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Abstract

We prove the tight lower bound $4n - 4$ on the size of tangent visibility graphs on n pairwise disjoint bounded obstacles in the euclidean plane, and we give a simple description of the configurations of convex obstacles which realize this lower bound.

Keywords: Visibility graphs; Triangulations; Pseudotriangles; Pseudo-triangulations; Convex hulls; Relative convex hulls; Plane trees; Maps; Davenport–Schinzel sequences

1. Introduction

Visibility and shortest path problems in a scene consisting of disjoint polygons in the plane have been studied extensively. Recently the scope of this research has been extended to scenes of disjoint convex plane sets (convex obstacles for short). One of the combinatorial questions concerns the complexity of such scenes. Our starting point is the following question: what is the minimal number of *free bitangents* shared by n convex obstacles? A *bitangent* is a closed line segment whose supporting line is tangent to two obstacles at its endpoints; it is called *free* if it lies in *free space* (i.e., the complement of the union of the relative interiors of the obstacles). The endpoints of these bitangents split the boundaries of the obstacles into a sequence of arcs; these arcs and the bitangents are the edges of the so-called tangent visibility graph. The size of the tangent visibility graph is defined to be the number of free bitangents, so our question asks for the minimal size of tangent visibility graphs. Visibility graphs (for polygonal obstacles) were introduced by Lozano-Pérez and Wesley [11] for planning collision-free paths among polyhedral obstacles; in the plane a shortest euclidean path between two points runs via edges of the tangent visibility graph of the collection of obstacles augmented with the source and target points. Since then numerous papers have been devoted to the problem of their efficient construction
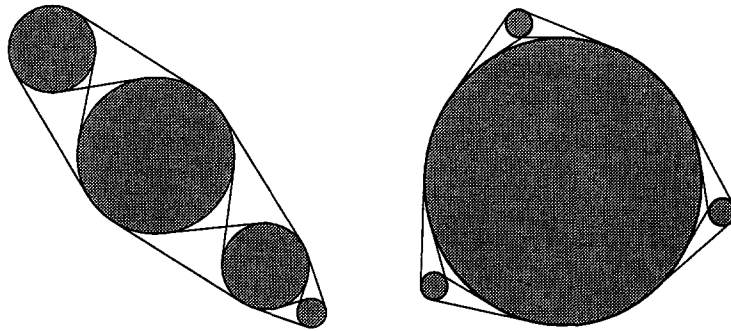

Fig. 1. Configurations of 4 obstacles with $4 \times 4 - 4 = 12$ free bitangents.

([1,4,5,8,10,13,17,18,21,22]) as well as their characterization (see [12] and the references cited therein). The more recent papers [14–16] consider the problem of the efficient computation of tangent visibility graphs for curved obstacles. This paper is concerned with the problem of characterizing the minimal tangent visibility graphs and classifying the corresponding configurations; such configurations are called, in this paper, *minimal* configurations. The answer to our question is given in the following theorem (we assume that the obstacles are closed, bounded, and are not reduced to points).

Theorem 1.1. *The number of free bitangents shared by n pairwise disjoint convex obstacles is at least $4n - 4$; this bound is tight.*

Configurations of $n(= 4)$ convex obstacles with exactly $4n - 4(= 12)$ bitangents are depicted in Fig. 1. These examples are easily extented to any value of n. The $4n - 4$ lower bound has been established previously in the case where the obstacles are line segments by Shen and Edelsbrunner [19] (see also [2,20]). Here we give a different proof based on the notion of *pseudo-triangulation* introduced in [14]. In fact we prove the following stronger result.

Theorem 1.2. *Consider a collection O of n pairwise disjoint convex obstacles. The following assertions are equivalent.*
(1) *The weak visibility graph of O is a tree.*
(2) *The number of free bitangents of O is minimal (i.e., $4n - 4$).*
(3) *The size of the convex hull of O is maximal (i.e., $2n - 2$).*

Recall that the *weak visibility graph* is the graph whose nodes are the obstacles and whose edges are pairs of obstacles such that there is a free line segment with endpoints lying on the obstacles. The *size* of the convex hull is the number of bitangents appearing on its boundary.

To discuss the characterization/classification problem we use the notion of *visibility type* (introduced in [15]). The visibility type might be considered as a combinatorial version of the tangent visibility graph where, for each obstacle, we take into account the circular order of the free bitangents incident to this obstacle. More precisely, let $O = O_1 \cup O_2 \cup \cdots \cup O_n$ be the union of n pairwise disjoint convex obstacles; for the sake of simplicity we assume that each obstacle is strictly convex and has a smooth boundary (equivalently its boundary has a well-defined tangent line at every point, and a well-defined touching point for every tangent line). Let b be a bitangent of O with endpoints p_i and

p_j lying on the boundary of O_i and O_j, respectively; we define the *type* of the bitangent b directed from p_i to p_j to be the pair $(\varepsilon, \varepsilon')$ with $\varepsilon = +$ or $-$ ($\varepsilon' = +$ or $-$) depending on whether O_i (O_j) lies, locally at the touch point p_i (p_j), to the left or to the right of the supporting line of the directed bitangent b. Now let B be a set of bitangents of O, and let P be the set of endpoints of bitangents in B. We define two permutations θ and σ on P by the two following conditions: (1) the line segment $[p, \theta(p)]$ is a bitangent in B (observe that θ is an involution), and (2) the point $\sigma(p)$ is the first point in P encountered when walking counterclockwise along the boundary of the obstacle on which lies p, starting at p. Finally, for p in P, we define $\ell(p)$ to be the type of the bitangent $[p, \theta(p)]$ directed from p to $\theta(p)$. We denote by $T_B(O)$ the (combinatorial) map (P, σ, θ) augmented with the labeling ℓ; elements of P are usually called darts. $T_B(O)$ can also be considered as a topological map: its vertices are the cycles of the permutation σ, its edges are the pairs $\{p, \theta(p)\}$, and its faces are the cycles of the permutation $\sigma \circ \theta$ (see [3,9] for background material on combinatorial and topological maps).

By definition the visibility type of O is the labeled map $T_B(O)$ where B is the set of free bitangents of O; the visibility type is denoted by $V(O)$. The visibility type of a collection of two convex obstacles is depicted in Fig. 2.

Given a visibility type (P, θ, σ) one can easily recover the corresponding tangent visibility graph: this is the graph whose set of nodes is P and whose set of edges is the set of pairs $\{p, \sigma(p)\}$ and $\{p, \theta(p)\}$, where p ranges over P. We do not know if, conversely, the tangent visibility graph determines the visibility type (up to reorientation of the plane). However, it follows easily from our analysis that the notion of visibility type and the notion of tangent visibility graph are equivalent for the class of minimal configurations (with smooth and strictly convex obstacles). Our characterization/classification result is the following.
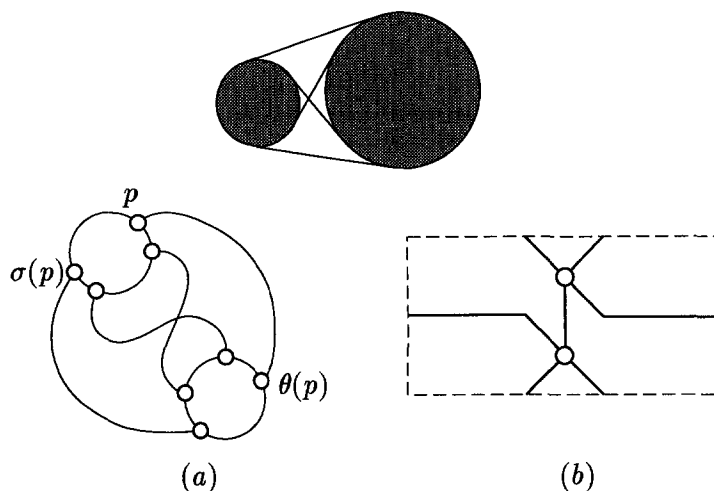
Fig. 2. (a) Representation of the visibility type of two disjoint convex obstacles: the cycles of σ are represented by circles in a conventional way (counterclockwise for instance) and the cycles of θ are represented by arcs. (b) The corresponding topological map, obtained by contraction of each circle to a point, lies on the torus (opposite sides of the parallelogram are identified in the usual way). The labels of the darts p, $\sigma(p)$, $\sigma^2(p)$, $\sigma^3(p)$ are $(-,-)$, $(+,+)$, $(+,-)$, $(-,+)$.

Theorem 1.3. *The set of minimal visibility types on n disjoint convex obstacles is in 1-1 correspondence with the set of plane labeled trees on n nodes. Furthermore the realization space of a minimal visibility type is connected.*

It is worth noting that in general the realization space of a visibility type is not connected (to see this recall that realization spaces of order types of points are in general not connected [6], and note that order types of points are visibility types of convex obstacles such that stabbing lines of triplets of obstacles do not exist).

The paper is organized as follows. In Section 2 we introduce the notion of pseudo-triangulation and we prove the three theorems above. In Section 3 we generalize Theorems 1.1 and 1.2 to configurations of obstacles which are not necessarily convex. A classification of the corresponding minimal configurations is left open.

2. Minimal tangent visibility graphs of convex obstacles

Let $O = O_1 \cup O_2 \cup \cdots \cup O_n$ be a configuration of n pairwise disjoint convex obstacles. As mentioned in the introduction we assume that each obstacle is strictly convex and has a smooth boundary (equivalently its boundary has a well-defined tangent line at every point, and a well-defined touching point for every tangent line). An *extremal* point of an obstacle is a boundary point at which the tangent line to the boundary is horizontal.

2.1. Pseudo-triangulation

A *pseudotriangle* is a simply connected bounded subset T of $\mathbb{R}^2$ such that (i) the boundary ∂T is a sequence of three convex curves that are tangent at their endpoints, and (ii) T is contained in the triangle formed by the three endpoints of these convex curves (see Fig. 3). Observe that there is a well-defined tangent line to the boundary of a pseudotriangle with a given (unoriented) direction. A *pseudo-triangulation* of the set of obstacles is the subdivision of the plane induced by the obstacles and a maximal (with respect to the inclusion relation) family of pairwise noncrossing free bitangents. It is
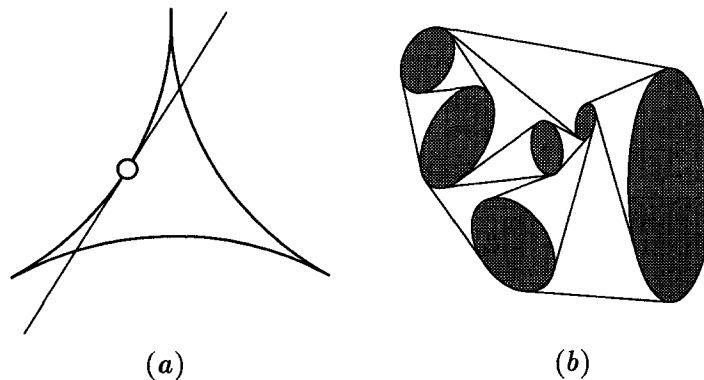

(a) (b)

Fig. 3. (a) A pseudotriangle and (b) a pseudo-triangulation.

clear that a pseudo-triangulation always exists and that the bitangents of the boundary of the convex hull of the obstacles are edges of any pseudo-triangulation. A pseudo-triangulation of a collection of six obstacles is depicted in Fig. 3.

Lemma 2.1. *The bounded free faces of any pseudo-triangulation are pseudotriangles.*

Proof. Let B be a family of noncrossing bitangents containing the bitangents of the boundary of the convex hull of the collection of obstacles. Assume that some free bounded face of the subdivision is not a pseudotriangle; from which we shall derive that B is not maximal. This means that this face is not simply connected or that its exterior boundary contains at least 4 cusp points. In both cases we add to B a bitangent as follows. Take a minimal length curve homotopy equivalent to the curve formed by the part of the exterior boundary of the face that goes through all cusp points of the exterior boundary but one. This curve contains a free bitangent not in B; hence B is not maximal. □

Lemma 2.2. *Consider a pseudo-triangulation of a collection of n disjoint convex obstacles induced by a maximal family B of free bitangents and let F_i be the set of pseudotriangles with exactly i bitangents on their boundaries. Then we have*

$$|B| = 3n - 3, \tag{1}$$

$$|F_2| + |F_3| + \cdots = 2n - 2, \tag{2}$$

$$2|F_2| + 3|F_3| + \cdots = 6n - 6 - h, \tag{3}$$

$$|F_3| + 2|F_4| + \cdots = 2n - 2 - h, \tag{4}$$

where h is the number of bitangents on the boundary of the convex hull of the collection.

Proof. Each pseudotriangle contains in its boundary exactly 1 extremal point (namely the touching point of the horizontal tangent line to the boundary of the pseudotriangle); since there are $2n - 2$ extremal points in bounded free space (= free space inside the convex hull of the collection of obstacles) there are exactly $2n - 2$ pseudotriangles; this proves Eq. (2). The first equation is then an easy application of Euler's relation for planar graphs. To see this observe that the set of vertices (of the pseudo-triangulation) consists of all endpoints of bitangents. In particular every vertex has degree 3. Furthermore the number of edges, that lie on the boundary of some object, is equal to the number of vertices. Finally the total number of bounded regions is equal to the sum of the number of pseudotriangles and the number (n) of obstacles.

The third equation is obtained by counting the number of incidences between the faces and the bitangents of the pseudo-triangulation. The last equation is a linear combination of the two preceding it. □

From Eq. (4) we deduce that $2n - 2$ is an upper bound for h; Fig. 1 shows that this upper bound is tight. An alternative argument is the following. The number h is also the size of the circular sequence of obstacles that appear on the convex hull (we call this sequence the *combinatorial* convex hull of the collection of obstacles). Since the obstacles are pairwise disjoint this circular sequence is a circular Davenport–Schinzel sequence on n symbols and parameter 2 (i.e., factors aa and subwords $abab$ are forbidden). It is well known (and easy to verify) that such a circular sequence has length at most $2n - 2$. Conversely any circular Davenport–Schinzel sequence (not necessarily maximal) on

n symbols with parameter 2 can be realized as the combinatorial convex hull of n pairwise disjoint obstacles. The argument is very simple. Let $i_1 \ldots i_h$ be a circular Davenport–Schinzel sequence on the alphabet $\{1, \ldots, n\}$ with parameter 2. Now label in clockwise order the h vertices of a regular h-gon by the indices of the sequence $i_1 \ldots i_h$. The convex hulls O_i of the points labeled i are pairwise disjoint (because subwords $abab$ are forbidden) obstacles whose combinatorial convex hull is exactly $i_1 \ldots i_h$. Finally we note the following simple fact.

Lemma 2.3. *Consider a pseudo-triangulation of a collection of obstacles, and let F_2 be the set of pseudotriangles with exactly 2 bitangents on their boundaries. Then a pseudotriangle in F_2 is adjacent to at most one other pseudotriangle in F_2.*

2.2. Proof of the main results

Lemma 2.4. *The number of free bitangents of a collection of n disjoint convex obstacles is at least $6n - 6 - h$, where h is the number of bitangents on the boundary of the convex hull of the collection.*

Proof. Consider a pseudo-triangulation of the set of obstacles induced by a maximal set B of pairwise noncrossing free bitangents. Let $b \in B$ and suppose that b lies inside the convex hull. This bitangent is the common boundary of two pseudotriangles, say T_1 and T_2. Exactly one cusp point of T_i, say A_i, does not belong to the convex boundary of T_i that contains the bitangent b. Now a shortest path, inside $T_1 \cup T_2$, between A_1 and A_2 contains a free bitangent b' that crosses b, and only b among the bitangents in B (see Fig. 4 for an illustration). Hence there are at least as many free bitangents as there are incidences among pseudotriangles, viz $\sum_{i \geqslant 2} i |F_i| = 6n - 6 - h$, see Lemma 2.2. □

Proof of Theorem 1.1. Lemma 2.4 implies (the first part of) Theorem 1.1 since, as we have observed in the previous section, $2n - 2$ is an upper bound for h. □
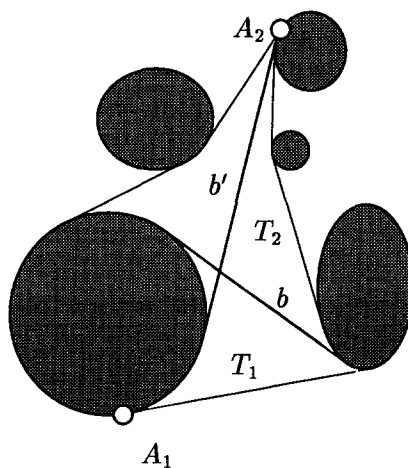

Fig. 4. Bitangent b' crosses b.

Proof of Theorem 1.2. Since the number of bitangents between two convex obstacles is 4 it is clear that the size of a tangent visibility graph is bounded above by 4 times the number of edges of the weak visibility graph. Assuming (1) (i.e., the weak visibility graph is a tree) it follows that the size of the tangent visibility graph is bounded above by $4n - 4$; since $4n - 4$ is a lower bound, the size of the tangent visibility graph is exactly $4n - 4$. This proves that (1) implies (2). The fact that (2) implies (3) is an obvious consequence of Lemma 2.4 and the fact that $2n - 2$ is an upper bound for the size of the convex hull. Now we prove that (3) implies (1). According to Eq. (4) of Lemma 2.2 we have $|F_i| = 0$ for $i \geqslant 3$, i.e., the $2n - 2$ pseudotriangles of any pseudo-triangulation have exactly two bitangents on their boundaries. It follows (see Lemma 2.3) that the connected components of bounded free space are pseudoquadrangles (i.e., the union of two adjacent pseudotriangles). There are $n - 1$ of these pseudoquadrangles. Each of these connected components is incident to exactly 2 obstacles and induces exactly one edge of the weak visibility graph. Therefore the weak visibility graph is a tree. □

Proof of Theorem 1.3. Let $O = O_1 \cup \cdots \cup O_n$ be a collection of n disjoint convex obstacles with minimal visibility type $V(O)$. According to the argument in the proof of Theorem 1.2 there is exactly one edge of the weak visibility graph per connected component of bounded free space. Therefore there is a unique (topological) embedding of the weak visibility graph in the plane such that the counterclockwise ordering of the edges incident to a node i coincides with the counterclockwise ordering of the connected components of bounded free space incident to the corresponding obstacle O_i. We denote this (topological) plane tree by $T(O)$. Clearly the plane tree $T(O)$ determines the order type $V(O)$, and conversely. It remains to show that any plane labeled tree on n nodes, say T, is the tree $T(O)$ of some collection O of n disjoint convex obstacles. Let $i_1 i_2 \ldots i_{2n-2}$ be the circular sequence of nodes encountered when we follow the boundary of the external face of the plane tree in a counterclockwise direction. This is a circular Davenport–Schinzel sequence on n symbols and parameter 2. As we have already observed such a sequence is realizable as the combinatorial convex hull of a collection O of n disjoint convex obstacles. Clearly $T = T(O)$. Finally a simple induction argument shows that the realization space of a minimal visibility type is connected. □

It follows easily from the above analysis that the visibility type of a minimal configuration on n obstacles can be recovered from its tangent visibility graph by searching in the tangent visibility graph $n - 1$ disjoint occurrences of the following subgraph (their number is $n - 1 + 2f$, where f is the number of leafs of the corresponding plane tree):

$$\boxed{\bowtie}$$
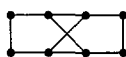
which represent the $n - 1$ connected components of bounded free space (see Fig. 5); details are left to the reader.

3. Extension to nonconvex obstacles

In this section we extend our analysis to configurations of obstacles which are not necessarily convex. For our purpose, in this section an *obstacle* is a bounded closed set whose boundary is an
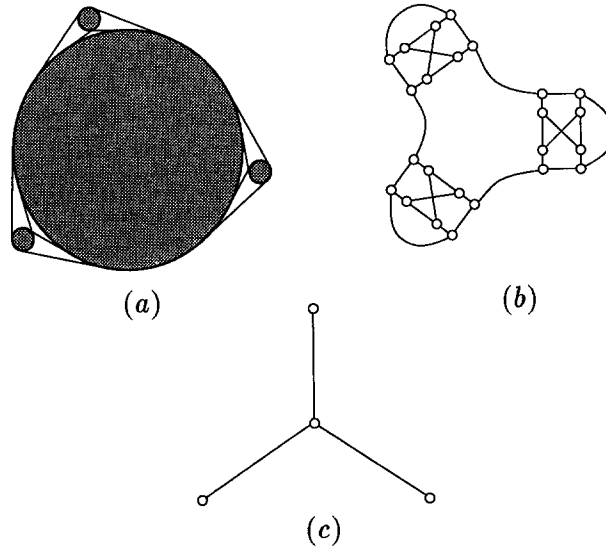
Fig. 5. (a) A minimal configuration, (b) its tangent visibility graph and (c) its weak visibility graph.

injective smooth closed regular curve (i.e., an injective curve $\gamma : \mathcal{S}^1 \mapsto \mathbb{R}^2$ whose derivative satisfies $\gamma'(t) \neq 0$ for all $t \in \mathcal{S}^1$). Given a configuration $O = O_1 \cup \cdots \cup O_n$ of n pairwise disjoint obstacles, we denote by C_0 its convex hull, and by C_i the relative convex hull of O_i with respect to O, i.e., the interior of the shortest curve in the closure of $\mathbb{R}^2 \setminus O$ homotopy equivalent to the boundary of O_i (this shortest curve is not necessarily injective; its interior can be defined as the set of points in the plane whose winding number with respect to the curve is equal to $+1$ [7]). We denote by h_i ($i \geqslant 0$) the number of bitangents (counting multiplicities) on the boundary of C_i, and by l_i ($i \geqslant 1$) the number of connected components of $C_i \setminus O_i$. (Note that a bitangent might involve only one obstacle.) Set $l(O) = \sum_{i=1}^{n} l_i$ and $h(O) = \sum_{i=0}^{n} h_i$. The complement in $\mathbb{R}^2$ of the union of the C_i ($i \geqslant 1$) is called *free* space, the union $\bigcup_{i=1}^{n}(C_i \setminus O_i)$ is called *semi-free* space, and the complement in C_0 of the union of the C_i ($i \geqslant 1$) is called the *relative convex hull* of the family of obstacles. We denote by $h'(O)$ the number of bitangents lying on the boundary of the relative convex hull. Observe that $h(O) = h'(O) + \omega(O)$, where $\omega(O)$ is the number of free bitangents incident on both sides upon semi-free space (these bitangents are counted twice in $h(O)$). See Fig. 6(a) for an illustration of these notions.

3.1. The lower bound

As in the case of convex obstacles we define a pseudo-triangulation to be a subdivision of the plane induced by the obstacles and a maximal family of pairwise noncrossing free bitangents. Clearly a pseudo-triangulation always exists and the corresponding maximal family of free bitangents contains the $h'(O)$ bitangents of the relative convex hull (see Fig. 6(b)).

Lemma 3.1. *The free bounded faces of any pseudo-triangulation are pseudotriangles. Furthermore the number of pseudotriangles in a pseudo-triangulation is equal to $2n - 2$.*
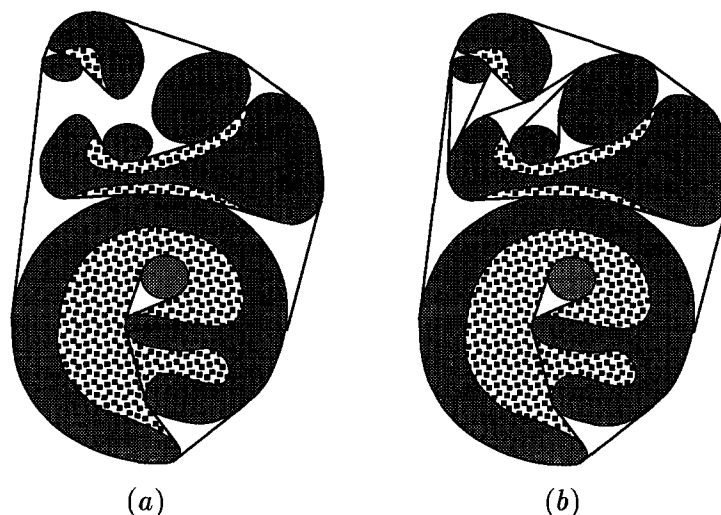
(a) (b)

Fig. 6. (a) Relative convex hull (semi-free space is the dotted region) and (b) pseudo-triangulation of a configuration of 7 obstacles (its bounded faces are the obstacles, the connected components of semi-free space, and pseudotriangles).

Proof. The first part is proven as in Lemma 2.1. For the second part we claim that the number of extremal points on the boundary of bounded free space is $2n - 2$, as in the convex case. Let us say that an extremal point on the boundary of obstacles is *red* (*green*) if it lies on a convex (concave) arc. Obviously every green point lies in the boundary of semi-free space. Since the number of red points on a given obstacle is equal to the number of green points plus 2, the number of red points exceeds the number of green points by $2n$. Now observe that each semi-free bounded face of a pseudo-triangulation contains in its boundary exactly the same number of green points and red points. Furthermore the boundary of the convex hull contains exactly two red points. These last two observations imply that the number of pseudotriangles is equal to the excess of red points, minus two. This proves our lemma. □

Consider now a pseudo-triangulation of the collection of obstacles, induced by a maximal family B of free bitangents. Let F_i be the set of pseudotriangles of the pseudo-triangulation with i bitangents on their boundaries. From the previous lemma we deduce, arguing as in the proof of Lemma 2.2, that

$$|B| = 3n - 3 + l(O), \tag{5}$$

$$|F_2| + |F_3| + \cdots = 2n - 2, \tag{6}$$

$$2|F_2| + 3|F_3| + \cdots = 6n - 6 + 2l(O) - h(O), \tag{7}$$

$$|F_3| + 2|F_4| + \cdots = 2n - 2 + 2l(O) - h(O). \tag{8}$$

It follows from (8) that $2n + 2l(O) - 2$ is an upper bound for $h(O)$. It is not hard to verify that this upper bound is tight for fixed n ($\geqslant 2$) and $l(O)$, see Fig. 7. Similarly $2n + 2l(O) - 2$ is an upper bound for $h'(O)$, since $h'(O) = h(O) - \omega(O)$.

We recall also that a pseudotriangle in F_2 is adjacent to at most one pseudotriangle in F_2.
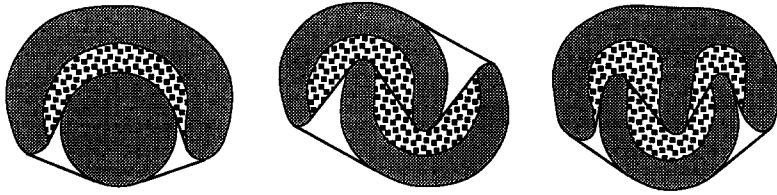
Fig. 7. Configurations of two obstacles with $l(O) = 1, 2, 3$, and maximal value of h.

Theorem 3.1. *Consider a collection O of n pairwise disjoint obstacles. Then the number of free bitangents of the collection is at least $6n - 6 + 2l(O) - h(O) + \omega(O)$. Furthermore the following assertions are equivalent.*

(1) *The number of free bitangents of O is minimal (i.e., $4n - 4$).*

(2) *The size of the relative convex hull of O is maximal (i.e., $h(O) = h'(O) = 2(n + l(O) - 1)$).*

(3) *The connected components of the relative convex hull of O are pseudotriangles and/or pseudoquadrangles (= union of two adjacent pseudotriangles of size two).*

In case at least one (and hence all) of the conditions (1)–(3) hold, the number of pseudotriangles is $2l(O)$ and the number of pseudoquadrangles is $n - 1 - l(O)$.

Proof. The proof of the first part is similar to that of Lemma 2.4. First observe that there are $\omega(O)$ free bitangents that are not incident upon any pseudotriangle. Secondly, as in the convex case, there are at least $\sum_i i|F_i|$ free bitangents incident upon or inside the pseudotriangles. Therefore the number of free bitangents is at least $\omega(O) + \sum_i i|F_i|$ which, according to (7), is equal to $6n - 6 + 2l(O) - h(O) + \omega(O)$. This proves the first part of the lemma. Since $2(n + l(O) - 1)$ is an upper bound for $h(O)$, it follows that $4n - 4 + \omega(O)$ is a lower bound for the number of free bitangents of O. Now we prove the second part. If the number of free bitangents is equal to its minimal value $4n - 4$, it follows from the first part that $\omega(O) = 0$ and $h'(O) = h(O) = 2n - 2 + 2l(O)$. This proves that (1) implies (2). Assume now that the size of the relative convex hull is maximal, i.e., $h'(O) = h(O) = 2n + 2l(O) - 2$. It follows that $\omega(O) = 0$ and, according to Eq. (8), that $|F_3| = |F_4| = \cdots = 0$. Hence every pseudotriangle has exactly 2 bitangents in its boundary. It follows that the connected components of the relative convex hull are pseudotriangles and pseudoquadrangles. This proves that (2) implies (3). Furthermore, if the connected components of the relative convex hull are pseudotriangles (in number n_1) and pseudoquadrangles (in number n_2) the number of free bitangents (which lie necessarily in the connected components of free space) is exactly $2n_1 + 4n_2$, i.e., $4n - 4$, since $n_1 + 2n_2 = 2n - 2$. This proves that (3) implies (1). Finally, from $2(n_1 + n_2) = 2n - 2 + 2l(O)$ and $n_1 + 2n_2 = 2n - 2$, we deduce that $n_1 = 2l(O)$ and $n_2 = n - 1 - l(O)$. □

3.2. A zoo of minimal configurations

New minimal configurations appear with nonconvex obstacles, see Fig. 8. From the above analysis (see Theorem 3.1) we can easily deduce that there is a 1–1 correspondence between the set of minimal visibility types and the set of maximal (for a given value of $l(O)$ between 0 and $n - 1$) combinatorial relative convex hulls (= labeled maps $T_B(O)$ where B is the set of bitangents of O which appears
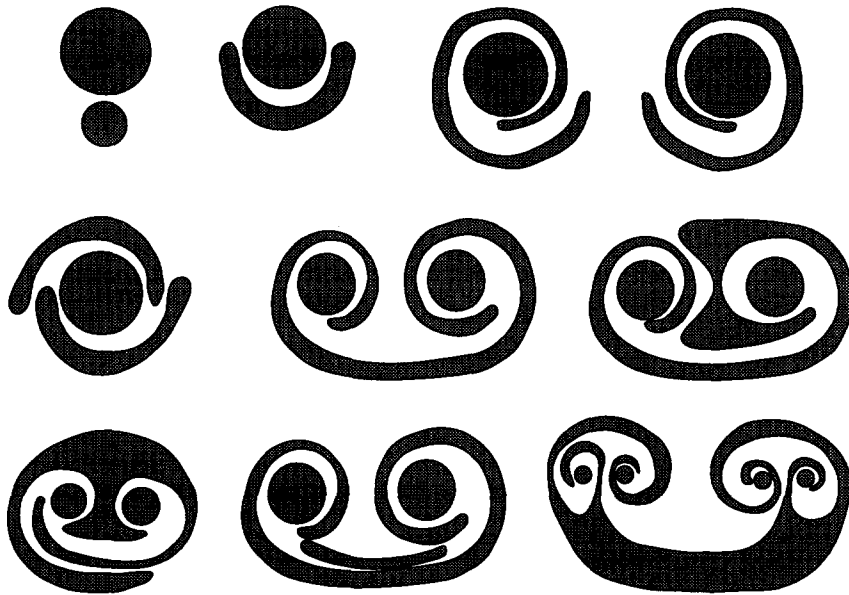
Fig. 8. Minimal configurations on 2, 3 and 5 obstacles.

in the relative convex hull of O). In the case of convex obstacles ($l(O) = 0$) the set of maximal combinatorial convex hulls is in 1–1 correspondence with the set of plane trees (see Section 2). The case of non convex obstacles seems to be much harder to analyze; one reason is that we have no obvious "canonical" configuration with a given minimal visibility type. However we conjecture that these labeled maps are still recognizable in polynomial time and that the realization space of a minimal visibility type is still connected.

4. Conclusion

We have proven that $4n - 4$ is a tight lower bound for the size of tangent visibility graphs on n obstacles. We have also given a simple description of the corresponding minimal configurations of convex obstacles. Our main tool is the notion of pseudo-triangulation. It is expected that a better understanding of this notion will give insights in the classification problem of tangent visibility graphs.

Acknowledgements

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