Group Invariance in Mathematical Morphology

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Group Invariance in Mathematical Morphology

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Abstract In this paper we discuss how invariance of operators arising in binary mathematical morphology can be achieved for the collection of groups commonly denoted as 'the computer vision groups'. We present an overview, starting with set mappings such as dilations, erosions, openings and closings, which are invariant under the group of Euclidean translations. This can be trivially extended to other abelian groups such as the group of rotations and scalar multiplications ('polar morphology'). Then we go to arbitrary transitive group actions on the plane. All these cases are discussed within a common framework, using the theory of morphological operators on homogeneous spaces developed previously by the author.

Keywords: Mathematical morphology, image processing, transitive group action, Minkowski operations, complete lattice, invariance, computer vision.

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1 Introduction

Mathematical morphology is a set-theoretical approach to image analysis [5, 13], studying image transformations with a simple geometrical interpretation. A two-dimensional binary image is modeled as a subset X of the plane. To analyze the image it is probed by translating small subsets B, called *structuring elements*, of various forms and sizes over the image plane and recording the locations where certain relations between the image X and translates of the structuring element B are satisfied. In this way one obtains image transformations which are invariant under the Euclidean translation group. The basic 'object space' is the Boolean algebra of subsets of the the image plane. In the case of grey-level images a lattice formulation is required, see [3, 14]. We will restrict ourselves to binary images in this paper.

Now translation invariance is not always appropriate (nor sufficient). In computer vision an important question is how to take the projective geometry of the imaging process into account. Here one requires invariance under other groups, such as the Euclidean motion group, the similarity group, the affine group or the

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projective group, which are all non-commutative groups. For general questions of invariance in computer vision, see for example [6]. In this paper we discuss the construction of morphological operators invariant under the groups mentioned above, and give some examples of such transformations. This is based on a generalization of classical morphology to arbitrary homogeneous spaces developed by the author [8, 9]. The essential difficulty to be solved here was the problem how to deal with the non-commutativity of the acting group. The simpler case of a space with an *abelian* acting group constitutes a straightforward generalization of Euclidean morphology, see [2, 11]. We mention two examples: (i) (Polar morphology) For images with a polar symmetry one needs image transformations invariant under the polar group generated by rotations and scalar multiplications, which is abelian. Here the size of the structuring element increases with increasing distance from the origin. (ii) (Perspective morphology) Consider the perspective transformation of a planar shape lying in an object plane V. Three-dimensional Euclidean motions of the plane V induce projective transformations on the image plane under perspective projection. Subgroups of the projective group apply when the motion of the planar object is constrained. An example is that of translations only within the object plane, in which case the induced group is abelian, and a group multiplication between points of the image plane can be defined. The interested reader is referred to [10] for more details.

The main goal of this paper is to show the existence — not always obvious — of nontrivial morphological transformations invariant under increasingly larger groups. Although this requires the development of a certain amount of mathematical machinery before we can consider the application to invariant feature extraction, the motivating ideas are quite simple. To illustrate this, let us take a glance ahead at Fig. 1 in Section 3. In Fig. 1(a) is shown a figure containing a number of quadrangles. A typical morphological operation is an opening, which extracts from the input image all structures which are 'similar' to the structuring element, in this case a square. Here 'similar' means: obtainable from the square by a certain group operation. Clearly this depends on the transformation group considered. For example, when the group is that of the translations, the opening extracts all translates of the square, see Fig. 1(b). When the group is enlarged, one gradually recovers the various geometric shapes present in the image, see Fig. 1(c-f). The practical significance of this is that the morphological operations to be applied to the image for feature extraction can be adapted to the type of geometric invariance which is deemed to be appropriate for the application under consideration.

The organization of the paper is as follows. First we introduce in Section 2 the general framework of mathematical morphology on Euclidean space, followed by its generalization to arbitrary homogeneous spaces. Then in Section 3 the application to invariant feature extraction is studied. A summary and conclusions are given in Section 4.

2 Morphology on homogeneous spaces

2.1 Euclidean morphology

For any set E, denote by $\mathcal{P}(E)$ the power set of E. Let $E = \mathbb{R}^n$ or $E = \mathbb{Z}^n$. The classical Minkowski addition and subtraction for subsets X, A of E are given by

$$X \oplus A = \bigcup_{a \in A} X_a, \quad X \ominus A = \bigcap_{a \in A} X_{-a}, \tag{1}$$

where

$$X_a = \tau_a(X) = \{x + a : x \in X\}, \tag{2}$$

is the translate of X over the vector $a \in E$, $x + y$ is the sum of x and y, and $-x$ the reflection of x. The transformations $\delta_A : X \mapsto X \oplus A$ and $\epsilon_A : X \mapsto X \ominus A$ are called a *dilation* and *erosion* by the structuring element A, respectively. It can be shown that $X \oplus A = \{h \in E : \check{A}_h \Uparrow X\}$, where $\check{A} = \{-a : a \in A\}$ is the *reflection* of A and $A \Uparrow B$ (A 'hits' B) is a general notation for $A \cap B \neq \emptyset$. An important property of dilation is:

$$Translation\ invariance: \qquad (X \oplus A)_h = X_h \oplus A. \tag{3}$$

A similar property holds for the erosion. Dilation and erosion are *increasing* mappings. (A mapping ψ is called increasing when for all $X, Y \in \mathcal{P}(E)$, $X \subseteq Y$ implies that $\psi(X) \subseteq \psi(Y)$.)

Other important increasing transformations are the opening and closing by a set A (X^c is the complement of X):

$$\textit{Opening}: \qquad X \circ A := (X \ominus A) \oplus A = \bigcup_{h \in E} \{A_h : A_h \subseteq X\} \tag{4}$$

$$\textit{Closing}: \qquad X \bullet A := (X \oplus A) \ominus A = \bigcap_{h \in E} \{(\check{A}^c)_h : (\check{A}^c)_h \supseteq X\}. \tag{5}$$

The opening is the union of all the translates of the structuring element which are included in the set X. Opening and closing are related by Boolean duality: $(X^c \circ A)^c = X \bullet \check{A}$.

Note that $X \circ A = \delta_A \epsilon_A(X)$, $X \bullet A = \epsilon_A \delta_A(X)$, i.e., an opening is the product of an erosion followed by a dilation, and vice versa for a closing.

2.2 Generalized Minkowski operators

On any group Γ one can define generalizations of the Minkowski operations [9]. By definition a *dilation* (*erosion*) is a mapping commuting with unions (intersections). For a fixed subset H (the structuring element) of Γ, define the dilation δ_H^λ and erosion ε_H^λ by

$$\delta_H^\lambda(G) := G \overset{\Gamma}{\oplus} H := \bigcup_{h \in H} Gh = \bigcup_{g \in G} gH, \tag{6}$$

$$\varepsilon_H^\lambda(G) := G \overset{\lambda}{\ominus} H := \bigcap_{h \in H} Gh^{-1}, \tag{7}$$

which generalizes the Minkowski addition and subtraction to non-commutative groups. Here $gH := \{gh : h \in H\}, \quad Gh := \{gh : g \in G\}$, with gh the group product of g and h, and h^{-1} the group inverse of h. Both mappings are *left-invariant*, i.e., $\delta_H^\lambda(gG) = g\delta_H^\lambda(G), \quad \varepsilon_H^\lambda(gG) = g\varepsilon_H^\lambda(G), \quad \forall g \in \Gamma$. This is the reason for the superscript 'λ' on the '$\ominus$' symbol.

Remark 1. Because of the non-commutativity of the set product there is another possibility to introduce a generalized dilation and erosion. The *right-invariant* dilation δ_H^ρ and erosion ε_H^ρ by the structuring element H are the mappings $\mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ defined by

$$\delta_H^\rho(G) := H \overset{\rho}{\oplus} G := \bigcup_{h \in H} hG = \bigcup_{g \in G} Hg,$$
$$\varepsilon_H^\rho(G) := G \overset{\rho}{\ominus} H := \bigcap_{h \in H} h^{-1}G.$$

At this point we would like to mention the connection to the theory of residuated lattices and ordered semigroups, which is explained in more detail in [9]. Only left-invariant dilations and erosions will be used in the sequel of this paper.

For later use we also define the *inverted* set G^{-1} of G by

$$G^{-1} = \{g^{-1} : g \in G\}. \tag{8}$$

Duality by complementation is expressed by the formula $(G \overset{\rho}{\oplus} H)^c = G^c \overset{\lambda}{\ominus} H^{-1}$.

Of fundamental importance is the concept of *adjunction*, cf. [3]. This requires the notion of complete lattices. A general introduction to lattice theory is Birkhoff [1].

Definition 1. *A* complete lattice $(\mathcal{L}, \leq)$ *is a partially ordered set* $\mathcal{L}$ *with order relation* $\leq$*, a supremum or join operation written* $\bigvee$ *and an infimum or meet operation written* $\bigwedge$*, such that every (finite or infinite) subset of* $\mathcal{L}$ *has a supremum (smallest upper bound) and an infimum (greatest lower bound). In particular there exist two universal bounds, the least element written* $O_\mathcal{L}$ *and the greatest element* $I_\mathcal{L}$*.*

In the case of the power lattice $\mathcal{P}(E)$ of all subsets of a set E, the order relation is set-inclusion $\subseteq$, the supremum is the union $\bigcup$ of sets, the infimum is the intersection $\bigcap$ of sets, the least element is the empty set $\emptyset$ and the greatest element is the set E itself.

An *atom* is an element X of a lattice $\mathcal{L}$ such that for any $Y \in \mathcal{L}, O_\mathcal{L} \leq Y \leq X$ implies that $Y = O_\mathcal{L}$ or $Y = X$. A complete lattice $\mathcal{L}$ is called *atomic* if every element of $\mathcal{L}$ is the supremum of the atoms less than or equal to it. It is called *Boolean* if (i) it satisfies the distributivity laws $X \sup(Y \inf Z) = (X \sup Y) \inf(X \sup Z)$ and $X \inf(Y \sup Z) = (X \inf Y) \sup(X \inf Z)$ for all $X, Y, Z \in \mathcal{L}$, and (ii) every element X has a unique complement X^c, defined by $X \sup X^c = I_\mathcal{L},\ X \inf X^c = O_\mathcal{L}$. The power lattice $\mathcal{P}(E)$ is an atomic complete Boolean lattice, and conversely any atomic complete Boolean lattice has this form.

Let $\mathcal{L}$ and $\tilde{\mathcal{L}}$ be complete lattices. A mapping $\psi : \mathcal{L} \to \tilde{\mathcal{L}}$ is called *increasing* (*isotone, order-preserving*) when $X \leq Y \Longrightarrow \psi(X) \leq \psi(Y)$ for all $X, Y \in \mathcal{L}$. A *dilation* $\delta : \mathcal{L} \to \tilde{\mathcal{L}}$ is a mapping commuting with suprema. An *erosion* $\epsilon : \mathcal{L} \to \tilde{\mathcal{L}}$ is a mapping commuting with infima.

An *automorphism* of $\mathcal{L}$ is a bijection $\psi : \mathcal{L} \to \mathcal{L}$ such that for any $X, Y \in \mathcal{L}$, $X \leq Y$ if and only if $\psi(X) \leq \psi(Y)$. When a group Γ is an automorphism group of both $\mathcal{L}$ and $\tilde{\mathcal{L}}$, a mapping $\psi : \mathcal{L} \to \tilde{\mathcal{L}}$ is called *Γ-invariant* or a *Γ-mapping* if it commutes with all $\tau \in \Gamma$, i.e., if $\psi(\tau(X)) = \tau(\psi(X))$ for all $X \in \mathcal{L}$, $\tau \in \Gamma$. Accordingly, we will speak below of Γ-dilations, Γ-erosions, etc. If no invariance under a group is required, one may set $\Gamma = \{id_{\mathcal{L}}\}$, the identity operator on $\mathcal{L}$.

Remark 2. In contrast to binary images, where one can work with the power lattice $\mathcal{P}(E)$, non-Boolean complete lattices are needed when one is interested in convex subsets of the plane or grey-level images [3, 14].

Definition 2. *Let $\epsilon : \mathcal{L} \to \tilde{\mathcal{L}}$ and $\delta : \tilde{\mathcal{L}} \to \mathcal{L}$ be two mappings, where $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are complete lattices. Then the pair (ϵ, δ) is called an* adjunction between $\mathcal{L}$ and $\tilde{\mathcal{L}}$, *if for every $X \in \tilde{\mathcal{L}}$ and $Y \in \mathcal{L}$, the following equivalence holds:*

$$\delta(X) \leq Y \iff X \leq \epsilon(Y). \tag{9}$$

If $\tilde{\mathcal{L}}$ coincides with $\mathcal{L}$ we speak of an adjunction on $\mathcal{L}$.

If (ϵ, δ) is an adjunction, then ϵ is an erosion and δ is a dilation (ϵ and δ are called 'adjoints' of each other).

The dilation δ_H^λ and erosion ε_H^λ as defined above form an adjunction on $\mathcal{P}(\Gamma)$. If Γ is the Euclidean translation group one recovers the operators δ_A, ϵ_A on $\mathcal{P}(E)$ of Section 2.1.

In the rest of this paper we restrict ourselves to Boolean lattices, as is appropriate for binary image processing.

2.3 Group actions

Let $\mathcal{X}$ be a non-empty set, Γ a transformation group on $\mathcal{X}$, that is, each element $g \in \Gamma$ is a mapping $g : \mathcal{X} \to \mathcal{X}$, satisfying

$$(i) \quad gh(x) = g(h(x)) \quad (ii) \quad e(x) = x, \tag{10}$$

where e is the unit element of Γ, and gh denotes the product of two group elements g and h. Instead of $g(x)$ we will also write gx. We say that Γ is a *group action* on $\mathcal{X}$ [7, 15]. The group Γ is called *transitive on $\mathcal{X}$* if for each $x, y \in \mathcal{X}$ there is a $g \in \Gamma$ such that $gx = y$, and *simply transitive* when this element g is unique. A *homogeneous space* is a pair $(\Gamma, \mathcal{X})$ where Γ is a group acting transitively on $\mathcal{X}$. Any transitive abelian permutation group Γ is simply transitive. If Γ acts on $\mathcal{X}$, the *stabilizer* of $x \in \mathcal{X}$ is the subgroup $\Gamma_x := \{g \in$

$\Gamma : gx = x\}$. Let ω be an arbitrary but fixed point of $\mathcal{X}$, henceforth called the *origin.* The stabilizer Γ_ω will be denoted by Σ from now on:

$$\Sigma := \Gamma_\omega = \{g \in \Gamma : g\omega = \omega\}. \tag{11}$$

The set of group elements which map ω to a given point x is called a *left coset* and denoted by $g_x\Sigma := \{g_x s : s \in \Sigma\}$. Here g_x is a representative (an arbitrary element) of this coset.

2.4 Examples

In the following we present three examples. In each case Γ denotes the group and $\mathcal{X}$ the corresponding set.

Example 1. $\mathcal{X}$ = Euclidean space $\mathbb{R}^n$, Γ = the Euclidean translation group $\mathbf{T}$. $\mathbf{T}$ is abelian, therefore it can be identified with $\mathcal{X}$ [9]. Elements of $\mathbf{T}$ can be parameterized by vectors $h \in \mathbb{R}^n$, with τ_h the translation over the vector h: $\tau_h x = x + h, \quad h, x \in \mathbb{R}^n$.

Example 2. $\mathcal{X} = \mathbb{R}^2 \setminus \{0\}$, Γ = the abelian group generated by rotations and scalar multiplication w.r.t. the origin. In this case points of $\mathcal{X}$ can be given in polar coordinates $(r, \theta), r > 0, 0 \leq \theta < 2\pi$. Again Γ can be identified with $\mathcal{X}$ and the group multiplication is $(r_1, \theta_1) * (r_2, \theta_2) = (r_1 r_2, \theta_1 + \theta_2)$, cf. [11].

Example 3. $\mathcal{X}$ = Euclidean space $\mathbb{R}^n$ $(n \geq 2)$, Γ = the Euclidean motion group $\mathbf{M} := E^+(3)$ (proper Euclidean group, group of rigid motions) [8]. The subgroup leaving a point p fixed is the set of all rotations around that point. $\mathbf{M}$ is not abelian. The collection of translations forms the Euclidean translation group $\mathbf{T}$. The stabilizer, denoted by $\mathbf{R}$, equals the group S^1 of rotations around the origin. Let τ_h denote the translation over the vector $h \in \mathbb{R}^2$ and ρ_ϕ^p the rotation over an angle ϕ around the point p. Any element of $\mathbf{M}$ can be written in the form $\gamma_{h,\phi}$ where $\gamma_{h,\phi} = \tau_h \rho_\phi^0, \quad h \in \mathbb{R}^2, \ \phi \in \ S^1$, that is, a rotation around the origin followed by a translation. For a simple geometrical representation, see [8, 9].

2.5 Morphological operations on homogeneous spaces

Our interest is in defining Γ-invariant morphological operators $\mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$, where $(\Gamma, \mathcal{X})$ is a homogeneous space.

Definition 3 ((Γ-invariance)). *A mapping $\psi : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ is called Γ-*invariant *or a Γ-*mapping *if $\psi(gX) = g\psi(X)$ for all $X \in \mathcal{P}(\mathcal{X}), \ g \in \Gamma$.*

Accordingly, we will speak below of Γ-dilations, Γ-erosions, etc. Let the 'origin' ω be an arbitrary point of $\mathcal{X}$.

Definition 4. *The* lift $\vartheta : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\Gamma)$ *and* canonical projection $\pi : \mathcal{P}(\Gamma) \to \mathcal{P}(\mathcal{X})$ *are defined by*

$$\vartheta(X) = \{g \in \Gamma : g\omega \in X\}, \quad X \subseteq \mathcal{X} \tag{12}$$

$$\pi(G) = \{g\omega : g \in G\}, \quad G \subseteq \Gamma. \tag{13}$$

The mapping ϑ associates to each subset X all group elements which map the origin ω to an element of X. For the case of the Euclidean motion group (Example 3), these formulas specialize to

$$\vartheta(X) = \bigcup_{x \in X} \tau_x \mathbf{R} = \tau(X) \overset{\mathbf{M}}{\oplus} \mathbf{R}, \tag{14}$$

where $\tau(X) := \{\tau_x : x \in X\}$. The mapping π associates to each subset G of Γ the collection of all points $g\omega$ where g ranges over G. We also need to introduce a modified projection as follows.

Definition 5. *Let π be the projection (13) and $\tilde{\epsilon}_\Sigma$ the erosion $\tilde{\epsilon}_\Sigma(G) = G \overset{\lambda}{\ominus} \Sigma$. Then $\pi_\Sigma : \mathcal{P}(\Gamma) \to \mathcal{P}(\mathcal{X})$ is the* modified projection *defined by*

$$\pi_\Sigma = \pi \tilde{\epsilon}_\Sigma. \tag{15}$$

The operators ϑ, π and π_Σ have several useful properties [9]. The most important ones are:

1. $\pi,\ \vartheta,\ \pi_\Sigma$ are increasing and Γ-invariant;
2. ϑ and π commute with unions, ϑ and π_Σ commute with intersections;
3. $\pi\vartheta = id_{\mathcal{P}(\mathcal{X})}$; $\pi_\Sigma\vartheta = id_{\mathcal{P}(\mathcal{X})}$; ($id_\mathcal{L}$ is the identity operator on $\mathcal{L}$)
4. $X \subseteq Y \iff \vartheta(X) \subseteq \vartheta(Y)$;
5. (ϑ, π) forms an adjunction between $\mathcal{P}(\mathcal{X})$ and $\mathcal{P}(\Gamma)$;
6. (π_Σ, ϑ) forms an adjunction between $\mathcal{P}(\Gamma)$ and $\mathcal{P}(\mathcal{X})$.

There is a general construction of Γ-invariant operators [8,9]. Given a mapping ψ on $\mathcal{P}(\mathcal{X})$ we 'lift' it to a mapping $\tilde{\psi}$ on $\mathcal{P}(\Gamma)$. Then we apply the results of Section 2.2 on $\mathcal{P}(\Gamma)$ and finally 'project' the results back to $\mathcal{P}(\mathcal{X})$. The following results are quoted from [9].

Definition 6. *Let Γ be a permutation group on $\mathcal{X}$, with Σ the stabilizer of the origin ω in $\mathcal{X}$. A subset X of $\mathcal{X}$ is called Σ-*invariant *if $X = \overline{X}$, where $\overline{X} := \Sigma X = \bigcup_{s \in \Sigma} sX$. If X is not Σ-invariant, $\overline{X}$ is called the Σ-*invariant extension *of X.*

Proposition 1 ((Representation of dilations and erosions)).
The pair (ϵ, δ) is a Γ-invariant adjunction on $\mathcal{P}(\mathcal{X})$ if and only if, for some $Y \in \mathcal{P}(\mathcal{X})$, it is true that

$$\delta(X) = \delta_Y^\Gamma(X) := \pi[\vartheta(X) \overset{\Gamma}{\oplus} \vartheta(Y)] = \pi[\vartheta(X) \overset{\Gamma}{\oplus} \vartheta(\overline{Y})] \tag{16}$$

$$\epsilon(X) = \epsilon_Y^\Gamma(X) := \pi_\Sigma[\vartheta(X) \overset{\lambda}{\ominus} \vartheta(Y)] = \pi[\vartheta(X) \overset{\lambda}{\ominus} \vartheta(\overline{Y})], \tag{17}$$

where $\overline{Y}$ is the Σ-invariant extension of Y. In particular, $(\epsilon_Y^\Gamma, \delta_Y^\Gamma)$ is invariant under the substitution $Y \to \overline{Y}$.

The proposition above shows that any dilation on $\mathcal{P}(\mathcal{X})$ can be reduced to a dilation δ_Y^Γ involving a *Σ-invariant* structuring element Y; the same is true for erosions. Next we consider openings and closings.

Definition 7. *The* structural *opening* $\gamma_Y^\Gamma(X)$ *and closing* $\phi_Y^\Gamma(X)$ *by a subset* $Y \subseteq \mathcal{X}$ *are defined by*

$$\gamma_Y^\Gamma(X) = \bigcup_{g\in\Gamma}\{gY : gY \subseteq X\}, \tag{18}$$

$$\phi_Y^\Gamma(X) = \bigcap_{g\in\Gamma}\{gY : gY \supseteq X\}. \tag{19}$$

In words, $\gamma_Y^\Gamma(X)$ is the union of all translates gY which are included in X.

An important consequence of the above proposition is that the morphological opening $\delta_Y^\Gamma\epsilon_Y^\Gamma$ and closing $\epsilon_Y^\Gamma\delta_Y^\Gamma$ with Y an arbitrary subset of $\mathcal{X}$, are also invariant under the substitution $Y \to \overline{Y}$.

Example 4. Let X be a union of line segments of varying sizes in the plane and Y a line segment of size L with center at the origin. Let the acting group Γ equal the translation-rotation group $\mathbf{M}$. Then $\gamma_Y^{\mathbf{M}}(X)$ consists of the union of all segments in X of size L or larger, but $\delta_Y^{\mathbf{M}}\epsilon_Y^{\mathbf{M}}(X) = \gamma_{\Sigma Y}^{\mathbf{M}}(X) = \emptyset$, since $\overline{Y}$ is a disc of radius $L/2$ and does not fit anywhere in X.

So in general we cannot build the opening γ_Y^Γ from a Γ-erosion ϵ_Y^Γ on $\mathcal{P}(\mathcal{X})$ followed by a Γ-dilation δ_Y^Γ on $\mathcal{P}(\mathcal{X})$, in contrast to the classical case of the translation group ($\Gamma = \mathbf{T}$), cf. Section 2.1. However, if erosions and dilations between the distinct lattices $\mathcal{P}(\mathcal{X})$ and $\mathcal{P}(\Gamma)$ are allowed, openings and closings can be decomposed into products of erosion and dilation.

Proposition 2 ((Decomposition of structural openings)).
The structural opening $\gamma_Y^\Gamma : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ *defined by (18) is the projection of the* Γ*-opening* $\tilde{\gamma}_{\vartheta(Y)}$ *on* $\mathcal{P}(\Gamma)$*, i.e.*

$$\gamma_Y^\Gamma(X) = \big(\pi\tilde{\delta}_{\vartheta(Y)}\tilde{\epsilon}_{\vartheta(Y)}\vartheta\big)(X) = \pi\Big(\{\vartheta(X) \overset{\lambda}{\ominus} \vartheta(Y)\} \overset{\Gamma}{\oplus} \vartheta(Y)\Big). \tag{20}$$

So, γ_Y^Γ *is the product of a* Γ*-erosion* $\epsilon^\uparrow : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\Gamma)$ *followed by a* Γ*-dilation* $\delta^\downarrow : \mathcal{P}(\Gamma) \to \mathcal{P}(\mathcal{X})$*, where* $(\epsilon^\uparrow, \delta^\downarrow) := (\tilde{\epsilon}_{\vartheta(Y)}\vartheta, \pi\tilde{\delta}_{\vartheta(Y)})$ *is a* Γ*-adjunction between* $\mathcal{P}(\mathcal{X})$ *and* $\mathcal{P}(\Gamma)$*.*

A similar representation holds for structural closings [9]. By a general result from [12], every Γ-opening γ on $\mathcal{P}(\mathcal{X})$ is a union of structural openings γ_Y^Γ, where Y ranges over a subset $\mathcal{Y} \subseteq \mathcal{P}(\mathcal{X})$. Combining this with Proposition 2 we therefore can decompose any Γ-opening into Γ-openings of the form $\pi\tilde{\delta}_{\vartheta(Y)}\tilde{\epsilon}_{\vartheta(Y)}\vartheta$.

3 Invariant feature extraction

Using the general theory outlined above we now consider several group actions on the plane, where the group is chosen from the set of 'computer vision groups'. For all cases we process the following two images: (i) the 'solid' image shown in Fig. 3(a) (collection of filled sets); (ii) the 'quadrangle' image shown in Fig. 1(a) (collection of non-filled quadrangles). As the image transformation we take the opening γ_Y^Γ, where the structuring element Y is an open disc (interior of a circle) or a square (without interior), respectively.

The Euclidean translation group This is the classical case. Only translates of the structuring element are allowed. So the opening will extract translated copies of the structuring element. In Fig. 3(b) we show the result of opening the solid image by an open disc. It is seen that corners are removed, small bridges are broken, etc. The corresponding 'top-hat transform' [13], defined as the set difference $X \setminus \gamma_Y^\Gamma$, is shown in Fig. 3(c).

The opening with a square of the quadrangle image is shown in Fig. 1(b).

The Euclidean motion group Now translated and rotated versions of the structuring element are extracted. In the case of the opening by a disc nothing new is obtained compared to the previous case. The opening with a square of the quadrangle image is shown in Fig. 1(c).

The similarity group Now also scaled copies of the structuring element are allowed. This means that the dilation of a nonempty set X by a disc will be the whole space $\mathbb{R}^2$, and the adjoint erosion of X will be the empty set. But the opening of X is nontrivial, cf. Section 2. In Fig. 3(d) we show the result of opening the solid image by a disc. What we obtain now is the *interior* of the sets, since discs of arbitrary nonzero radii are fitted into the image. The corresponding top-hat transform now results in the edges of the image, see Fig. 3(e).

The opening with a square of the quadrangle image is shown in Fig. 1(d).

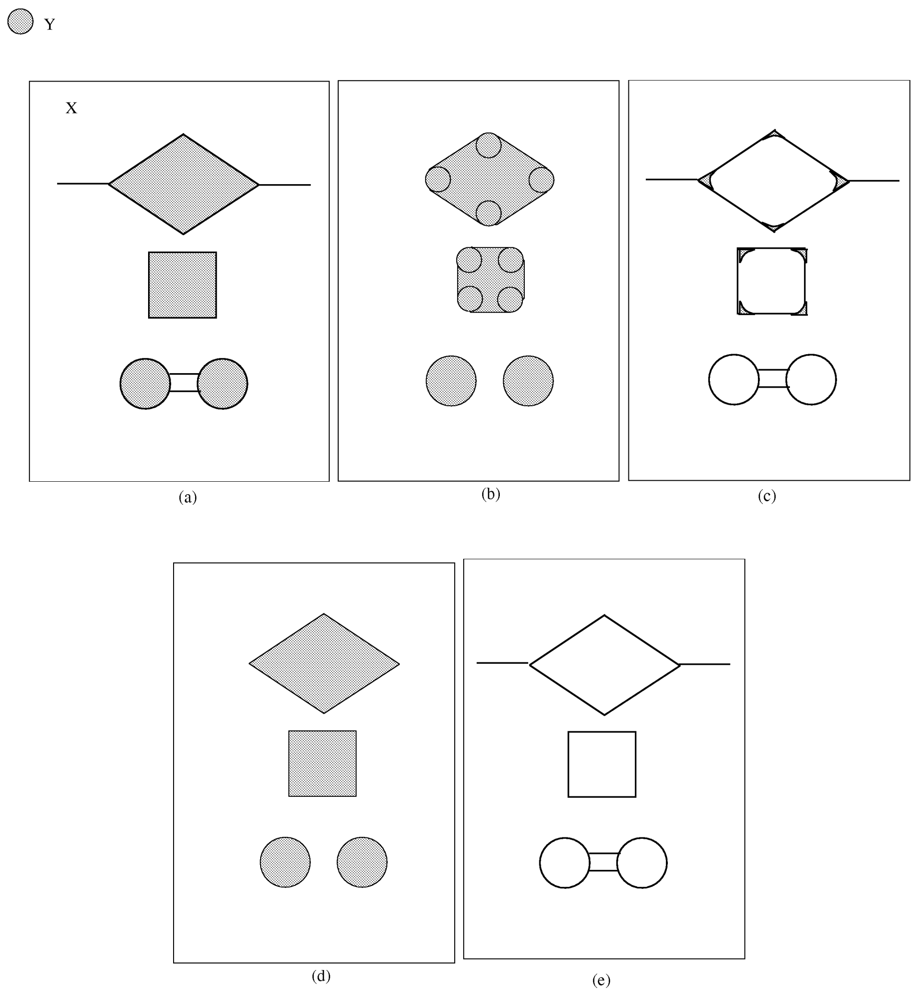
The affine group Since every parallelogram can be mapped to every other parallelogram by an affine mapping, the opening with a square extracts all parallelograms from the image. The result is shown in Fig. 1(e).

An interesting application of affine morphology is the extraction of parameters of an iterated function system (IFS) used in fractal image modeling, see [4]. Notice though that the operators constructed in that paper are not affine-invariant in the sense of Definition 3.

The projective group Since every quadrangle can be mapped to every other quadrangle by a projective transformation, the opening with a square extracts all quadrangles from the image. Therefore the result is equal to the original, see Fig. 1(f).

4 Conclusions

We have shown how invariance of operators arising in binary mathematical morphology can be achieved for the collection of groups commonly denoted as 'the computer vision groups': (i) the Euclidean motion group; (ii) the similarity group; (iii) the affine group; (iv) the projective group. An outline has been presented of the construction of set mappings such as dilations, erosions, openings and closings, which are invariant under a given group. It has been shown that, although dilations and erosions on the image plane may become trivial if the acting group becomes larger, nontrivial operations such as openings or closings exist which are useful for invariant object feature extraction.



Processing the 'solid' image with an open disc as structuring element: (a) original image X, structuring element Y; (b) Opening by the translation group; (c) Top-hat transform of (b); (d) Opening by the similarity group; (e) Top-hat transform of (d).

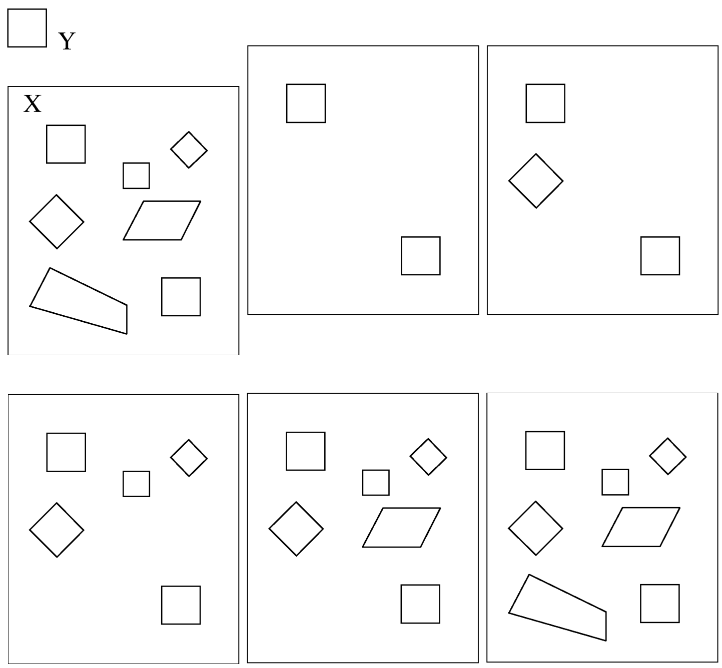


Figure 1. *Opening of the quadrangle image X shown in (a) by a square structuring element Y, using as acting group: (b) Translation group; (c) Motion group; (d) Similarity group; (e) Affine group; (f) Projective group.*

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