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POSITIVITY AND STORAGE FUNCTIONS FOR QUADRATIC DIFFERENTIAL FORMS

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1 Introduction

Differential equations and one–variable polynomial matrices play an essential role in describing dynamics of systems. When studying functions of the dynamical variables, as in Lyapunov theory, in the study of dissipation and passivity, as in electric circuit theory, or in specifying performance criteria in optimal control, we invariably encounter quadratic expressions in the variables and their derivatives. As we shall see, two–variable polynomial matrices are the proper mathematical tool to express these quadratic functionals. Dynamical equations expressed through one–variables polynomial matrices and functionals expressed through two–variable polynomial matrices fit, we aim to illustrate throughout this paper, as a glove fits a hand.

2 Quadratic differential forms

Let $\mathbb{R}^{q_1 \times q_2}[\zeta, \eta]$ denote the set of real polynomial matrices in the (commuting) indeterminates ζ and η. Explicitely, an element $\Phi \in \mathbb{R}^{q_1 \times q_2}[\zeta, \eta]$ is thus given by

$$\Phi(\zeta, \eta) = \sum_{k, \ell} \Phi_{k\ell} \zeta^k \eta^\ell \qquad (2.1)$$

The sum in (2.1) ranges over the non–negative integers and is assumed to be finite, and $\Phi_{k\ell} \in \mathbb{R}^{q_1 \times q_2}$. Such a Φ induces a *bilinear differential form* (BLDF), that is, the map $L_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{q_1}) \times \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{q_2}) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ defined by

$$(L_\Phi(v, w))(t) := \sum_{k, \ell} \left(\frac{d^k v}{dt^k}(t) \right)^T \Phi_{k\ell} \left(\frac{d^\ell w}{dt^\ell}(t) \right) \qquad (2.2)$$

If $q_1 = q_2 \; (=: q)$ then Φ induces a *quadratic differential form* (QDF) $Q_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ defined by

$$Q_\Phi(w) := L_\Phi(w, w) \qquad (2.3)$$

Define the star operator * by

$$^* : \mathbb{R}^{q_1 \times q_2}[\zeta, \eta] \to \mathbb{R}^{q_2 \times q_1}[\zeta, \eta]; \quad \Phi^*(\zeta, \eta) := \Phi^T(\eta, \zeta)$$

where ^{T} denotes transposition. Obviously $L_\Phi(v, w) = L_{\Phi^*}(w, v)$. If $\Phi \in \mathbb{R}^{q \times q}[\zeta, \eta]$ satisfies $\Phi = \Phi^*$ then Φ will be called *symmetric*. The symmetric elements of $\mathbb{R}^{q \times q}[\zeta, \eta]$ will be denoted by $\mathbb{R}_s^{q \times q}[\zeta, \eta]$. Clearly $Q_\Phi = Q_{\Phi^*} = Q_{\frac{1}{2}(\Phi + \Phi^*)}$. This shows that when considering quadratic differential forms we can hence in principle restrict attention to Φ's in $\mathbb{R}_s^{q \times q}[\zeta, \eta]$. However, both bilinear and quadratic forms will be of interest to us. Associated with $\Phi \in \mathbb{R}^{q_1 \times q_2}[\zeta, \eta]$ we can form the matrix

$$\tilde{\Phi} = \begin{bmatrix} \Phi_{00} & \Phi_{01} & \cdots & \cdot & \cdots \\ \Phi_{10} & \Phi_{11} & \cdots & \cdot & \cdots \\ \vdots & \vdots & & \vdots & \\ \cdot & \cdot & \cdots & \Phi_{k\ell} & \cdots \\ \vdots & \vdots & & \vdots & \end{bmatrix} \qquad (2.4)$$

Note that, although $\tilde{\Phi}$ is an infinite matrix, all but a finite number of its elements are non–zero. We can factor $\tilde{\Phi}$ as $\tilde{\Phi} = \tilde{N}^T \tilde{M}$, with $\tilde{N}$ and $\tilde{M}$ infinite matrices having a finite number of rows and all but a finite number of elements equal to zero. This decomposition leads, after pre–multiplication by $[I_{q_1} \; I_{q_1}\zeta \; I_{q_1}\zeta^2 \cdots]$ and post–multiplication by $\text{col}[I_{q_2} \; I_{q_2}\eta \; I_{q_2}\eta^2 \cdots]$, to the following factorization of Φ:

$$\Phi(\zeta, \eta) = N^T(\zeta) M(\eta) \qquad (2.5)$$

This decomposition is not unique, but if we take $\tilde{N}$ and $\tilde{M}$ surjective, then their number of rows is equal to the rank of $\tilde{\Phi}$. The factorization (2.5) will then be called a *canonical factorization* of Φ. Associated with (2.5) we obtain the following expresion for the BLDF L_Φ:

$$L_\Phi(w_1, w_2) = (N(\frac{d}{dt})w_1)^T M(\frac{d}{dt})w_2. \qquad (2.6)$$

Next we will discuss the case that Φ is symmetric. Clearly, $\Phi = \Phi^*$ iff $\tilde{\Phi}$ is symmetric. In that case it can be factored as $\tilde{\Phi} = \tilde{M}^T \Sigma_M \tilde{M}$ with $\tilde{M}$ an infinite matrix having a finite number of rows and all but a finite number of elements equal to zero, and Σ_M a signature matrix, i.e., a matrix of the form

$$\begin{bmatrix} I_{r_+} & 0 \\ 0 & -I_{r_-} \end{bmatrix}$$

This decomposition leads to the following decomposition of Φ:

$$\Phi(\zeta, \eta) = M^T(\zeta)\Sigma_M M(\eta). \qquad (2.7)$$

Also this decomposition is not unique but if we take $\tilde{M}$ surjective, then Σ_M will be unique. We will denote this Σ_M as Σ_Φ and the resulting pair (r_-, r_+) by (ϕ_-, ϕ_+). This pair will be called the *inertia* of Φ. The resulting factorization

$$\Phi(\zeta, \eta) = M^T(\zeta)\Sigma_\Phi M(\eta) \qquad (2.8)$$

will be called a *symmetric canonical factorization* of Φ. Of course, a symmetric canonical factorization is not unique. However, they can all be obtained from one by replacing $M(\xi)$ by $UM(\xi)$ with $U \in \mathbb{R}^{\text{rank}(\tilde{\Phi}) \times \text{rank}(\tilde{\Phi})}$ such that $U^T\Sigma_\Phi U = \Sigma_\Phi$.

Associated with (2.7), we obtain the following decomposition of Q_Φ into a sum of positive and negative squares:

$$Q_\Phi(w) = \|P(\frac{d}{dt})w\|^2 - \|N(\frac{d}{dt})w\|^2 \qquad (2.9)$$

where $N, P \in \mathbb{R}^{\bullet \times q}[\xi]$ are obtained by partitioning $\tilde{M}$ conform Σ_M as:

$$\tilde{M} = \begin{bmatrix} \tilde{P} \\ \tilde{N} \end{bmatrix} \qquad (2.10)$$

For a given symmetric $\Phi(\zeta, \eta)$ we will also be interested in the symmetric two-variable polynomial matrix $|\Phi|(\zeta, \eta)$, the *absolute value* of Φ, defined as follows. For a given real symmetric matrix A define its absolute value, $|A|$, as follows. Factor $A = \tilde{U}^T \Lambda \tilde{U}$, where Λ is a diagonal matrix with the non-zero eigenvalues of A on its diagonal and where $\tilde{U}\tilde{U}^T = I$, and define $|A| := \tilde{U}^T|\Lambda|\tilde{U}$ with $|\Lambda|$ defined in the obvious way. Now let $\tilde{\Phi}$ be the symmetric matrix associated with $\Phi(\zeta, \eta)$. Let $|\tilde{\Phi}|$ be the absolute value of $\tilde{\Phi}$. Next, define $|\Phi|(\zeta, \eta)$ as the symmetric two-variable polynomial matrix associated with $|\tilde{\Phi}|$:

$$|\Phi|(\zeta, \eta) := \begin{bmatrix} I \\ \zeta I \\ \zeta^2 I \\ \vdots \end{bmatrix}^T |\tilde{\Phi}| \begin{bmatrix} I \\ \eta I \\ \eta^2 I \\ \vdots \end{bmatrix}$$

One of the convenient things of identifying BLDF's and QDF's with two-variable polynomial matrices is that they allow a very convenient calculus. One instance of this is differentiation. Obviously if L_Φ is a BLDF, so will be $\frac{d}{dt}L_\Phi$, and if Q_Φ is a QDF, so will be $\frac{d}{dt}Q_\Phi$. The result of differentiation is easily expressed in terms of the two-variable polynomial matrices and leads to the dot operator $\bullet$ defined as

$$\bullet : \mathbb{R}^{q_1 \times q_2}[\zeta, \eta] \to \mathbb{R}^{q_2 \times q_1}[\zeta, \eta];$$

$$\overset{\bullet}{\Phi}(\zeta, \eta) := (\zeta + \eta)\Phi(\zeta, \eta)$$

It is easily calculated that

$$\frac{d}{dt}L_\Phi = L_{\overset{\bullet}{\Phi}} \text{ and } \frac{d}{dt}Q_\Phi = Q_{\overset{\bullet}{\Phi}} \qquad (2.11)$$

In the sequel, an important role will be played by certain one–variable polynomial matrices obtained from two–variable polynomial matrices by means of the delta operator ∂, defined as

$$\partial : \mathbb{R}^{q_1 \times q_2}[\zeta, \eta] \to \mathbb{R}^{q_1 \times q_2}[\xi]; \quad \partial\Phi(\xi) := \Phi(-\xi, \xi)$$

Introduce the asterisk operator $\star$ acting on matrix polynomials by

$$\star : \mathbb{R}^{q_1 \times q_2}[\xi] \to \mathbb{R}^{q_1 \times q_2}[\xi]; \quad R^\star(\xi) := R^T(-\xi).$$

A poynomial matrix $M \in \mathbb{R}^{q \times q}[\xi]$ is called *para-Hermitian* if $M = M^\star$. Note that $(\partial\Phi)^\star = \partial(\Phi^\star)$. Hence if $\Phi \in \mathbb{R}^{q \times q}_s[\zeta, \eta]$, then $\partial\Phi$ is para-Hermitian.

In addition to studying BLDF's and QDF's as maps to $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ we will be interested in their integrals. In order to make sure that those integrals exist, we will assume in this case that the arguments have compact support. As is common we will denote by $\mathfrak{D}(\mathbb{R}, \mathbb{R}^q) := \{w \in C^\infty(\mathbb{R}, \mathbb{R}^q) \mid w$ has compact support$\}$. Let $\Phi \in \mathbb{R}^{q_1 \times q_2}[\zeta, \eta]$. Then obviously $L_\Phi : \mathfrak{D}(\mathbb{R}, \mathbb{R}^{q_1}) \times \mathfrak{D}(\mathbb{R}, \mathbb{R}^{q_2}) \to \mathfrak{D}(\mathbb{R}, \mathbb{R})$. Now consider the integral

$$\int L_\Phi : \mathfrak{D}(\mathbb{R}, \mathbb{R}^{q_1}) \times \mathfrak{D}(\mathbb{R}, \mathbb{R}^{q_2}) \to \mathbb{R} \qquad (2.12)$$

defined as

$$\int L_\Phi(v, w) := \int_{-\infty}^{+\infty} L_\Phi(v, w)dt \qquad (2.13)$$

The notation $\int Q_\Phi$ follows readily from this.
The question of when the map $\int L_\Phi$ is zero will now be studied.

Theorem 2.1 *Let* $\Phi \in \mathbb{R}^{q_1 \times q_2}[\zeta, \eta]$. *Then the following statements are equivalent:*

1. $\int L_\Phi = 0$.

2. There exists a $\Psi \in \mathbb{R}^{q_1 \times q_2}[\zeta, \eta]$ *such that* $\Phi = \overset{\bullet}{\Psi}$, *equivalently, such that* $L_\Phi = \frac{d}{dt}L_\Psi$. *Obviously,* Ψ *is given by*

$$\Psi(\zeta, \eta) = \frac{\Phi(\zeta, \eta)}{\zeta + \eta}. \qquad (2.14)$$

3. $\partial\Phi = 0$, *i.e.,* $\Phi(-\xi, \xi) = 0$.

The same equivalence holds for QDF's. Simply assume $\Phi \in \mathbb{R}^{q \times q}_s[\zeta, \eta]$ *and replace the L's by Q's in 1 and 2.*

The importance of this theorem is that condition (3) gives a very convenient way of checking (1) or (2). It is a necessary and sufficient condition for verifying independence of path of the integral

$$\int_{t_1}^{t_2} L_\Phi(v, w)dt \qquad (2.15)$$

With *path independence* we mean that the result of the integral (2.15) depends only on the values of v and w and (a finite number of) their derivatives at $t = t_1$ and $t = t_2$, but not on the intermediate path used to connect these endpoints. It is easily seen that this is equivalent to condition (1) of theorem 3.1. Path integrals and path independence feature prominently in Brockett's work in the sixties (see [2], [3]), and indeed some of our results can be viewed as streamlined versions of this work. We will now briefly discuss positivity of QDF's.

Definition 2.2 *Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$. We will call the QDF Q_Φ non–negative, denoted $\Phi \geq 0$, if $Q_\Phi(w) \geq 0$ for all $w \in C^\infty(\mathbb{R}, \mathbb{R}^q)$, and positive, denoted $\Phi > 0$, if $\Phi \geq 0$ and if the only $w \in C^\infty(\mathbb{R}, \mathbb{R}^q)$ for which $Q_\Phi(w) = 0$ is $w = 0$.*

Using the matrix representation of Φ, it is easy to see that $\Phi \geq 0$ iff there exists $D \in \mathbb{R}^{\bullet \times q}[\xi]$ such that $\Phi(\zeta, \eta) = D^T(\zeta)D(\eta)$. Simply factor $\tilde{\Phi}$ as $\tilde{\Phi} = \tilde{D}^T\tilde{D}$, and take $D(\xi) = \tilde{D} \operatorname{col}[I_q \ \ I_q\xi \ \ I_q\xi^2 \cdots]$. Moreover $\Phi > 0$ iff this D has the property that $D(\lambda)$ is of rank q for all $\lambda \in \mathbb{C}$. In other words, iff the image representation $w = D(\frac{d}{dt})\ell$ defined by D is observable.

3 Average-positivity

Up to now, we have considered positivity of QDF's and its use in establishing stability through Lyapunov functions. However, in many applications, especially in control theory, we are interested in an average type of positivity. In section 2, we already observed when $\int Q_\Phi$ is zero. We will now study when it is positive.

Definition 3.1 *Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$. The QDF Q_Φ (or simply Φ) is said to be*

1. *average-nonnegative, denoted $\int Q_\Phi \geq 0$, if $\int_{-\infty}^{+\infty} Q_\Phi(w)dt \geq 0$ for all $w \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$,*

2. *average-positive, denoted by $\int Q_\Phi > 0$, if $\int Q_\Phi \geq 0$ and if $\int_{-\infty}^{+\infty} Q_\Phi(w)dt = 0$ implies $w = 0$,*

3. *strongly average positive, denoted $\int Q_\Phi \overset{per}{>} 0$, if for all non-zero periodic $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ there holds $\frac{1}{T}\int_0^T Q_\Phi(w)dt > 0$, where T denotes the period of w,*

Proposition 3.2 *Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$. Then*

(i) *$(\int Q_\Phi \geq 0) \Longleftrightarrow (\partial\Phi(i\omega) \geq 0 \ \forall \omega \in \mathbb{R})$.*

(ii) *$(\int Q_\Phi > 0) \Longleftrightarrow (\partial\Phi(i\omega) \geq 0 \quad \forall \omega \in \mathbb{R} \text{ and } \det(\partial\Phi) \neq 0)$.*

(iii) *$(\int Q_\Phi \overset{per}{>} 0) \Longleftrightarrow (\partial\Phi(i\omega) > 0 \ \forall \omega \in \mathbb{R})$.*

Concerning the equivalence (ii), note that $\partial\Phi(i\omega) \geq 0 \ \forall \omega \in \mathbb{R}$ and $\det(\partial\Phi) \neq 0$ is equivalent with: $\partial\Phi(i\omega) > 0$ for all but finitely many $\omega \in \mathbb{R}$.

Intuitively, we think of $Q_\Phi(w)$ as the power going into a physical system. In many applications, the power will indeed be a quadratic differential form of some system variables. For example, in mechanical systems, it is $\sum_k F_k \frac{dq_k}{dt}$ with F_k the external force acting on, and q_k the position of the k-th pointmass; in electrical circuits it will be $\sum_k V_k I_k$ with V_k the potential and I_k the current into the circuit at the k-th terminal. Note that in these examples the variables themselves will be related. When this relation is expressed as an image representation, then we will obtain a general QDF for the power delivered to a system. Average nonnegativity states that the net flow of energy going into the system is non-negative: the system dissipates energy. Of course at some times energy flows into the system, while at other times it flows out. This outflow is due to the fact that energy is stored. However, because of dissipation, the rate of increase of storage cannot exceed the supply. This interaction between supply, storage, and dissipation will now be formalized.

Definition 3.3 *Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ induce the QDF Q_Φ. The QDF Q_Ψ induced by $\Psi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ is said to be a storage function for Φ if*

$$\frac{d}{dt}Q_\Psi \leq Q_\Phi \tag{3.1}$$

A QDF Q_Δ induced by $\Delta \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ is said to be a dissipation function for Φ if

$$\Delta \geq 0 \text{ and } \int Q_\Phi = \int Q_\Delta \tag{3.2}$$

Proposition 3.4 *The following conditions are equivalent*

1. *$\int Q_\Phi \geq 0$*

2. *Φ admits a storage function*

3. *Φ admits a dissipation function.*

Moreover, there is a one-one relation between storage and dissipation functions, Ψ and Δ, respectively, defined by

$$\frac{d}{dt}Q_\Psi(w) = Q_\Phi(w) - Q_\Delta(w)$$

equivalently, $\overset{\bullet}{\Psi} = \Phi - \Delta$, i.e.,

$$\Psi(\zeta, \eta) = \frac{\Phi(\zeta, \eta) - \Delta(\zeta, \eta)}{\zeta + \eta} \tag{3.3}$$

Of course, we should expect that a storage function is related to the memory, to the state. The question, however, is: *the state of which system?* After all, we are considering a QDF, not a dynamical system. But we will see that the factorization of Φ as

$$\Phi(\zeta, \eta) = M^T(\zeta)\Sigma_M M(\eta) \tag{3.4}$$

discussed earlier in section 2 allows us to introduce a state for the QDF Q_Φ. Indeed, (3.4) induces the dynamical system in image representation

$$v = M(\frac{d}{dt})w \qquad (3.5)$$

Note that in (3.5) we are considering w as the latent variable, and v as the manifest one. We are hence considering the behavior of the possible trajectories v. Assume that M has r rows, i.e., that $M \in \mathbb{R}^{r \times q}[\xi]$. Thus, (3.5) defines a system $\mathfrak{B} \in \mathcal{L}^r$ with $\mathfrak{B} = \text{im}(M(\frac{d}{dt}))$ and $M(\frac{d}{dt})$ viewed as a map from $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ to $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^r)$. This system hence has a state representation. Assume that

$$x = X(\frac{d}{dt})w \qquad (3.6)$$

induces a state representation. Thus $X \in \mathbb{R}^{\bullet \times q}[\xi]$ is a polynomial matrix defining a state map for $\mathfrak{B}$ (see [8]). Let $\Psi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$. Then the QDF Q_Ψ is said to be a *state function* (relative to the state of Φ) if there exists a real (symmetric) matrix P such that

$$Q_\Psi(w) = \|X(\frac{d}{dt})w\|_P^2 \qquad (3.7)$$

It is said to be a *state/supply function* if there exists a real (symmetric) matrix E such that

$$Q_\Psi(w) = \| \begin{bmatrix} M(\frac{d}{dt}) \\ X(\frac{d}{dt}) \end{bmatrix} w\|_E^2 \qquad (3.8)$$

where, as always, $\|a\|_A^2$ denotes $a^T A a$. We have the following important result.

Theorem 3.5 *Let $\int Q_\Phi \geq 0$, and let $\Psi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ be a storage function for Φ, i.e. $\overset{\bullet}{\Psi} \leq \Phi$. Then Ψ is a state function. Let $\Delta \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ be a dissipation function for Φ. Then Δ is a state/supply function. In fact, if $X \in \mathbb{R}^{\bullet \times q}[\xi]$ is a state map for Φ, then there exist real symmetric matrices P and E such that*

$$\Psi(\zeta, \eta) = X^T(\zeta)PX(\eta) \qquad (3.9)$$

$$\Delta(\zeta, \eta) = \begin{bmatrix} M(\zeta) \\ X(\zeta) \end{bmatrix}^T E \begin{bmatrix} M(\eta) \\ X(\eta) \end{bmatrix} \qquad (3.10)$$

Equivalently

$$L_\Psi(w_1, w_2) = (X(\frac{d}{dt})w_1)^T P X(\frac{d}{dt})w_2$$

$$L_\Delta(w_1, w_2) = \begin{bmatrix} M(\frac{d}{dt})w_1 \\ X(\frac{d}{dt})w_1 \end{bmatrix}^T E \begin{bmatrix} M(\frac{d}{dt})w_2 \\ X(\frac{d}{dt})w_2 \end{bmatrix}$$

for all $w_1, w_2 \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$.

Let $\Gamma \in \mathbb{R}^{q \times q}[\xi]$ be para-Hermitian: $\Gamma^\star = \Gamma$. A $F \in \mathbb{R}^{q \times q}[\xi]$ is said to induce a *symmetric factorization* of Γ if $\Gamma(\xi) = F^T(-\xi)F(\xi)$. It is said to be a *symmetric Hurwitz factorization* if F is square and Hurwitz, and a *symmetric anti-Hurwitz factorization* if $F^\star$ is square and Hurwitz. It is easy to see that for a factorization to exist we need to

have $\Gamma(i\omega) \geq 0$ $\forall \omega \in \mathbb{R}$ and for a (anti-)Hurwitz one to exist we must have $\Gamma(i\omega) > 0$ $\forall \omega \in \mathbb{R}$. The converses are also true, but not at all trivial in the matrix case. This result is well-known.

Theorem 3.6 *Let $\int Q_\Phi \geq 0$. Then there exist storage functions Ψ_- and Ψ_+ for Φ such that any other storage function Ψ for Φ satisfies*

$$\Psi_- \leq \Psi \leq \Psi_+ \qquad (3.11)$$

If $\int Q_\Phi \overset{per}{>} 0$ then Ψ^- and Ψ^+ may be constructed as follows. Let H and A be respectively Hurwitz and anti-Hurwitz factorizations of $\partial \Phi$. Then

$$\Psi_+(\zeta, \eta) = \frac{\Phi(\zeta, \eta) - A^T(\zeta)A(\eta)}{\zeta + \eta} \qquad (3.12)$$

and

$$\Psi_-(\zeta, \eta) = \frac{\Phi(\zeta, \eta) - H^T(\zeta)H(\eta)}{\zeta + \eta} \qquad (3.13)$$

4 Half-line positivity

In section 5, we studied QDF's for which $\int_{-\infty}^{+\infty} Q_\Phi(w)dt \geq 0$. The intuitive idea was that this expresses that the net supply (of 'energy') is into the system: energy is being absorbed and dissipated in the system. There are, however, situations where at any moment in time the system will have absorbed energy, i.e., $\int_{-\infty}^{t} Q_\Phi(w)(\tau)d\tau \geq 0$ for all $t \in \mathbb{R}$. For example, electrical circuits at rest will be in a state of minimum energy, and therefore the energy delivered *up to any time* will be nonnegative. This type of positivity will be studied in this section. It plays a crucial role in H_∞- problems (see [9]).

Definition 4.1 *Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$. The QDF Q_Φ (or simply Φ) is said to be* half-line nonnegative, *denoted by $\int^t Q_\Phi \geq 0$, if $\int_{-\infty}^0 Q_\Phi(w)dt \geq 0$ for all $w \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$, and* half-line positive, *denoted $\int^t Q_\Phi > 0$, if in addition $\int_{-\infty}^0 Q_\Phi(w)dt = 0$ implies $w(t) = 0$ for $t \leq 0$.*

Note that half-line nonnegativity implies average nonnegativity, and that half line positivity implies average positivity.
Write $\Phi(\zeta, \eta) = M^T(\zeta)\Sigma_M M(\eta)$ and partition M conform Σ_M as

$$M = \begin{bmatrix} P \\ N \end{bmatrix} \qquad (4.1)$$

so that $\Phi(\zeta, \eta) = P^T(\zeta)P(\eta) - N^T(\zeta)N(\eta)$ and hence $Q_\Phi(w) = \|P(\frac{d}{dt})w\|^2 - \|N(\frac{d}{dt})w\|^2$. In the following, for $\lambda \in \mathbb{C}$, let $\bar{\lambda}$ denote its complex conjugate.

Proposition 4.2 *Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$. Then*

(i) $(\int^t Q_\Phi \geq 0) \Rightarrow (\Phi(\bar{\lambda}, \lambda) \geq 0 \ \forall \lambda \in \mathbb{C}, \Re e(\lambda) \geq 0)$

(ii) $(\int^t Q_\Phi > 0) \Rightarrow (\Phi(\bar{\lambda},\lambda) \geq 0 \ \forall \lambda \in \mathbb{C}, \Re e(\lambda) \geq 0)$ *and* $\det(\partial\Phi) \neq 0)$.

As noted before, it immediately follows from the definitions that half-line non-negativity implies average non-negativity, etc. Thus proposition 3.4 implies the existence of a storage function. It is the non-negativity of the storage function that allows us to conclude the half-line property:

Theorem 4.3 *Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta,\eta]$. Then the following statements are equivalent.*

1. *$\int^t Q_\Phi \geq 0$,*

2. *there exists a storage function $\Psi \geq 0$ for Φ,*

3. *Φ admits a storage function, and the storage function Ψ_+ defined in theorem 5.7 satisfies $\Psi_+ \geq 0$.*

In order to check half-line non-negativity, one could in principle proceed as follows. Verify that $\Phi(-i\omega, i\omega) \geq 0$ for all $\omega \in \mathbb{R}$, compute Ψ_+, and check whether $\Psi_+ \geq 0$. In some situations, it is actually possible to verify this condition in a more immediate fashion. For example, when $\Phi(\zeta,\eta) = \Phi_0$, a constant matrix, with $\Phi_0 \geq 0$ (trivial, but that is the case that occurs in standard LQ-theory!), or when in (4.1) P is square and $\det(P) \neq 0$. Then, under the assumption that a storage function exists (equivalently: $N^T(-i\omega)N(i\omega) \leq P^T(-i\omega)P(i\omega)$ for all $\omega \in \mathbb{R}$), all storage functions are actually non-negative if one of them is non-negative. In fact, in this case the following theorem holds:

Theorem 4.4 *Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta,\eta]$. Assume it is factored as $\Phi(\zeta,\eta) = P^T(\zeta)P(\eta) - N^T(\zeta)N(\eta)$ with P square and $\det(P) \neq 0$. Let $X \in \mathbb{R}^{\bullet \times q}[\xi]$ be a minimal state map for the $\mathfrak{B}$ given in image representation by (4.1). The following statements are equivalent:*

1. *$\int^t Q_\Phi \geq 0$,*

2. *$\Phi(\bar{\lambda},\lambda) \geq 0$ for all $\lambda \in \mathbb{C}$, $\Re e(\lambda) \geq 0$,*

3. *NP^{-1} has no poles in $\Re e(\lambda) \geq 0$ and $\Phi(-i\omega, i\omega) \geq 0$ for all $\omega \in \mathbb{R}$.*

4. *there exists a storage function $\Psi \geq 0$ for Φ,*

5. *there exists a storage function for Φ and every storage function Ψ for Φ satisfies $\Psi \geq 0$,*

6. *there exists a real symmetric matrix $K > 0$ such that $Q_K(w) := \|X(\frac{d}{dt})w\|_K^2$ is a storage function for Φ,*

7. *there exists a storage function for Φ and every real symmetric matrix K such that $Q_K(w) := \|X(\frac{d}{dt})w\|_K^2$ is a storage function for Φ satisfies $K > 0$.*

Furthermore, if $\begin{bmatrix} P \\ N \end{bmatrix}$ is observable, then any of the above statements is equivalent with

3'. *P is Hurwitz and $\Phi(-i\omega, i\omega) \geq 0$ for all $\omega \in \mathbb{R}$.*

5 Observability

One of the noticeable features of QDF's is that a number of interesting systems theory concepts generalize very nicely to QDF's. We have already seen that the state of a symmetric canonical factorization of Φ functions as the state of the QDF Q_Φ. In this section we will introduce observability of a QDF.

For $\Phi \in \mathbb{R}^{q_1 \times q_2}[\zeta,\eta]$ and $w_1 \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{q_1})$ fixed, the linear map $w_2 \mapsto L_\Phi(w_1, w_2)$ will be denoted by $L_\Phi(w_1, \bullet)$. For $w_2 \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{q_2})$ fixed, the linear map $w_1 \mapsto L_\Phi(w_1, w_2)$ will be denoted by $L_\Phi(\bullet, w_2)$. The BLDF Φ will be called observable if $L_\Phi(w_1, \bullet)$ and $L_\Phi(\bullet, w_2)$ determine w_1 and w_2 uniquely. Equivalently:

Definition 5.1 *Let $\Phi \in \mathbb{R}^{q_1 \times q_2}[\zeta,\eta]$. We will call Φ observable if for all $w_1 \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{q_1})$ and for all $w_2 \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{q_2})$ we have*

$$L_\Phi(w_1, \bullet) = 0 \ \Leftrightarrow \ w_1 = 0$$

and

$$L_\Phi(\bullet, w_2) = 0 \ \Leftrightarrow \ w_2 = 0.$$

The following theorem gives necessary and sufficient conditions for observability purely in terms of the two-variable polynomial matrix Φ, and in terms of the (one-variable) polynomial matrix M occuring in any canonical factorization of Φ.

Theorem 5.2 *Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta,\eta]$ and let $\Phi(\zeta,\eta) = M^T(\zeta)\Sigma_\Phi M(\eta)$ be a symmetric canonical factorization. Then the following statements are equivalent:*

1. *Φ is observable*

2. *$L_\Phi(w, \bullet) = 0 \ \Leftrightarrow \ w = 0$*

3. *for every $\lambda \in \mathbb{C}$, the rows of $\Phi(\lambda,\xi) \in \mathbb{R}^{q_1 \times q_2}[\xi]$ are linearly independent over $\mathbb{C}$*

4. *$M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$, equivalently, the image representation $v = M(\frac{d}{dt})w$ is observable*

5. *$|\Phi| > 0$*

6. *$|\Phi|(\bar{\lambda},\lambda) > 0$ for all $\lambda \in \mathbb{C}$*

6 Strict positivity

Throughout this section we will assume that $\Phi \in \mathbb{R}_s^{q \times q}[\zeta,\eta]$ is observable. We will now introduce and develop the notion of strict positivity. The concept of strict half-line positivity given here is very analogous to that used by Meinsma [4].

Definition 6.1 *Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta,\eta]$ be observable. We will call the QDF Q_Φ strictly positive, denoted $\Phi \gg 0$ if there exists $\epsilon > 0$ such that $\Phi - \epsilon|\Phi| \geq 0$. We will call*

it strictly average positive, *denoted by* $\int Q_\Phi \gg 0$ *if there exists* $\epsilon > 0$ *such that*

$$\int_{-\infty}^{+\infty} Q_\Phi(w)dt \geq \epsilon \int_{-\infty}^{+\infty} Q_{|\Phi|}(w)dt \qquad (6.1)$$

for all $w \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$. *We will call it* strictly half-line positive, *denoted* $\int^t Q_\Phi \gg 0$, *if there exists an* $\epsilon > 0$ *such that*

$$\int_{-\infty}^{0} Q_\Phi(w)dt \geq \epsilon \int_{-\infty}^{0} Q_{|\Phi|}(w)dt. \qquad (6.2)$$

for all $w \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$.

Note that (because of observability) strict positivity implies positivity, and similar for the other cases.

7 A Pick matrix condition for half-line positivity

It is surprisingly difficult to establish some type of analogue of proposition 5.2 for half-line positivity, and earlier attempts ([5], [6], [1]) turned out to be flawed. In proposition 6.4 such an analogue of proposition 5.2 was given but only in the special case that $\Phi(\zeta, \eta) = P^T(\zeta)P(\eta) - N^T(\zeta)N(\eta)$ with $\det(P) \neq 0$. In this section we will give a necessary and sufficient condition for strict half-line positivity in terms of Φ.

As is well-known, the Pick matrix plays an important role in system and circuit theory. We will derive a Pick-matrix type test for non-negativity of Ψ_+. This test is perhaps the most original result of this paper. For simplicity we will consider only the case of strict half-line positivity. First, however, we need to define the Pick-type matrix which may be computed from a $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$. Let $F \in \mathbb{R}^{q \times q}[\xi]$, and assume that $\det(F) \neq 0$. We will call F *semi-simple* if for all $\lambda \in \mathbb{C}$ the dimension of the kernel of $F(\lambda)$ is equal to the multiplicity of λ as a root of $\det(F)$. Note that F is certainly semi-simple if $\det(F)$ has distinct roots.

We will now define the matrix T_Φ. Since the expression is much simpler in the semi-simple case, in this short paper we we will only explain that case. For the general case we refer to the full paper [].

Definition 7.1 (*semi-simple case*): *Let* $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ *be observable, and assume that* $\det(\partial\Phi)$ *has no roots on the imaginary axis. Let* $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ *be the roots of* $\det(\partial\Phi)$ *with positive real part and let* $a_1, a_2, \ldots, a_n \in \mathbb{C}^q$ *be linearly independent and such that* $\partial\Phi(\lambda_i)a_i = 0$, *and such that the* a_k's *associated with the same* λ_i *span* $\ker(\partial\Phi(\lambda_i))$. *Then*

$$T_\Phi := \left[\frac{\bar{a}_i^T \Phi(\bar{\lambda}_i, \lambda_j)a_j}{\bar{\lambda}_i + \lambda_j} \right]_{i,j=1,\ldots,n}. \qquad (7.1)$$

The next theorem is the most refined result of this paper. It shows, on the one hand, the relation between strict half-line positivity and positivity of a storage function, and, on the other hand, the relation with the positivity of the Pick matrix T_Φ.

We have seen in theorem 5.5 that a storage function will be a quadratic state function, i.e., $Q_\Phi(w)$ is of the form $x^T K x$, $K = K^T$, with $x = X(\frac{d}{dt})w$ a minimal state map for Φ. We will call this state function *positive definite* if $K > 0$.

Theorem 7.2 : *Let* $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ *be observable. The following are equivalent:*

1. $\int^t Q_\Phi \gg 0$,

2. *(a)* $\int Q_\Phi \gg 0$
   (b) there exists a storage function that is a positive definite state function,

3. *(a)* $\exists \epsilon > 0$ *such that* $\Phi(-i\omega, i\omega) \geq \epsilon|\Phi|(-i\omega, i\omega)$ *for all* $\omega \in \mathbb{R}$,
   (b) $T_\Phi > 0$.

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