{l})\,\alpha(l))$$
$$= \alpha(l)\,a^{\pm}(l)(f, \alpha(\bar{l})),$$

where

$$a^{\pm}(l) = (A(l)\,\alpha(\bar{l}), \alpha(l)), \qquad l \in \mathbb{C}^{\pm}.$$

Thus $A(l)$ is an integral operator

$$A(l)\,f(x) = \int_a^b A(x, y, l)\,f(y)\,dy, \qquad f \in \mathfrak{H}, \quad l \in \mathbb{C}_0,$$

where

$$A(x, y, l) = \alpha(x, l)\,a^{\pm}(l)\,\alpha^*(y, \bar{l}), \qquad l \in \mathbb{C}^{\pm},$$
$$(a^+(l))^* = a^-(\bar{l}), \qquad l \in \mathbb{C}^+.$$

Consequently $R(l) = G(l) + A(l)$ is an integral operator

$$R(l)\,f(x) = \int_a^b K(x, y, l)\,f(y)\,dy, \qquad f \in \mathfrak{H}, \quad l \in \mathbb{C}_0,$$

with kernel

$$K(x, y, l) = G(x, y, l) + A(x, y, l).$$

If

$$N_B(l) = \{G(l)(\sigma + l\tau) + \tau \mid \{\sigma, \tau\} \in B\},$$

then we claim that

$$\nu(S^* - lI) = \nu(S_0^* - lI) \dotplus N_B(l), \tag{6.1}$$

where the algebraic sum is direct. Indeed, if $u = G(l)(\sigma + l\tau)$, then $S_0^* u = lu + \sigma + l\tau$, and

$$\{u, S_0^* u\} + \{\tau, -\sigma\} = \{u + \tau, l(u + \tau)\} \in S^*,$$

which shows that $N_B(l) \subset \nu(S^* - lI)$, and consequently $\nu(S_0^* - lI) \dotplus N_B(l) \subset \nu(S^* - lI)$. The linear map $\kappa \colon B \to N_B(l)$ given by $\kappa(\{\sigma, \tau\}) = G(l)(\sigma + l\tau) + \tau$ is bijective. For if $G(l)(\sigma + l\tau) + \tau = 0$ then $\tau \in \mathfrak{D}(S_0^*)$, $S_0^* \tau = -\sigma$, which implies $\{\tau, -\sigma\} \in S_0^* \cap -B^{-1} = \{\{0, 0\}\}$. The same argument shows that the sum is direct. Since $\dim N_B(l) = \dim B = p$, $\dim \nu(S_0^* - lI) = \omega^{\pm}$, we have $\dim[\nu(S_0^* - lI) \dotplus N_B(l)] = \omega^{\pm} + p = \dim \nu(S^* - lI)$, resulting in (6.1).

Let $\theta^1(l)$ be a $1 \times \omega^{\pm}$ matrix, $l \in \mathbb{C}^{\pm}$, whose elements form a basis for $\nu(S_0^* - lI)$, and let

$$\theta^2(l) = G(l)(\sigma + l\tau) + \tau,$$

where now $\{\sigma, \tau\}$ is a $1 \times p$ matrix whose elements form a basis for B. Then the elements of

$$\theta(l) = (\theta^1(l) : \theta^2(l)), \qquad l \in \mathbb{C}^{\pm},$$

constitute a basis for $\nu(S^* - lI)$. We note that $\theta^1(l)$, $v(l) = G(l)(\sigma + l\tau)$ satisfy the differential equations

$$(L - l)\,\theta^1(l) = 0, \qquad (L - l)\,v(l) = \sigma + l\tau,$$

and we now proceed to express these solutions in terms of an entire basis for the solutions of these equations. Let c be fixed, $a < c < b$, and let

$$s^1(x, l) = (s_1(x, l),..., s_n(x, l)), \qquad u(x, l) = (u_{n+1}(x, l),..., u_{n+p}(x, l)),$$

be the unique matrices satisfying

$$\begin{aligned} (L - l)\,s^1(l) &= 0, & \tilde{s}^1(c, l) &= I_n, & l \in \mathbb{C}, \\ (L - l)\,u(l) &= \sigma + l\tau, & \tilde{u}(c, l) &= O_n^p, & l \in \mathbb{C}. \end{aligned} \qquad (6.2)$$

We denote by $s(l)$ the $1 \times (n + p)$ matrix given by

$$s(l) = (s^1(l) : s^2(l)), \qquad s^2(l) = u(l) + \tau. \qquad (6.3)$$

If $w(x, l) = (s^1(x, l) : u(x, l))$, then the matrix-valued function $\tilde{w}$ is continuous on $\iota \times \mathbb{C}$, and, for each fixed $x \in \iota$, it is entire. That $\tilde{s}^1$ has these properties follows from the existence theorem. Now u may be expressed in terms of s^1 via

$$u(x, l) = s^1(x, l)\,\mathscr{S}^{-1} \int_c^x (s^1(y, \bar{l}))^* [\sigma(y) + l\tau(y)]\,dy,$$

where $\mathscr{S}$ is the invertible matrix

$$\mathscr{S} = [s^1(l)\,s^1(\bar{l})](x),$$

which is independent of x and l. This representation shows that $\tilde{u}$ also has the properties stated. It is now clear that there exist matrices $d_1(l)$ and $d_2(l)$ such that

$$\theta^1(l) = s^1(l)\,d_1(l), \qquad \theta^2(l) = s^1(l)\,d_2(l) + s^2(l),$$

or $\theta(l) = s(l)\,d(l)$ for some matrix $d(l)$.

Returning now to the integral operator $A(l) = R(l) - G(l)$, we may express its kernel in terms of $s(l)$ as follows:

$$A(x, y, l) = s(x, l) a(l) s^*(y, \bar{l}), \qquad l \in \mathbb{C}_0,$$

where $a(l)$ is an $(n + p) \times (n + p)$ matrix. The kernel of $G(l)$ may be written as

$$G(x, y, l) = K_0(x, y, l) + G_1(x, y, l),$$

where

$$K_0(x, y, l) = \bar{K}_0(y, x, \bar{l}) = \tfrac{1}{2} s^1(x, l) \, \mathscr{S}^{-1}(s^1(y, \bar{l}))^*, \qquad x \geqslant y.$$
$$G_1(x, y, l) = s^1(x, l) g(l) (s^1(y, \bar{l}))^*,$$

for some $n \times n$ matrix $g(l)$. Thus the kernel of $R(l)$ may be represented as

$$K(x, y, l) = K_0(x, y, l) + K_1(x, y, l),$$

where

$$K_1(x, y, l) = G_1(x, y, l) + A(x, y, l) = s(x, l) \Psi(l) s^*(y, \bar{l}),$$

and $\Psi(l)$ is an $(n + p) \times (n + p)$ matrix given by

$$\Psi(l) = a(l) + \begin{pmatrix} g(l) & O_n^p \\ O_p^n & O_p^p \end{pmatrix}.$$

THEOREM 6.1. *The matrix-valued function Ψ has the following properties:*

(a) *Ψ is analytic for $l \in \mathbb{C}_0$,*

(b) *$\Psi^*(l) = \Psi(\bar{l})$,*

(c) *Im $\Psi(l)$/Im $l \geqslant 0$, where Im $\Psi = (\Psi - \Psi^*)/2i$.*

In order to prove Theorem 6.1 we require the following lemma, which provides for a weak approximation to $\{\sigma, \tau\}$.

LEMMA. *Given the $1 \times p$ matrix $\{\sigma, \tau\}$, whose elements form a basis for B, there exists a matrix $\{\varphi^0, L\varphi^0\} = (\{\varphi_1^0, L\varphi_1^0\},..., \{\varphi_p^0, L\varphi_p^0\})$ such that*

$$\varphi^0 = (\varphi_1^0,..., \varphi_p^0) \in C_0^n(\iota), \tag{6.4}$$

and

$$(\{\sigma, \tau\}, \{\varphi^0, L\varphi^0\}) = (\sigma, \varphi^0) + (\tau, L\varphi^0) = I_p . \tag{6.5}$$

Proof. Let $\{f_j , S_0 f_j\} = Q_0 \{\sigma_j , \tau_j\}$, where Q_0 is the orthogonal projection of $\mathfrak{H}^2$ onto S_0 , and let $\{f, S_0 f\}$ be the $1 \times p$ matrix with components $\{f_j , S_0 f_j\}, j = 1,..., p$. The projection Q_0 is a bijective map of B onto $Q_0 B$,

since if $Q_0\{\sigma_0, \tau_0\} = \{0, 0\}$ then $\{\sigma_0, \tau_0\} \in B \cap S_0^{\perp} = \{\{0, 0\}\}$. Thus the components of $\{f, S_0 f\}$ form a basis for $Q_0 B$, and this implies that

$$(\{f, S_0 f\}, \{f, S_0 f\}) = (f, f) + (S_0 f, S_0 f)$$

is nonsingular. Now the set

$$\{\{\varphi, L\varphi\} \mid \varphi \in C_0^n(\iota)\}$$

is dense in S_0, and the determinant is a continuous function of its elements. Hence we can find a matrix-valued function $\{\varphi^1, L\varphi^1\}$, $\varphi^1 \in C_0^n(\iota)$, so close to $\{f, S_0 f\}$ (in the sense that $\| f - \varphi^1 \|^2 + \| S_0 f - L\varphi^1 \|^2$ is small) that the matrix

$$A_1 = (f, \varphi^1) + (S_0 f, L\varphi^1) = (\sigma, \varphi^1) + (\tau, L\varphi^1)$$

is nonsingular. The matrix $\{\varphi^0, L\varphi^0\} = \{\varphi^1, L\varphi^1\}(A_1^{-1})^*$ satisfies the conditions (6.4) and (6.5).

The principal application of the lemma is given by the following corollary, which follows directly from an application of Green's formula.

COROLLARY. *If $s(l)$ is given by* (6.2), (6.3), *and* $\{\varphi^0, L\varphi^0\}$ *satisfies* (6.4), (6.5), *then*

$$(s(l), (L - \bar{l})\varphi^0) = (O_p{}^n : I_p), \tag{6.6}$$

that is,

$$(s^1(l), (L - \bar{l})\varphi^0) = O_p{}^n, \qquad (s^2(l), (L - \bar{l})\varphi^0) = I_p. \tag{6.7}$$

Proof of Theorem 6.1. Let J be a finite subinterval of ι containing c in its interior, and let $h \in C_0^n(J)$ be such that $0 \leqslant h(x) \leqslant 1$, $h(c) = 1$. For such h we put

$$\tilde{h}^+ = (h, (-1)h', \ldots, (-1)^{n-1} h^{(n-1)}).$$

This $\tilde{h}^+$ is a "formal adjoint" to $\tilde{h}$ in the sense that $(\tilde{f}, h) = (f, \tilde{h}^+)$ for all $f \in C^n(\iota)$. We let

$$s_0(l) = (s_0{}^1(l) : s_0{}^2(l)),$$

where

$$s_0{}^1(l) = \tilde{h}^+ - (L - \bar{l})\,\varphi^0(\tilde{h}^+, \tau), \qquad s_0{}^2(l) = (L - \bar{l})\,\varphi^0\gamma,$$

$$\gamma = \int_a^b h(x)\,dx > 0.$$

Thus

$$s_0(l) = \eta + (L - \bar{l})\chi^0, \tag{6.8}$$

where

$$\eta = (\tilde{h}^+ : O_1{}^p), \qquad \chi^0 = (-\varphi^0(\tilde{h}^+, \tau): \varphi^0\gamma).$$

From the definition of $s_0(l)$ and the properties (6.2), (6.7), it follows that

$$(s(l), s_0(l)) = (\Sigma(l), h), \tag{6.9}$$

where $\Sigma(l)$ is the $(n + p) \times (n + p)$ matrix given by

$$\Sigma(l) = \begin{pmatrix} \tilde{s}^1(l) & \tilde{u}(l) \\ O_p{}^n & I_p \end{pmatrix}.$$

It is clear that Σ is continuous on $\iota \times \mathbb{C}$, for fixed $x \in \iota$ it is entire, and $\Sigma(c, l) = I_{n+p}$. From these properties it follows that

$$\gamma^{-1}(\Sigma(l), h) \to I_{n+p} = \gamma^{-1}(\Sigma(c, l), h),$$

as the length $| J |$ of J tends to zero, uniformly for l in any compact subset of $\mathbb{C}$. It follows that if $| J |$ is small enough, then $(s(l), s_0(l)) = (\Sigma(l), h)$ is invertible for any $h \in C_0{}^n(J)$ of the type mentioned above. We now assume J has been chosen in this way.

For $f \in C_0(\iota)$ we define

$$R_0(l) f(x) = \int_a^b K_0(x, y, l) f(y) \, dy,$$

$$R_1(l) f(x) = \int_a^b K_1(x, y, l) f(y) \, dy = s(x, l) \, \Psi(l)(f, s(\bar{l})).$$

If $r(l) = (R(l)f, g)$, $r_0(l) = (R_0(l)f, g)$, $r_1(l) = (R_1(l)f, g) = r(l) - r_0(l)$, for fixed $f, g \in C_0(\iota)$, then r is analytic on $\mathbb{C}_0$, r_0 is entire, and thus r_1 is analytic on $\mathbb{C}_0$. The equality

$$(R_1(l) s_0(\bar{l}), s_0(l)) = (\Sigma(l), h) \, \Psi(l)(h, \Sigma(\bar{l})), \tag{6.10}$$

which follows from (6.9), shows that

$$\Psi(l) = (\Sigma(l), h)^{-1}(R_1(l) s_0(\bar{l}), s_0(l))(h, \Sigma(\bar{l}))^{-1}. \tag{6.11}$$

From (6.8) it is clear that $(R_1(l) s_0(\bar{l}), s_0(l))$ is analytic on $\mathbb{C}_0$, and since $(\Sigma(l), h)^{-1}$, $(h, \Sigma(\bar{l}))^{-1}$ are analytic, we see that Ψ is analytic on $\mathbb{C}_0$, proving (a). The equalities

$$(R_1(l) s_0(\bar{l}), s_0(l))^* = (s_0(l), R_1(l) s_0(\bar{l})) = (R_1(\bar{l}) s_0(l), s_0(\bar{l})),$$

and (6.11), now show that $\Psi^*(l) = \Psi(\bar{l})$, which is (b).

We turn to the proof of (c). This depends upon the following inequalities for matrices:

$$0 \leqslant (R(l) \, s_0(\bar{l}) - \chi^0, R(l) \, s_0(\bar{l}) - \chi^0)$$

$$= (R(l) \, s_0(\bar{l}), R(l) \, s_0(\bar{l})) - (R(l) \, s_0(\bar{l}), \chi^0) - (\chi^0, R(l) \, s_0(\bar{l})) + (\chi^0, \chi^0)$$

$$\leqslant \left(\frac{[R(l) - R(\bar{l})]}{l - \bar{l}} \, s_0(\bar{l}), s_0(\bar{l}) \right) - (R(l) \, s_0(\bar{l}), \chi^0) - (\chi^0, R(l) \, s_0(\bar{l})) + (\chi^0, \chi^0),$$

$$\tag{6.12}$$

where the latter inequality is a consequence of (5.2)(iii). An easy computation shows that

$$\left(\frac{[R_1(l) - R_1(\bar{l})]}{l - \bar{l}} \, s_0(\bar{l}), s_0(\bar{l}) \right) - (R_1(l) \, s_0(\bar{l}), \chi^0) - (\chi^0, R_1(l) \, s_0(\bar{l}))$$

$$= \frac{1}{l - \bar{l}} \left[(R_1(l) \, s_0(\bar{l}), s_0(l)) - (s_0(l), R_1(l) \, s_0(\bar{l})) \right]$$

$$= \frac{1}{l - \bar{l}} \left[(\Sigma(l), h) \, \Psi(l)(h, \Sigma(l)) - (\Sigma(\bar{l}), h) \, \Psi(\bar{l})(h, \Sigma(l)) \right]$$

$$= \int_a^b \int_a^b Q_1(x, y, l) \, h(x) \, h(y) \, dx \, dy,$$

where

$$Q_1(x, y, l) = (1/(l - \bar{l}))[\Sigma(x, l) \, \Psi(l) \, \Sigma^*(y, l) - \Sigma(x, \bar{l}) \, \Psi(\bar{l}) \, \Sigma^*(y, \bar{l})].$$

We note that $Q_1(c, c, l) = [\Psi(l) - \Psi(\bar{l})]/(l - \bar{l})$. Similarly we have

$$\left(\frac{[R_0(l) - R_0(\bar{l})]}{l - \bar{l}} \, s_0(\bar{l}), s_0(\bar{l}) \right) - (R_0(l) \, s_0(\bar{l}), \chi^0)) - (\chi^0, R_0(l) \, s_0(\bar{l})) + (\chi_0, \chi_0)$$

$$= \left(\frac{[R_0(l) - R_0(\bar{l})]}{l - \bar{l}} \, \eta, \eta \right) = \int_a^b \int_a^b Q_0(x, y, l) \, h(x) \, h(y) \, dx \, dy.$$

The matrix $Q_0(x, y, l)$ has the form

$$Q_0(x, y, l) = \begin{pmatrix} q_0(x, y, l) & O_n{}^p \\ O_p{}^n & O_p{}^p \end{pmatrix},$$

where

$$(q_0(x, y, l))_{jk} = (\partial^{j+k-2} H_0 / \partial x^{j-1} \, \partial y^{k-1})(x, y, l),$$

and

$$H_0(x, y, l) = (1/(l - \bar{l}))[K_0(x, y, l) - K_0(x, y, \bar{l})].$$

From the structure of H_0 it follows that $Q_0(c, c, l) = O_n{}^n$. Thus (6.12) yields

$$0 \leqslant \int_a^b \int_a^b Q(x, y, l) \, h(x) \, h(y) \, dx \, dy,$$

where $Q = Q_0 + Q_1$. Since, for sufficiently small $|J|$, this is true for all $h \in C_0{}^n(J)$ satisfying $0 \leqslant h(x) \leqslant 1$, $h(c) = 1$, we see that

$$0 \leqslant Q(c, c, l) = (\Psi(l) - \Psi(\bar{l}))/(l - \bar{l}),$$

which is (c). This completes the proof of Theorem 6.1.

The argument leading from (5.4) to an explicit representation for the generalized spectral family F, corresponding to the self-adjoint extension H of S, now follows along the lines given in [2, Theorems 4–7]. We briefly sketch the reasoning. From Theorem 6.1 it follows that Ψ has an integral representation

$$\Psi(l) = \alpha + l\beta + \int_{-\infty}^\infty \frac{\lambda l + 1}{\lambda - l} \, d\sigma(\lambda), \qquad l \in \mathbb{C}_0,$$

where α, β are constant hermitian matrices, $\beta \geqslant 0$, and σ is a nondecreasing hermitian matrix-valued function of bounded variation on $\mathbb{R}$. This representation implies that the matrix-valued function ρ given by

$$\rho(\lambda) = \lim_{\epsilon \to +0} \frac{1}{\pi} \int_0^\lambda \operatorname{Im} \Psi(\nu + i\epsilon) \, d\nu \tag{6.13}$$

exists, is nondecreasing, and of bounded variation on any finite subinterval of $\mathbb{R}$. We now use (5.4), namely,

$$(F(\Delta)f, f) = \lim_{\epsilon \to +0} \frac{1}{\pi} \int_\Delta \operatorname{Im}(R(\nu + i\epsilon)f, f) \, d\nu,$$

for $f \in C_0(\iota)$. For such f we let $\hat{f}(\nu) = (f, s(\nu))$, $\nu \in \mathbb{R}$. Then the structure of K_0 implies that $\operatorname{Im}(R_0(\nu + i\epsilon)f, f) \to 0$, as $\epsilon \to +0$, uniformly for $\nu \in \Delta$, and so we just have to consider $\operatorname{Im}(R_1(\nu + i\epsilon)f, f)$. We have

$$\operatorname{Im}(R_1(\nu + i\epsilon)f, f) = (\hat{f}(\nu))^* \operatorname{Im} \Psi(\nu + i\epsilon) \hat{f}(\nu)$$
$$+ \Gamma(\nu, \epsilon, f) - \Gamma(\nu, -\epsilon, f),$$

where

$$\Gamma(\nu, \epsilon, f) = (1/2i)[(\hat{f}(\nu + i\epsilon))^* \Psi(\nu + i\epsilon) \hat{f}(\nu - i\epsilon) - (\hat{f}(\nu))^* \Psi(\nu + i\epsilon) \hat{f}(\nu)]$$
$$= (1/2i)\{[\hat{f}(\nu + i\epsilon) - \hat{f}(\nu)]^* \Psi(\nu + i\epsilon) \hat{f}(\nu)$$
$$+ (\hat{f}(\nu + i\epsilon))^* \Psi(\nu + i\epsilon)[\hat{f}(\nu - i\epsilon) - \hat{f}(\nu)]\}.$$

From a theorem due to Helly it follows that

$$\lim_{\epsilon \to +0} \frac{1}{\pi} \int_{\varDelta} (\hat{f}(\nu))^* \operatorname{Im} \Psi(\nu + i\epsilon) \hat{f}(\nu) \, d\nu = \int_{\varDelta} (\hat{f}(\nu))^* \, d\rho(\nu) \, \hat{f}(\nu).$$

As to $\Gamma(\nu, \epsilon, f)$ we note that

$$|\hat{f}(\nu \pm i\epsilon) - \hat{f}(\nu)| \leqslant \| s(\nu \pm i\epsilon) - s(\nu)\|_0 \| f \|,$$

where f vanishes outside J_0, and

$$\| g \|_0^2 = \int_{J_0} g^*(x) g(x) \, dx.$$

Since

$$s(x, \nu \pm i\epsilon) - s(x, \nu) = w(x, \nu \pm i\epsilon) - w(x, \nu),$$

and w is continuous on $\iota \times \mathbb{C}$, and entire for each fixed $x \in \iota$, we see that for all sufficiently small $\epsilon > 0$,

$$\| s(\nu + i\epsilon) - s(\nu)\|_0 \leqslant k\epsilon, \qquad \nu \in \varDelta, \tag{6.14}$$

for some constant k depending only on J_0 and $\varDelta$. The integral representation of Ψ implies that

$$\int_{\varDelta} | \Psi_{ij}(\nu \pm i\epsilon)| \, d\nu = O(\log(1/\epsilon)), \qquad \epsilon \to +0. \tag{6.15}$$

Thus (6.14) and (6.15) show that

$$\frac{1}{\pi} \int_{\varDelta} \Gamma(\nu, \pm\epsilon, f) \, d\nu = O(\epsilon \log(1/\epsilon)), \qquad \epsilon \to +0;$$

in particular, this integral tends to 0 as $\epsilon \to +0$. We have now shown that

$$(F(\varDelta)f, f) = \int_{\varDelta} (\hat{f}(\nu))^* \, d\rho(\nu) \, \hat{f}(\nu), \qquad f \in C_0(\iota),$$

and this readily implies the following result.

THEOREM 6.2. *Let H be any self-adjoint subspace extension of S in $\mathfrak{R}^2$, $\mathfrak{H} \subset \mathfrak{R}$, with corresponding generalized resolvent R and generalized spectral family F given by (5.4). If $s(x, l)$ is defined by (6.2), (6.3) and the matrix ρ is given by (6.13), then*

$$F(\varDelta)f = \int_{\varDelta} s(\nu) \, d\rho(\nu) \, \hat{f}(\nu), \qquad f \in C_0(\iota), \tag{6.16}$$

where the endpoints of $\varDelta$ are continuity points for F, and $\hat{f}(\nu) = (f, s(\nu))$.

Let ζ, η represent vector-valued functions from $\mathbb{R}$ to $\mathbb{C}^{n+p}$ (considered as $(n + p) \times 1$ matrices), and define

$$(\zeta, \eta) = \int_{-\infty}^{\infty} \eta^*(\nu) \, d\rho(\nu) \, \zeta(\nu).$$

Since ρ is nondecreasing we have $(\zeta, \zeta) \geqslant 0$ and we can define $\| \zeta \| = (\zeta, \zeta)^{1/2}$. The Hilbert space $\hat{\mathfrak{H}}$ is then given by

$$\hat{\mathfrak{H}} = \mathfrak{L}^2(\rho) = \{\zeta \mid \| \zeta \| < \infty\}.$$

The eigenfunction expansion result then takes the following form.

THEOREM 6.3. *Let* $H = H_s \oplus H_\infty$ *be as in Theorem 6.2, and* $f \in \mathfrak{H}$. *Then* $\hat{f}$, *where*

$$\hat{f}(\nu) = \int_a^b s^*(x, \nu) f(x) \, dx, \tag{6.17}$$

converges in norm in $\hat{\mathfrak{H}} = \mathfrak{L}^2(\rho)$, *and*

$$F(\infty)f = \int_{-\infty}^{\infty} s(\nu) \, d\rho(\nu) \, \hat{f}(\nu), \tag{6.18}$$

where this integral converges in norm in $\mathfrak{H} = \mathfrak{L}^2(\iota)$. *Moreover,* $(F(\infty)f, g) = (\hat{f}, \hat{g})$ *for all* $f, g \in \mathfrak{H}$. *In particular, the map* $V: \mathfrak{H} \to \hat{\mathfrak{H}}$ *given by* $Vf = \hat{f}$ *is a contraction* $(\| Vf \| \leqslant \| f \|)$. *It is an isometry* $(\| Vf \| = \| f \|)$ *for* $f \in \mathfrak{H} \cap H(0)^\perp = \mathfrak{H} \ominus PH(0)$, *and*

$$f = \int_{-\infty}^{\infty} s(\nu) \, d\rho(\nu) \, \hat{f}(\nu), \qquad f \in \mathfrak{H} \ominus PH(0). \tag{6.19}$$

Proof. Recall that $F(\infty) = PP_s$, where P, P_s are the projections of $\mathfrak{K}$ onto $\mathfrak{H}$ and $H(0)^\perp$, respectively. The validity of (6.18) for $f \in C_0(\iota)$ follows from (6.16) and the fact that $\| F(\Delta)f - F(\infty)f \| \to 0$ as $\Delta \to \mathbb{R}$. Since, for $f \in C_0(\iota)$,

$$(F(\Delta)f, f) = \int_\Delta \hat{f}^*(\nu) \, d\rho(\nu) \, \hat{f}(\nu) \to (F(\infty)f, f),$$

as $\Delta \to \mathbb{R}$, we see that $\| \hat{f} \|^2 = (F(\infty)f, f) \leqslant \| f \|^2$. The denseness of $C_0(\iota)$ in $\mathfrak{H}$ permits us to extend these results to all $f \in \mathfrak{H}$, and polarization yields $(F(\infty)f, g) = (\hat{f}, \hat{g})$. For $f \in \mathfrak{H} \cap H(0)^\perp$, we have $F(\infty)f = f$, which shows that V is an isometry when restricted to $\mathfrak{H} \ominus PH(0)$ and that (6.19) is valid.

The operators $F(\infty)$ and V imply a splitting of $\mathfrak{H}$ and $V\mathfrak{H}$. If

$$\mathfrak{H}_0 = \{f \in \mathfrak{H} \mid F(\infty)f = f\}, \qquad \mathfrak{H}_1 = \{f \in \mathfrak{H} \mid F(\infty)f = 0\},$$

.

then we have the following result. (Please note that this $\mathfrak{H}_0$ is *not* the $\mathfrak{H}_0$ introduced just prior to Theorem 2.3.)

THEOREM 6.4. *The spaces $\mathfrak{H}_0$, $\mathfrak{H}_1$ are also characterized as*

$$\mathfrak{H}_0 = \mathfrak{H} \cap H(0)^\perp = \{f \in \mathfrak{H} \mid \| Vf \| = \|f\|\}, \tag{6.20}$$

$$\mathfrak{H}_1 = \mathfrak{H} \cap H(0) = \{f \in \mathfrak{H} \mid Vf = 0\}. \tag{6.21}$$

Thus $\mathfrak{H}_0 \perp \mathfrak{H}_1$ and the splitting $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1 \oplus \mathfrak{H}_2$, where $\mathfrak{H}_2 = \mathfrak{H} \ominus (\mathfrak{H}_0 \oplus \mathfrak{H}_1)$, implies that $V\mathfrak{H} = V\mathfrak{H}_0 \oplus V\mathfrak{H}_2$.

Proof. As noted above $\mathfrak{H} \cap H(0)^\perp \subset \mathfrak{H}_0$. If $f \in \mathfrak{H}_0$, then $f \in \mathfrak{H}$ and $PP_s f = f$, or $P(I - P_s)f = 0$. Thus

$$\|(I - P_s)f\|^2 = ((I - P_s)f, f) = (P(I - P_s)f, f) = 0,$$

showing that $P_s f = f$, or $f \in \mathfrak{H} \cap H(0)^\perp$, and hence $\mathfrak{H}_0 = \mathfrak{H} \cap H(0)^\perp$. We have $(F(\infty)f, f) = (Vf, Vf)$, and so $f \in \mathfrak{H}_0$ implies $\| Vf \| = \|f\|$. Conversely, if $(Vf, Vf) = (F(\infty)f, f) = (f, f)$, then

$$\|(I - F(\infty))f\|^2 = \| F(\infty)f \|^2 - \|f\|^2 = \| PP_s f \|^2 - \|f\|^2 \leqslant 0$$

shows that $F(\infty)f = f$, or $f \in \mathfrak{H}_0$. Thus (6.20) is established. Replacing P_s in the above argument by $I - P_s$ we obtain the first equality in (6.21). If $f \in \mathfrak{H}_1$ then $(Vf, Vf) = (F(\infty)f, f) = 0$, or $Vf = 0$. Conversely, suppose $Vf = 0$. Then $(F(\infty)f, f) = 0$ and

$$\| F(\infty)f \|^2 = \|(I - F(\infty))f\|^2 - \|f\|^2 = \| P(I - P_s)f \|^2 - \|f\|^2 \leqslant 0$$

implies that $F(\infty)f = 0$, or $f \in \mathfrak{H}_1$. This gives (6.21). Now clearly $V\mathfrak{H} = V\mathfrak{H}_0 \dotplus V\mathfrak{H}_2$, and we claim that $V\mathfrak{H}_0 \perp V\mathfrak{H}_2$. Indeed, if $f_0 \in \mathfrak{H}_0$, $f_2 \in \mathfrak{H}_2$ then $(Vf_0, Vf_2) = (F(\infty)f_0, f_2) = (f_0, f_2) = 0$, since $\mathfrak{H}_0 \perp \mathfrak{H}_2$.

Remarks on Theorem 6.4. If either $\mathfrak{D}(S)$ is dense in $\mathfrak{H}$, or H is an operator, then $\mathfrak{H} = \mathfrak{H}_0$. The first assertion follows from the fact that $\mathfrak{D}(S) \subset \mathfrak{H} \cap \mathfrak{D}(H) \subset \mathfrak{H} \cap H(0)^\perp = \mathfrak{H}_0$, and then $(\mathfrak{D}(S))^c = \mathfrak{H} \subset \mathfrak{H}_0$ implies $\mathfrak{H} = \mathfrak{H}_0$. For the second, if H is an operator, then $H(0) = \{0\}$ and hence $\mathfrak{R} = H(0)^\perp$, or $\mathfrak{H}_0 = \mathfrak{H} \cap H(0)^\perp = \mathfrak{H}$. Thus nontrivial $\mathfrak{H}_1$, $\mathfrak{H}_2$ can exist only for a nondensely defined S and a subspace (nonoperator) extension H in a $\mathfrak{R}$ properly containing $\mathfrak{H}$. A simple example where such $\mathfrak{H}_1$, $\mathfrak{H}_2$ exist is as follows. Let T be the maximal operator for id/dx on $\mathfrak{R} = \mathfrak{L}^2(-1, 1)$. We let $\mathfrak{H} = \mathfrak{L}^2(0, 1)$, and identify $\mathfrak{H}$ with the set of all $f \in \mathfrak{R}$ such that $f(x) = 0$ for $-1 \leqslant x < 0$. Let $\varphi(x) = 0$ for $-1 \leqslant x < 0$ and $\varphi(x) = 1$ for

$0 \leqslant x \leqslant 1$, and let $\psi(x) = x - \frac{1}{2}$ for $-1 \leqslant x \leqslant 1$. We define $S \subset \mathfrak{H}^2$ and $H \subset \mathfrak{R}^2$ by

$$S = \{\{f, Tf\} \mid f \in \mathfrak{D}(T) \cap \mathfrak{H}, f(0) = f(1) = 0, (f, \varphi) = (f, \psi) = 0\},$$

$$H = \{\{f, Tf + c\varphi + d\psi\} \mid f \in \mathfrak{D}(T), f(-1) = f(1),$$
$$(f, \varphi) = (f, \psi) = 0, c, d \in \mathbb{C}\}.$$

It is clear that S is a symmetric operator in $\mathfrak{H}$. Since

$$H_1 = \{\{f, Tf\} \mid f \in \mathfrak{D}(T), f(-1) = f(1)\}$$

is self-adjoint in $\mathfrak{R}$, it follows that H is a self-adjoint subspace in $\mathfrak{R}^2$, which is an extension of S (see [4, Section 5]). Now $H(0) = \{\varphi, \psi\}$ and $PH(0) = \{\varphi, \psi_0\}$, where $\psi_0(x) = 0$ for $-1 \leqslant x < 0$ and $\psi_0(x) = x - \frac{1}{2}$ for $0 \leqslant x \leqslant 1$. Thus $\mathfrak{H}_0 = \mathfrak{H} \ominus PH(0) = \mathfrak{H} \ominus \{\varphi, \psi_0\}$, $\mathfrak{H}_1 = \mathfrak{H} \cap H(0) = \{\varphi\}$, $\mathfrak{H}_2 = \{\psi_0\}$. Here $\{\varphi\}$, $\{\varphi, \psi\}$ represent the subspaces spanned by φ and φ, ψ, respectively.

It is easy to check that $PH(0) \subset S^*(0)$, which has a finite dimension. Thus $\mathfrak{H}_1 \oplus \mathfrak{H}_2 = PH(0)$ is finite-dimensional and $\mathfrak{H}_0 = \mathfrak{H} \ominus PH(0)$ has finite codimension.

THEOREM 6.5. *We have* $V\mathfrak{H}_0 = \hat{\mathfrak{H}}$ *if and only if* F *is the spectral family for a self-adjoint subspace extension of* S *in* $\mathfrak{H}^2$ *itself.*

Proof. Suppose $V\mathfrak{H}_0 = \hat{\mathfrak{H}}$. Then Theorem 6.3 shows that $V\mathfrak{H}_2 = \{0\}$, and since V is bijective from $\mathfrak{H}_0 \oplus \mathfrak{H}_2$ onto $V\mathfrak{H}$ it follows that $\mathfrak{H}_2 = \{0\}$. Thus $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$, and $F(\infty)$ is the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{H}_0$. The relation (6.16) implies that $(F(\varDelta)f, g) = (\chi_\varDelta \hat{f}, \hat{g})$, for all $f, g \in \mathfrak{H}$, where $\chi_\varDelta(\lambda) = 1$ for $\lambda \in \varDelta$, $\chi_\varDelta(\lambda) = 0$, $\lambda \notin \varDelta$. For $g \in \mathfrak{H}_1$, $\hat{g} = 0$, and thus this relation shows that $F(\varDelta)f \in \mathfrak{H}_0$ for all $f \in \mathfrak{H}$. Moreover, if $g \in \mathfrak{H}_0$, then

$$(F(\varDelta)f, g) = (VF(\varDelta)f, Vg) = (\chi_\varDelta Vf, Vg),$$

and the equality $V\mathfrak{H}_0 = \hat{\mathfrak{H}}$, shows that

$$VF(\varDelta)f = \chi_\varDelta \hat{f}, \qquad f \in \mathfrak{H}.$$

From this it follows that

$$F(\varDelta)f = E(\varDelta)f, \qquad f \in \mathfrak{H}.$$

Indeed, we have

$$\| E(\varDelta)f \|^2 = (E(\varDelta)f, f) = (F(\varDelta)f, f) = (\chi_\varDelta \hat{f}, \hat{f})$$
$$= (\chi_\varDelta \hat{f}, \chi_\varDelta \hat{f}) = \| F(\varDelta)f \|^2,$$

and this implies that $\|(I - P) E(\Delta)f\| = 0$, or $F(\Delta)f = E(\Delta)f$. In particular,

$$F(\Delta)f = E_s(\Delta)f, \qquad f \in \mathfrak{H}_0,$$
$$= 0, \qquad f \in \mathfrak{H}_1,$$

and F restricted to $\mathfrak{H}_0$ is a spectral family for a self-adjoint operator H_0 in $\mathfrak{H}_0$. Its domain is

$$\mathfrak{D}(H_0) = \left\{ f \in \mathfrak{H}_0 \,\bigg|\, \int_{-\infty}^{\infty} \lambda^2 d(F(\lambda)f, f) = \int_{-\infty}^{\infty} \lambda^2 d(E_s(\lambda)f, f) < \infty \right\},$$

and

$$H_0 f = \int_{-\infty}^{\infty} \lambda \, dF(\lambda)f = \int_{-\infty}^{\infty} \lambda \, dE_s(\lambda)f, \qquad f \in \mathfrak{D}(H_0).$$

It is easy to see that $V H_0 V^{-1} = \Lambda$, the self-adjoint operator of "multiplication by λ" on $\hat{\mathfrak{H}}$. We define

$$H_1 = H_0 \oplus \{\{0\} \oplus \mathfrak{H}_1\}.$$

The subspace $H_1 \subset \mathfrak{H}^2$ is self-adjoint, for H_0 is self-adjoint in $\mathfrak{H}_0^2$ and $\{0\} \oplus \mathfrak{H}_1$ is self-adjoint in $\mathfrak{H}_1^2$. We claim that $H_1 = H \cap \mathfrak{H}^2$. Clearly $H_0 \subset H_s$, and $\{0\} \oplus \mathfrak{H}_1 = \{0\} \oplus (\mathfrak{H} \cap H(0)) \subset \{0\} \oplus H(0) = H_\infty$. Therefore $H_1 \subset H \cap \mathfrak{H}^2$. If $\{f, H_s f + \chi\} \in H \cap \mathfrak{H}^2$, then $f \in \mathfrak{D}(H_s) \cap \mathfrak{H} = \mathfrak{D}(H_0)$, and $H_s f + \chi = H_0 f + \chi \in \mathfrak{H}$. Thus $\chi \in H(0) \cap \mathfrak{H} = \mathfrak{H}_1$, showing that $H \cap \mathfrak{H}^2 \subset H_1$, and thus $H_1 = H \cap \mathfrak{H}^2$. Since $S \subset H$, we have $S \subset H_1$ and we have shown that H_1, whose spectral family is clearly F, is a self-adjoint subspace extension of S in $\mathfrak{H}^2$.

Now suppose $H = H_s \oplus H_\infty \subset \mathfrak{H}^2$, that is, $F = E$ is the spectral family for a self-adjoint extension H of S in $\mathfrak{H}^2$. Then $\mathfrak{H}_0 = H(0)^\perp$, $\mathfrak{H}_1 = H(0)$, $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$, and $F(\infty) = E(\infty) = P_s$, the projection of $\mathfrak{H}$ onto $\mathfrak{H}_0$. The self-adjoint operator H_s in $\mathfrak{H}_0$ is such that

$$V H_s f = \Lambda V f, \qquad f \in \mathfrak{D}(H_s), \tag{6.22}$$

where Λ is the self-adjoint "multiplication by λ" operator in $\hat{\mathfrak{H}}$, that is,

$$\mathfrak{D}(\Lambda) = \left\{ \zeta \in \hat{\mathfrak{H}} \,\bigg|\, \int_{-\infty}^{\infty} \lambda^2 \zeta^*(\lambda) \, d\rho(\lambda) \, \zeta(\lambda) < \infty \right\},$$
$$\Lambda \zeta(\lambda) = \lambda \zeta(\lambda), \qquad \zeta \in \mathfrak{D}(\Lambda).$$

Indeed, for $f \in \mathfrak{D}(H_s)$,

$$\| V H_s f \|^2 = \| H_s f \|^2 = \int_{-\infty}^{\infty} \lambda^2 d(E(\lambda)f, f)$$
$$= \int_{-\infty}^{\infty} \lambda^2 \hat{f}^*(\lambda) \, d\rho(\lambda) \, \hat{f}(\lambda) = \| \Lambda V f \|^2,$$

and for $g \in \mathfrak{H}_0$,

$$(VH_s f, Vg) = (H_s f, g) = \int_{-\infty}^{\infty} \lambda d(E(\lambda)f, g)$$

$$= \int_{-\infty}^{\infty} \lambda \hat{g}^*(\lambda) \, d\rho(\lambda) \, \hat{f}(\lambda) = (\Lambda Vf, Vg).$$

Thus $VH_s f = \Lambda Vf + \zeta$, $\zeta \in \hat{\mathfrak{H}} \ominus V\mathfrak{H}^0$. But then

$$\| \Lambda Vf \|^2 = \| VH_s f \|^2 + \| \zeta \|^2 = \| \Lambda Vf \|^2 + \| \zeta \|^2$$

implies $\zeta = 0$, proving (6.22). Now $E(\Delta)f \in \mathfrak{D}(H_s)$ for any $f \in \mathfrak{H}$ and any interval Δ. We claim that

$$VE(\Delta)f = \chi_\Delta Vf, \qquad f \in \mathfrak{H}. \tag{6.23}$$

The proof is similar to that for (6.22). On the one hand,

$$(VE(\Delta)f, Vg) = (E(\Delta)f, g) = (\chi_\Delta Vf, Vg), \qquad f \in \mathfrak{H}, \quad g \in \mathfrak{H}_0,$$

implies $VE(\Delta)f = \chi_\Delta Vf + \zeta$, $\zeta \in \hat{\mathfrak{H}} \ominus V\mathfrak{H}_0$. But

$$\| VE(\Delta)f \|^2 = \| E(\Delta)f \|^2 = (E(\Delta)f, f) = (\chi_\Delta \hat{f}, \hat{f}) = \| \chi_\Delta \hat{f} \|^2$$

then shows that $\zeta = 0$. From (6.22) and (6.23) it follows that

$$VH_s E(\Delta)f = \Lambda \chi_\Delta Vf, \qquad f \in \mathfrak{H}. \tag{6.24}$$

In order to prove that $V\mathfrak{H}_0 = \hat{\mathfrak{H}}$ in case $H \subset \mathfrak{H}^2$, we show that $\xi \in \hat{\mathfrak{H}}$, $(\xi, Vg) = 0$ for all $g \in \mathfrak{H}_0$ implies that $\xi = 0$. For any $f \in C_0(\iota)$ we have $E(\Delta)f \in \mathfrak{H}_0$ and hence

$$0 = (\xi, VE(\Delta)f) = (\xi, \chi_\Delta \hat{f}) = \int_\Delta (s(\lambda), f) \, d\rho(\lambda) \, \xi(\lambda)$$

for all subintervals Δ of $\mathbb{R}$. Let $f = s_0(0) = \eta + L\chi^0$, where $s_0(l)$ is defined by (6.8). We have

$$(s(\lambda), s_0(0)) = (s(\lambda), s_0(\lambda)) + (s(\lambda), s_0(0) - s_0(\lambda))$$
$$= (\Sigma(\lambda), h) + \lambda(s(\lambda), \chi^0),$$

and therefore

$$\int_\Delta (\Sigma(\lambda), h) \, d\rho(\lambda) \, \xi(\lambda) + \int_\Delta \lambda(s(\lambda), \chi^0) \, d\rho(\lambda) \, \xi(\lambda) = 0.$$

The second integral is just $(\xi, VH_sE(\Delta)\chi^0)$ by (6.24), and is thus zero. Now we have

$$\int_\Delta (\Sigma(\lambda), h)\, d\rho(\lambda)\, \xi(\lambda) = 0$$

for all Δ, and since $\gamma^{-1}(\Sigma(\lambda), h) \to I_{n+p}$, as $|J| \to 0$, uniformly on Δ, we see that

$$\int_\Delta d\rho(\lambda)\, \xi(\lambda) = 0$$

for all Δ. This implies that $(\xi, \zeta) = 0$ for all $\zeta \in \hat{\mathfrak{H}}$ whose components are step functions and which vanish outside compact subsets of $\mathbb{R}$. These ζ are dense in $\hat{\mathfrak{H}}$, and therefore $\xi = 0$ as desired. Note that the self-adjoint operator VH_sV^{-1} in $V\mathfrak{H}_0 = \hat{\mathfrak{H}}$ is such that $VH_sV^{-1} \subset \Lambda$, by (6.22), and hence $VH_sV^{-1} = \Lambda$.

The argument used in the proof of Theorem 6.5 to show that $V\mathfrak{H}_0 = \hat{\mathfrak{H}}$ in case $H \subset \mathfrak{H}^2$ can be used to prove the essential uniqueness of the matrix-valued function ρ of Theorem 6.2.

THEOREM 6.6. *Let ρ_1, ρ_2 be two $(n + p) \times (n + p)$ matrix-valued functions on $\mathbb{R}$ such that*

$$(F(\Delta)f, g) = \int_\Delta \hat{g}^*(\lambda)\, d\rho_j(\lambda)\, \hat{f}(\lambda), \qquad j = 1, 2,$$

for all $f, g \in C_0(\iota)$ and all intervals Δ whose endpoints are continuity points of F. Then

$$\int_\Delta d\rho_1(\lambda) = \int_\Delta d\rho_2(\lambda)$$

for all such intervals Δ.

Proof. For $f, g \in C_0(\iota)$ we have

$$\int_\Delta \lambda^k d(F(\lambda)f, g) = \int_\Delta \lambda^k \hat{g}^*(\lambda)\, d\rho_j(\lambda)\, \hat{f}(\lambda); \qquad j = 1, 2, \quad k = 0, 1, 2.$$

Thus if $\rho = \rho_1 - \rho_2$, and $f, g \in C_0(\iota)$,

$$\int_\Delta \lambda^k \hat{g}^*(\lambda)\, d\rho(\lambda)\, \hat{f}(\lambda) = 0, \qquad k = 0, 1, 2. \tag{6.25}$$

We apply (6.25) with $k = 0$, $g = s_0(0)$, which implies

$$\hat{g}(\lambda) = (s_0(0), s(\lambda)) = (h, \Sigma(\lambda)) + \lambda(\chi^0, s(\lambda)),$$

and hence

$$\int_{\Delta} (\Sigma(\lambda), h) \, d\rho(\lambda) \, \hat{f}(\lambda) = 0, \tag{6.26}$$

since

$$\int_{\Delta} \lambda(s(\lambda), \chi^0) \, d\rho(\lambda) \, \hat{f}(\lambda) = 0,$$

using (6.25) with $k = 1$, $g = \chi^0$. Now we apply (6.25) with $k = 1$, $f = \chi^0$, $g = s_0(0)$. This results in

$$\int_{\Delta} \lambda(\Sigma(\lambda), h) \, d\rho(\lambda)(\chi^0, s(\lambda)) = 0 \tag{6.27}$$

since

$$\int_{\Delta} \lambda^2(s(\lambda), \chi^0) \, d\rho(\lambda)(\chi^0, s(\lambda)) = 0,$$

using (6.25) with $k = 2$, $f = g = \chi^0$. Now let $f = s_0(0)$ in (6.26). We get

$$\int_{\Delta} (\Sigma(\lambda), h) \, d\rho(\lambda)(h, \Sigma(\lambda)) = 0, \tag{6.28}$$

where we have used (6.27). Now recall that $\gamma^{-1}(\Sigma(\lambda), h) \to I_{n+p}$, as $|J| \to 0$, uniformly on Δ. Using this in (6.28) we obtain

$$\int_{\Delta} d\rho(\lambda) = 0,$$

which proves the theorem.

7. Eigenfunction Expansions (Continued)

In this section we present different proofs of Theorems 6.2 and 6.5. We repeat some of the definitions given in Section 6. Let c be fixed, $a < c < b$, and let

$$s^1(x, l) = (s_1(x, l),..., s_n(x, l)), \qquad u(x, l) = (u_{n+1}(x, l),..., u_{n+p}(x, l))$$

be the unique matrix solutions of

$$(L - l) \, s^1(l) = 0, \qquad \tilde{s}^1(c, l) = I_n, \qquad l \in \mathbb{C},$$
$$(L - l) \, u(l) = \sigma + l\tau, \qquad \tilde{u}(c, l) = O_n{}^p, \qquad l \in \mathbb{C},$$

where $\{\sigma, \tau\}$ is a $1 \times p$ matrix whose entries form a basis for B. Let $s^2(l) = u(l) + \tau$ and $s(l) = (s^1(l) : s^2(l))$.

Let $k(x, y, l)$, $x, y \in \iota$, $l \in \mathbb{C}$, be the solution of

$$(L - l) k(x, y, l) = 0, \qquad \tilde{k}(y, y, l) = (0, ..., 0, 1/p_n(y))^t, \qquad l \in \mathbb{C}.$$

Then $K(l) z(x) = \int_c^x k(x, y, l) z(y) \, dy$, $z \in \mathfrak{H}$, is the uniquely determined solution of

$$(L - l) f(l) = z, \qquad \tilde{f}(c, l) = O_n^1, \qquad l \in \mathbb{C}.$$

Since $k(x, y, l)$ is continuous in (x, y, l) for $y \leqslant x$ and for $y > x$, it follows that for each compact interval $J \subset \iota$ and for each $l \in \mathbb{C}$ there exists a constant $b(J, l) > 0$, bounded for l in compact sets, such that

$$\| K(l) z \|_J \leqslant b(J, l) \| z \|, \tag{7.1}$$

where

$$\| f \|_J = \left(\int_J |f(x)|^2 \, dx \right)^{1/2}.$$

Let $H = H_s \oplus H_\infty$ be a self-adjoint extension of S in $\mathfrak{R}^2$, as described in Section 5. Let $R_s(l) = (H_s - l)^{-1}$, $l \in \mathbb{C}_0$, be the resolvent of the self-adjoint operator H_s in $H(0)^\perp = \mathfrak{R} \ominus H(0)$. Then $R_s(l)$ is a bounded operator defined on all of $H(0)^\perp$ and it is easily verified that for all $h \in H(0)^\perp$,

$$\{PR_s(l)h, \, lPR_s(l)h + Ph\} \in S_0^* \dotplus -B^{-1},$$

where P is the orthogonal projection of $\mathfrak{R}$ onto $\mathfrak{H}$. Thus for each $h \in H(0)^\perp$ there exist a unique $f \in \mathfrak{D}(S_0^*)$ and a unique $p \times 1$ matrix a of complex constants, both depending on $l \in \mathbb{C}_0$, such that

$$\{PR_s(l)h, \, lPR_s(l)h + Ph\} = \{f, S_0^* f\} + \{\tau, -\sigma\}a. \tag{7.2}$$

We define

$$\Gamma^1(PR_s(l)h) = \tilde{f}(c), \qquad \Gamma^2(PR_s(l)h) = a,$$

$$\Gamma(PR_s(l)h) = \begin{pmatrix} \Gamma^1(PR_s(l)h) \\ \Gamma^2(PR_s(l)h) \end{pmatrix}.$$

LEMMA 7.1. *For each* $l \in \mathbb{C}_0$, *the map* $h \to \Gamma(PR_s(l)h)$ *from* $H(0)^\perp$ *into* $\mathbb{C}^{n+p}$ *is linear and continuous.*

Proof. Clearly, the indicated map is linear. We shall prove it is continuous. Let $h \in H(0)^\perp$. Then (7.2) implies that

$$Ph = (L - l)(f - u(l)a).$$

Since $Ph = (L - l) K(l) Ph$, we have that

$$(L - l)(f - K(l) Ph - u(l)a) = 0.$$

Using the initial conditions for $K(l) Ph$, $u(l)$ and the fact that by (7.2) $f = PR_s(l)h - \tau a$, we find that

$$PR_s(l)h - K(l) Ph = s(l) \Gamma(PR_s(l)h). \tag{7.3}$$

Let $\{\varphi^0, L\varphi^0\}$ be a $1 \times p$ matrix satisfying the Lemma preceding the proof of Theorem 6.1. Then, using (7.3) and (6.6) we find that

$$(PR_s(l)h - K(l) Ph, (L - \bar{l})\varphi^0) = \Gamma^2(PR_s(l)h).$$

From this equality, the continuity of $PR_s(l)$, and (7.1), it follows that there exists a constant $c_2(l) > 0$ such that

$$| \Gamma^2(PR_s(l)h)|_p \leqslant c_2(l) \| h \|, \tag{7.4}$$

where $| \ |_p$ denotes the norm of $\mathbb{C}^p$. We rewrite (7.3) to obtain

$$s^1(l) \Gamma^1(PR_s(l)h) = PR_s(l)h - K(l)h - s^2(l) \Gamma^2(PR_s(l)h). \tag{7.5}$$

Let J be a compact subinterval of ι such that $c \in J$. Then the $n \times n$ matrix

$$(s^1(l), s^1(l))_J = \int_J (s^1(x, l))^* s^1(x, l) \, dx$$

is invertible. From (7.5) it then follows that

$$\Gamma^1(PR_s(l)h) = [(s^1(l), s^1(l))_J]^{-1}(PR_s(l)h - K(l)h - s^2(l) \Gamma^2(PR_s(l)h), s^1(l))_J .$$

Since the right-hand side of (7.5) is continuous on $H(0)^\perp$, it follows from the above equality that there exists a constant $c_1(l) > 0$ such that

$$| \Gamma^1(PR_s(l)h)|_n \leqslant c_1(l) \| h \|. \tag{7.6}$$

The inequalities (7.4) and (7.6) show that the map $h \to \Gamma(PR_s(l)h)$ is continuous on $H(0)^\perp$.

Lemma 7.1 and the Riesz representation theorem imply that there exists a $1 \times (n + p)$ matrix $G(l)$ whose entries belong to $H(0)^\perp$ such that

$$\Gamma(PR_s(l)h) = (h, G(l)), \qquad l \in \mathbb{C}_0, \quad h \in H(0)^\perp.$$

Without loss of generality we may and do assume that H is *minimal*, i.e., that the set $\{E(\lambda)f \mid f \in \mathfrak{H}, \ \lambda \in \mathbb{R}\} \cup \mathfrak{H}$ is fundamental in $\mathfrak{R}$ (cf. [7]),

where $\{E(\lambda) \mid \lambda \in \mathbb{R}\}$ is the spectral family of projections in $\mathfrak{R}$ for the subspace H. The assumption implies that $\mathfrak{R}$, and therefore $H(0)^{\perp}$ also, is separable. Consequently H_s has only countably many eigenvalues λ_ν, $\nu = 0, \pm 1,...$, listed as often as the multiplicity of the eigenvalue requires. Let y_ν, $\nu = 0, \pm 1,...$, be a corresponding orthonormal system of eigenfunctions of H_s in $H(0)^{\perp}$. We decompose $E_s(\lambda) \in E_s$, the spectral family of orthogonal projections in $H(0)^{\perp}$ for H_s, as follows:

$$E_s(\lambda) = P_s(\lambda) + Q_s(\lambda),$$

where

$$Q_s(\lambda)f = \sum_{\lambda_\nu \leqslant \lambda} (E_s(\lambda_\nu) - E_s(\lambda_\nu - 0))f$$

$$= \sum_{\lambda_\nu \leqslant \lambda} (f, y_\nu)\, y_\nu, \qquad f \in H(0)^{\perp},$$

and

$$P_s(\lambda) = E_s(\lambda) - Q_s(\lambda).$$

Hence $\{P_s(\lambda) \mid \lambda \in \mathbb{R}\}$ is a continuous and $\{Q_s(\lambda) \mid \lambda \in \mathbb{R}\}$ is a right-continuous family of projections in $H(0)^{\perp}$; the first family is related to the continuous part and the second family is related to the discrete part of the spectrum of H_s. Furthermore, for $\lambda, \mu \in \mathbb{R}$, we have

$$P_s(\lambda) \leqslant P_s(\mu), \qquad Q_s(\lambda) \leqslant Q_s(\mu), \qquad \lambda \leqslant \mu,$$
$$P_s(\lambda)\, Q_s(\mu) = Q_s(\lambda)\, P_s(\mu) = 0,$$

and

$$P_s(\lambda)H_s = H_sP_s(\lambda), \qquad Q_s(\mu)H_s = H_sQ_s(\mu).$$

For $h \in \mathfrak{D}(H_s)$ we have by the spectral theorem,

$$(H_s - l)h = \int_{-\infty}^{\infty} (\lambda - l)\, d(P_s(\lambda) - P_s(0))h$$
$$+ \sum_\nu (\lambda_\nu - l)(h, y_\nu)\, y_\nu, \qquad l \in \mathbb{C}. \tag{7.7}$$

LEMMA 7.2. Let $h \in H(0)^{\perp}$ and $v(\lambda) = (P_s(\lambda) - P_s(0))h$, $\lambda \in \mathbb{R}$. Then

$$Pv(\lambda) = \int_0^\lambda s(\mu)\, d\Gamma(Pv(\mu)).$$

Proof. The function v from $\mathbb{R}$ into $\mathfrak{D}(H_s)$ is continuous and $v(0) = 0$. Let $l \in \mathbb{C}_0$ be fixed and let

$$u(\lambda) = (H_s - l)\, v(\lambda) = \int_0^\lambda (\mu - l)\, dv(\mu)$$
$$= (\lambda - l)\, v(\lambda) - \int_0^\lambda v(\mu)\, d\mu \tag{7.8}$$

(cf. (7.7)). Then u, too, is a continuous function from $\mathbb{R}$ into $\mathfrak{D}(H_s)$ and $u(0) = 0$. Hence $\Gamma(Pv(\cdot)) = \Gamma(PR_s(l)\,u(\cdot))$ is a continuous function from $\mathbb{R}$ into $\mathbb{C}^{n+p}$. This implies that

$$w(\lambda) = \int_0^\lambda s(\mu)\, d\Gamma(Pv(\mu))$$

exists and

$$w(\lambda) = s(\lambda)\,\Gamma(Pv(\lambda)) - \int_0^\lambda ((\partial/\partial\mu)\,s(\mu))\,\Gamma(Pv(\mu))\,d\mu.$$

A simple calculation shows that

$$(L - \lambda)(w(\lambda) - \tau\Gamma^2(Pv(\lambda))) = (\sigma + \lambda\tau)\,\Gamma^2(Pv(\lambda)) - \int_0^\lambda w(\mu)\,d\mu.$$

On the other hand it follows from (7.8) that

$$P(H_s - \lambda)\,v(\lambda) = -\int_0^\lambda Pv(\mu)\,d\mu,$$

and hence that

$$(L - \lambda)(Pv(\lambda) - \tau\Gamma^2(Pv(\lambda))) = (\sigma + \lambda\tau)\,\Gamma^2(Pv(\lambda)) - \int_0^\lambda Pv(\mu)\,d\mu,$$

since $\{Pv(\lambda), PH_sv(\lambda)\} \in S_0{}^* \dotplus -B^{-1}$. Combining these two results and putting

$$z(\lambda) = Pv(\lambda) - w(\lambda), \qquad h(\lambda) = -\int_0^\lambda z(\mu)\,d\mu,$$

we get

$$(L - \lambda)\,z(\lambda) = h(\lambda).$$

Since

$$\tilde{z}(\lambda)(c) = (Pv(\lambda) - \tau\Gamma^2(Pv(\lambda)))^\sim(c) - (w(\lambda) - \tau\Gamma^2(Pv(\lambda)))^\sim(c)$$

$$= \Gamma^1(Pv(\lambda)) - \int_0^\lambda \tilde{s}^1(c, \mu)\,d\Gamma^1(Pv(\mu))$$

$$= 0,$$

it follows that $z(\lambda) = K(\lambda)\,h(\lambda)$. Using (7.1) we obtain that for each compact $J \subset \iota$,

$$\| z(\lambda)\|_J \leqslant b(J, \lambda)\,\| h(\lambda)\|_J$$

$$\leqslant |\lambda|\,b(J, \lambda)\left|\int_0^\lambda \| z(\mu)\|_J^2\,d\mu\right|^{1/2}.$$

Gronwall's lemma and the continuity of $z(\lambda)(x)$ as a function of $x \in \iota$ imply that $z(\lambda) = 0$ on ι for all $\lambda \in \mathbb{R}$. Thus $Pv(\lambda) = w(\lambda)$, $\lambda \in \mathbb{R}$, and we have proved the lemma.

Let $l_0 \in \mathbb{C}_0$ be fixed and let t^1 be the $1 \times (n + p)$ matrix-valued function on $\mathbb{R}$ with components in $\mathfrak{D}(H_s)$ defined by

$$t^1(\lambda) = (H_s - \bar{l}_0)(P_s(\lambda) - P_s(0))\, G(l_0), \qquad \lambda \in \mathbb{R}.$$

Clearly we have $t^1(0) = O_1^{n+p}$. We define the $(n + p) \times (n + p)$ matrix-valued function ρ^1 on $\mathbb{R}$ by

$$\rho^1(\lambda) = \Gamma(Pt^1(\lambda)) = (\Gamma(Pt_1^1(\lambda)) : \cdots : \Gamma(Pt_{n+p}^1(\lambda))),$$

where

$$t^1(\lambda) = (t_1^1(\lambda),..., t_{n+p}^1(\lambda)), \qquad \lambda \in \mathbb{R}.$$

THEOREM 7.3. *The matrix-valued function ρ^1 is hermitian nondecreasing, and continuous on $\mathbb{R}$ and $\rho^1(0) = O_{n+p}^{n+p}$. For each $h \in \mathfrak{H}$ and $\alpha, \beta \in \mathbb{R}$,*

$$P(P_s(\beta) - P_s(\alpha))\, P_s h = \int_\alpha^\beta s(\lambda)\, d_\lambda \left(h, \int_0^\lambda s(\mu)\, d\rho^1(\mu) \right),$$

where P_s is the orthogonal projection from $\mathfrak{K}$ onto $H(0)^\perp$.

Proof. (Note the distinction between P_s and the $P_s(\alpha)$; $P_s = E_s(\infty) = P_s(\infty) + Q_s(\infty)$.) Let $\lambda, \mu \in \mathbb{R}$ and choose $\alpha, \beta \in \mathbb{R}$ such that $0, \lambda, \mu \in [\alpha, \beta]$. Then

$$t^1(\lambda) - t^1(\mu) = (P_s(\lambda) - P_s(\mu))(H_s - \bar{l}_0)(P_s(\beta) - P_s(\alpha))\, G(l_0) \quad (7.9)$$

and

$$
\begin{aligned}
\rho^1(\lambda) - \rho^1(\mu) &= \Gamma(P(t^1(\lambda) - t^1(\mu))) \\
&= ((H_s - l_0)(t^1(\lambda) - t^1(\mu)),\, G(l_0)) \\
&= (t^1(\lambda) - t^1(\mu),\, (P_s(\lambda) - P_s(\mu))(H_s - \bar{l}_0)(P_s(\beta) - P_s(\alpha))\, G(l_0)) \\
&= (t^1(\lambda) - t^1(\mu),\, t^1(\lambda) - t^1(\mu)). \quad (7.10)
\end{aligned}
$$

By (7.9), t^1 is continuous and thus by (7.10) ρ^1 is hermitian, nondecreasing and continuous on $\mathbb{R}$. Clearly, $\rho^1(0) = O_{n+p}^{n+p}$. From (7.9) with $\mu = 0$, Lemma 7.2 (with h, $v(\lambda)$ replaced by $(H_s - \bar{l}_0)(P_s(\beta) - P_s(\alpha))\, G(l_0)$, $t^1(\lambda)$) and the definition of ρ^1, we deduce that

$$Pt^1(\lambda) = \int_0^\lambda s(\mu)\, d\rho^1(\mu). \quad (7.11)$$

Thus we have

$$\Gamma(P(P_s(\lambda) - P_s(0)) P_s h) = ((H_s - l_0)(P_s(\lambda) - P_s(0)) P_s h, G(l_0))$$
$$= (h, Pt^1(\lambda))$$
$$= \left(h, \int_0^\lambda s(\mu) \, d\rho^1(\mu)\right).$$

Again we apply Lemma 7.2 (with h replaced by $P_s h$) and obtain

$$P(P_s(\lambda) - P_s(0)) P_s h = \int_0^\lambda s(\tau) \, d_\tau \left(h, \int_0^\tau s(\mu) \, d\rho^1(\mu)\right).$$

From this equality the second part of the theorem easily follows.

With the same $l_0 \in \mathbb{C}_0$ as above we define the $1 \times (n + p)$ matrix-valued function t^2 on $\mathbb{R}$ with components in $\mathfrak{D}(H_s)$ by

$$t^2(\lambda) = (H_s - l_0)(Q_s(\lambda) - Q_s(0)) \, G(l_0), \qquad \lambda \in \mathbb{R},$$

and we define the $(n + p) \times (n + p)$ matrix-valued function ρ^2 on $\mathbb{R}$ by

$$\rho^2(\lambda) = \Gamma(Pt^2(\lambda)), \qquad \lambda \in \mathbb{R}.$$

THEOREM 7.4. *The matrix-valued function ρ^2 is hermitian, nondecreasing and right continuous on $\mathbb{R}$ and $\rho^2(0) = O_{n+p}^{n+p}$. For each $h \in \mathfrak{H}$ and $\alpha, \beta \in \mathbb{R}$,*

$$P(Q_s(\beta) - Q_s(\alpha)) P_s h = \int_\alpha^\beta s(\lambda) \, d_\lambda \left(h, \int_0^\lambda s(\mu) \, d\rho^2(\mu)\right).$$

Proof. The proof of the first part of this theorem can be given along the same lines as the proof of the first part of Theorem 7.3. For, if $0, \lambda, \mu \in [\alpha, \beta]$ then

$$t^2(\lambda) - t^2(\mu) = (Q_s(\lambda) - Q_s(\mu))(H_s - l_0)(Q_s(\beta) - Q_s(\alpha)) \, G(l_0)$$

and

$$\rho^2(\lambda) - \rho^2(\mu) = (t^2(\lambda) - t^2(\mu), t^2(\lambda) - t^2(\mu)). \tag{7.12}$$

We shall prove the second part of the theorem by showing that

$$P(Q_s(\lambda) - Q_s(0)) P_s h = \int_0^\lambda s(\tau) \, d_\tau \left(h, \int_0^\tau s(\mu) \, d\rho^2(\mu)\right). \tag{7.13}$$

This equality evidently implies the equality of the theorem.

Let $u \in H(0)^{\perp}$. Then for $\lambda \geqslant \mu$,

$$(H_s - l_0)(H_s - \bar{l}_0)(Q_s(\lambda) - Q_s(\mu))u$$
$$= \sum_{\mu < \lambda_\nu \leqslant \lambda} (\lambda_\nu - l_0)(u, (\lambda_\nu - l_0) y_\nu) y_\nu . \qquad (7.14)$$

Hence the series in (7.14) converges in $H(0)^{\perp}$. An application of the continuous linear function $\Gamma(PR_s(l_0)\cdot)$ (cf. Lemma 7.1) to this series yields the convergence in $\mathbb{C}^{n+p}$ of

$$\sum_{\mu < \lambda_\nu \leqslant \lambda} \Gamma(Py_\nu)(u, (\lambda_\nu - l_0) y_\nu). \qquad (7.15)$$

As u is arbitrary in $H(0)^{\perp}$, it follows that the series

$$\sum_{\mu < \lambda_\nu \leqslant \lambda} (\lambda_\nu - l_0) y_\nu \Gamma(Py_\nu)^* \qquad (7.16)$$

is weakly convergent in $H(0)^{\perp} \oplus \cdots \oplus H(0)^{\perp}$ ($n + p$ copies). Since y_ν, $\nu = 0, \pm 1,...$, is an orthonormal system in $H(0)^{\perp}$ the series in fact converges in the norm. Again we apply $\Gamma(PR_s(l_0)\cdot)$ and obtain the convergence of the series of $(n + p) \times (n + p)$ matrices

$$\sum_{\mu < \lambda_\nu \leqslant \lambda} \Gamma(Py_\nu) \Gamma(Py_\nu)^*.$$

From (7.14) (with $u = G(l_0)$) we get for $\lambda \geqslant \mu$,

$$(H_s - l_0)(t^2(\lambda) - t^2(\mu)) = \sum_{\mu < \lambda_\nu \leqslant \lambda} (\lambda_\nu - l_0) y_\nu \Gamma(Py_\nu)^*. \qquad (7.17)$$

Applying $\Gamma(PR_s(l_0)\cdot)$ to both sides of (7.17) we obtain

$$\rho^2(\lambda) - \rho^2(\mu) = \Gamma(Pt^2(\lambda) - Pt^2(\mu)) = \sum_{\mu < \lambda_\nu \leqslant \lambda} \Gamma(Py_\nu) \Gamma(Py_\nu)^*.$$

The series on the right-hand side equals

$$\Gamma(PR_s(l_0)(Q_s(\lambda) - Q_s(\mu)) \sum_{\sigma < \lambda_\nu \leqslant \lambda} (\lambda_\nu - l_0) y_\nu \Gamma(Py_\nu)^*),$$

where $\sigma \leqslant \mu < \lambda$. Letting $\mu \to \lambda$ we obtain

$$\rho^2(\lambda) - \rho^2(\lambda - 0) = \sum_{\lambda_\nu = \lambda} \Gamma(Py_\nu) \Gamma(Py_\nu)^*.$$

Since

$$Py_\nu = s(\lambda_\nu) \Gamma(Py_\nu), \qquad \nu = 0, \pm 1,..., \qquad (7.18)$$

we have

$$\int_0^\tau s(\mu)\, d\rho^2(\mu) = \sum_{0<\lambda_\nu\leqslant\tau} s(\lambda_\nu)\, \Gamma(Py_\nu)\, \Gamma(Py_\nu)^*$$

$$= \sum_{0<\lambda_\nu\leqslant\tau} Py_\nu\Gamma(Py_\nu)^* \tag{7.19}$$

(the last series converges since it equals the series obtained from the application of $PR_s(l_0)$ to the series (7.16)).

Now in (7.15) let $u = R_s(\bar{l}_0)\, P_s h$. Using the fact that $R_s(l_0)^* = R_s(\bar{l}_0)$, we see that the series $\sigma(h, \lambda)$, defined by

$$\sigma(h, \lambda) - \sigma(h, \mu) = \sum_{\mu<\lambda_\nu\leqslant\lambda} \Gamma(Py_\nu)(h, y_\nu), \qquad \mu < \lambda,$$

$$\sigma(h, 0) = O_{n+p}^1\,,$$

converges in $\mathbb{C}^{n+p}$. A reasoning similar to the one used above leads to

$$\sigma(h, \lambda) - \sigma(h, \lambda - 0) = \sum_{\lambda_\nu=\lambda} \Gamma(Py_\nu)(h, y_\nu).$$

From this, (7.18) and (7.19), we derive that for $\lambda \geqslant 0$,

$$\int_0^\lambda s(\tau)\, d_\tau \left(h, \int_0^\tau s(\mu)\, d\rho^2(\mu)\right) = \int_0^\lambda s(\tau)\, d\sigma(h, \tau)$$

$$= \sum_{0<\lambda_\nu\leqslant\lambda} s(\lambda_\nu)\, \Gamma(Py_\nu)(h, y_\nu)$$

$$= \sum_{0<\lambda_\nu\leqslant\lambda} (h, y_\nu)\, Py_\nu$$

$$= P \sum_{0<\lambda_\nu\leqslant\lambda} (h, y_\nu)\, y_\nu$$

$$= P(Q_s(\lambda) - Q_s(\mu))\, P_s h.$$

For $\lambda < 0$ a similar derivation may be given. Hence (7.13) is valid and thus the proof of the theorem is complete.

We observe that if we apply $PR_s(l_0)$ to both sides of (7.17) (with $\mu = 0$, $\lambda = \tau$) then by (7.19) we obtain

$$Pt^2(\tau) = \int_0^\tau s(\mu)\, d\rho^2(\mu). \tag{7.20}$$

Combining Theorems 7.3 and 7.4, using the definition of $\{F(\lambda) \mid \lambda \in \mathbb{R}\}$, the generalized spectral family for S corresponding to H, we obtain that if $h \in \mathfrak{H}$, $\alpha, \beta \in \mathbb{R}$, then

$$(F(\beta) - F(\alpha))h = \int_\alpha^\beta s(\lambda)\, d_\lambda \left(h, \int_0^\lambda s(\mu)\, d\rho(\mu)\right),$$

where $\rho(\mu) = \rho^1(\mu) + \rho^2(\mu)$, $\mu \in \mathbb{R}$. If $h = f \in C_0(\iota)$ then (6.16) holds. Thus Theorem 6.2 has been proved.

We define $\hat{\mathfrak{H}} = \mathfrak{L}^2(\rho)$, the map $V: \mathfrak{H} \to \hat{\mathfrak{H}}$ and the subspaces $\mathfrak{H}_0$, $\mathfrak{H}_1$ and $\mathfrak{H}_2$ as in Section 6. Since the proofs of Theorems 6.3 and 6.4 are based on the above formula for $F(\lambda)$, proofs of these theorems using the approach presented in this section would be exactly the same and are therefore not repeated. However, a proof of Theorem 6.5 can be based on the machinery we have built up in this section and we now give this proof.

Second proof of Theorem 6.5. Let $t(\lambda) = t^1(\lambda) + t^2(\lambda)$, $\lambda \in \mathbb{R}$. Then t is a $1 \times (n + p)$ matrix-valued function on $\mathbb{R}$ whose components $t_j \in \mathfrak{D}(H_s) \subset H(0)^\perp$. By $\varDelta$ we shall denote finite left open and right closed subintervals of $\mathbb{R}$. If $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, are the endpoints of $\varDelta \subset \mathbb{R}$, then we put $t(\varDelta) = t(\beta) - t(\alpha)$ and $\rho(\varDelta) = \rho(\beta) - \rho(\alpha)$. Since $(t^1(\lambda), t^2(\mu)) = O_{n+p}^{n+p}$, we have on account of (7.10) and (7.12) that

$$\rho(\varDelta \cap \varDelta_1) = (t(\varDelta), t(\varDelta_1)), \qquad \varDelta, \varDelta_1 \subset \mathbb{R}. \tag{7.21}$$

From (7.11) and (7.20) it follows that

$$Pt(\varDelta) = \int_\varDelta s(\mu) \, d\rho(\mu). \tag{7.22}$$

Let $\chi(j, \varDelta)$ be the $(n + p) \times 1$ matrix-valued function in $\hat{\mathfrak{H}}$ whose jth component equals the characteristic function of $\varDelta$ while all other entries are zero, $j = 1, ..., n + p$, $\varDelta \subset \mathbb{R}$. The collection X of all such functions is fundamental in $\hat{\mathfrak{H}}$. We define $T: X \to H(0)^\perp$ by

$$T\chi(j, \varDelta) = t_j(\varDelta).$$

From the previous observation and from (7.21) we deduce that T may be extended by continuity to an isometry from $\hat{\mathfrak{H}}$ into $H(0)^\perp$. We denote this isometry by T also. We claim that $T = V^{-1}$ on $V\mathfrak{H}_0 \subset \hat{\mathfrak{H}}$.

To prove this claim, let $h \in \mathfrak{H}_0 \cap C_0(\iota)$, $\epsilon > 0$ and J, a compact subinterval of ι, be fixed. We choose $\varDelta \subset \mathbb{R}$ such that

$$\left\| h - \int_\varDelta s(\lambda) \, d\rho(\lambda)(Vh)(\lambda) \right\|_J = \left\| (I - F(\varDelta))h \right\|_J \leqslant \epsilon \tag{7.23}$$

and

$$\| PT(Vh)_\varDelta - PTVh \|_J \leqslant \epsilon, \tag{7.24}$$

where $(Vh)_\varDelta$ equals Vh on $\varDelta$ and O_{n+p}^1 outside $\varDelta$. We observe that for J and this fixed interval $\varDelta$ there exists a constant $M \geqslant 0$ such that

$$\left\| \left(\int_\varDelta s(x, \lambda) \, d\rho(\lambda) \, s(x, \lambda)^* \right)^{1/2} \right\|_J \leqslant M.$$

Let η_k, $k = 1, 2, \dots$, be a sequence of linear combinations of elements in X having support in Δ such that $\eta_k \to (Vh)_\Delta$, as $k \to \infty$. On account of (7.22) we have

$$PT\eta_k = \int_\Delta s(\lambda) \, d\rho(\lambda) \, \eta_k(\lambda). \tag{7.25}$$

We choose k so large that

$$\left\| \int_\Delta s(\lambda) \, d\rho(\lambda)(Vh)(\lambda) - \int_\Delta s(\lambda) \, d\rho(\lambda) \, \eta_k(\lambda) \right\|_J \leqslant M \, \|(Vh)_\Delta - \eta_k \|_\rho \leqslant \epsilon$$

$$. \tag{7.26}$$

and

$$\| PT\eta_k - PT(Vh)_\Delta \|_J \leqslant \epsilon, \tag{7.27}$$

where $\| \; \|_\rho$ denotes the norm in $\hat{\mathfrak{H}}$. Using (7.25) and the triangle inequality we see that $\| h - PTVh \|_J$ is bounded by the sum of the terms on the left-hand sides of the inequalities (7.23), (7.24), (7.26) and (7.27). Therefore $\| h - PTVh \|_J \leqslant 4\epsilon$, and since ϵ and J have been arbitrarily chosen, $PTVh = h$ in $\mathfrak{H}_0 \cap C_0(\iota)$. Since T on $\hat{\mathfrak{H}}$ and V on $\mathfrak{H}_0$ are isometries, $\| PTVh \| = \| h \| = \| TVh \|$ and thus $TVh = h$. Using the continuity of T and V we see that $TVh = h$ holds for all $h \in \mathfrak{H}_0$, which shows that our claim is true.

Assume that $V\mathfrak{H}_0 = \hat{\mathfrak{H}}$. Then $t_j(\Delta) = V^{-1}\chi(j, \Delta) \in \mathfrak{H}_0$, and since X is fundamental in $\hat{\mathfrak{H}}$, the set $\{t_j(\Delta) \mid \Delta \in \mathbb{R}, j = 1, \dots, n + p\}$ is fundamental in $\mathfrak{H}_0$. From the definitions of t^1, t^2 and t it follows that $E_s(\Delta_1) \, t_j(\Delta) = t_j(\Delta \cap \Delta_1)$, $\Delta, \Delta_1 \subset \mathbb{R}, j = 1, \dots, n + p$, which shows that $E_s(\lambda)$, $\lambda \in \mathbb{R}$, maps $\mathfrak{H}_0$ into $\mathfrak{H}_0$, and since the surjectivity of V implies that $\mathfrak{H}_2 = \{0\}$ we have that

$$\{E(\lambda)f \mid f \in \mathfrak{H}, \lambda \in \mathbb{R}\} \cup \mathfrak{H} = \{E_s(\lambda)h \mid h \in \mathfrak{H}_0, \lambda \in \mathbb{R}\} \cup \mathfrak{H}_1 \subset \mathfrak{H} \subset \mathfrak{R}.$$

Now, H is minimal, which means that the set on the left-hand side is fundamental in $\mathfrak{R}$. Thus we see that $\mathfrak{R} = \mathfrak{H}$, and hence that the "only if" part of Theorem 6.5 holds true.

To prove the converse, let H be a self-adjoint subspace extension of S in $\mathfrak{H}^2$. Then $\mathfrak{H}_0 = \mathfrak{H} \ominus H(0)$ and T is an isometry which maps $\hat{\mathfrak{H}}$ into $\mathfrak{H}_0$. Let $\eta \in \hat{\mathfrak{H}}$ be such that $(Vh, \eta)_\rho = 0$ for all $h \in \mathfrak{H}_0$, where $(\; , \;)_\rho$ denotes the inner product in $\hat{\mathfrak{H}}$. Then, since $T = V^{-1}$ on $V\mathfrak{H}_0$ and thus

$$0 = (Vh, \eta)_\rho = (TVh, T\eta) = (h, T\eta),$$

we see that $T\eta = 0$, which implies $\eta = 0$. Hence V is surjective.

Some special cases of Theorem 6.3. Let $H = H_s \oplus H_\infty$ be a minimal self-adjoint subspace extension of S in $\mathfrak{R}^2$ and suppose that H_s has a pure

point spectrum. Then, for all $\lambda \in \mathbb{R}$, $P_s(\lambda) = 0$ and hence the matrix ρ entering in the eigenfunction expansion equals ρ^2 which consists of step functions only. It follows from the definitions of $F(\lambda)$, $Q_s(\lambda)$ and (7.18) that for all $h \in \mathfrak{H} \cap C_0(\iota)$ and $\alpha, \beta \in \mathbb{R}$, $\alpha \leqslant \beta$,

$$
\begin{aligned}
(F(\beta) - F(\alpha))h &= P(Q_s(\beta) - Q_s(\alpha)) \, P_s h \\
&= \sum_{\alpha < \lambda_\nu \leqslant \beta} (h, y_\nu) \, Py_\nu \\
&= \sum_{\alpha < \lambda_\nu \leqslant \beta} \Gamma(Py_\nu)^* \, \hat{h}(\lambda_\nu) \, s(\lambda_\nu) \, \Gamma(Py_\nu),
\end{aligned}
$$

and hence

$$
PP_s h = \sum_\nu \Gamma(Py_\nu)^* \, \hat{h}(\lambda_\nu) \, s(\lambda_\nu) \, \Gamma(Py_\nu),
$$

where the series converges in $\mathfrak{H} = \mathfrak{L}^2(\iota)$.

If H is an extension of S in $\mathfrak{H}^2$, then H is automatically minimal and we must have $\omega^+ = \omega^-$. If furthermore, $\omega^+ = \omega^- = n$, the order of the differential operator L, then it is clear from (6.2) and (6.3) that the components of the matrix-valued functions $s(l) = (s^1(l) : s^2(l))$ are in $\mathfrak{L}^2(\iota)$ for all $l \in \mathbb{C}_0$. From the representation of the kernel $K = K_0 + K_1$ of the resolvent R of H in $\mathfrak{H}^2$ in terms of $s(l)$, described in Section 6, it follows that $R(l)$ is an integral operator of Hilbert–Schmidt type. This implies that the spectrum of H_s is a pure point spectrum. In particular this is true for self-adjoint problems in the regular case which is described in detail in Section 4.

For the regular case let H be a self-adjoint extension of S described as in Theorem 4.1, and let $\lambda \in \mathbb{R}$ and $y \in \mathfrak{H} = \mathfrak{L}^2(\iota)$ be such that $H_s y = \lambda y$. We shall indicate how the eigenvalue λ and $\Gamma(y)$ can be determined.

We split the matrices $\{\sigma, \tau\}$, whose elements form a basis for B, and $u(l)$ defined by (6.2) into two parts:

$$
\{\sigma, \tau\} = (\{\sigma^1, \tau^1\} : \{\sigma^2, \tau^2\}), \qquad u(l) = (u^1(l) : u^2(l)),
$$

where $\{\sigma^1, \tau^1\}$ is as in Theorem 4.1, the elements of $\{\sigma^2, \tau^2\}$ form a basis for B_2 such that $L\tau^2 + \sigma^2 = (\Phi_0 : \Phi_1)$, $(L - l) \, u^1(l) = \sigma^1 + l\tau^1$, and $(L - l) \, u^2(l) = \sigma^2 + l\tau^2$, $l \in \mathbb{C}$. We define $t(l) = (s^1(l) : u^1(l) : u^2(l) + \tau^2)$.

By Theorem 4.1 the eigenfunction y has the form $y = h + \tau^1 c_1$ and by (4.9),

$$
h + \tau^1 c_1 = s(\lambda) \begin{pmatrix} d \\ c_1 \\ (Lh - \sigma^1 c_1, \Phi_0) \\ -(C : D : -A_2{}^* : G^* + A_2{}^* T_1) \, h^1 - (h + \tau^1 c_1, \Psi) \end{pmatrix}
$$

$$
\tag{7.28}
$$

for some constant $n \times 1$ matrix d. By (7.18), where now P is the identity operator on $\mathfrak{H}$, we have

$$h + \tau^1 c_1 = s(\lambda)\, \Gamma(y). \qquad \bullet \qquad (7.29)$$

We claim that $\Gamma(y)$ equals the $q \times 1$ matrix on the right-hand side of (7.28). To see this, it suffices to prove that if

$$s^1(\lambda)a + (u^1(\lambda) + \tau^1)b + (u^2(\lambda) + \tau^2)c = 0$$

for some constant $n \times 1$, $(p - m) \times 1$ and $m \times 1$ matrices a, b and c, then all three matrices are zero. Now, since the elements of $t(l)$ belong to $\mathfrak{D}(S_0{}^*)$, the equality implies $\tau^1 b \in \mathfrak{D}(S_0{}^*)$. From the definition of B_1 this implies $b = 0$. Consequently, $s^1(\lambda)a + (u^2(\lambda) + \tau^2)c = 0$. Applying $(L - \lambda)$ to both sides of this equality, we get that $(\Phi_0 : \Phi_1)c = 0$, and hence that $c = 0$ since the elements of $(\Phi_0 : \Phi_1)$ are linearly independent. Finally $s^1(\lambda)a = 0$ implies $a = 0$ on account of the initial conditions (6.2) for $s^1(l)$.

From the claim we deduce that

$$c_1 = \mathscr{J}_1\Gamma(y), \qquad (7.30)$$

and

$$(C : D : -A_2{}^* : G^* + A_2{}^*T_1)h^1 + (h + \tau^1 c_1\, , \Psi) + \mathscr{J}_2\Gamma(y) = 0, \quad (7.31)$$

where

$$\mathscr{J}_1 = (O_{p-m}^n : I_{p-m} : O_{p-m}^m), \qquad \mathscr{J}_2 = (O_{m-s}^{q+s-m} : I_{m-s}).$$

From (7.29) and (7.30) we deduce that

$$h = t(\lambda)\, \Gamma(y),$$

which implies that

$$h^1 = \begin{pmatrix} \check{t}(a, \lambda) \\ \check{t}(b, \lambda) \\ (\{t(\lambda), Lt(\lambda)\}, \{\sigma^1, \tau^1\}) \\ \mathscr{J}_1 \end{pmatrix} \Gamma(y). \qquad (7.32)$$

Let $\Lambda(\lambda)$ be the $q \times q$ matrix defined by

$$\Lambda(\lambda) = \begin{pmatrix} (s(\lambda), \Phi_0) \\ M\check{t}(a, \lambda) + N\check{t}(b, \lambda) - A_1{}^*(\{t(\lambda), Lt(\lambda)\}, \{\sigma^1, \tau^1\}) \\ + (F^* + A_1{}^*T_1)\, \mathscr{J}_1 + (s(\lambda), Z) \\ \mathscr{J}_2 + C\check{t}(a, \lambda) + D\check{t}(b, \lambda) - A_2{}^*(\{t(\lambda), Lt(\lambda)\}, \{\sigma^1, \tau^1\}) \\ + (G^* + A_2{}^*T_1)\, \mathscr{J}_1 + (s(\lambda), \Psi) \end{pmatrix}.$$

Then it follows from the first two defining relations for $\mathfrak{D}(H)$ in (4.8) combined with (7.28), (7.32) and from (7.31) combined with (7.20), (7.32) that

$$\det \varLambda(\lambda) = 0, \tag{7.33}$$

and

$$\varLambda(\lambda)\, \varGamma(y) = 0. \tag{7.34}$$

Hence we have shown that if $H_s y = \lambda y$ for some $\lambda \in \mathbb{R}$ then (7.33) and (7.34) hold. Conversely, if for some $\lambda \in \mathbb{R}$ (7.33) holds then λ is an eigenvalue for H_s, and if for some nonzero constant $q \times 1$ matrix $\varGamma$ we have $\varLambda(\lambda)\varGamma = 0$, then $y = s(\lambda)\varGamma$ is an eigenfunction associated with λ and $\varGamma = \varGamma(y)$. This can be shown by tracing the above argument in the opposite direction.

REFERENCES

1. E. A. CODDINGTON, The spectral matrix and Green's function for singular self-adjoint boundary value problems, *Canad. J. Math.* **6** (1954), 169–185.
2. E. A. CODDINGTON, Generalized resolutions of the identity for symmetric ordinary differential operators, *Ann. of Math.* **68** (1958), 378–392.
3. E. A. CODDINGTON, Extension theory of formally normal and symmetric subspaces, *Mem. Amer. Math. Soc.* **134** (1973).
4. E. A. CODDINGTON, Self-adjoint subspace extensions of nondensely defined symmetric operators, *Advances in Math.* **14** (1974), 309–332.
5. E. A. CODDINGTON, Self-adjoint problems for nondensely defined ordinary differential operators and their eigenfunction expansions, *Advances in Math.* **15** (1974), 1–40.
6. A. DIJKSMA AND H. S. V. DE SNOO, Eigenfunction expansions for nondensely defined differential operators, *J. Differential Equations*, **17** (1975), 198–219.
7. A. DIJKSMA AND H. S. V. DE SNOO, Self-adjoint extensions of symmetric subspaces, *Pacific J. Math.* **54** (1974), 71–100.
8. I. HALPERIN, Closures and adjoints of linear differential operators, *Ann. of Math.* **38** (1937), 880–919.
9. T. KATO, "Perturbation Theory for Linear Operators," Springer-Verlag, New York, 1966.
10. A. M. KRALL, Stieltjes differential-boundary operators, *Proc. Amer. Math. Soc.* **41** (1973), 80–86.
11. J. LOCKER, Self-adjointness for multi-point differential operators, *Pacific J. Math.* **45** (1973), 561–570.
12. A. E. TAYLOR, "Introduction to Functional Analysis," Wiley, New York, 1958.
13. O. VEJVODA AND M. TVRDÝ, Existence of solutions to a linear integro-boundary-differential equation with additional conditions, *Ann. di Mat. Pura ed Appl.* **89** (1971), 169–216.
14. H. J. ZIMMERBERG, Linear integro-differential-boundary-parameter problems, *Ann. di Mat. Pura ed Appl.* **177** (1975), 241–256.