Symmetries in string theory

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Chapter 4

Physical spectra in string theory

In this chapter we compute cohomology classes of the BRST operator for a number of string models. Within the formalism of BRST quantization, these cohomology classes build the spectrum of physical states. As usual, we start with a discussion of the bosonic string. In section 4.2 we turn to W-string spectra. To illustrate the general structure of such spectra and some methods of computation, we consider in section 4.3 the BRST analysis of the W_4 string in detail. In the last section of this chapter we review some of the relations that exist between strings based on different world-sheet gauge symmetries.

4.1 The bosonic string

First we should stress that all considerations in this chapter are concerned with free strings only. That is, we compute physical spectra of free strings, but no correlation functions. Let us just briefly motivate the term 'vertex operator' that is used for operators which create physical states. First, Weyl transformations can be used to map any two-dimensional surface representing a string scattering process to a compact surface with the same number of handles (quantum loops), but on which the external strings are mapped to points. The quantum numbers of the external string states are then to be described by local operators of the two-dimensional quantum field theory inserted at these points. These operators, which create the external string states, are precisely the physical operators or vertex operators to be discussed in this chapter. String scattering amplitudes then involve the correlation functions of products of vertex operators in the conformal field theory. See for example [104, 90] for discussions of string scattering.

The physical spectrum of the critical bosonic string has been known since the early days of string theory. From the previous chapter, we know that the holomorphic part of the BRST operator is given by $Q = \oint \frac{dz}{2\pi i} j$ with

$$j = c(T + \tfrac{1}{2}T_{gh}) \,, \qquad T_{gh} = -2b\partial c - \partial bc \,, \tag{4.1}$$

where the matter energy-momentum tensor involves the 26 string coordinates, $T = -\frac{1}{2}\partial X_\mu \partial X^\mu$. A basis of operators on which we act with the BRST operator to determine the physical spectrum, is given by

$$F(b, c, \partial X)e^{ip\cdot X} \,, \tag{4.2}$$

where F is an arbitrary polynomial in the ghost variables and ∂X^μ plus their derivatives. All expressions are assumed to be normal-ordered. We will call an operator (or state) 'physical' if it belongs to the BRST cohomology, irrespective of its ghost number.

The cohomology analysis is most easily done in a level by level computation[1], where one starts from level 0 spanned by the operators $F(b, c, \partial X)$ of lowest conformal weight. As has been mentioned in the previous chapter, there are two such operators: c and $c\partial c$, having ghost numbers $G = 1$ and $G = 2$, respectively. These are the only ghost numbers that can occur at level 0. The BRST variations are[2]

$$\begin{aligned} &[Q, ce^{ip\cdot X}] = -(\tfrac{1}{2}p^2 - 1)c\partial ce^{ip\cdot X} \,, \\ &[Q, c\partial ce^{ip\cdot X}] = 0 \,, \end{aligned} \tag{4.3}$$

where we compute $[Q, A]$ using OPEs. From the relation between equal-time commutators and OPEs described in chapter 2, in particular equations (2.41) and (2.37) (where in the present case the infinitesimal transformation parameter ϵ is an anticommuting constant and we consider transformations generated by the BRST current $j(z)$ instead of conformal transformations generated by $T(z)$), we readily see that the BRST commutator is the first order pole of the OPE $j(z)A(w)$. Since no $G = 0$ operators exist at level 0, the operators $ce^{ip\cdot X}$ cannot be BRST exact. Therefore, we conclude that $ce^{ip\cdot X}$ is physical for $p^2 = 2$. These operators create tachyonic states, with $M^2 = -p^2 = -2$,

$$|0; p> \equiv \lim_{z,\bar{z}\to 0} c(z)e^{ip\cdot X(z,\bar{z})}|0> \,, \tag{4.4}$$

where $|0>$ is the $sl(2)$-invariant vacuum. The operators $c\partial ce^{ip\cdot X}$ are trivially BRST invariant, because no $G = 3$ level 0 operators exist. However, as is clear from (4.3), they are only non-exact for $p^2 = 2$. These operators then generate copies of the tachyonic states (4.4) at the next ghost number. Their role has already been discussed in the previous chapter as providing states that give nonzero inner product with the standard states (4.4). We can consider the operators $ce^{ip\cdot X}$ and $c\partial ce^{ip\cdot X}$ as corresponding to the same physical operator in a different picture. The picture changing operator is given by

$$P_a = [Q, a_\mu X^\mu] = a_\mu c\partial X^\mu \,, \tag{4.5}$$

[1] The BRST operator respects the grading by level since it has itself zero conformal dimension and it does not change momentum p_μ.

[2] We use the notation $[,]$ for both commutators and anticommutators. As usual, when two operators of odd ghost number are considered, an anticommutator is understood.

for some polarization vector a_μ. Since P_a is BRST invariant[3], the normal-ordered product of P_a with a BRST invariant operator is also BRST invariant. We obtain the $G=2$ tachyon operator by taking the normal-ordered product of P_a (assuming $a_\mu p^\mu \neq 0$) with the $G=1$ tachyon operator,

$$c\partial c e^{ip\cdot X} \propto \oint \frac{dz}{2\pi i} \frac{P_a(z) c e^{ip\cdot X}(w)}{(z-w)} \,. \tag{4.6}$$

At the first excited level, where the operators $F(b,c,\partial X)$ have conformal weight zero, the lowest ghost number is $G=0$, corresponding to the purely exponential operators $e^{ip\cdot X}$. It is easy to see that BRST invariance requires $p_\mu = 0$, which is the statement that the unit operator is physical. The corresponding state is the $sl(2)$-invariant vacuum. This is a discrete state, it is only physical for a single momentum. The $sl(2)$-invariant vacuum together with the picture changing operators[4] plus their conjugates (the conjugate of the identity operator is $\partial^2 c\partial c c$) create all discrete states of the chiral sector of the bosonic string [88, 109]. The next ghost number at the first excited level, $G=1$, admits as the most general operator

$$V = (a_\mu c\partial X^\mu + x\partial c)e^{ip\cdot X} \,, \tag{4.7}$$

where x is a free parameter. The BRST variation is

$$[Q,V] = -(\tfrac{1}{2}p^2 a_\mu - ixp_\mu)c\partial c\partial X^\mu e^{ip\cdot X} - (-\tfrac{i}{2}a_\mu p^\mu - x)c\partial^2 c e^{ip\cdot X} \,. \tag{4.8}$$

Therefore, the BRST invariant combination has $x = -\frac{i}{2}a\cdot p$, and the polarization vector must satisfy $p^2 a_\mu = 2ixp_\mu$. However, this operator is BRST exact if a_μ is proportional to p_μ. Thus, the physical operators are given by $a_\mu c\partial X^\mu e^{ip\cdot X}$ with $p^2 = a\cdot p = 0$. Furthermore we have the equivalence relation $a_\mu \simeq a_\mu + \alpha p_\mu$ for any constant α.

Let us interpret these results. For an open string, there are no separate left and right-moving sectors, and the operator given in equation (4.1) is the full BRST operator. The level 1 physical operators (apart from the discrete ones) correspond to massless states ($p^2=0$) with transverse polarizations ($a\cdot p = 0$), and an equivalence relation $a_\mu \simeq a_\mu + \alpha p_\mu$. This leaves the 24 positive norm states expected for a massless vector particle in 26 dimensions.

For the closed string, we recall that there is also an anti-holomorphic sector that we usually ignore since it is treated in exactly the same way as the holomorphic sector. The BRST operator is in fact given by the sum of holomorphic and antiholomorphic BRST operators as in (3.21). The cohomology in the anti-holomorphic sector is isomorphic to that in the holomorphic sector and the total cohomology is obtained by tensoring

[3] One might think that P_a is BRST trivial as well. This is not the case though, since we do not include the non-derivative fields X^μ in the BRST complex. However, see [7] for a discussion of an extended BRST complex which does include X^μ (i.e. its center of mass operator x^μ besides the modes α_n^μ). In this extended complex, there is no doubling of operators since the $G=2$ copies are BRST exact. Moreover, the 26 picture changing operators themselves are no longer physical.

[4] The picture changing operators are actually part of the generic $p^2=0$ spectrum. However, they are singular in the sense that they cannot be reached from generic light-like excitations by Lorentz transformation. They are also left out in the proof of the no-ghost theorem [109].

physical states from both sectors. Since the exponential factors of physical operators are common to both sectors, or in other words the left and right-moving momenta are assumed to be equal (see equation (2.26)), the levels of excitation in both sectors are required to be the same by the $L_0 - \bar{L}_0 = 0$ constraint. This is the only constraint that connects left and right-moving sectors. For the first excited level this yields the operators

$$e_{\mu\nu} c\bar{c} \partial X^\mu \bar{\partial} X^\nu e^{ip\cdot X} \,. \tag{4.9}$$

They correspond to massless states, $p^2 = 0$, and the polarization tensor satisfies $p^\mu e_{\mu\nu} = p^\nu e_{\mu\nu} = 0$. The BRST equivalence relation reads $e_{\mu\nu} \simeq e_{\mu\nu} + p_\mu k_\nu + k'_\mu p_\nu$ with $p\cdot k = p\cdot k' = 0$. The resulting physical states can be decomposed under the transverse rotation group $SO(24)$ into a traceless symmetric tensor, antisymmetric tensor and invariant. These correspond to the graviton, antisymmetric tensor and dilaton, respectively. Note that the free spectrum of the open string does *not* contain a massless spin-two particle. However, if interactions are taken into account, two open strings may join to form a closed string. Therefore, a theory of open strings contains closed strings as well and thus also the graviton.

It is interesting to note that for $c = 26$ all states of the form (2.63) with $h = -1$ correspond to null states of physical weight one. Together with the null states $L_{-1}|h = 0>$ they arrange for the decoupling of longitudinal excitations in the spectrum. This is one example of the interplay between space-time gauge symmetry in string theory and its underlying conformal field theory.

States at excitation levels two or higher are massive. This may be seen from the mass-shell formula which differs from the classical formula (2.32) by the normal-ordering constant in L_0 and $\bar{L}_0$,

$$M^2 = -p^2 = -2 + 2\sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n = 2(N-1)\,, \tag{4.10}$$

where α_n are the Fourier modes of the string coordinates, and N is the total energy level of these harmonic oscillators. Reintroducing $\alpha' = \frac{1}{2\pi T}$, we find that the mass-shell condition reads $M^2 = \frac{4}{\alpha'}(N-1)$ in the case of closed strings, and $M^2 = \frac{1}{\alpha'}(N-1)$ in the case of open strings. The massive physical states fill out representations of $SO(25)$, and the maximum spin at level N is N for open strings and $2N$ for closed strings. Thus we have the inequalities $J \le \alpha' M^2 + 1$ for open strings and $J \le \frac{1}{2}\alpha' M^2 + 2$ for closed strings, corresponding to the well-known Regge trajectories.

All physical operators, apart from the identity operator and its conjugate, take the form

$$S = cV(X,p) = cP(\partial X^\mu) e^{ip\cdot X} \,. \tag{4.11}$$

We ignore for the moment their copies at the next ghost number, and also restrict ourselves, as usual, to the holomorphic sector of the theory. Using the Wick rule (2.77), one can show that the first order pole of the OPE $j(z)S(w)$[5] is given by

$$[jS]_1 = -\partial c c V + \sum_{n\ge 1} \partial^n c c [TV]_{n+1} \,. \tag{4.12}$$

[5]We use the notation $[AB]_n$ for the operator in the n^{th} order pole of $A(z)B(w)$.

It follows that the condition of BRST invariance requires the operator $V(X,p)$ to be primary of conformal dimension one with respect to T, as claimed before in equation (3.33). The complete operator S is then a primary of the *total* energy-momentum tensor of vanishing weight. Thus the problem of constructing the complete cohomology of the bosonic string is the same as the problem of identifying all dimension one primary operators $V(X,p)$ or their corresponding highest weight states $|V>$. However, among these states there are still zero norm states that are BRST exact. All nonzero norm states can be generated by the so-called spectrum generating algebra of DDF operators [68]. This algebra is isomorphic to the algebra of transverse oscillators α_n^i. Application of the DDF operators to the tachyonic ground state results in the DDF states which have been proven to span the complete spectrum of the bosonic string in the case that $D=26$ and $a=1$. More precisely, any physical state $|\phi>$ can be uniquely decomposed as

$$|\phi>=|f>+|n>, \tag{4.13}$$

where $|f>$ is a DDF state and $|n>$ is a null state. A null state is a spurious state[6] that is physical as well. In the language of CFT, a spurious state is a Virasoro descendant and a physical state is a dimension one primary state. A state which is both a descendant and primary is null and decouples completely from the theory. In the BRST formalism, a DDF state $|f>$ corresponds to a certain representative of a cohomology class, whereas a null state $|n>$ corresponds to a BRST trivial state. From the fact that the DDF operators only generate excitations in the transverse directions, it follows that the spectrum contains no negative norm states. This is the no-ghost theorem [52, 102, 188]. For a more complete discussion, see [104]. The no-ghost theorem in the BRST quantization was established in [85, 88].

4.2 W-strings

The purpose of this and the next section is to describe the general structure of the physical spectrum of a class of W-strings. This class consists of the W_N strings where the W_N algebra is realized by a Miura realization. We mainly restrict ourselves to the critical case, where the fields of the Miura realization carry a total central charge $c_{\rm crit}$ given by minus the ghosts' contribution

$$c=c_{\rm crit}=2\sum_{k=2}^{N}(6k^2-6k+1)=2(N-1)(2N^2+2N+1)\,. \tag{4.14}$$

The results for W_N strings turn out to be somewhat disappointing in the sense that their spectra appear to be quite similar to that of the ordinary bosonic string. This is a consequence of a special property of the Miura realizations, as described in subsection 3.3.2. As explained there, the currents of the Miura realization for W_N can be re-expressed in terms of W_{N-1} currents and one explicit scalar field ϕ_{N-1}. Repeating this

[6] A spurious state is a state of the form $|n>=L_{-1}|1>+L_{-2}|2>$ for some states $|1>,|2>$ and is therefore orthogonal to any physical state.

procedure, we obtain a W_N realization in terms of an energy-momentum tensor and $N-2$ scalar fields $\{\phi_2, \phi_3, ...\phi_{N-1}\}$. It turns out that the physical state conditions 'freeze' the scalar fields $\{\phi_2, \phi_3, ...\phi_{N-1}\}$, and leave a set of ordinary Virasoro type string sectors with different intercepts [164, 65, 136]. This is most easily understood in the nested basis. The BRST charge inherits the nested structure (at least classically) given in (3.81). Starting the cohomology computation with the highest-spin part Q_N^N of the BRST operator and going downwards to Q_N^3, we see that this successively fixes excitations and momenta corresponding to $\phi_{N-1}\,, \phi_{N-2}\,, ...\phi_2$. In the next section we intend to clarify this in the example of the W_4 string.

It is interesting to see what are the central charges of the embedded W_n algebras ($n \leq N$) in the Miura realization of the W_N string. The condition for criticality (4.14) fixes the free parameter of the Miura realization, α_0,

$$c = (N-1)(1+2N(N+1)(\alpha_0)^2) = c_{\text{crit}} \quad \Rightarrow \quad (\alpha_0)^2 = \frac{(2N+1)^2}{2N(N+1)}\,. \tag{4.15}$$

This then fixes the background charges of the scalar fields, and the contribution of ϕ_n to the central charge becomes

$$c_{\phi_n} = 1 + 3(2N+1)^2 \frac{n(n+1)}{N(N+1)}\,, \tag{4.16}$$

for $n = 1, 2, ...N-1$. The total central charge of the fields $\{\phi_{N-1}; c_N, b_N\}$ is now

$$c_N^N = c_{\phi_{N-1}} - 2(6N^2 - 6N + 1) = \frac{2(N-2)}{N+1} = c_{N-1,N}\,, \tag{4.17}$$

which is precisely the central charge of the first unitary W_{N-1} minimal model. We recall from equation (2.87) that the central charges of unitary W_N minimal models are given by

$$c_{N,q} = (N-1)\left(1 - \frac{N(N+1)}{q(q+1)}\right)\,, \quad q \geq N\,. \tag{4.18}$$

For $N=3$, for example, $c_3^3 = \frac{1}{2}$ is the central charge of the Ising model, the first unitary Virasoro minimal model. More generally, the central charge of the fields $\{\phi_{n-1}, ...\phi_{N-1}; c_n, b_n, ...c_N, b_N\}$, which may be considered to correspond to the subsector of the $N-n+1$ highest-spin currents of the W_N algebra [20], adds up to

$$\begin{aligned} c_N^n &= \sum_{k=n-1}^{N-1} c_{\phi_k} - 2\sum_{k=n}^{N}(6k^2 - 6k + 1) \\ &= (n-2)\left(1 - \frac{n(n-1)}{N(N+1)}\right) = c_{n-1,N}\,, \end{aligned} \tag{4.19}$$

which is the central charge of the $(N, N+1)$ unitary W_{n-1} minimal model. So these central charge counting arguments suggest that a critical W_N string is related to a series of $(N, N+1)$ W_n minimal models with $n = 2, 3, ...N-1$ [136, 20]. This is indeed what is found also in cohomology computations, as we will now discuss.

For simplicity, we concentrate on the W_3 string. Explicit results for W_4 will be discussed in the next section. For $N > 4$, the explicit form of the BRST operator is not known. However, from the results for W_3 and W_4, the general pattern for critical W_N strings based on Miura realizations seems to be clear. More on W_N strings may be found in the review papers [161, 192, 115].

The $c = 100$ realization needed for a nilpotent BRST operator is given in (3.75) if we choose $\alpha_0^2 = \frac{49}{24}$. The freedom to choose the sign for α_0 corresponds to the simple OPE automorphism $A \to -A$. We write $A = \partial\phi$ and take the energy-momentum tensor T_X to be realized by D scalar fields X^μ,

$$T_X = -\frac{1}{2}\partial X_\mu \partial X^\mu + a_\mu \partial^2 X^\mu \,, \tag{4.20}$$

with central charge $c_X = D + 12a_\mu a^\mu = \frac{1}{4}c + \frac{1}{2} = 25\frac{1}{2}$. It is clear that at least one of the scalars X^μ must have a nonzero background charge.

A background charge q in $T_\phi = -\frac{1}{2}\partial\phi\partial\phi + q\partial^2\phi$ corresponds to a coupling of ϕ to the world-sheet curvature scalar,

$$S_0 \to S_0 + \frac{q}{4\pi}\int d^2\sigma \sqrt{h} R^{(2)}\phi \,, \tag{4.21}$$

as observed before in the case of the Liouville field of the non-critical string. If we consider a correlation function on the sphere

$$< \prod_k V_k(p_k) >= \int \mathcal{D}\phi e^{-S_0 - \frac{q}{4\pi}\int d^2\sigma\sqrt{h}R^{(2)}\phi} \prod_k V_k(p_k) \,, \tag{4.22}$$

where the vertex operators $V_k(p_k)$ have exponential parts $e^{ip_k\phi}$, the change of variable $\phi \to \phi + a$ yields the Ward identity

$$\sum_k p_k = -2iq \,. \tag{4.23}$$

This follows from the Gauss-Bonnet theorem, $\frac{1}{4\pi}\int d^2\sigma\sqrt{h}R^{(2)} = 2(1-g)$ with g the genus of the Riemann surface that represents the world-sheet. We see then that scalars with real background charges, such as the scalars in the usual Miura realization, are supposed to have imaginary momenta. Rescaling such scalars by $\sqrt{-1}$ gives real momenta but changes the sign of the OPE and therefore suggests that they may be thought of as timelike coordinates[7]. One introduces so-called screening charges, to be inserted in correlation functions, to ensure that (4.23) is satisfied. For the two-scalar W_3 Miura realization ($D = 1$ in (4.20)), screening currents are given by

$$S_i^\pm = e^{i\alpha_\pm \vec{e}_i\cdot\vec{\phi}} \,, \tag{4.24}$$

[7]However, in the case of 'frozen' scalar fields whose allowed momenta are discrete, a space-time interpretation, if any, is not clear. Perhaps they can be viewed as corresponding to compact directions. See [162] for a discussion.

where $\vec{e}_i$, $i = 1, 2$ are the simple roots of $sl(3)$ and $\alpha_\pm$ are determined from the requirement that the currents are spin-one primaries,

$$\alpha_\pm = \tfrac{i}{\sqrt{2}}(\alpha_0 \pm \sqrt{\alpha_0^2 - 2})\,. \tag{4.25}$$

We use the basis in which $\vec{\phi} = (X, \phi)$. The screening charges commute with the generators of the W_3 algebra. It turns out that the momenta of cohomology classes of the two-scalar W_3 string are all of the form [166]

$$\vec{p} = \sum_{i=1}^{2} (n_i^+ \alpha_+ + n_i^- \alpha_-)\vec{e}_i\,, \tag{4.26}$$

where $n_i^\pm$ are integers and $\vec{p} = (p_X, p_\phi)$. This guarantees (at least for negative integers $n_i^\pm$) that with appropriate insertions of screening charges momentum conservation (4.23) can always be satisfied.

W_3 string ground states are created by

$$V = c_2 \partial c_3 c_3 e^{i\vec{p}\cdot\vec{\phi}}\,. \tag{4.27}$$

The BRST invariance condition for standard operators (operators with ghost structure as in (4.27)), is the condition that they are W_3 primary with spin-two and spin-three weights (intercepts) $(h, w) = (4, 0)$. Since this amounts to a quadratic and a cubic equation in the momenta, there are six physical values (p_X, p_ϕ). They form a multiplet under the following action of the Weyl group $\mathcal{W}$ of $sl(3)$,

$$w \cdot \vec{p} \equiv w(\vec{p} - i\sqrt{2}\alpha_0\vec{\rho}) + i\sqrt{2}\alpha_0\vec{\rho}\,, \qquad w \in \mathcal{W}\,. \tag{4.28}$$

These are the transformations which leave (h, w) invariant. The six-to-one map from momenta to weights is an artefact of the Miura transformation [78, 32].

As argued in the previous chapter, the BRST analysis is simplified considerably after an appropriate canonical transformation. We want the BRST operator in (3.60) and (3.61) to become a sum of two terms with spin-two and spin-three ghost numbers $(G_2, G_3) = (1, 0)$ and $(G_2, G_3) = (0, 1)$. To accomplish this, it turns out that we have to perform the quantum canonical transformation generated by $G = \frac{4i}{3\sqrt{29}}\partial\phi c_3 b_2 - \frac{7i}{3\sqrt{58}}\partial c_3 b_2$. The change in the fields is calculated using (3.80), where in the present case only the first three terms contribute. After another OPE-preserving rescaling of the spin-three ghost fields, the BRST operator takes the form $Q = Q_0 + Q_1$, with

$$\begin{aligned} Q_0 &= \oint \frac{dz}{2\pi i} c_2 \left(T + T_{c_3,b_3} + \tfrac{1}{2}T_{c_2,b_2}\right)\,, \\ Q_1 &= \oint \frac{dz}{2\pi i} c_3 \left((\partial\phi)^3 + 3\sqrt{3}\alpha_0\partial^2\phi\partial\phi + \tfrac{19}{8}\partial^3\phi + \tfrac{9}{2}\partial\phi b_3\partial c_3 + \tfrac{3\sqrt{3}}{2}\partial b_3\partial c_3\right)\,. \end{aligned} \tag{4.29}$$

As a simple consequence of the graded structure, we note that $Q_0^2 = Q_1^2 = \{Q_0, Q_1\} = 0$. This form of the BRST operator was first given in [135].

It can be shown[8] that all physical operators with standard ghost structure are of the form

$$V = c_2 \partial c_3 c_3 e^{-\frac{1}{2\sqrt{2}} p\phi} Y_\Delta(X)\,, \tag{4.30}$$

where $Y_\Delta(X)$ is primary under T_X with weight Δ. Physical operators (4.30) belong to one of three sectors, namely

$$p = 6\,,\ \Delta = 1\,; \qquad p = 7\,,\ \Delta = \frac{15}{16}\,; \qquad p = 8\,,\ \Delta = 1\,. \tag{4.31}$$

Thus we see that the scalar ϕ is 'frozen' in the sense that the momentum in the ϕ direction is restricted to three discrete values, and no ϕ excitations are physical [162]. Since the total weight of physical operators is zero, the 'highest-spin part' $\partial c_3 c_3 e^{ip\phi}$ of (4.30) can only have weight 0 or $\frac{1}{16}$. Note that in the case of the two-scalar W_3 string, the momentum of X is also discrete.

We recall from equation (4.17) that the fields $(\phi; b_3, c_3)$ together have central charge $c_3^3 = \frac{1}{2}$, the central charge of the first (non-trivial) unitary Virasoro minimal model, the Ising model. The appearance of an Ising model structure is now becoming clear by noticing that operators of weight 0 (corresponding to the identity operator) and $\frac{1}{16}$ also appear in the Ising model. The third primary operator of the Ising model has conformal weight $h = \frac{1}{2}$ and appears in the W_3 cohomology as a non-standard operator (an operator with non-standard ghost structure), see below. The fields X^μ constitute an effective space-time sector, and we see that the standard ghost structure operators give rise to effective space-time sectors with intercepts 1 and $\frac{15}{16}$. The sector with intercept 1 is almost the same as the standard bosonic string spectrum. The difference is that here we have $c_X = 25\frac{1}{2}$ instead of 26. Note that $Y_\Delta(X)$ can be any $h = \Delta$ primary. In the ordinary critical string, physical states are built from excitations in $D-2$ transverse directions. In the effective space-time sectors of the W_3 string, however, fewer states decouple, and excitations in $D-1$ directions are physical. Unitarity requires the effective space-time intercepts to be of the form $\Delta = 1 - h_{p,q}(m = 3)$, where the conformal weights $h_{p,q}$ of unitary representations (in this case of the Ising model) are given in (2.65). For an explanation, see references [191, 192].

Besides states of standard ghost structure, one can also consider states of non-standard ghost structure. Among them are states that correspond to the $h = \frac{1}{2}$ operator of the Ising model. The simplest such operator is the level 1 operator

$$c_2 c_3 e^{-\frac{1}{2\sqrt{2}} p\phi} Y_\Delta(X)\,, \tag{4.32}$$

with $p = 4$ and $\Delta = \frac{1}{2}$. Operators of non-standard ghost structure are usually associated with vanishing null states of the W_3 algebra. In this context, let us note that precisely the W_3 modules with momenta (weights) as in (4.26) are degenerate, i.e. contain null vectors.

The operators in (4.30) and (4.32) by no means exhaust the BRST cohomology. The physical spectrum turns out to contain an infinite number of operators with different

[8] See the review papers [161, 192] and references therein.

$(\phi; b_3, c_3)$-dependence for all three effective space-time sectors. The $(\phi; b_3, c_3)$-dependent parts of physical operators can be found quite easily from the cohomology of the Q_1 operator in (4.29) alone. The Q_1 cohomology is the first term in a spectral sequence that can be associated with the double complex defined by all fields and the differential $Q = Q_0 + Q_1$. The Q_1 cohomology has been shown to provide a new realization of the Ising model [116]. Singular vectors are divided out in the sense that they are Q_1 exact. The infinite number of copies of each primary are all connected by screening operators. However, from the W_3 string point of view there is no reason to identify them. The Q_1 cohomology acting on the Fock space generated by the fields $(\phi; b_3, c_3)$ *plus* the identifications using screening operators would seem to play a role similar to the Felder reduction that provides irreducible Virasoro minimal model realizations from Fock space representations.

Note that all physical operators described up to now have the factorized form

$$c_2 U(\phi; b_3, c_3) V_\Delta(X) \,, \tag{4.33}$$

where $U(\phi; b_3, c_3)$ are Ising model operators and $V(X)$ are primaries with weights dual to those of U. The operators (4.33) create continuous momentum states of the W_3 string. 'Continuous momentum' here refers to the momenta p_μ of X^μ in the multi-scalar case $(D > 1)$. In addition to continuous momentum states, the physical spectrum also contains discrete states which are physical only for $p_\mu = 0$ or $p_\mu = -2ia_\mu$. For $p_\mu = 0$ the corresponding operators take the form [161]

$$V = c_2 U_1(\phi; b_3, c_3) + U_2(\phi; b_3, c_3) \,, \tag{4.34}$$

where U_2 is a $h = 0$ primary in the Q_1 cohomology. The identity operator is the special case with $U_1 = 0$ and $U_2 = \mathbf{1}$. Note that whereas the operators (4.33) have standard *spin-two* ghost structure, the operators corresponding to discrete states do not have this property. In [140], two invertible discrete physical operators were found. These are guaranteed to give BRST non-trivial physical states when normal-ordered with any physical operator. They have been used to compute the complete spectrum of the W_3 string for the two-scalar as well as for the multi-scalar case [140].

A generic feature of the cohomology is that ghost number G physical states have partners at ghost number $2 - G$. This corresponds to Hermitian conjugation under which the BRST operator is invariant. Conjugation pairs the G and $2 - G$ sectors in the ghosts' Fock space and changes the momenta (p_μ, p_ϕ) of scalar fields to $(-p_\mu - 2ia_\mu, -p_\phi + 2i\sqrt{3}\alpha_0)$. Furthermore, all states described thus far are so-called prime states. Acting with picture changing operators on prime states gives additional states at the next few ghost numbers [165]. For the bosonic string we have seen that the picture changing operator accounts for the doubling of the states. For the two-scalar W_3 string, two independent picture changing operators $a_X = [Q, X]$ and $a_\phi = [Q, \phi]$ generate, starting from a prime state $|P>$, quartets of physical states $\{|P>, a_X|P>, a_\phi|P>, a_X a_\phi|P>\}$ at ghost numbers $\{G, G+1, G+1, G+2\}$.

The appearance of Ising model operators in the spectrum suggests that the critical W_3 string is related to the non-critical Virasoro string with the Ising model as its matter sector. Indeed, the Q_1 cohomology represents the Ising model whose operators are then dressed by the 'Liouville' scalar X (in the two-scalar case) to operators of vanishing total conformal dimension in the Q cohomology. Moreover, in both the non-critical string spectrum [133] and the critical W_3 spectrum [140], the ghost numbers of cohomology classes range from $-\infty$ to $+\infty$. It is also known [70] that W_3 constraints appear as Dyson-Schwinger equations in the two-matrix model corresponding to the Ising model coupled to 2d-gravity. In the same way, critical W_N strings are believed to be intimately related to the $(N, N+1)$ unitary Virasoro minimal model coupled to 2d-gravity. For $N \to \infty$ one would then expect some connection between $c = 1$ matter coupled to 2d-gravity (often referred to as the two-dimensional string) and the critical W_∞ string. It is indeed known that the two-dimensional string has a W_∞ symmetry structure [8, 128, 194].

The critical W_N string based on the Miura realization is not only related to the non-critical Virasoro string with $(N, N+1)$ minimal matter, but also to a series of non-critical W_n strings for all $3 < n < N$, as argued before by counting central charges. The Q_N^{n+1} cohomology (cf. equation (3.81)) is conjectured to realize the unitary $(N, N+1)$ W_n minimal model and is coupled to W_n gravity through the transition to the total cohomology. For W_4 we give details below.

What we have been calling the critical W_N string is in fact more like the analogue of pure gravity and it is perhaps better to call it pure W_N gravity. Indeed, the W_N symmetry of the $sl(N)$ Toda action expected to describe quantum W_N gravity in a conformal gauge is realized by the $(N-1)$-scalar Miura realization. Pure gravity is described by the Liouville scalar with a background charge such that $c_L = 26$. Its cohomology classes are known to extend through all ghost numbers [133]. Similarly, the critical W_3 string described above also has cohomology classes at all ghost numbers. As mentioned before, direct W-extensions of the 26-dimensional critical bosonic string do not seem to exist since scalar field realizations of W_N algebras always involve background charges.

4.2.1 Non-critical W-strings

Non-critical W_N strings describe W_N matter coupled to W_N gravity and provide generalizations of the critical W_N string which corresponds to the special case of W_N gravity coupled to trivial ($c = 0$) matter. Results on the spectrum of non-critical W_N strings have been given in [30, 31, 46, 18, 47, 48].

The matter sector is usually taken to be a W_N minimal model. The (p, q) W_N minimal models have central charges

$$c_{p,q} = (N-1)\left(1 - N(N+1)\frac{(p-q)^2}{pq}\right), \tag{4.35}$$

for positive integers p and q. The unitary models have $q = p + 1 > N + 1$ and their central charges were given before in (2.87). The central charge of the W_N gravity (Toda) theory is then given by $c_{p,-q}$, since

$$c_{p,q} + c_{p,-q} = c_{\text{crit}} . \tag{4.36}$$

The BRST charge for the non-critical W_3 string in its canonical form was given in equation (3.66). To compute the physical spectrum we need a realization of the matter and Liouville currents. For the Liouville sector it is natural to take the usual two-scalar Miura realization of the W_3 algebra. To facilitate computation of the cohomology, the matter minimal model is usually also represented by the two-scalar Miura realization of the appropriate central charge. However, one has to perform a further reduction in order to obtain a minimal model. This should be similar to the Felder BRST reduction of the Coulomb gas realizations for Virasoro minimal models. However, Fock space resolutions of irreducible modules for minimal models are very complex in the case of W-algebras. (In the case of W_3 such resolutions were constructed in [89, 44].) Nevertheless, a complete classification of physical states for a $W[g]$ minimal model coupled to $W[g]$ gravity has been conjectured in [46]. For $g = sl(N)$ this corresponds to W_N minimal matter coupled to W_N gravity.

The BRST analysis for the non-critical W_3 string can again be simplified using the redefinition that leads, at the classical level, to a nested subalgebra structure. In [18] the non-critical W_3 string is investigated using the redefined BRST operator of [21], which is the sum of two nilpotent operators Q_0 and Q_1. The minimal model structure of the cohomology is then elucidated. In particular, it is shown in [18] that the Q_1 cohomology is closely related to a (p, q) Virasoro minimal model if one chooses for the matter sector a (p, q) W_3 minimal model. This generalizes the connections to minimal models of critical W_N strings. One might also wonder what happens if one takes a (non-unitary) $(p, q) = (2, 3)$ W_3 model as matter sector with central charge -2. This might lead to the trivial $c = 0$ Virasoro minimal model in the Q_1 cohomology. In fact, this particular W_3 non-critical string model is used in [23] and we come back to it in section 4.4.

Another interesting class of non-critical W_N strings is obtained if the matter theory is realized by $N-1$ free scalar fields. For $N = 2$ this corresponds to the $D = 2$ string whose physical states have been calculated in [134]. Interesting algebraic structures have been found in the $D = 2$ string, as described, for example, in [194]. For $N = 3$, the $D = 4$ W_3 string has been extensively studied in [47]. In this work the algebraic structure of the cohomology is emphasized.

4.3 An example: the W_4 string

This section closely follows [40]. In order to study the physical spectrum of the W_4 string, we need the BRST operator for the W_4 algebra. It was given in [113, 200], but a more convenient form of the BRST operator was found in [20]. Let us summarize how it was constructed.

The energy-momentum tensor in the three scalar Miura realization is

$$T_M = -\tfrac{1}{2}\partial\vec{\phi}\cdot\partial\vec{\phi} - \sqrt{2}\alpha_0\vec{\rho}\cdot\partial^2\vec{\phi}, \tag{4.37}$$

where $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$ and $\vec{\rho}$ is the Weyl vector of $sl(4)$. We use the representation $\vec{\rho} = \frac{1}{2}(\sqrt{2}, \sqrt{6}, \sqrt{12})$, and denote by $q_i = \sqrt{2}\alpha_0\rho_i$, $i = 1, 2, 3$, the background charges of the scalar fields. The central charge is

$$c_M = 3 + 24(\alpha_0)^2\rho^2 = 3 + 120(\alpha_0)^2\,. \tag{4.38}$$

As described in the previous chapter, in the classical limit it is possible to redefine the generators such that the algebra is brought to a special form with a nested subalgebra structure [20]. The energy-momentum tensor is not affected by the redefinition. The BRST charge associated to the resulting classical algebra inherits the same nested structure. Quantization by parametrising all possible quantum corrections and demanding nilpotency leads to a BRST operator which still has this nested structure, and therefore, as we will see, is convenient for studying the spectrum. It should also be possible to relate the operators in the original and redefined basis by a (quantum) canonical transformation, as in the case of the W_3 string. Explicitly, the BRST current is

$$\begin{aligned} j_2 &= c_4\{(\partial\phi_3)^4 + 4q_3\partial^2\phi_3(\partial\phi_3)^2 + \tfrac{41}{5}(\partial^2\phi_3)^2 + \tfrac{124}{15}\partial^3\phi_3\partial\phi_3 \\ &\quad + \tfrac{46}{135}q_3\partial^4\phi_3\} - 8(\partial\phi_3)^2c_4\partial c_4 b_4 + \tfrac{16}{9}q_3\partial^2\phi_3 c_4\partial c_4 b_4 \\ &\quad + \tfrac{32}{9}q_3\partial\phi_3 c_4\partial^2 c_4 b_4 + \tfrac{4}{5}c_4\partial^3 c_4 b_4 - \tfrac{16}{3}c_4\partial c_4\partial^2 b_4\,, \end{aligned} \tag{4.39}$$

$$\begin{aligned} j_1 &= c_3\{(\partial\phi_2)^3 + \tfrac{3}{4}\partial\phi_2(\partial\phi_3)^2 + \tfrac{5\sqrt{2}}{8}(\partial\phi_3)^3 + 3q_2\partial\phi_2\partial^2\phi_2 \\ &\quad + \tfrac{3}{2}q_3\partial\phi_2\partial^2\phi_3 + \tfrac{9}{2}q_2\partial\phi_3\partial^2\phi_3 + \tfrac{93}{40}\partial^3\phi_2 + \tfrac{69\sqrt{2}}{10}\partial^3\phi_3\} \\ &\quad - \tfrac{9}{2}\partial\phi_2 c_3\partial c_3 b_3 + \tfrac{3}{2}q_2 c_3\partial^2 c_3 b_3 - \tfrac{243}{64}c_3\partial c_3 b_4 \\ &\quad - \tfrac{9}{2}\partial\phi_2 c_3 c_4\partial b_4 - 6\partial\phi_2 c_3\partial c_4 b_4 + \tfrac{9}{2}q_2 c_4\partial^2 c_3 b_4 \\ &\quad + \tfrac{3}{2}q_2 c_3\partial^2 c_4 b_4 - \tfrac{9\sqrt{2}}{2}\partial\phi_3 c_4\partial c_3 b_4 - 3\sqrt{2}\partial\phi_3 c_3\partial c_4 b_4 + j_2\,, \end{aligned} \tag{4.40}$$

$$j = c_2(T_M + T_{c_3,b_3} + T_{c_4,b_4} + \tfrac{1}{2}T_{c_2,b_2}) + j_1\,. \tag{4.41}$$

Here (c_k, b_k) is the conjugate ghost pair of the spin-k symmetry with conformal dimension $(1-k,k)$ with respect to the corresponding energy-momentum tensor $T_{c_k,b_k} = -kb_k\partial c_k + (1-k)\partial b_k c_k$. The total W_4 BRST operator is $Q = \oint \frac{dz}{2\pi i} j(z)$ and as the way of representing it in equations (4.39-4.41) suggests, it involves two other nilpotent BRST charges: $Q_2 = \oint \frac{dz}{2\pi i} j_2(z)$ is a BRST operator corresponding to a spin-four symmetry, and $Q_1 = \oint \frac{dz}{2\pi i} j_1(z)$ is a BRST operator corresponding to a symmetry generated by spin-three and spin-four currents. We have

$$(Q_2)^2 = (Q_1)^2 = (Q)^2 = 0\;; \qquad (Q_{Vir})^2 = \{Q_{Vir}, Q_1\} = 0\,, \tag{4.42}$$

where we defined $Q_{Vir} = Q - Q_1$. Note that Q_{Vir} is just the usual Virasoro BRST operator when we consider the spin-(3,4) ghost systems to belong to the matter part. The spin-four part Q_2 was obtained before in [139]. We should also mention that momenta and ghost numbers of physical operators are not affected by the redefinition leading to (4.39-4.41). However, the explicit expressions for physical operators are expected to be much simpler in the new basis, as in the W_3 string case [135].

The BRST operator (4.41) is nilpotent provided the total central charge of matter plus ghosts vanishes. This requires T_M to have central charge $c_M = 246$ implying $(\alpha_0)^2 = \frac{81}{40}$. Then we obtain what we call a critical W_4 string. For a non-critical string one would expect another sector with W_4 symmetry. Unfortunately, however, the redefinition described above can only be applied to one of both sectors [20] which means that the description of the non-critical W_4 string would be much more involved.

Screening operators play an important role in the physical state analysis. The standard W_4 screening currents [78] are

$$S_i^{\pm} = e^{i\alpha_{\pm}\vec{e}_i\cdot\vec{\phi}}\,, \tag{4.43}$$

where $\vec{e}_i$ are the $sl(4)$ simple roots, and $\alpha_{\pm}$ are given in (4.25). The screening charges $\oint \frac{dz}{2\pi i}S_i^{\pm}$ commute with the W_4 generators in the Miura realization, and they will appear in modified form in the discussion of the Q-cohomology. Besides, there will be more screening operators that simplify the classification of physical states. In general, for a physical operator $\mathcal{O}$ of zero conformal weight, one can find an associated screening current $S_{\mathcal{O}}$ via the descent equation

$$[Q, S_{\mathcal{O}}(z)] = \partial\mathcal{O}(z)\,, \tag{4.44}$$

where Q is the BRST charge under consideration. The corresponding screening charge $\oint \frac{dz}{2\pi i}S_{\mathcal{O}}(z)$ will then commute with Q.

Picture changing operators are defined by

$$P_i(z) = [Q, \phi_i(z)]\,. \tag{4.45}$$

In the usual BRST complex these are physical $h = 0$ primaries. Applying a picture changing operator to a physical state, i.e. taking the normal-ordered product, either gives zero or another physical state.

As in (4.26), the momenta of physical states are multiples of the momenta of the screening currents (4.43):

$$\vec{p} = \sum_{i=1}^{3}(n_i^+\alpha_+ + n_i^-\alpha_-)\vec{e}_i\,. \tag{4.46}$$

All physical operators described below indeed have momenta on this lattice. It is then convenient to rewrite the momenta as

$$p_i = \frac{iq_i}{27}\tilde{p}_i\,, \tag{4.47}$$

because (4.46) now implies that

$$\tilde{p}_1 \in 3\mathbb{Z}\,; \quad \tilde{p}_2 \in \mathbb{Z}\,; \quad \tilde{p}_3 \in 2\mathbb{Z}\,. \tag{4.48}$$

Note that the momenta p_i are imaginary. In the following we will usually refer to $\tilde{p}_i$ as the momentum.

We now proceed to determine the cohomology of Q in steps, starting with the Q_2 cohomology which imposes the spin-four constraint.

4.3.1 The Q_2 cohomology

Since the BRST current j_2 only depends on a single scalar field ϕ_3 and the spin-four ghost variables (c_4, b_4), we only need to consider operators built from these fields. These fields together have central charge $\frac{4}{5}$, the central charge of the $(p,p')=(4,5)$ W_3 unitary minimal model.

The Q_2 physical states at some low energy levels are given implicitly in a discussion of the $W_{2,4}$ string in [139, 138]. The extra Virasoro constraint of the $W_{2,4}$ algebra does little more than dress the primary Q_2 physical operators to operators of total spin zero. In [141], the complete cohomology of the critical $W_{2,4}$ string is given. Ignoring the Virasoro constraint, i.e. ignoring the 'Liouville dressings', this cohomology seems to be equivalent to the W_3 primary part of the Q_2 cohomology obtained below, apart from some descendants which couple to the Virasoro fields in the $W_{2,4}$ cohomology.

The ghost vacuum is given by acting on the $sl(2)$-invariant vacuum with $\partial^2 c_4 \partial c_4 c_4$. In the following, we will always write down operators that are supposed to act on the $sl(2)$-invariant vacuum. First consider operators of the form

$$V_0^0 = \partial^2 c_4 \partial c_4 c_4 e^{ip_3\phi_3} \,. \tag{4.49}$$

The lower index denotes the level. The upper index refers to the ghost number G of the state created by this operator. Thus the $sl(2)$-invariant vacuum is assigned ghost number -3 for the moment. Level 1 states at lowest ghost number ($G=-1$) are created by

$$V_1^{-1} = \partial c_4 c_4 e^{ip_3\phi_3} \,. \tag{4.50}$$

The physical states of lowest ghost number at a particular level are easy to find. Since they can't be Q_2 exact, one only has to impose the vanishing of their Q_2 variation. We restrict ourselves to operators on levels 0 and 1, since this will turn out to be enough to understand the full structure of the Q_2 cohomology. Imposing the physical state conditions on (4.49) and (4.50), we obtain the results listed in table 1.

V_0^0	$\tilde{p}_3$	h	w
	24	0	0
	30	0	0
	26	1/15	1
	28	1/15	-1

V_1^{-1}	$\tilde{p}_3$	h	w
	16	1/15	-1
	18	2/5	0
	20	2/3	-26

Table 1. *Level 0 and 1 physical states in the Q_2 cohomology. Momenta are denoted by $\tilde{p}_3$, see (4.47). The last two columns give the weights h and w with respect to the spin-two and three generators of the $c=\frac{4}{5}$ W_3 algebra.*

Note that the physical values of p_3 agree with equations (4.46-4.48) (we only consider the third component of (4.46)). The last two columns give the weights of the physical states with respect to the generators of a W_3 algebra. It turns out that the physical states in the Q_2 cohomology can be organized in representations of the $c = \frac{4}{5}$ W_3 algebra whose generators (T, W) are physical operators at levels 8 and 9 with zero momentum. The Virasoro generator T is the energy-momentum tensor built from the fields $(\phi_3; c_4, b_4)$, and the spin-three generator is given by [138]

$$\begin{aligned} W \;=\; & \sqrt{\tfrac{2}{13}}\{\tfrac{5}{3}(\partial\phi_3)^3 + 5q_3\partial^2\phi_3\partial\phi_3 + \tfrac{25}{4}\partial^3\phi_3 + 20\partial\phi_3 b_4\partial c_4 \\ & +12\partial\phi_3\partial b_4 c_4 + 12\partial^2\phi_3 b_4 c_4 + 5q_3\partial b_4\partial c_4 + 3q_3\partial^2 b_4 c_4\}\,. \end{aligned} \tag{4.51}$$

They generate the $c = \frac{4}{5}$ W_3 algebra with standard normalization, up to an extra primary spin-four operator, which turns out to be a multiple of the Q_2 exact operator $V = \{Q_2, b_4\}$. It was noticed by the authors of [138] that after bosonizing the spin-four ghost pair, this realization of the W_3 algebra coincides with a special two-scalar realization found in [22]. As a side-remark, we note that this $c = \frac{4}{5}$ realization is unique in the sense that it has one real and one imaginary background charge, the latter belonging to the 'ghost scalar'.

The physical states in table 1 are all primary with respect to the W_3 algebra, with L_0 and W_0 eigenvalues h and w, respectively. For convenience, the weights w have been rescaled as in [138]. The Virasoro weights are given in terms of p_3 as

$$h = \tfrac{1}{2}(p_3)^2 - iq_3p_3 + l - 6\,, \tag{4.52}$$

where l is the level. The spin-three weight is a cubic polynomial in p_3 and depends on the detailed structure of the operator.

Let us now compare the Q_2 spectrum with the spectrum of primaries in a $c = \frac{4}{5}$ W_3 minimal model. The spectrum of conformal weights in a generic (p, p') W_3 minimal model is given by (see e.g. [49])

$$\begin{aligned} h(r_1, r_2; s_1, s_2) \;=\; & -\frac{(p-p')^2}{pp'} \\ & + \frac{1}{3pp'}\{\sum_{i\le j=1}^{2}(p'(r_i+1) - p(s_i+1))(p'(r_j+1) - p(s_j+1))\}\,, \end{aligned} \tag{4.53}$$

where the non-negative integers r_i, s_i run over the range

$$0 \le r_1 + r_2 \le p - 3\,; \quad 0 \le s_1 + s_2 \le p' - 3\,. \tag{4.54}$$

Note that the level 0 states in table 1 correspond to the 'diagonal' entries of the $(p, p') = (4, 5)$ Kac table (4.53), since $h(0,0;0,0) = 0$ and $h(0,1;0,1) = h(1,0;1,0) = \frac{1}{15}$. The weights $\frac{2}{5}$ and $\frac{2}{3}$ of the level 1 states are also in the set (4.53). Moreover, at levels 0 and 1 together, all conformal weights of the $(4, 5)$ W_3 minimal model occur. It is also interesting to note that the maximum possible conformal dimension of an operator at a particular level, $h_{max} = \frac{1}{2}q_3^2 + l - 6$ (see (4.52)), forbids the appearance of $h = \frac{2}{5}$ and $h = \frac{2}{3}$ operators on level 0.

The spin-three weights w corresponding to the Virasoro weights $h(r_1, r_2; s_1, s_2)$ in a (p, p') W_3 minimal model are given by [49]

$$\begin{aligned} w(r_1, r_2; s_1, s_2) &= C(p,p')(p'(r_1 - r_2) - p(s_1 - s_2)) \\ \times(p'(2r_1 + r_2 + 3) &- p(2s_1 + s_2 + 3))(p'(r_1 + 2r_2 + 3) - p(s_1 + 2s_2 + 3)) \,, \end{aligned} \tag{4.55}$$

where $C(p, p')$ depends on the normalization of the spin-three current. The w-values in table 1 are indeed in agreement with the minimal model values (4.55). Under the $\mathbb{Z}_2$ transformation $(r_1, r_2; s_1, s_2) \to (r_2, r_1; s_2, s_1)$ h is invariant, while w changes sign. We observe that the level 0 states occur in such $\mathbb{Z}_2$ pairs.

From table 1 it is clear that a physical state with $(h, w) = (\frac{2}{3}, +26)$ is missing at levels 0 and 1. However, we only discussed states of lowest ghost number. In particular, any state of ghost number G has a conjugate state at ghost number $1 - G$ at the same level, and it turns out that the $(\frac{2}{3}, +26)$ state occurs at level 1, $G = 2$. It is in fact the conjugate of the $(\frac{2}{3}, -26)$ state. More generally, the $\mathbb{Z}_2$ symmetry mentioned above is part of the conjugation of a physical operator. This completes the identification of all minimal model primaries in the Q_2 cohomology, at levels 0 and 1.

For the purpose of finding physical states at higher levels and different ghost numbers we introduce the following screening operators,

$$S = b_4 e^{ip_3\phi_3}\,, \qquad \text{with } \tilde{p}_3 = -6\,, \tag{4.56}$$

$$R = \partial c_4 c_4 e^{ip_3\phi_3}\,, \qquad \tilde{p}_3 = 30\,, \tag{4.57}$$

$$\bar{R} = \partial c_4 c_4 e^{ip_3\phi_3}\,, \qquad \tilde{p}_3 = 24\,. \tag{4.58}$$

They are spin-one primaries whose charges commute with Q_2. It is not difficult to see that R and $\bar{R}$ are the screening currents associated to the level 0, $h = 0$ physical operators (see table 1) via the descent equation (4.44). With these screening charges it is possible to obtain new physical states by acting on the level 0 and 1 states described above. The OPEs of T and W with the screening currents are total derivatives (in the case of R and $\bar{R}$ this is true up to Q_2 exact terms), which means that W_3 primaries are mapped to W_3 primaries of the same (h, w) under the action of the screening charges.

We follow [86], where a similar discussion for the W_3 string can be found. For the action of n screening charges on a physical state of momentum p to be well-defined, the following expression must be an integer,

$$P_n \equiv n - 1 + \sum_{i<j=1}^{n} p_{s_i} p_{s_j} + p \sum_{i=1}^{n} p_{s_i}\,, \tag{4.59}$$

with screening momenta p_{s_i}. Using this, one can show that, for example, the action of S on $V_0^0[\tilde{p}_3 = 30]$ is well-defined. However, this action[9] is trivial in the sense that it gives zero, and to obtain a new physical state we have to make use of the picture

[9] By the action of a screening operator S on a physical operator V we mean the commutator $\oint \frac{dz}{2\pi i} S(z)V(w)$, whereas the action of a picture changing operator P is the normal-ordered product $\oint \frac{dz}{2\pi i} \frac{P(z)V(w)}{z-w}$, with an integration contour around w.

changing operator $P_3(z) = [Q_2, \phi_3(z)]$. Taking the normal-ordered product of P_3 with $V_0^0[\tilde{p}_3 = 30]$ and then acting with S gives the physical state $V_0^0[\tilde{p}_3 = 24]$. Generalizing this, we can write down infinite series of operators by analogy with the W_3 case [86, 116]. If we define $V(0,0) = V_0^0[\tilde{p}_3 = 30]$, the series with $(h, w) = (0, 0)$ may be written as

$$\bar{V}(0,n) = SP_3V(0,n)\,, \quad V(0,n) = (S)^4P_3\bar{V}(0,n-1)\,. \tag{4.60}$$

The other series are

$$\begin{aligned}
& V_-(\tfrac{1}{15},0) \equiv V_0^0[\tilde{p}_3 = 28]\,, & (4.61)\\
\bar{V}_-(\tfrac{1}{15},n) &= (S)^2P_3V_-(\tfrac{1}{15},n)\,, \quad V_-(\tfrac{1}{15},n) = (S)^3P_3\bar{V}_-(\tfrac{1}{15},n-1)\,; \\
& V_+(\tfrac{1}{15},0) \equiv V_0^0[\tilde{p}_3 = 26]\,, & (4.62)\\
\bar{V}_+(\tfrac{1}{15},n) &= (S)^3P_3V_+(\tfrac{1}{15},n)\,, \quad V_+(\tfrac{1}{15},n) = (S)^2P_3\bar{V}_+(\tfrac{1}{15},n-1)\,; \\
& V(\tfrac{2}{5},0) \equiv V_1^{-1}[\tilde{p}_3 = 18]\,, & (4.63)\\
\bar{V}(\tfrac{2}{5},n) &= (S)^2P_3V(\tfrac{2}{5},n)\,, \quad V(\tfrac{2}{5},n) = (S)^3P_3\bar{V}(\tfrac{2}{5},n-1)\,; \\
& V_-(\tfrac{2}{3},0) \equiv V_1^{-1}[\tilde{p}_3 = 20]\,, & (4.64)\\
\bar{V}_-(\tfrac{2}{3},n) &= SP_3V_-(\tfrac{2}{3},n)\,, \quad V_-(\tfrac{2}{3},n) = (S)^4P_3\bar{V}_-(\tfrac{2}{3},n-1)\,; \\
& V_+(\tfrac{2}{3},0) \equiv V_1^{1}[\tilde{p}_3 = 34]\,, & (4.65)\\
\bar{V}_+(\tfrac{2}{3},n) &= (S)^4P_3V_+(\tfrac{2}{3},n)\,, \quad V_+(\tfrac{2}{3},n) = SP_3\bar{V}_+(\tfrac{2}{3},n-1)\,.
\end{aligned}$$

The notation, not to be confused with the previous notation V_l^G with level and ghost number indices, is $V_\pm(h, n)$, where h is the spin and $\pm$ indicates the sign of the spin-three weight w (see table 1). Although the actions of the screening operator S in general do not seem to have inverses, one can act on any operator in (4.60-4.65) with R, thus extending the series to negative n,

$$\begin{aligned}
V_\pm(h,n-1) &= RP_3V_\pm(h,n)\,, \\
\bar{V}_\pm(h,n-1) &= RP_3\bar{V}_\pm(h,n)\,.
\end{aligned} \tag{4.66}$$

We observe that the operator $V(0,1)$ is the identity, and $\bar{V}(0,1)$ is another $h = 0$ operator at the same ghost number $G = -3$ relative to the tachyonic operators (4.49). To summarize, we list all these operators with their momentum, ghost number and level in table 2.

We did not prove that all these operators are BRST non-trivial (they are certainly BRST closed). The operators in table 2 are supposed to be the prime operators [165] from which new physical operators are obtained by normal-ordering with the picture changing operator P_3. Thus all operators in the Q_2 cohomology come in doublets.

operator	$\tilde{p}_3$	G	level
$V(0,n)$	$30-30n$	$-3n$	$\frac{1}{2}(3n(5n-1))$
$\bar{V}(0,n)$	$24-30n$	$-3n$	$\frac{1}{2}(3n(5n+1))$
$V_-(\frac{1}{15},n)$	$28-30n$	$-3n$	$\frac{1}{2}(n(15n-1))$
$\bar{V}_-(\frac{1}{15},n)$	$16-30n$	$-1-3n$	$\frac{1}{2}(15n^2+11n+2)$
$V_+(\frac{1}{15},n)$	$26-30n$	$-3n$	$\frac{1}{2}(n(15n+1))$
$\bar{V}_+(\frac{1}{15},n)$	$8-30n$	$-2-3n$	$\frac{1}{2}(15n^2+19n+6)$
$V(\frac{2}{5},n)$	$18-30n$	$-1-3n$	$\frac{1}{2}(15n^2+9n+2)$
$\bar{V}(\frac{2}{5},n)$	$6-30n$	$-2-3n$	$\frac{1}{2}(15n^2+21n+8)$
$V_-(\frac{2}{3},n)$	$20-30n$	$-1-3n$	$\frac{1}{2}(15n^2+7n+2)$
$\bar{V}_-(\frac{2}{3},n)$	$14-30n$	$-1-3n$	$\frac{1}{2}(15n^2+13n+4)$
$V_+(\frac{2}{3},n)$	$34-30n$	$1-3n$	$\frac{1}{2}(15n^2-7n+2)$
$\bar{V}_+(\frac{2}{3},n)$	$10-30n$	$-2-3n$	$\frac{1}{2}(15n^2+17n+6)$

Table 2. *Operators in the Q_2 cohomology.*

From equations (4.60-4.65) one observes that the action of five S screening charges (together with two picture changes) is special. It lowers $\tilde{p}_3$ by 30 and G by 3. Indeed, a screening operator exists which does the same in one go (together with one picture change), namely

$$S_x = \partial^3 b_4 \partial^2 b_4 \partial b_4 b_4 e^{ip_3\phi_3}\,, \quad \text{with } \tilde{p}_3 = -30\,. \tag{4.67}$$

This screening operator is also used in [141]. It has a well-defined action on all physical operators. The physical operator x associated with S_x through the descent equation turns out to be (up to an irrelevant constant factor)

$$x(z) = \oint \frac{dw}{2\pi i} S_x(w) P_3(z)\,, \tag{4.68}$$

which we identify as $V(0,2)$ in (4.60). We will not give x explicitly; it is a complicated expression consisting of 50 terms. We may recover S_x from x via

$$S_x(w) = (b_4)_{-1}(w)x(w) \equiv \oint \frac{dz}{2\pi i}(z-w)^2 b_4(z)x(w)\,. \tag{4.69}$$

The operator x has a physical inverse x^{-1}, such that the normal-ordered product of x with x^{-1} is a nonvanishing multiple of the identity. This inverse is precisely the physical level 0 operator $V_0^0[\tilde{p}_3 = 30]$. We may write it as

$$x^{-1} = \oint \frac{dw}{2\pi i} R(w) P_3(z)\,. \tag{4.70}$$

Thus we have identified three members of the family (4.60): $V(0,0) = x^{-1}$, $V(0,1) = \mathbf{1}$ and $V(0,2) = x$. We also see that RP_3 acts as the inverse of $S_x P_3$.

In [140], analogous invertible operators enabled the computation of the complete cohomology of the critical W_3 string by the observation that normal-ordered products of arbitrary powers of x or x^{-1} with a physical operator give new non-trivial physical operators. The situation here is somewhat different in that the operators in the Q_2 cohomology do not all have vanishing total conformal weight, due to the lack of a Virasoro constraint. As this is an essential argument used in [140], we have no complete proof here that the operators of table 2 plus their W_3 descendants and picture changed versions generate the full Q_2 cohomology.

Equation (4.69) seems to give the general procedure to obtain the screening current associated to a $h = 0$ physical operator in the Q_2 cohomology. For $S_{\mathcal{O}} \equiv (b_4)_{-1}\mathcal{O}$, with $\mathcal{O}$ an arbitrary $h = 0$ physical operator, one has

$$[Q_2, S_{\mathcal{O}}(w)] = V_{-1}(w)\mathcal{O}(w) = \oint \frac{dz}{2\pi i}(z-w)^2 V(z)\mathcal{O}(w)\,, \tag{4.71}$$

where, as before, V is the spin-four current $\{Q_2, b_4\}$, and the RHS is indeed a (multiple of) $\partial\mathcal{O}$ in the cases examined.

4.3.2 The Q_1 cohomology

We now take the next nilpotent BRST operator, Q_1, and study its cohomology. It is the part of the total W_4 BRST current (4.41) which does not involve the Virasoro sector. It imposes only a spin-three and a spin-four constraint. The Fock space must now be extended to include also the scalar ϕ_2 and the spin-three ghosts. Together, the fields $(\phi_2, \phi_3; c_3, b_3; c_4, b_4)$ have central charge $c = \frac{7}{10}$ which is the central charge of the $(p, p') = (4, 5)$ unitary Virasoro minimal model.

Operators in the Q_1 cohomology can be computed from operators in the Q_2 cohomology in a systematic way using a spectral sequence argument (for a review, see e.g [45]). Taking the spin-three ghost number G_3 as an extra grading on the complex of scalar plus ghost Fock spaces, one can decompose Q_1 in three parts with $G_3 = 0, 1$ and 2:

$$Q_1 = d_0 + d_1 + d_2\,, \tag{4.72}$$

where the $G_3 = 0$ part, d_0, is Q_2. There is only one term in Q_1 with $G_3 = 2$, namely $d_2 = -\frac{243}{64} c_3 \partial c_3 b_4$. This term prevents the complex from being a double complex. The remaining terms have $G_3 = 1$ and form d_1. The first term of the spectral sequence $(E_r, \delta_r)_{r=0}^{\infty}$ associated to this gradation, is the Q_2 cohomology

$$\begin{aligned} E_1 &= H(Q_2, \mathcal{F}(\phi_2, \phi_3; c_3, b_3; c_4, b_4)) \\ &= \mathcal{F}(\phi_2; c_3, b_3) \otimes H(Q_2, \mathcal{F}(\phi_3; c_4, b_4))\,, \end{aligned} \tag{4.73}$$

where the second equality follows from the fact that Q_2 acts trivially on any of the fields $(\phi_2; c_3, b_3)$. So we can start with a Q_2 physical operator and extend it (if possible) to a Q_1 physical operator by computing the next terms in the spectral sequence. The successive terms that are added to the original Q_2 physical operator have increasing spin-three ghost number G_3 (but of course the same total ghost number). At low levels the spectral sequence will collapse after a few terms due to the small range of ghost numbers available there, but at higher levels the procedure becomes increasingly laborious.

Having said this, we found it just as convenient to compute operators in the Q_1 cohomology by imposing the complete Q_1 physical condition at once. We used the Mathematica package OPEdefs [185] for computing OPEs. Still, it is useful to observe from the above-mentioned arguments that the Q_1 physical operators are extensions of Q_2 physical operators, so that only the ϕ_2 momentum and the spin-three ghost structure remain to be determined from the Q_1 physical condition.

The level 0 operators are now of the form

$$W_0^0 = \partial c_3 c_3 \partial^2 c_4 \partial c_4 c_4 e^{ip_2\phi_2 + ip_3\phi_3} . \tag{4.74}$$

The notation is the same as in (4.49) except that we use W for operators in the Q_1 cohomology. Level 1 operators that create states with the lowest ghost number $G = -1$ can now be linear combinations of two terms of different ghost structure:

$$W_1^{-1} = (x_1 c_3 \partial^2 c_4 \partial c_4 c_4 + x_2 \partial c_3 c_3 \partial c_4 c_4) e^{ip_2\phi_2 + ip_3\phi_3} . \tag{4.75}$$

Table 3 lists the momenta for which the level 0 and level 1 operators are physical.

W_0^0	$\tilde{p}_3$	$\tilde{p}_2$	h
	24	24	0
		27	3/80
		30	0
	26	22	0
		28	1/10
		31	3/80
	28	23	3/80
		26	1/10
		32	0
	30	24	0
		27	3/80
		30	0

W_1^{-1}	$\tilde{p}_3$	$\tilde{p}_2$	h
	16	23	3/80
		26	1/10
		32	0
	18	18	1/10
		27	7/16
		36	1/10
	20	19	7/16
		22	3/5
		40	0
	24	12	1/10
		15	7/16
	26	16	3/5
	28	11	3/80
	30	12	1/10
		15	7/16

Table 3. *Level 0 and 1 operators in the Q_1 cohomology.*

All cohomology classes are one-dimensional. The level 1 operators with $\tilde{p}_3$-values 16, 18 and 20 have $x_1 = 0$ in (4.75) while the other level 1 operators have a nonzero ratio $\frac{x_1}{x_2}$.

The last column in table 3 shows the total conformal weight of the physical operators with respect to the $c = \frac{7}{10}$ energy-momentum tensor, which is itself a physical operator at level 11. Thus the physical states are organized into $c = \frac{7}{10}$ Virasoro representations. Unitarity then requires that all primary physical operators have conformal dimensions of the corresponding Kac table. This appears to be the case. In particular, the level 0 physical operators correspond to the diagonal entries of the Kac table. The multiplicities of operators of fixed weight can be understood from Weyl group transformations [136] (cf. (4.28)). The presence of non-diagonal operators at level 0 is impossible because of the maximum conformal weight

$$h_{max}(l) = \tfrac{1}{2}(q_2^2 + q_3^2) + l - 9 = \tfrac{9}{80} + l\,. \tag{4.76}$$

At level 1, primary operators corresponding to the first off-diagonal in the Kac table, with $h = \frac{7}{16}$ and $h = \frac{3}{5}$, are allowed by (4.76), and they are indeed in the Q_1 cohomology as can be seen from table 3. Also observe that there is no physical operator at levels 0 and 1 corresponding to the outermost entry in the Kac table, $h = \frac{3}{2}$. From (4.76) it is clear that such an operator can exist only at levels $l \geq 2$. So it is natural to look for this missing operator at level 2. At this level, the ghost number can take values $-2 \leq G \leq 4$. To see if there is a $h = \frac{3}{2}$ physical state, it suffices to consider only $G \leq 0$, since for $G \geq 1$ the spectrum consists of conjugates of $G \leq 1$ states with the same conformal weight. The lowest ghost number operator at level 2 has the form $W_2^{-2} = c_3 \partial c_4 c_4 e^{ip_2\phi_2 + ip_3\phi_3}$ and is physical for two values of the momenta (p_2, p_3) giving rise to two $h = \frac{3}{80}$ operators. For $G = -1$ there is no $h = \frac{3}{2}$ cohomology either. However, for $G = 0$ there is a one-dimensional $h = \frac{3}{2}$ cohomology class, with momentum $(\tilde{p}_2, \tilde{p}_3) = (34, 20)$. It may be represented by $\partial^3 c_3 \partial c_3 c_3 \partial c_4 c_4 e^{ip_2\phi_2 + ip_3\phi_3}$ which is primary up to Q_1 exact terms. Also, there is a cohomology class with the conjugate momentum[10] (and thus also $h = \frac{3}{2}$), $(\tilde{p}_2, \tilde{p}_3) = (20, 34)$. One can understand the appearance of states at $G = 0$ in pairs with conjugate momenta as follows. First note that states in the Q_1 cohomology occur in quartets with ghost numbers $(G, G+1, G+1, G+2)$, where the state at lowest ghost number is called the prime state [165], and the other states are obtained by applying the picture changing operators to this prime state (remember that we have two independent picture changing operators in the Q_1 cohomology description). Besides, any state at ghost number G has a conjugate state at ghost number $2 - G$ with the conjugate momentum. Combining these observations, we see that prime states at $G = 0$ occur in pairs with conjugate momenta.

We have now identified all operators of the $c = \frac{7}{10}$ minimal model in the Q_1 cohomology at levels 0,1 and 2. The next objective is to show that all physical operators (at least the ones found so far) of the same conformal weight are related to each other through the action of screening operators and picture changes. Therefore, let us introduce a number of useful screening charges, which are now required to commute with Q_1. First of all, the operator S in equation (4.56) is still a screening current in the Q_1 cohomology, as is

[10] We recall that the momentum conjugate to $\vec{p}$ is $-\vec{p} + 2i\vec{q}$, where $\vec{q}$ is the background charge vector. In the tilded variables it means that $54 - \tilde{p}_i$ is conjugate to $\tilde{p}_i$.

S_x. The Q_2 screening operators R and $\bar{R}$ commute with Q_1 only after adding an extra term,

$$R = (\partial c_4 c_4 + \tfrac{15}{88} q_2 c_3 c_4) e^{ip_3\phi_3}\,, \quad \tilde{p}_3 = 30\,, \tag{4.77}$$

$$\bar{R} = (\partial c_4 c_4 + \tfrac{15}{56} q_2 c_3 c_4) e^{ip_3\phi_3}\,, \quad \tilde{p}_3 = 24\,. \tag{4.78}$$

New screening currents are given by (the notation will become clear in the next subsection)

$$T_3^- = (1 - \tfrac{256}{729} q_2 b_3 c_4) e^{i\vec{p}\cdot\vec{\phi}}\,, \quad \tilde{\vec{p}} \equiv (\tilde{p}_2, \tilde{p}_3) = (-8, 8)\,, \tag{4.79}$$

$$T_3^+ = (1 - \tfrac{32}{729} q_2 b_3 c_4) e^{i\vec{p}\cdot\vec{\phi}}\,, \quad \tilde{\vec{p}} = (-10, 10)\,. \tag{4.80}$$

Screening currents with positive $\tilde{p}_2$-values are

$$R' = c_3 e^{ip_2\phi_2}\,, \quad \tilde{p}_2 = 30\,, \tag{4.81}$$

$$\bar{R}' = c_3 e^{ip_2\phi_2}\,, \quad \tilde{p}_2 = 24\,. \tag{4.82}$$

Of course, many more screening currents at higher or lower ghost numbers exist, but we expect that they can be represented by composite actions of the given ones, together with P_2 and/or P_3 picture changes.

The $h = 0$ physical operators are obtained through the action of the associated screening currents on a picture changed version of the identity operator, so they can all be viewed as different screened versions of the identity. Also, operators of table 3 with the same conformal weight can be connected to each other more directly by the action of certain combinations of the screening charges given above. It is more important, however, to find operators which can normal-order with any physical operator and thereby create new physical operators. The operator x that was found in the previous section is easily extended to the Q_1 cohomology, since the associated screening current S_x is still given by (4.67). Again, it can also be expressed as

$$x(z) = \oint \frac{dw}{2\pi i} S_x(w) P_3(z)\,, \tag{4.83}$$

where now $P_3 = [Q_1, \phi_3]$ contains some additional terms compared with the P_3 operator in the Q_2 discussion. Its inverse now is the level 3 physical operator

$$\begin{aligned} x^{-1} = {} & (\partial^2 c_4 \partial c_4 c_4 + \tfrac{45}{56} \partial\phi_2 c_3 \partial c_4 c_4 - \tfrac{45\sqrt{2}}{56} \partial\phi_3 c_3 \partial c_4 c_4 - \tfrac{5}{56} q_2 c_3 \partial^2 c_4 c_4 \\ & + \tfrac{5}{28} q_2 \partial c_3 \partial c_4 c_4 + \tfrac{3645}{19712} \partial c_3 c_3 c_4) e^{ip_3\phi_3}\,, \end{aligned} \tag{4.84}$$

with $\tilde{p}_3 = 30$. This is the operator corresponding to the screening current (4.77) via the descent equation, and it also equals the commutator of R with P_3. Conversely, we get back R as $(b_3)_{-1} x^{-1}$.

There should also be similar operators y and y^{-1} with nonzero ϕ_2 momentum. We expect y to be a level 37 physical operator with momentum $(\tilde{p}_2, \tilde{p}_3) = (-40, -20)$. Such an operator can normal-order with any physical operator. This can be easily checked using (4.59), and by noting that $\tilde{p}_2 + \tilde{p}_3$ is a multiple of 3 for all physical operators (this also follows from (4.46)). We did not try to construct the operator y. However, y^{-1} should be the level 1 physical operator with $(\tilde{p}_2, \tilde{p}_3) = (40, 20)$ of table 3.

The overall picture of the Q_1 cohomology is then the following. Physical operators come in minimal model modules of the $c = \frac{7}{10}$ Virasoro algebra realized in terms of the scalar fields (ϕ_2, ϕ_3) and the ghost fields $(c_3, b_3; c_4, b_4)$. There seems to be an infinite number of representatives of each minimal model primary (but only a finite number at fixed ghost number). We expect that all primaries belonging to the Q_1 cohomology can be written as normal-ordered products of powers of the operators x, y and their inverses acting on a set of physical operators at some low-lying levels. Since the ghost numbers and momenta of the operators x, y, x^{-1}, y^{-1} are known, we can predict the ghost numbers, level numbers, and momenta of Q_1 cohomology classes as in table 2 for the Q_2 cohomology.

4.3.3 The complete cohomology

We now consider the cohomology of $Q = Q_1 + Q_{Vir}$ on the full Fock space generated by the three scalar fields and the three conjugate ghost pairs. Because of the Virasoro constraint, the Q cohomology contains only $h = 0$ primaries. This is different from the Q_2 and Q_1 cohomologies which also contain descendants of minimal model primaries.

Since Q_1 and Q_{Vir} anticommute, see (4.42), they define a double complex. Note that Q_1 does not involve the fields $(\phi_1; c_2, b_2)$, and since Q_1 physical operators have already been computed, we take a spectral sequence where the first term is the Q_1 cohomology. This spectral sequence provides a systematic procedure to obtain operators in the Q cohomology by adding to Q_1 physical operators terms with higher spin-two ghost number G_2 (but the same total ghost number). Physical operators in the total cohomology are then given by

$$\mathcal{O} = \sum_{i=k}^{\infty} \mathcal{O}_i \,, \tag{4.85}$$

where the first term in the sum, $\mathcal{O}_k$ with $G_2 = k$, is an operator in the Q_1 cohomology, and the higher G_2 terms are defined by $[Q_1, \mathcal{O}_{i+1}] = -[Q_{Vir}, \mathcal{O}_i]$. At small values of the level, the sum in (4.85) will only have a few terms (e.g. only one term at level 0 and $G = 0$).

The 'tachyon operators' (level 0, $G = 0$) take the form

$$X_0^0 = c_2 \partial c_3 c_3 \partial^2 c_4 \partial c_4 c_4 e^{ip_1\phi_1 + ip_2\phi_2 + ip_3\phi_3} \,. \tag{4.86}$$

They are physical for 24 values of the momenta which form a multiplet of the $sl(4)$ Weyl group [136]. For the explicit values of the physical momenta, we refer to [40]. The 24 physical operators X_0^0 correspond to the 12 Q_1 operators W_0^0 'dressed up' with the c_2 ghost and the ϕ_1 part of the exponential, to operators of vanishing total conformal dimension (thus giving two possible p_1-values for each Q_1 operator). The level 0 operators of the W_4 string were already constructed by Das, Dhar and Rama [65] in

1992. The W_4 BRST operator was not known at that time, and their computation was based on an assumption about the existence of the 'cosmological constant operator'.

At level 1 and lowest ghost number $G = -1$, we can form the operators

$$X_1^{-1} = (x_1 \partial c_3 c_3 \partial^2 c_4 \partial c_4 c_4 + x_2 c_2 c_3 \partial^2 c_4 \partial c_4 c_4 + x_3 c_2 \partial c_3 c_3 \partial c_4 c_4) e^{i\vec{p}\cdot\vec{\phi}} . \tag{4.87}$$

All momenta at which such operators become physical have been listed in [40]. The set of physical level 1 operators can be divided in continuous and discrete momentum operators. The continuous momentum operators correspond to Q_1 operators W_1^{-1} that have been dressed to operators of the total cohomology. They have $x1 = 0$, thus their spin-two ghost structure is standard. The discrete momentum operators have nonzero x_1, x_2 and x_3, thus their spin-two ghost structure is non-standard. The latter operators have $p_1 = 0$. This is all in agreement with (or rather analogous to) the observations made for the W_3 string in equations (4.33) and (4.34).

Next we compute some screening charges, which commute with Q. First, we note that all Q_1 screening currents are Q screening currents as well. So S_x is still a screening current, and its associated physical operator is still given by the relation (4.83), where now $P_3 = [Q, \phi_3]$ contains two additional terms relative to $[Q_1, \phi_3]$. The physical operator x^{-1} can be found using the spectral sequence argument described at the beginning of this subsection. We find that it is given by (4.84) with the following modification:

$$x^{-1} \;\rightarrow\; x^{-1} - (\tfrac{15}{28} c_2 \partial c_4 c_4 + \tfrac{225}{2464} q_2 c_2 c_3 c_4) e^{ip_3\phi_3} , \quad \tilde{p}_3 = 30 . \tag{4.88}$$

In the Q cohomology, this is a level 4 operator. Similar physical operators y, y^{-1}, z, z^{-1} are also expected to exist, where y is supposed to have momentum $(0, -40, -20)$ and z should have nonzero ϕ_1 momentum in order to connect states with different p_1-values.

We find four new screening currents involving b_2 and ϕ_1,

$$\begin{aligned}
T_2^- &= (1 + \tfrac{2}{3} q_2 b_2 c_3) e^{i\vec{p}\cdot\vec{\phi}} , && \tilde{\vec{p}} \equiv (\tilde{p}_1, \tilde{p}_2, \tilde{p}_3) = (-12, 12, 0) , \\
T_2^+ &= (1 + \tfrac{5}{6} q_2 b_2 c_3) e^{i\vec{p}\cdot\vec{\phi}} , && \tilde{\vec{p}} = (-15, 15, 0) , \\
T_1^- &= e^{ip_1\phi_1} , && \tilde{p}_1 = 24 , \\
T_1^+ &= e^{ip_1\phi_1} , && \tilde{p}_1 = 30 .
\end{aligned} \tag{4.89}$$

The operators $T_i^{\pm}$, $i = 1, 2, 3$, (recall the Q_1 screening currents in (4.79) and (4.80)) have exactly the same momenta as the standard screening currents $S_i^{\pm}$ given in (4.43). In fact, $T_1^{\pm} = S_1^{\pm}$. The other screening currents have been modified by a ghost contribution. This is a consequence of the redefinition that we carried out to obtain the W_4 BRST charge (4.41), since in this redefinition the scalar fields and ghosts are mixed to some extent [136, 20]. In [65] it was noted that the tachyonic physical operators are precisely the composites that can be formed out of the screening currents $S_i^{\pm}$.

Now that we have included the Virasoro constraint, it is trivial to obtain screening currents associated to physical operators, since the descent equation (4.44) is solved by $S_{\mathcal{O}}(w) = \oint \frac{dz}{2\pi i} b_2(z) \mathcal{O}(w)$.

In [46] a classification of physical states for a $W[g]$ minimal model coupled to $W[g]$ gravity is given. These results have already been seen to agree with those of [140] in the case of the two-scalar W_3 string (or pure W_3 gravity), see [46, 18].

If we take $g = sl(4)$ and the trivial ($c = 0$) W_4 minimal model, we are able to compare with our results. Non-trivial cohomology classes exist, according to [46], at the following values of the momentum[11]:

$$\vec{p} = w^{-1}(\alpha_- \sigma \vec{\rho} - \alpha_+ \vec{\rho}) + i\sqrt{2}\alpha_0 \vec{\rho}\,, \tag{4.90}$$

where $\vec{\rho}$ and the parameters $\alpha_0, \alpha_\pm$ are as before (see (4.25)), and w is an element of the $sl(4)$ Weyl group $\mathcal{W}$ while σ can be an element of the $\widehat{sl(4)}$ affine Weyl group $\widehat{\mathcal{W}}$. The ghost number at which the state with momentum (4.90) occurs is given by $-l_w(\sigma)$, where $l_w(\sigma)$ is the twisted length of σ,

$$l_w(\sigma) = \lim_{N\to\infty} \left(l(t_{-Nw\rho}\sigma) - l(t_{-Nw\rho}) \right), \tag{4.91}$$

and l is the ordinary length of an affine Weyl group element. In order to compute the twisted length, one should decompose the translation $t_{-Nw\rho}$ into the simple affine Weyl reflections $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ and then look for the cancellations that take place between $t_{-Nw\rho}$ and σ. For $\sigma \in \mathcal{W}$, (4.91) reduces to $l_w(\sigma) = l(w^{-1}\sigma) - l(w^{-1})$. The action of σ_0 on ρ should be taken here as $\sigma_0\rho = \sigma_\theta\rho + 5\theta$, where θ is the highest root of $sl(4)$. For completeness we give the decompositions of the translations associated with the simple roots:

$$\begin{aligned} t_{e_1} &= \sigma_2\sigma_3\sigma_0\sigma_3\sigma_2\sigma_1\,, \\ t_{e_2} &= \sigma_3\sigma_1\sigma_0\sigma_1\sigma_3\sigma_2\,, \\ t_{e_3} &= \sigma_2\sigma_1\sigma_0\sigma_1\sigma_2\sigma_3\,. \end{aligned} \tag{4.92}$$

For $\sigma = \mathbf{1}$, (4.90) yields all level 0 physical states corresponding to X_0^0, when w runs over the 24 elements of $\mathcal{W}$. Whereas the Weyl group action in (4.90) does not change the level, the affine Weyl group action does. If we let σ run over the simple Weyl reflections $\{\sigma_1, \sigma_2, \sigma_3\}$ and w over all elements in $\mathcal{W}$, we obtain all momenta and ghost numbers of the level 1 prime physical states of which the ones with $G = -1$ agree with our findings for the operators X_1^{-1}. Thus we find complete agreement with the results of [46] at levels 0 and 1.

The affine Weyl elements can be decomposed into ordinary Weyl transformations and translations in the co-root lattice, $\sigma = t_\beta w$. The translations associated with the simple roots (4.92) correspond to the following changes in the momenta,

$$\begin{aligned} \Delta_1\tilde{\vec{p}} &= (120, 0, 0)\,, \\ \Delta_2\tilde{\vec{p}} &= (-60, 60, 0)\,, \\ \Delta_3\tilde{\vec{p}} &= (0, -40, 40)\,. \end{aligned} \tag{4.93}$$

[11] It can be checked that they are compatible with the selection rule (4.46).

Physical operators with such momenta are supposed to be invertible and they can be used to classify the complete cohomology in terms of a set of low level physical operators. Recall that the operator x with momentum $\tilde{\vec{p}} = (0, 0, -30)$ can also be used for this purpose. Using x and the operators corresponding to the simple root translations, one finds that an alternative basis of x-like operators has momenta $(0, 0, -30)$, $(0, -40, -20)$ and $(-60, -20, -10)$.

As before, we only considered prime operators. Seven other sectors of operators can be obtained by acting with the picture changing operators P_1, P_2 and P_3.

The energy-momentum tensor for ϕ_1 can be replaced by an arbitrary effective energy-momentum tensor T_{eff} with the same central charge. Some three-scalar states will then generalize to continuous momentum multi-scalar states, some will generalize to discrete momentum multi-scalar states and some may not be generalized at all to multi-scalar states. An effective space-time exponential may be replaced by any effective space-time operator with the same OPE under T_{eff} to obtain other physical operators. See [140] for a discussion of this multi-scalar generalization in the case of W_3.

It is expected [20] that the W_N BRST charge can be decomposed in a way similar to the W_4 BRST charge. This is certainly true at the classical level. The studies of W_3 and W_4 strings make the following picture of minimal models in the W_N string very plausible. Imposing the spin-N constraint results in an $(N, N+1)$ unitary W_{N-1} minimal model. In the next step, where the spin-$(N-1)$ constraint is added, the operators are dressed to operators of the $(N, N+1)$ W_{N-2} minimal model. This goes on in the same way, resulting in an $(N, N+1)$ Virasoro minimal model in the Q_1 cohomology, and the total cohomology is obtained from the double complex with BRST charges Q_1 and Q_{Vir}. This agrees with the counting of central charges as discussed earlier. A similar discussion for *non-critical* W_N strings may be found in [18].

Some results on higher-spin strings based on $W_{2,N}$ algebras have been obtained in [139, 138, 141, 142]. A complication noted by the authors of these papers is that for $N \geq 5$, the central charge of the spin-N sector, corresponding to a W_{N-1} minimal model, becomes greater or equal to one, as can be seen in equation (4.17). Consequently, the number of effective space-time sectors which couple to the spin-N fields is no longer finite. This complication is not expected to occur for the W_N string, since there is a sequence of W_k minimal models, the last one being the $(N, N+1)$ Virasoro minimal model which of course has $c < 1$ so that there is only a finite number of effective space-time intercepts.

4.4 Relations between strings based on different gauge symmetries

In the previous sections we discussed some relations between (non-critical) W_N strings for different N. It was argued that in the cohomology of the (p, q) non-critical W_N string all (p, q) non-critical W_n strings with $2 \leq n < N$ naturally appear. This is not to say that they all are equivalent. In fact, the minimal model operators appearing in the spectra of W_N strings are repeated at an infinite number of different ghost numbers. In principle, this degeneracy could be lifted by using certain screening operators with nonzero ghost number to identify all the copies of a particular minimal model operator. However, it is not natural to do so. See [18] for a discussion.

In this section we briefly describe further relations between string theories based on different world-sheet gauge symmetries. In [29] it was shown that the bosonic string may be viewed as a special background for the $N=1$ superstring, and that the $N=1$ superstring may be viewed as a special background for the $N=2$ superstring.

Let us review some of the arguments for the $N=0 \subset N=1$ case. Starting from a critical bosonic string with a $c_{bos}=26$ energy-momentum tensor T_{bos}, one obtains a realization of the $N=1$ superconformal algebra by introducing fermionic (i.e. anticommuting) fields (b_1, c_1) of spin $(3/2, -1/2)$ and defining [29]

$$\begin{aligned} T &= T_{bos} - \tfrac{3}{2} b_1 \partial c_1 - \tfrac{1}{2} \partial b_1 c_1 + \tfrac{1}{2} \partial^2 (c_1 \partial c_1)\,, \\ G &= b_1 + c_1 \left(T_{bos} + \partial c_1 b_1\right) + \tfrac{5}{2} \partial^2 c_1\,. \end{aligned} \tag{4.94}$$

The fields (b_1, c_1) can be viewed as twisted versions of spin-two ghost fields (b, c). Then the second term in G has the structure of the BRST current of the bosonic string. Note that this particular realization of the $N=1$ superconformal algebra acts nonlinearly due to the first term in G.

The currents T and G generate the $N=1$ superconformal algebra with critical central charge $c=15$. Thus the system $\{T_{bos}, (b_1, c_1)\}$ can serve as a background for the $N=1$ fermionic string. To impose the constraints T and G, one needs to introduce the usual anticommuting ghosts (b, c) and the commuting ghosts (β, γ), respectively. It is then shown in [29] that correlation functions of the fermionic string in this specific background reduce to corresponding bosonic string correlation functions essentially because the path integral over the (β, γ) fields cancels that over the (b_1, c_1) fields.

The arguments of [29] use the assumption that all physical operators of the bosonic string with energy-momentum tensor T_{bos} are of the standard form. However, the equivalence of the class of $N=1$ strings based on the realizations (4.94) to the bosonic string based on T_{bos} holds also in the case where other operators of non-standard ghost structure are present, as in non-critical string theories. This follows from the work of Ishikawa and Kato [121] in which a similarity transformation is used to prove that the cohomology of the $N=1$ BRST operator is isomorphic to the cohomology of the bosonic string BRST operator. Indeed, for a certain generating function R [121],

$$e^R Q_{N=1} e^{-R} = Q_{bos} + Q_{top}\,, \tag{4.95}$$

where

$$Q_{top} = \oint \frac{dz}{2\pi i} (-\tfrac{1}{2} b_1 \gamma)\,, \tag{4.96}$$

and Q_{bos} is the standard bosonic string BRST operator with energy-momentum tensor T_{bos}. The cohomology of $Q_{N=1}$ is simply the direct product of the cohomology of Q_{bos} and Q_{top}. From the form of Q_{top} and the fact that (b_1, c_1) and (β, γ) are conjugate pairs, it immediately follows that the cohomology of Q_{top} consists of only the ground state in the $\{(b_1, c_1), (\beta, \gamma)\}$ system. Decoupling of all excitations in these fields is due to the quartet mechanism of Kugo and Ojima. Moreover, (4.95) is a similarity transformation and therefore preserves the form of all OPEs. As a result, any correlation function in the $N = 1$ theory reduces to the corresponding one in the $N = 0$ theory[12].

Further embeddings of strings into string models with larger world-sheet gauge symmetry have since been found. The embedding described above has been generalized to a hierarchy of superstrings [14] where N-extended superstrings may be viewed as a special class of $(N + 1)$-extended superstrings. In [28], the embeddings are generalized to non-critical superstrings and in [132] a hierarchy of w-strings[13] is obtained.

A very interesting idea behind all these embeddings is that there might exist a universal string theory which includes all the others by special choices of vacua. This universal string theory would then be the most symmetrical one, and less symmetric string theories arise by spontaneous symmetry breaking, i.e. by the choice of a certain vacuum. However, all realizations used in the embeddings are of a very special kind. In particular, the symmetries of the more symmetrical string theory are always nonlinearly realized, as in (4.94).

It is known that nonlinear realizations[14] for some symmetry algebra G may be induced from realizations of some smaller algebra H. In the case that G is a finite dimensional group and H a subgroup, the BRST charge for the nonlinear realization of G is related via a similarity transformation to the BRST charge of H [131, 146, 81]. Thus it seems that the existence of hierarchies of string theories with different world-sheet symmetries is purely a consequence of the fact that the algebras are nonlinearly realized. In other words, one can start from the bosonic string and add fields such that a nonlinear realization of some larger symmetry is obtained. Gauging the extra symmetry in effect eliminates the new degrees of freedom and therefore gives back the original theory. This mechanism of enlarging the symmetry is rather trivial and makes the significance of string embeddings unclear. See also the discussion in [155].

In the case of W_N strings we also described similarity transformations like (4.95). In particular, for the W_3 string a similarity transformation (or canonical transformation, it preserves OPEs) turns the BRST operator into a sum of two nilpotent terms $Q = Q_0 + Q_1$, as described above equation (4.29). There are some differences, however. One is that the W_3 symmetry is realized nonlinearly in another sense, namely higher than quadratically. Indeed, we know that Q_1 does not correspond to a topological sector of the theory, rather its cohomology is a realization of the Ising model. Another difference is that the critical value $c_{Vir} = 26$ does not lead, through the Miura realization, to the critical value $c_{W_3} = 100$ but rather to $c_{W_3} = 102$. The latter difference can be eliminated if we consider a non-critical W_3 string with a $c_L = -2$ Liouville sector and therefore a $c_M = 102$ matter sector [23]. Then this matter sector can be realized by the usual 26 free scalar fields of the bosonic string plus an additional scalar field. For

[12]However, there may be problems for a complete identification on higher genus surfaces where (super)moduli play a role.

[13]The w-strings considered in [132] are based on linear versions of the W_N algebras.

[14]In this section we mean by 'nonlinear' that there are transformations with terms of zeroth order in the fields. For example, G in (4.94) acts nonlinearly in this sense on c_1.

example, the matter energy-momentum tensor is given by

$$T_M = -\tfrac{1}{2}\partial X_\mu \partial X^\mu - \tfrac{1}{2}\partial\phi\partial\phi + \tfrac{5}{2}\partial^2\phi\,. \tag{4.97}$$

After performing our usual redefinition in the matter sector, the BRST current can be cast into the form $j = j_0 + j_1$ with

$$\begin{aligned} j_0 &= c_2\left(T_M + T_L + T_{c_3,b_3} + \tfrac{1}{2}T_{c_2,b_2}\right)\,, \\ j_1 &= c_3 F(T_L, W_L; \phi; c_3, b_3)\,. \end{aligned} \tag{4.98}$$

The explicit expression for F is given in [21, 23]; it is not important here. From the form of the BRST operator $Q = Q_0 + Q_1$ we can already see that this non-critical W_3 string contains the complete critical bosonic string spectrum in its cohomology. Indeed, we can rewrite $Q = Q_{Vir} + Q_R$, where Q_{Vir} is the standard BRST operator of the bosonic string. Since Q_R does not depend on X^μ and b_2, any bosonic string physical state of the form $c_2 V(X^\mu)$ is automatically Q invariant. However, the results of [18] that show how the Q_1 cohomology in the case that the Liouville sector is a (p,q) minimal model reduces to the (p,q) Virasoro minimal model, cannot be directly applied here since there does not seem to be a W_3 minimal model at $c_L = -2$. Although $c = -2$ corresponds to $(p,q) = (3,2)$ in the formula for minimal model central charges, a corresponding Kac table of minimal model primaries does not exist. One could still restrict the Liouville sector to its identity operator only[15], just to see to what model it leads in the Q_1 cohomology. It does not lead to the $(3,2)$ $(c = 0)$ trivial minimal Virasoro model. Instead, the Q_1 cohomology involves operators of dimensions $\{n, \frac{1}{8}+n, \frac{5}{8}+n\}$ for non-negative integers n. Applying naively the fusion rules of BPZ, it can be seen that dimension $0, \frac{1}{8}, \frac{5}{8}$ primaries are part of a closed fusion algebra. However, only the dimension 0 operators yield the bosonic string spectrum in the complete cohomology. It is clear that the relation described here between a special W_3 string realization and the bosonic string is by no means an equivalence. For more details we refer to [23].

Other attempts of embedding the bosonic string in a W-string, apart from the hierarchy of linearized w_N strings [132], have been described in [27, 13, 137, 143]. In [13], the critical bosonic string is realized as a particular background of a string based on the linearized W_3 algebra W_3^{lin} [130]. The nonlinear W_3 algebra obtained from this critical W_3^{lin} algebra by a redefinition has central charge 102, the same value as in the matter sector of the non-critical W_3 string discussed above. Related work in [137] shows that both the $c_M = 100$ critical and $c_M = 102$ non-critical BRST operators can be altered without losing nilpotence by adding an extra term to the Q_1 operator. They then become equivalent up to a similarity transformation. Moreover, with this extra term, which corresponds to a nonlinearly (in the sense of 0th order in the fields) realized W_3 symmetry, they also become equivalent to the bosonic string BRST operator plus a topological part. The latter part decouples a quartet of fields leaving precisely the critical bosonic string. The W_3 realizations used in [137] are obtained using the linearized approach of [130]. They involve a bosonic bc system and are, therefore, different from the usual Miura realizations.

[15] However, this is not a modular invariant restriction.