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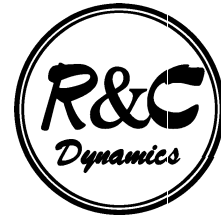
INTEGRABLE AND NON-INTEGRABLE DEFORMATIONS OF THE SKEW HOPF BIFURCATION

In the skew Hopf bifurcation a quasi-periodic attractor with nontrivial normal linear dynamics loses hyperbolicity. Periodic, quasi-periodic and chaotic dynamics occur, including motion with mixed spectrum. The case of 3-dimensional skew Hopf bifurcation families of diffeomorphisms near integrability is discussed, surveying some recent results in a broad perspective. One result, using KAM-theory, deals with the persistence of quasi-periodic circles. Other results concern the bifurcations of periodic attractors in the case of resonance.

1. Introduction

The present paper presents some aspects of a certain class of bifurcations of a quasi-periodic circle attractor in a family of diffeomorphisms, the *skew Hopf bifurcations*. These are a variation on the well-known *reducible Hopf bifurcations* (see e. g. [11], [23], [24]): there, a quasi-periodic torus attractor (that is an invariant normally hyperbolic torus carrying quasi-periodic dynamics) of dimension n bifurcates to a quasi-periodic torus attractor of dimension $n + 1$ — in particular, an quasi-periodic circle bifurcates to a quasi-periodic 2-torus. In the simplest skew Hopf bifurcation, which will be considered exclusively in the following, a quasi-periodic circle attractor bifurcates to a weakly chaotic torus attractor. The transition from order to chaos occurs in one step, without any repetition.

2. Setting of the problem

This section gives a general outline of the questions considered in the present paper.

A few general remarks on bifurcation problems are made in order to establish the motivation of these questions. Investigating a given class of bifurcations is likely to be easier in the presence of some symmetry or other special property. Bifurcations in the class under consideration possessing the "maximal" amount of symmetry or special properties usually can be analyzed directly; they will be called integrable, in analogy to integrable systems of Hamiltonian dynamics (see below and [3]). However, this notion is a little vague, and has to be specified for each separate class.

To obtain general information about the class of bifurcations, small, but otherwise arbitrary perturbations are applied to integrable bifurcations (that is, a family, not necessarily special, in a small

neighbourhood of the integrable bifurcation family is considered), and the effects of the perturbations are investigated.

2.1. Integrable skew Hopf bifurcation families

Integrable skew Hopf bifurcation families are introduced and analyzed.

Definition. An *integrable skew Hopf bifurcation family* is a family of diffeomorphisms of the phase space $M = S^1 \times \mathbb{R}^2$, where $S^1 = \mathbb{T}^1 = \dfrac{\mathbb{R}}{2\pi\mathbb{Z}}$, given by:

$$\phi_p(x, y) = \left(x + \omega(p) + f(|y|^2, p)|y|^2, \left(\beta + g(|y|^2, p)|y|^2\right) E_k(x)y\right) . \tag{2.1}$$

Here $x \in S^1$, $y \in \mathbb{R}^2$, and $|.|$ denotes the Euclidean norm: $|y|^2 = y_1^2 + y_2^2$; the parameter p takes values in the *parameter space* P, which is an open neighbourhood of 0 in $\mathbb{R}^q$. The functions ω, β, f and g all take values in $\mathbb{R}$; moreover, $\beta(p) > 0$ for all p.

The integer k takes values in $\mathbb{Z}\backslash\{0\}$, and the map $E_k(x)$ is of the form:

$$E_k(x) = \begin{pmatrix} \cos kx & -\sin kx \\ \sin kx & \cos kx \end{pmatrix} .$$

A general remark about regularity: if not specified otherwise, all functions in this paper are assumed to be *infinitely differentiable* (or *smooth*).

Some initial remarks are in order.

REMARK.

1. If $k \neq 0$, the matrix $E_k(x)$ cannot be reduced to a constant (i. e. it cannot be made independent of x) by any coordinate transformation, and it is this which accounts for the "skewness" of the system.

2. For $k = 0$ a reducible diffeomorphism is obtained.

3. If a coordinate change $(x, y) \mapsto (-x, y)$ is performed, the family ϕ_p is conjugated to another integrable skew Hopf family, with $-k$ instead of k. Hence it is sufficient to consider the case that $k > 0$, which is assumed from here on.

4. The above is the lowest dimensional case for which a skew Hopf bifurcation can be obtained.

The theory of skew Hopf bifurcation families as discussed in this paper can be applied to invariant circles in higher dimensional dynamical systems, if the method of Bibikov [5] is adapted. At present the theorem on reduction of the dynamics to a centre manifold [34] cannot be used, since parts of the theory do not yet allow a straightforward generalization to the case of finite differentiability (contrary to e. g. [11]). The reader is referred to [17].

Symmetries

An integrable skew Hopf bifurcation family admits two symmetries. Let ρ_θ denote the following $\mathbb{T}^1$-action on M:

$$\rho_\theta(x, y) = \left(x, \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} y\right) .$$

Note that ϕ_p is equivariant with respect to ρ_θ for all $\theta \in [0, 2\pi)$:

$$\rho_\theta \circ \phi_p = \phi_p \circ \rho_\theta .$$

Equivariance with respect to ρ will be called (*normal*) *rotational symmetry* in the following.

The second symmetry appears if the rotational symmetry is reduced out. For this, *cylindrical coordinates* (x, r, s) are introduced by setting:

$$(x, y_1, y_2) = (x, r \cos s, r \sin s) \,.$$

In these coordinates, ϕ_p takes the form:

$$\phi_p(x, r, s) = \big(x + \omega(p) + f(r^2, p), \beta r + g(r^2, p)r, s + kx\big) \,. \tag{2.2}$$

The *reduced family* $\tilde{\phi}_p$ is given by:

$$\tilde{\phi}_p(x, r) = \big(x + \omega(p) + f(r^2, p), \beta r + g(r^2, p)r\big) \,.$$

Let τ_α denote the following $\mathbb{T}^1$-action:

$$\tau_\alpha(x, r) = (x + \alpha, r) \,.$$

Note that the reduced family $\tilde{\phi}_p$ is equivariant with respect to τ_α for all $\alpha \in [0, 2\pi)$. Equivariance with respect to τ will be called (*internal*) *translational symmetry* in the following. The integrable skew Hopf family ϕ_p is said to admit translational symmetry, in the sense that the reduced family $\tilde{\phi}_p$ is equivariant with respect to τ_α.

Bifurcation analysis in the integrable case

For the integrable skew Hopf family, bifurcation analysis is straightforward. Note that because of the rotational symmetry, the set $S = \{(x, y) \in M : y = 0\}$ is an invariant circle. Its stability is governed by $\beta(p)$: for $\beta(p) < 1$, the invariant circle is normally attracting, while for $\beta(p) > 1$, it is normally repelling.

If $p_* \in P$ is such that:

$$\beta(p_*) = 1 \,, \tag{2.3}$$

then p_* is a bifurcation value. Without loss of generality, it may be assumed that $p_* = 0$. Note that generically:

$$\frac{\partial \beta}{\partial p}(0) \neq 0 \,; \tag{2.4}$$

and that then condition (2.3) determines a bifurcation submanifold of codimension 1 in a neighbourhood of 0, which separates this neighbourhood in regions of attraction $\mathscr{A}$ and repulsion $\mathscr{R}$ of the invariant circle.

Assume that the following generic condition holds as well:

$$g(0, 0) = c \neq 0 \,. \tag{2.5}$$

By the implicit function theorem, there is another, possibly smaller, neighbourhood of 0, and a unique function $T(p)$ defined on that neighbourhood, such that:

$$\beta(p) + g\Big(T(p), \; p\Big) = 1 \,.$$

For those p such that $T(p) > 0$, there is an invariant torus T_p, which bifurcates from the invariant circle at $p = 0$, given by:

$$T_p = \big\{(x, y) \in M : |y|^2 = T(p)\big\} \,.$$

If $c > 0$, this is a repelling torus existing for $p \in \mathscr{A}$, and the bifurcation is *subcritical*. If $c < 0$, T_p is attracting, it exists for $p \in \mathscr{R}$ and the bifurcation is *supercritical*.

Dynamics on the bifurcating torus

The dynamics ψ_p on T_p can be read off the expression (2.2) for ϕ_p in cylindrical coordinates, restricted to $\{(x, r, s) : r^2 = T(p)\}$:

$$\psi_p(x, s) = (x + \alpha(p), s + kx),$$

where $\alpha(p) = \omega(p) + f(T(p), p)$.

If $\alpha(p) = \frac{2\pi m}{n}$, with m and n integers, $n \neq 0$, the torus foliates into a collection of circles, invariant under ψ_p^n.

Otherwise, the motion on the torus is ergodic. Note that two points on T_p whose distance is less than some δ, $0 < \delta < 1$, eventually move apart from each other if their x-coordinates differ; hence, the motion is sensitive on initial conditions, unlike quasi-periodic motion. However, the distance grows only linearly in time, not exponentially; therefore we propose to call this kind of motion *weakly chaotic*.

In the ergodic case, let U be the Koopman operator associated to ψ_p, defined on square integrable complex valued functions f on the torus:

$$Uf(x, s) = f(x + \alpha(p), s + kx).$$

In [26] it is shown that the spectrum of U has a pure point component as well as an absolutely continuous part. The map is said to have mixed spectrum.

2.2. Plan

The aim of this paper is to give an overview of the results obtained on the skew Hopf bifurcation in [16], [17], [60], [61].

The analysis of the integrable case given above consists of two steps:

1. Symmetry considerations show the existence of an invariant circle. Dynamics on the circle are resonant or quasi-periodic.

2. Analyzing the dynamics in a neighbourhood of the invariant circle establishes the existence of an invariant 2-torus, bifurcating from the invariant circle, and carrying weakly chaotic dynamics.

The integrable case admits two symmetries: the rotational symmetry ρ_θ and the translational symmetry τ_α, where the latter can only be defined if the former is present. Hence it is natural to proceed in two steps:

1. To break the translational symmetry by a small perturbation which respects the rotational symmetry.

2. To break both symmetries by adding another small generic perturbation.

The first part of this program has carried out in [16] and [60], to a fairly complete degree. The second part however has been realized only partially (in [17], [60]), owing to what seem to be some fundamental mathematical difficulties.

Questions

Four questions are posed to guide the investigation. The first is analogous to the first step of the above analysis of the symmetric case, inasmuch as it concerns quasi-periodic circle dynamics.

> *Under what conditions does an invariant quasi-periodic circle persist in a near-integrable skew Hopf bifurcation?*

This question is addressed in [16] for case 1 (where it is trivial), and in [17] for case 2. In order to investigate the persistence of the invariant circles in case 2, the skew Hopf family has to be unfolded; that is, one has to consider perturbations depending on sufficiently many parameters. It turns out that the invariant circle persists for subfamilies of the unfolding. To obtain the result, an extension of ordinary KAM-theory is developed, which is the main mathematical result of [17], and which we shall partially summarize and explain below.

The second question, the matter of invariant 2-tori (compare with step 2 in the symmetric case) is considered in [16] (case 1) and Chapter 3 of [61] (case 2):

> *Under what conditions does an invariant 2-torus bifurcate from an invariant circle in a skew Hopf family?*

As mentioned before, these two questions have been given a rather full answer in case 1; however, in case 2, they are incomplete, holding only for "good" subfamilies of the unfolding. Therefore:

> *What happens outside the "good" subfamilies?*

In Chapter 4 of [61] the region one "knows nothing about" is indicated, and numerical simulations of systems in that region are given. The resulting simulations are presented below. They show a complicated bifurcation scenario, which we only partially understand. Here a numerical exploriation as in [14] is desirable.

Finally, there is the question of resonance:

> *What happens in the case of resonant dynamics on the invariant circle?*

In [60] a model system is considered for both case 1 and 2, which we think to be sufficiently general to exhibit most relevant phenomena of resonant skew Hopf systems. It turns out that (amongst others) the degenerate Hopf bifurcation investigated by Chenciner does occur, and so all its complexity, Cantori, homoclinic bifurcations, etc., features in the model system as well. The results of [17], [60] will be given below. This analysis is also valid for the non-skew case.

3. Concepts

This section introduces some general concepts which play a central part in the following. Most of these notions are well-kwown, but they are included for completeness.

3.1. Structural stability and persistence

A bifurcation is *structurally stable*, if all families in a neighbourhood of a given family, for which the bifurcation occurrs, can be conjugated to that family: all families near the given family behave in the same way. This is the strongest form of stability, and usually it cannot be proved easily (if at all: see [46]). If structural stability is too hard to prove, the *persistence* of typical properties of the integrable system may be investigated as the next most important factor. A property of a given family is said to be persistent, if it holds for all nearby families. E. g., the existence of certain invariant manifolds (circles, tori) is a property whose persistence is frequently being investigated.

3.2. Quasi-periodicity

Within the theory of general dynamical systems, there is a fairly elaborate theory on the persistence of quasi-periodic invariant circles and tori under small perturbations (see for instance [9]). We recall basic definitions and introduce our notation.

Periodic solutions

Quasi-periodic solutions of differential equations are natural generalizations of *periodic solutions*, which correspond to invariant circles in phase space, with dynamics described by:

$$\dot{x} = \omega \, .$$

Here $x \in S^1 = \mathbb{T}^1$ is a suitable coordinate, and $\omega \in \mathbb{R}$ is called the *frequency*. The solution $x(t)$ with initial condition $\xi \in S^1$ takes the form:

$$x(t) = \xi + \omega t \mod 2\pi \, .$$

Quasi-periodic solutions

A *quasi-periodic* solution corresponds, in phase space, to an invariant n-dimensional torus $\mathbb{T}^n \cong \dfrac{\mathbb{R}^n}{2\pi\mathbb{Z}^n}$, with coordinates $(x_1, \ldots, x_n)$. If on this torus the coordinates can be chosen such that:

$$\dot{x}_j = \omega_j \, , \tag{3.1}$$

for all $1 \leqslant j \leqslant n$, then the dynamics are referred to as *parallel* or *conditionally periodic*.

The vector $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n$ is called the *frequency vector*. Let $(\,\cdot\,, cdot\,)$ denote the usual inner product on $\mathbb{R}^n$; if ω satisfies the *non-resonance condition*:

$$(\boldsymbol{k}, \omega) = \sum_{j=1}^{n} \boldsymbol{k}_j \, \omega_j \neq 0 \quad \text{for all } \boldsymbol{k} \in \mathbb{Z}^n \backslash \{0\} \tag{3.2}$$

the flow is called *quasi-periodic*; otherwise it is called *resonant*. An invariant torus in phase space carrying quasi-periodic dynamics is often called a *quasi-periodic torus* for short.

The solution $x(t)$ of equation (3.1) with initial condition $x(0) = \xi$ takes the form:

$$x_j(t) = \xi_j + \omega_j t \mod 2\pi \, .$$

Condition (3.2) ensures that solution curves fill the torus densely: a quasi-periodic flow is *ergodic*.

By taking a Poincaré section of a quasi-periodic torus at, say,

$$\Sigma = \{ x \in \mathbb{T} \colon x_n = 0 \} \, ,$$

a diffeomorphism ϕ on the $(n-1)$-dimensional torus Σ is obtained. Note that the image $\phi(\xi)$ of $\xi \in \Sigma$ under the Poincaré map is given by:

$$\phi(\xi) = x\left(\frac{2\pi}{\omega_n}\right) = \left(\xi_1 + 2\pi\frac{\omega_1}{\omega_n}, \ldots, \xi_{n-1} + 2\pi\frac{\omega_{n-1}}{\omega_n}, 0\right) \, .$$

Setting $\alpha_j = \dfrac{2\pi\omega_j}{\omega_n}$, the non-resonance condition now reads:

$$\sum_{j=1}^{n-1} \boldsymbol{k}_j \alpha_j + 2\pi \boldsymbol{k}_n \neq 0 \, .$$

Generalizing this, a map $\phi \colon \mathbb{T}^n \to \mathbb{T}^n$ is called a *quasi-periodic diffeomorphism*, if for suitable coordinates:

$$\phi(x) = x + \alpha \mod 2\pi\mathbb{Z}^n \, ,$$

where $\alpha \in \mathbb{R}^n$ satisfies the following non-resonance condition:

$$(\boldsymbol{k}, \alpha) \neq 2\pi p \,,$$

for any $\boldsymbol{k} \in \mathbb{Z}^n \backslash \{0\}$ and any $p \in \mathbb{Z}$. As in the vector field case, the orbit of any point under a quasi-periodic diffeomorphism fills the n-torus densely, and the motion is ergodic.

In the case of the integrable skew Hopf family, the invariant circle S carries quasi-periodic dynamics if the non-resonance condition is satisfied, which is equivalent to:

$$\omega \neq 2\pi \frac{p}{q} \,,$$

for any $q \in \mathbb{Z} \backslash \{0\}$, $p \in \mathbb{Z}$.

Diophantine frequencies

In the theory of perturbations of quasi-periodic motions, the set of *Diophantine frequencies* plays a key role. For diffeomorphisms it is defined as follows. First let $\gamma, \tau > 0$ and define the set $\mathbb{R}_c^n = \mathbb{R}_c^n(\gamma, \tau)$ as:

$$\mathbb{R}_c^n = \{\alpha \in \mathbb{R}^n \colon |(\boldsymbol{k}, \alpha) - 2\pi p| \geqslant \gamma |\boldsymbol{k}|^{-n-\tau} \text{ for all } \boldsymbol{k} \in \mathbb{Z}^n \backslash \{0\} \,, \ p \in \mathbb{Z}\} \,.$$

A frequency vector α is called *Diophantine* if it is contained in one of the $\mathbb{R}_c^n(\gamma, \tau)$ for some γ, τ. The set of Diophantine frequencies has full (Lebesgue) measure in $\mathbb{R}^n$.

For any function a from a set U into $\mathbb{R}_c^n$, the subset U_c is defined as:

$$U_c = U \cap a^{-1}(\mathbb{R}_c^n) \,.$$

If U is open and the map a is a submersion, then U_c is called a *Whitney-smooth family of manifolds, parameterized over the Cantor set* $\mathbb{R}_c^n$, or a *Cantor foliation*. The leaves of this foliation are subsets $a^{-1}(\alpha)$, $\alpha \in \mathbb{R}_c^n$, of U. Note that if γ is sufficiently small, the set $U_c \subset U$ has positive Lebesgue measure.

In the case of skew Hopf bifurcation families:

$$\mathbb{R}_c = \left\{\alpha \in \mathbb{R} \colon \left|\alpha - 2\pi \frac{p}{q}\right| \geqslant \gamma |q|^{-2-\tau} \text{ for all } q \in \mathbb{Z} \backslash \{0\} \,, \ p \in \mathbb{Z}\right\} \,.$$

Linearization of circle diffeomorphisms

In the following, a theorem by Herman on the linearization of quasi-periodic diffeomorphisms of the circle [32] will be used. This result is quoted briefly here, though not in its full generality.

Let the projection of $\mathbb{R}$ onto S^1 be denoted by $\tilde{\pi}$, that is:

$$\tilde{\pi}(x) = x \mod 2\pi \,.$$

Let ψ be a circle diffeomorphism $\psi \colon S^1 \to S^1$. A diffeomorphism $\Psi \colon \mathbb{R} \to \mathbb{R}$ such that $\tilde{\pi} \circ \Psi = \psi \circ \tilde{\pi}$, is called a *lift* of ψ. The *rotation number* $\rho(\Psi)$ of the lift Ψ is defined as:

$$\rho(\Psi) = \frac{1}{2\pi} \lim_{n \to \infty} \frac{\Psi^n(x_0) - x_0}{n} \,;$$

it is independent of x_0. The rotation number $\rho(\psi)$ of the circle diffeomorphism ϕ is then defined as:

$$\rho(\psi) = \rho(\Psi) \mod 1 \,.$$

where Ψ is any lift of ψ. The rotation number is independent of the choice of the lift. It is invariant under conjugacies and it depends continuously on ψ (see [27]).

Theorem ([32]). *Let $\psi \colon S^1 \to S^1$ be a smooth diffeomorphism of the circle, such that $\alpha = \rho(\psi)$ is Diophantine. Then there is a smooth change of variable, conjugating ψ to:*

$$x \mapsto x + \alpha \,.$$

3.3. Unfolding parameters

Mathematical models of physical situations usually feature parameters. Sometimes more para-meters than those provided by the physical context are needed to simplify the mathematical analysis. These additional *unfolding* parameters help to understand the structure of the family, and this in turn gives insight into the "physical" subfamilies. Furthermore, non-persistent properties of the original families might prove to be persistent in an unfolded family.

The following example may illustrate this idea. In the context of general perturbations to the integrable skew Hopf bifurcation family, the most important linearized homological equation has the following structure:

$$v(x + \omega) - e^{ikx}v(x) = g(x). \tag{3.3}$$

Here $v(x) \in \mathbb{C}$ corresponds to the coordinate transform, $g(x)$ to the perturbation to be transformed away, and e^{ikx} corresponds to the matrix $E_k(x)$; $k \in \mathbb{Z}\setminus\{0\}$. In [16] this equation was shown to have a $|k|$-dimensional complex obstruction (equivalent to a $2|k|$-dimensional real one).

Intuitively, this means the following. Let $\sum_{n \in \mathbb{Z}} g_n e^{inx}$ be the Fourier expansion of $g(x)$, and consider the coefficients g_n, $n \notin \{1, \dots, k\}$ as given. Then there are unique values $g_1^*, g_2^*, \dots, g_k^*$ which the coefficients $g_1, g_2, \dots, g_k$ have to take in order that equation (3.3) has a real analytic (or even continuous) solution $v(x)$. Since generically the coefficients $g_1, \dots, g_k$ will differ from these special values, in the general case a small, non-symmetric perturbation cannot be transformed away.

However, the situation is different if instead of $g(x)$ a function $G(x, \sigma)$ is put in the right hand side, which is such that the Fourier coefficients $G_n(\sigma) = g_n$ for $n \neq \{1, \dots, k\}$, while the coeffi-cients $G_1(\sigma), \dots, G_k(\sigma)$ are of the form:

$$G_1(\sigma) = g_1 + \sigma_1, \dots, G_k(\sigma) = g_k + \sigma_k.$$

Then equation (3.3) has a solution for $G(x, \sigma^*)$, where:

$$\sigma_1^* = g_1^* - g_1, \dots, \sigma_k^* = g_k^* - g_k.$$

The family $G(x, \sigma)$ is called an unfolding of $g(x)$, since $g(x)$ is a (zero-dimensional) subfamily of $G(x, \sigma)$:

$$g(x) = G(x, 0).$$

Put in another way, if a $|k|$-dimensional complex ($2|k|$-dimensional real) unfolding parameter σ is added (in the right way), the following property is persistent: there is a value of the parameter for which (3.3) has a solution.

Note that if the original right hand side already depends on a q-dimensional real parameter p such that the map:

$$p \mapsto (g_1(p), \dots, g_k(p)),$$

has an injective derivative, then only $2|k| - q$ real parameters have to be added to obtain a solution of (3.3).

Another class of examples is obtained as follows. Consider a 2-parameter family of dynamical systems that has a codimension-2 bifurcation, and consider a 1-parameter subfamily, such that the corresponding curve in the parameter plane passes close to the codimension-2 point. The following situation may arise: the 1-parameter family, considered by itself, displays a lot of apparently accidental complexity, while this is naturally explained in the 2-parameter setting, being caused by the nearby codimension-2 point.

3.4. Integrability

The persistence of quasi-periodic invariant tori is usually shown in families of diffeomorphisms that are close to *integrable* families. From a perturbation theory point of view, in [11] integrable families are defined as families that display toroidal symmetry (equivariance with respect to a free torus action). Because of that symmetry, there is a continuum of invariant parallel tori. Above, the integrable skew Hopf family has been introduced, which is equivariant (in a certain sense) with respect to two $\mathbb{T}^1$-actions.

The term "integrable" originates from classical mechanics (see e. g. [3]): the differential equations of an integrable Hamiltionian system can be solved explicitly by quadratures, if the integrals are known. More generally, systems are called "integrable" if they exhibit a sufficient amount of symmetry or other special properties to allow the dynamics to be determined completely. In the following pages, the term "integrable" will be used in this latter sense.

That these notions of integrability are closely related is shown by E. Noether's [48] famous theorem, which roughly states that, if a Hamiltonian system has a symmetry, it has a first integral as well (see [3]).

A much-studied example of a near-integrable *Hamiltonian* system is the solar system. As the mass of the sun is much larger than the mass of any of the planets, the interaction between the various planets can be set to zero in first approximation. The result is an integrable system, with all the planets moving around the sun in Keplerian ellipses. The solar system can then be considered as a near-integrable perturbation of that integrable system, if the (very small) interactions between the planets are added.

In the dissipative context, (formal) normal forms at (quasi-periodic) bifurcations often have extra symmetry. These normal forms display the required toroidal symmetry. Here the questions are: what happens if these normal forms are perturbed generically; and: which properties are persistent?

The Landau – Hopf scenario for the onset of turbulence in a fluid flow for instance assumes repeated Hopf bifurcation of a quasi-periodic attractor. For the n-th bifurcation, an n-torus branches off an $(n-1)$-torus, and a new frequency ω_n comes in. If n is large, the dynamics cannot be distinguished, for all practical purposes, from turbulent dynamics. This is a typical integrable scenario, whose persistence under small perturbation might be (and is) investigated (see [7], [11]).

3.5. KAM-theory

Techniques developed by Kolmogorov [36], Arnol'd [1], Moser [43], and many others ([7], [10], [11], [32], [50], [54]), to obtain smooth conjugacies of dynamical systems, collectively known as *KAM-theory*, prove the existence of quasi-periodic tori in near-integrable families. KAM-theory was first developed for Hamiltonian systems, but there are variants for dissipative systems, reversible systems, etc. In the case of dissipative systems, the tori persist for parameters taking values in certain large measure, Cantor-like sets. This kind of persistence is called *quasi-periodic stability (in the sense of measure)* and is weaker than structural stability, since the dynamics do not persist in their entirety. Sometimes unfolding parameters are needed to enable the application of KAM-theory, see [10], [11].

4. Context

This section outlines some links of the theory of near-integrable skew Hopf bifurcations (and the closely connected question of persistence of essentially nonreducible invariant circles) to previous work and other problems.

The skew Hopf bifurcation was introduced by Chenciner and Iooss [23], [24] as an alternative to the *reducible* quasi-periodic Hopf bifurcation, which features in the Landau – Hopf scenario of the onset of turbulence. Broer and Takens [16] were interested in the phenomenon that, in the (rotationally

symmetric) skew Hopf system they considered, an invariant torus does persist, carrying ergodic dynamics which has a mixed spectrum. Broer and Wagener [17] and Wagener [61] investigated the effects of small, nonsymmetric perturbations of that symmetric skew Hopf family, that is, of the dynamics in the near-integrable case. Of interest is the question whether the ergodicity and the mixed spectrum do persist. These "physical" questions are addressed in more detail in Subsection 4.1.

There is a second, rather more mathematical reason to study the skew Hopf bifurcation. Most of the known KAM-theory uses the fact that the invariant tori have a constant normal linear part, or can be reduced to a system with constant normal linear part by choosing appropriate coordinates. In the case of periodic solutions to a differential equation, the possibility of this reduction is a consequence of the well-known Floquet theory. In the skew Hopf bifurcation, this reduction is impossible on topological grounds. These matters are discussed in more detail in Subsection 4.2.

It has so far been unclear whether KAM-theory can be adapted to the kind of systems occurring in the skew Hopf bifurcation. As far as we know [17] is the first successful attempt to develop KAM-theory for such a non-reducible case. Technically the difference with the "classical" theory is that the linearized conjugacy equations are coupled instead of decoupled. This difficulty is overcome by the introduction of many extra unfolding parameters. After this, our approach turns out to be an adaptation of the "classical" case, see for instance [10], [11], [44].

4.1. Onset of turbulence

Independently, E. Hopf ([35]) and L. Landau ([40], and later in [41]), proposed the following scenario for one possible kind of onset of turbulence. The Couette–Taylor experiment is taken as an illustration, and the reader is referred to [29] and [8] as experimental references.

Couette–Taylor experiment

The Couette–Taylor system consists of an incompressible fluid contained between the walls of two long concentric cylinders. The outer cylinder is fixed, while the inner one rotates with an angular velocity Ω, which can be varied. It is assumed that the time evolution of this system can be modeled by the Navier–Stokes equations. Following the usage of the theory of these equations, dimensionless variables are introduced, and the properties of the flow turn out to depend on a dimensionless constant, the *Reynolds number Re*. This number depends (linearly) on the driving angular velocity Ω, and thus Re may be considered to be a parameter.

The fluid is completely characterized by a divergence-free velocity field v (the fluid is described by Eulerian coordinates); the set of all divergence-free velocity fields v satisfying the boundary values is the phase space X. The problem of existence and uniqueness of solutions $v(t)$ to the Navier–Stokes equations shall not be discussed, nor the precise nature of the phase space X: it is assumed that existence and uniqueness of solutions holds, and that positive time evolutions are bounded.

As long as Ω, and hence Re, are relatively small, initial disturbances decay, and the flow $v(t)$ approaches some constant flow v_0. This is then a *global point attractor* in the phase space X. If Re is increased, we may observe that the flow changes from a constant flow to a flow depending periodically on time. That is, the attractor of the system may bifurcate from a fixed point v_0 to a (closed) orbit, tracing out a circle in phase space. If a time series is made of some observable quantity of the system, the change to periodic flow is marked by the appearance of sharp peaks at a frequency ω_1 and its higher harmonics $2\omega_1$, $3\omega_1$ etc. in the Fourier transform of the time series (which was indeed observed in [29], as in other places; compare also the interesting numerical study [19]). This bifurcation is the (first) *Hopf bifurcation*, also called *Hopf–Poincaré–Andronov bifurcation*.

Landau – Hopf scenario

Landau and Hopf proposed a sequence of successive Hopf bifurcations for increasing parameter values Re_2, Re_3, ... In their scenario, at some parameter value Re_2 the invariant circle (which is a 1-torus) bifurcates to a 2-torus, which then at Re_3 bifurcates to a 3-torus etc. These tori carry quasi-periodic torus dynamics of the form:

$$\dot{\theta} = \omega = (\omega_1, \ldots, \omega_n) ,$$

with $\theta \in \mathbb{T}^n$, $\omega \in \mathbb{R}^n$ non-resonant.

In other words, at each bifurcation Re_n, a new frequency ω_n "comes in"; at a limit point Re_∞ of the Re_n, the dynamics finally jump from quasi-periodic with many frequencies (which is complicated, but not chaotic) to turbulent.

Ruelle – Takens scenario

The Landau – Hopf scenario was criticized by Ruelle and Takens in 1970 ([52], [53]). They showed that there might already be chaotic strange attractors on a 4-torus; later, with Newhouse in [47], they improved this result by showing that, actually, 3-tori might already carry chaotic dynamics. Thus, at the bifurcation point Re_3, quasi-periodic dynamics on $\mathbb{T}^2$ might bifurcate to chaotic dynamics on $\mathbb{T}^3$.

Quasi-periodic unfolding theory

The seeming contradiction between the turbulence scenarios of Landau – Hopf and Ruelle – Takens was reconciled in 1990 by the quasi-periodic unfolding theory of Broer, Huitema, Takens and Braaksma [11]. By considering unfolding parameters, chosen such that the internal frequencies ω and the "normal frequencies" ω^N depend sufficiently on these parameters, they showed that the Landau – Hopf and the Ruelle – Takens scenarios occupy different parts of the parameter space. They gave an idea of the structure of those sets: the Landau – Hopf dynamics occur on a set of large *measure*; Ruelle – Takens dynamics occur on an open set, which is *topologically* large (see for this duality [49]; see also [10]).

Skew Hopf bifurcation

Yet another possible bifurcation was found by Chenciner and Iooss in 1979 [23], [24]: they noticed that an invariant 2-torus might be essentially nonreducible. This led to the *skew Hopf bifurcation*, which was analyzed in the normal rotationally symmetric context by Broer and Takens in 1993 (see [16]), and which is the main interest of this paper.

As remarked above, in the integrable skew Hopf bifurcation, a quasi-periodic circle attractor bifurcates to a weakly chaotic torus attractor, whose Koopman operator has mixed spectrum. The results of [16] imply that these dynamics are persistent under rotationally symmetric perturbations. It is an interesting question whether they are persistent if a generic non-symmetric perturbation is applied. We conjecture that they are not.

Mixed spectrum has been found when investigating experimental data obtained from the Couette – Taylor experiment discussed above. The physical configuration of that experiment is rotationally symmetric. This motivated the search for a rotationally symmetric model system having mixed spectrum, and led to the symmetric skew Hopf family discussed in [16]. It is however not clear whether the skew Hopf family has significance in the interpretation of the data of the Couette – Taylor experiment.

In order to establish whether an actual experiment exhibits a skew Hopf bifurcation, the existence of a skew torus attractor has to be established. In [45] methods to compute the homology groups of the attractor on basis of time series of observed quantities are presented, which may help to distinguish a quasi-periodic 3-torus (of a vector field) to the skew quasi-periodic attractor of the vector field that corresponds to the skew torus attractor of the skew Hopf family of (Poincaré) diffeomorphisms.

4.2. Reducibility

This subsection introduces the concepts of reducibility and essential nonreducibility.

The diffeomorphism ϕ is called *reducible to Floquet form at the torus V* (or *Floquetizable* or *normalizable*) in the following case. Suppose there are suitable coordinates $(x, y) \in \mathbb{T}^n \times \mathbb{R}^m$ in a neighbourhood of V, such that $V = \{(x, y) : y = 0\}$ and the diffeomorphism ϕ has the following form:

$$\phi(x, y) = \left(x + \omega + \mathcal{O}(|y|), \Omega y + \mathcal{O}(|y|^2)\right) .$$

Here Ω is a constant invertible $m \times m$ matrix. The choice of coordinates implies a (non-unique) choice of the trivialization of the normal bundle of V.

As the property of being reducible is in fact a property of the *normal diffeomorphism* $N(\phi)$ associated to ϕ, which is used later on, a formulation of reducibility is given in terms of normal diffeomorphisms. The domain of $N(\phi)$ is the *normal bundle* $N(V)$ of V: if $T(X)$ and $T(V)$ denote the tangent bundles of X and V respectively, and if $T_V(X)$ is the restriction of $T(X)$ to V, then the normal bundle $N(V)$ is defined as the fibre-wise quotient:

$$N(V) = \frac{T_V(X)}{T(V)} .$$

As $\phi(V) = V$, it induces a diffeomorphism $N(\phi)$, the *normal diffeomorphism*, on $N(V)$, which is linear on each fibre.

The diffeomorphism ϕ is reducible at the torus V, if there are coordinates $(x, y) \in \mathbb{T}^n \times \mathbb{R}^m$ of the normal bundle $N(V)$, such that $V = \{(x, y) : y = 0\}$, and such that the normal diffeomorphism $N(\phi)$ has the form:

$$N(\phi)(x, y) = (x + \omega, \Omega y) ,$$

where Ω is a constant invertible $m \times m$-matrix. For more details, see [10], [11].

Linking number

In the discussion of essential non-reducibility below, the concept of *linking number* of two circles is needed. First this is introduced for disjoint circles S_1, S_2 of the following special form:

$$S_j = \left\{(x, g_j(x)) \in S^1 \times \mathbb{R}^2\right\} , \tag{4.1}$$

where $g_j : S^1 \to \mathbb{R}^2$ are continuous functions. Disjointness of the circles as point sets is equivalent to the requirement that $g_1(x) \neq g_2(x)$ for all $x \in S^1$. Define a function $f(x)$ by:

$$f(x) = \frac{g_2(x) - g_1(x)}{|g_2(x) - g_1(x)|} .$$

The map $f(x)$ takes values in $\{x \in \mathbb{R}^2 : |x| = 1\}$, which is diffeomorphic to S^1; hence f maps S^1 to itself. The linking number $\ell(S_1, S_2)$ of S_1 and S_2 is defined to be the degree of f; intuitively speaking, this is the number of times $f(x)$ performs a complete revolution around 0, taking into account the orientation.

Without proof, the topological fact is mentioned that the linking number of circles homotopic to those of the form (4.1) is invariant under isotopy in the class of diffeomorphisms.

Essential non-reducibility

A parallel torus is called *essentially nonreducible*, if the normal diffeomorphism $N(\phi)$ is not isotopic to a constant linear diffeomorphism in the class of all diffeomorphisms; that is, if there is no

family Φ_t for $t \in [0, 1]$, continuous in t, such that $\Phi_0 = N(\phi)$, $\Phi_1(x, y) = (x, Ay)$ with $A \in \mathrm{GL}(m, \mathbb{R})$, and Φ_t is a diffeomorphism on $N(V)$ for all $t \in (0, 1)$.

Note that reducible normal diffeomorphisms are isotopic to constant linear diffeomorphisms by definition. The terminology "*essentially* nonreducible" is motivated by the results of Herman [33], who gave examples of normal diffeomorphisms which were not reducible, but nevertheless isotopic to constant linear diffeomorphisms.

In the following, mainly essentially nonreducible systems of the following form are considered:

$$N(\phi)(x, y) = (x + \omega, E_k(x)y) .$$

Here $(x, y) \in \mathbb{T}^1 \times \mathbb{R}^2$ and $k \in \mathbb{Z}_{>0}$; the notation:

$$E_k(x) = \begin{pmatrix} \cos kx & -\sin kx \\ \sin kx & \cos kx \end{pmatrix}$$

is recalled. Note that the circle $S = \{(x, y) : y = (0, 0)\}$ is invariant, while $S' = \{(x, y) : y = (1, 0)\}$ is not. The linking number of S and S' is 0. However, the linking number of $\phi(S) = S$ and $\phi(S')$ is k ($\phi(S')$ winds k times around S). This implies that ϕ cannot be isotopic to the identity.

The reducibility problem

The *reducibility problem* can be formulated as follows: given (normal) diffeomorphisms of the form:

$$\phi(x, y) = (x + \omega, A(x)y) , \qquad (4.2)$$

for which $A(x)$ is ϕ reducible? In general, is there a "normal form classification" for this kind of system? See for some literature [28], [33], [38], [39].

5. The rotationally symmetric skew Hopf bifurcation

This section presents a summary of results obtained in [16] and [60] for the case that the translational symmetry τ_α is broken, while the rotational symmetry ρ_θ is retained. The next section considers the case that both symmetries are broken.

Equivariance with respect to ρ_θ implies immediately that the circle S is invariant, where $S = \{(x, y) \in M : y = 0\}$. In Subsection 5.1, results from [16] are presented, where a bifurcation analysis is given for the case of quasi-periodic dynamics on S. Subsection 5.2 treats the case of resonant dynamics on S.

5.1. Quasi-periodic dynamics

This subsection presents results from [16]. There, general rotationally symmetric deformations of integrable skew Hopf families were considered. These take the following form in cylinder coordinates:

$$\phi_p(x, r, s) = \left(x + F(x, r^2, p), r\, G(x, r^2, p), s + kx + H(x, r^2, p)\right) ,$$

with F, G, H functions on M, taking values in $\mathbb{R}$.

The main result of [16] is the normal form theorem for rotationally symmetric skew Hopf bifurcation families. Let $\omega(p)$ be the rotation number of the map:

$$x \mapsto x + F(x, 0, p) .$$

The equation $\omega(p) = \alpha$, for Diophantine α, defines a codimension-1 hypersurface Ω_α in P:

$$\Omega_\alpha = \{p \in P : \omega(p) = \alpha\} .$$

Denote the union of all such hypersurfaces by Ω_c.

Theorem. *Let $0 \in \Omega_c$. For all $N > 0$, there is a real analytic coordinate transform, conjugating ϕ_p to*:

$$\tilde{\phi}_p^N (x, r, s) = \left(x + \omega(p) + f(r^2, p)r^2 + R_1, \beta(p)r + g(r^2, p)r^3 + R_2 r, s + kx + h(r^2, p)r^2 + R_3 \right),$$

where $\left| R_j(x, r^2, p) \right| \leqslant C \left(|r|^{N+1} + |p|^{N+1} \right)$ *for* $j \in \{1, 2, 3\}$.

The proof can be found in [16].

Corollary. *There is a $\delta > 0$, such that for $|r| < \delta$ and $|p| < \delta$, the map $\tilde{\phi}_p^N$ of the previous proposition can be transformed into*:

$$\tilde{\phi}_p^N (x, r, s) = \left(x + \omega(p) + \tilde{f}(r^2, p)r^2 + \tilde{R}_1, \beta(p)r + \tilde{g}(r^2, p)r^3 + \tilde{R}_2 r, s + kx \right).$$

where $\left| R_j(x, r^2, p) \right| \leqslant C \left(|r|^{N+1} + |p|^{N+1} \right)$ *for* $j \in \{1, 2\}$.

Proof.

Observe that the equation:

$$ku + h(r^2, p)r^2 + R_3(x + u, r^2, p) = 0,$$

has a unique solution $u = u(x, r^2, p)$ for $|r| < \delta$, if δ is chosen small enough. Then apply the coordinate transform:

$$(x, r, s) \mapsto \left(x + u(x, r^2, p), r, s \right).$$

to obtain the claimed form of the diffeomorphism. ∎

At this point all tildes are dropped. In (x, y)-coordinates, ϕ_p^N takes the form:

$$\phi_p^N (x, y) = \left(x + \omega(p) + f(|y|^2, p)|y|^2 + R_1, (\beta(p) + g(|y|^2, p)|y|^2 + R_2)E_k(x)y \right).$$

Note that in the case that R_1, R_2 are identically 0, the system is actually integrable. This motivates the designation of (2.1) as the general form of an integrable skew Hopf bifurcation.

The map ϕ_p^N is an N-*th order normal form* of ϕ_p. Hence any rotationally symmetric deformation of an integrable skew Hopf bifurcation family, whose rotation number on the invariant circle S is Diophantine, is conjugate to a rotationally symmetric perturbation of arbitrarily high order in $|r| + |p|$ of an integrable skew Hopf bifurcation family. Note however that the domain of definition of ϕ_p^N may shrink as N tends to infinity.

Bifurcation analysis

The bifurcation analysis in the rotationally symmetric case is quite close to the analysis for the integrable case given above. For proofs of statements the reader is referred to [16].

The circle S is invariant by symmetry, and hyperbolic if $\beta(p) \neq 1$. Any $p_* \in P$ for which $\beta(p_*) = 1$ is a bifurcation value; it may be assumed that $\beta(0) = 1$. If the following generic condition holds:

$$\frac{\partial \beta}{\partial p}(0) \neq 0$$

then in a neighbourhood of $p = 0$, there is a smooth codimension-1 bifurcation manifold $\mathscr{B}$, given by:

$$\mathscr{B} = \left\{ p \in P \colon \beta(p) = 1 \right\}.$$

As before, $\mathscr{B}$ separates the neighbourhood in an attracting region $\mathscr{A}$ where $\beta < 1$, and a repelling region $\mathscr{R}$ where $\beta > 1$.

Assume a second generic condition:

$$g(0,0) = c \neq 0 \,.$$

This implies the existence of a unique invariant bifurcating torus $\tilde{T}_p$ for parameter p such that $(\beta - 1)c < 0$. The torus is of the following form:

$$\tilde{T}_p = \{(x, r, s) \colon r = \tilde{T}(x, p)\} \,;$$

the function $\tilde{T}(x, p)$ is close to $T = T(p)$, which is a solution to:

$$\beta(p) + g(T, p)T = 1 \,.$$

Existence and uniqueness of $\tilde{T}_p$ is proved in [16].

Define the map $\psi(x, p)$ by:

$$\psi(x, p) = x + \omega(p) + f\left(\tilde{T}(x, p), p\right)\tilde{T}(x, p) + R_1(x, \tilde{T}(x, p), p) \,.$$

On the invariant torus $\tilde{T}_p$, dynamics are of the form:

$$\phi : (x, s) \rightarrow (\psi(x, p), s + kx) \,.$$

Let $\alpha(p)$ be the rotation number of $x \mapsto \psi(x, p)$. Note that if $f(0, 0) \neq 0$, the rotation number $\alpha(p)$ varies as p moves away from the bifurcation manifold $\mathscr{B}$, and there is a subset of positive measure of Ω_c such that $\alpha = \alpha(p)$ is Diophantine. For such α, there is a suitable coordinate transformation [16], bringing the torus dynamics into the form:

$$(x, s) \mapsto (x + \alpha, s + kx) \,.$$

Thus, for the rotationally symmetric case, skew dynamics are persistent for parameters in a positive measure subset of P.

5.2. Resonant dynamics

In [60] the following model system is considered. It is obtained from a general rotationally symmetric system by dropping all but the lowest order terms in the Fourier – Taylor expansion:

$$\phi_{\omega,\mu}(x, y) = \left(x + \omega + a\sin x + c|y|^2, \left(1 + \mu + b\cos x + d|y|^2\right)E_k(x)y\right) \,.$$

Here $1 + \mu$ has taken the role of β.

This model system is chosen in such a way that it is of sufficient complexity to display relevant phenomena of the rotationally symmetric skew Hopf bifurcation at resonance, while being simple enough to be accessible to analysis.

The system has been investigated in a neighbourhood of $(\omega, \mu) = (0, 0)$: for these values of the parameters there are fixed points on the invariant circle S. Essentially, there are two types of bifurcation diagrams. If the signs of a, c and d are chosen such that:

$$a > 0 \,, \quad c > 0 \,, \quad \text{and} \quad d < 0 \,,$$

then these two types can be distinguished by the sign of b. They are given in Fig. 1 and 2.

In both bifurcation diagrams, five organizing codimension-2 points are found: two saddle-Hopf, two degenerate Hopf and one Bogdanov – Takens bifurcation point.

From each saddle-Hopf point, two lines of saddle-node bifurcations of points on the invariant circle S emanate, as well as two curves of Hopf bifurcations, also of points on the invariant circle. Close to each saddle-Hopf point on the Hopf curve, there is a degenerate Hopf point where the Hopf bifurcation
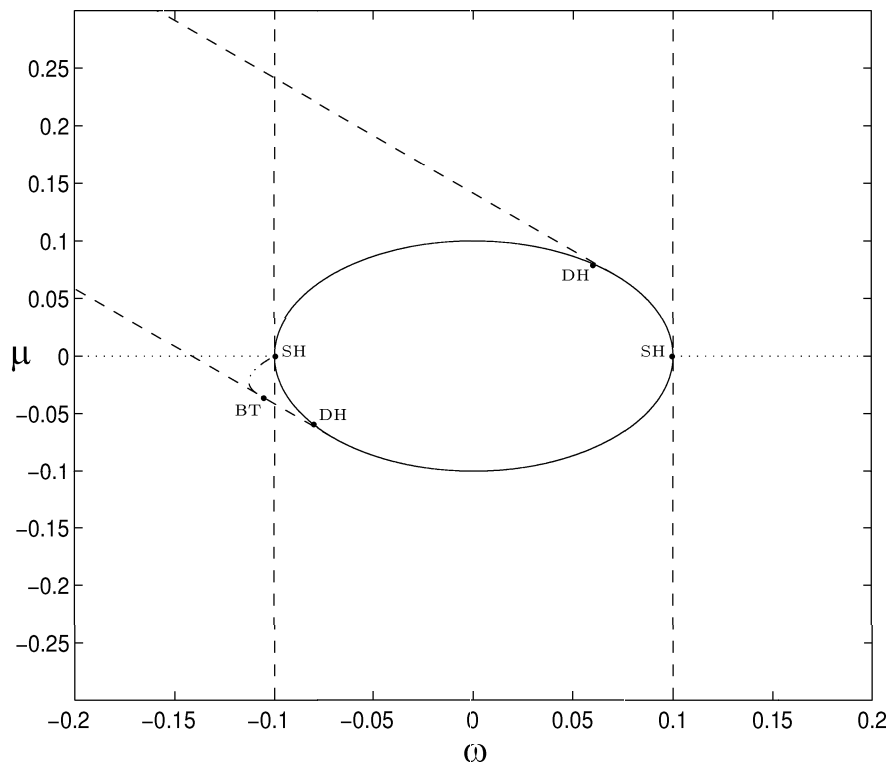
Fig. 1. The (ω, μ)-diagram for $b > 0$. The coding of the bifurcations is as follows. Codimension-2 bifurcation points: BT — Bogdanov–Takens; SH — Saddle–Hopf; DH — Degenerate Hopf. Codimension-1 bifurcation lines: dashed lines: saddle-node bifurcations; drawn lines: Hopf bifurcations of points; dash-dotted lines (emanating from BT points): quasi-periodic Hopf bifurcations of invariant circles; dotted lines: Hopf bifurcation of the invariant circle $y = 0$ (these lines are approximate only). For more details cf. [60]

type changes from sub- to supercritical, and from where a line of saddle-node bifurcations of invariant circles emanates. On one of these lines (determined by the sign of b) there is a Bogdanov–Takens point. From that point, a curve of quasi-periodic Hopf bifurcations emanates, which ends at one of the two saddle-Hopf points. These facts have been established by analysis of the local bifurcations in [60].

From the Bogdanov–Takens point, as well as from one of the saddle-Hopf points, homoclinic bifurcation curves emanate. Numerical evidence indicates that there is one homoclinic curve, connecting the two codimension-two points, but this kind of fact is notoriously hard to establish analytically.

6. The general case

This section presents results obtained in [17], [60] and [61], where small general perturbations to an integrable skew Hopf system were investigated.

6.1. Skew KAM theory

In both the integrable and the rotationally symmetric case, there is an invariant circle $S = S^1 \times \{0\}$. If general perturbations of an integrable skew Hopf family are considered, the existence of an invariant circle is no longer guaranteed. The following question was posed in Subsection 2.2:

Under what conditions does an invariant quasi-periodic circle persist in a near-integrable skew Hopf bifurcation?
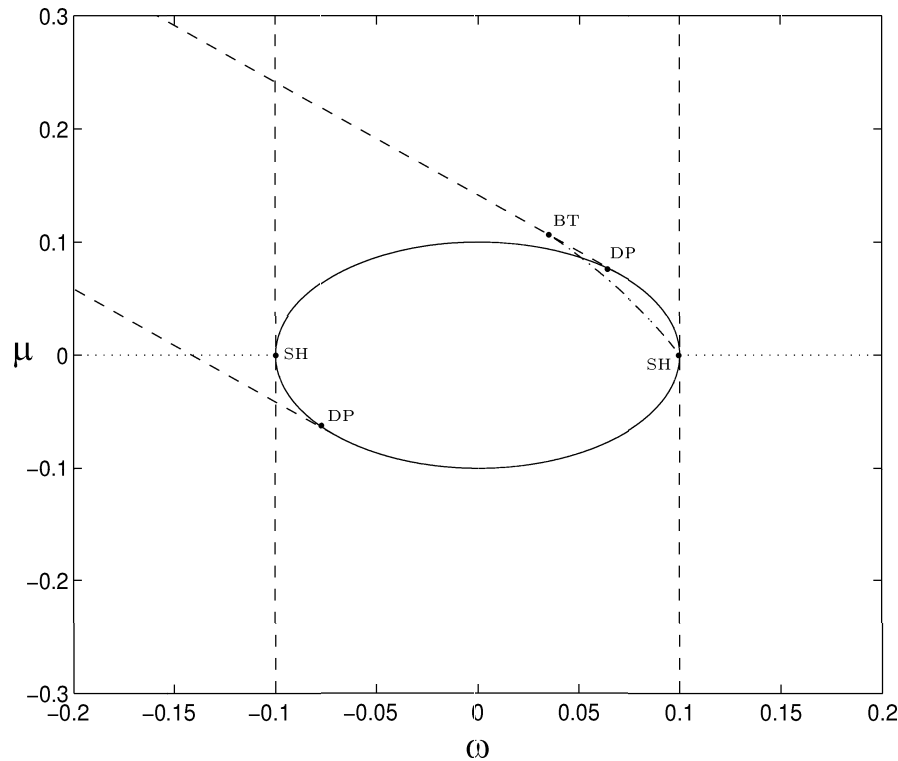
Fig. 2. A view of the (ω, μ)-bifurcation diagram of the reduced system for the case that $b < 0$. The legend is as in figure (1).

In [17] a partial answer has been given. The following map is considered there:

$$\phi_p(x, y) = (x + \omega(p) + f(x, y, p), \beta(p)E_k(x)y + g(x, y, p)) \; .$$

The functions f and g are assumed to be small in some real analytic norm, introduced below.

For convenience in the statement of following result, instead of parameterized families of diffeomorphisms ϕ_p on M, *vertical diffeomorphisms* ϕ on $M \times P$ are considered, where:

$$\phi(x, y, p) = (\phi_p(x, y), p) \; .$$

Unfolding

It turns out that it is convenient to consider the unfolding $\tilde{\phi}(x, y, \sigma) = \tilde{\phi}(x, y, p, \tilde{\beta}, m, \ell)$ of $\phi(x, y, p)$, where:

$$\tilde{\phi}(x, y, \sigma) = \Big(x + \omega(p) + f(x, y, p), E(x, \beta(p), \tilde{\beta})y + M(x, m) + L(x, \ell)y + g(x, y, p), \sigma\Big) \; ,$$

where:

$$E(x, \beta, \tilde{\beta}) = E_k(x) \begin{pmatrix} \beta & -\tilde{\beta} \\ \tilde{\beta} & \beta \end{pmatrix} \; ,$$

$$M(x, m) = \sum_{n=0}^{k-1} E_n(x) \begin{pmatrix} m_{2n+1} \\ m_{2n+2} \end{pmatrix} \; ,$$

and

$$L(x, \ell) = \sum_{n=0}^{2k-1} E_{n-k}(x) \begin{pmatrix} \ell_{2n+1} & \ell_{2n+2} \\ \ell_{2n+2} & -\ell_{2n+1} \end{pmatrix} \; .$$

Note that:

$$\tilde{\phi}(x, y, p, 0, 0, 0) = \phi(x, y, p) \; .$$

Generalized unfoldings

The main result of [17] makes a statement about a slightly larger class of vertical diffeomorphisms, generalizing the unfolding $\tilde{\phi}$ of ϕ. To introduce this class, some more definitions have to be made.

Let P be an open neighbourhood of 0 in $\mathbb{R}^q$ as before. For $r > 0$, the complex neighbourhood $(M \times P) + r$ of $M \times P$ is defined as:

$$(M \times P) + r = \left\{ z \in \frac{\mathbb{C}}{2\pi\mathbb{Z}} \times \mathbb{C}^2 \times \mathbb{C}^q \colon d(z, M \times P) < r \right\} .$$

Let ϕ denote the following vertical diffeomorphism on $M \times P$:

$$\phi(x, y, p) = (x + a(p) + f(x, y, p), E(x, b(p))y + M(x, m(p)) + L(x, \ell(p))y + g(x, y, p), p) , \qquad (6.1)$$

where:

$$w(p) = (a(p), b(p), m(p), \ell(p)) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^{2k} \times \mathbb{R}^{4k} ,$$

$f(x, y, p) \in \mathbb{R}$ and $g(x, y, p) \in \mathbb{R}^2$, and where E, M and L are as above. The functions f, g and w are real analytic which have a complex analytic extension to $(M \times P) + r$. Note that the unfolding $\tilde{\phi}$ given above is of this form.

The diffeomorphism ϕ is said to be *non-degenerate* at p_*, if the map $p \mapsto (a, b, m, \ell)$ has surjective derivative at p_*. Note that a necessary condition for non-degeneracy is that $q \geqslant 6k + 3$.

There exists a change of parameters:

$$W(p) = (w(p), \mu(p)) ,$$

such that after this change, $p = (a, b, m, \ell, \mu)$ can be considered as an independent parameter on some neighbourhood of $p_0 = W(p_*)$, which will be denoted by P also. The "extra" parameter μ is dropped here: it can easily be incorporated again.

Hence, instead of investigating a diffeomorphism ϕ of the form (6.1), non-degenerate at p_*, a diffeomorphism ϕ of the following form can be considered, with P an open neighbourhood of p_0:

$$\phi(x, y, p) = (x + a + f(x, y, p), E(x, b)y + M(x, m) + L(x, \ell)y + g(x, y, p), p) . \qquad (6.2)$$

A space of diffeomorphisms

Motivated by the above, let X_r denote the space of diffeomorphisms of $M \times P$ of the form (6.2). Introduce the norm $|f|_r$ of real analytic function $f \colon M \times P \to \mathbb{R}$ by:

$$|f|_r = \sup_{(M \times P) + r} |\tilde{f}(x, y, p)| ,$$

where $\tilde{f}$ is the complex analytic extension of f to $(M \times P) + r$. The norm $|g|_r$ is defined analogously.

Two diffeomorphisms ϕ_1, ϕ_2 will be considered to be δ-close if:

$$|f_1 - f_2|_r < \delta , \quad |g_1 - g_2|_r < \delta .$$

This defines the compact-open topology on X_r.

The perturbation theorem

Recall the definitions of normal bundle and normal conjugacy from Subsection 4.2, and the definition of the set $\mathbb{R}_c = \mathbb{R}_c(\gamma, \tau)$ from Subsection 3.2.

Theorem 1 ([17]). *Let $\phi^0 \in X_r$ be given by:*

$$\phi^0(x, y, p) = (x + a + \chi, E(x, b)y + M(x, m) + L(x, \ell)y + \psi, p),$$

where $\chi(x, y, p)$ and $\psi(x, y, p)$ have complex analytic extension to $(M \times P) + r$, and which are such that $\chi = \mathcal{O}(|y|)$ and $\psi = \mathcal{O}(|y|^2)$ as $|y| \to 0$.

Let $p_0 = (a_0, b_0, m_0, \ell_0)$ be such that:

$$b_0 = (\cos\theta, \sin\theta), \quad m_0 = 0, \quad \ell_0 = 0,$$

for some $\theta \in [0, 2\pi)$.

Let $\gamma, \tau > 0$ be fixed. Then there is a real analytic codimension-$6k$ manifold $\mathscr{E}$ through p_0 given by:

$$\mathscr{E} = \{p \in P \colon m = 0 \quad \text{and} \quad \ell = 0\},$$

a neighbourhood $\mathscr{N}$ of p_0 in $\mathscr{E}$, and a neighbourhood $\mathscr{V}$ of ϕ^0 in X_r, such that for all $\phi \in \mathscr{V}$, there is a map $\Phi \colon M \times \mathscr{N} \to M \times P$ with the following properties:

1. *Φ is a diffeomorphism onto its image, C^∞-close to the identity map.*

2. *Φ is affine (equal to its normal linear part) in y, and real analytic in x.*

3. *Φ preserves the projection to P, that is, Φ is of the form:*

   $$\Phi(x, y, p) = (\Phi_p(x, y), \Pi(p)),$$

   where $\Phi_p \colon M \times \mathscr{N} \to M$ and $\Pi \colon \mathscr{N} \to P$.

4. *For every $p \in \mathscr{N}_c = \{p \in \mathscr{N} \colon a \in \mathbb{R}_c\}$, the diffeomorphisms Φ_p normally conjugates ϕ_p^0 to $\phi_{\Pi(p)}$ at the invariant circle $S = S^1 \times \{0\} \subset M$.*

REMARK. The theorem states that the following diagram commutes for every $p \in \mathscr{N}_c$:

$$
\begin{array}{ccc}
N(S) & \xrightarrow{\ N(\phi_p^0)\ } & N(S) \\
\Big\downarrow{\scriptstyle N(\Phi_p)} & & \Big\downarrow{\scriptstyle N(\Phi_p)} \\
N(\Phi_p(S)) & \xrightarrow{\ N(\phi_{\Pi(p)})\ } & N(\Phi_p(S))
\end{array}
$$

More intuitively, the theorem can be rephrased as follows: for ϕ sufficiently close to ϕ^0, there are diffeomorphisms $\Phi_p(x, y)$ and $\Pi(p)$, such that the following holds. If $p_* \in \tilde{\mathscr{N}}_c = \Pi(\mathscr{N}_c)$, say $p_* = \Pi(\tilde{p})$, then:

$$\phi_*(x, y) = \phi(x, y, p_*),$$

is conjugated, by $(\Phi_{\tilde{p}})^{-1}$, to:

$$\tilde{\phi}(x, y) = \left(x + a(\tilde{p}) + \chi(x, y, \tilde{p}), \ E(x, b(\tilde{p})) + \psi(x, y, \tilde{p})\right),$$

where $\chi = \mathcal{O}(|y|)$ and $\psi = \mathcal{O}(|y|^2)$.

Note that the set $\tilde{\mathscr{N}}_c$ is a positive measure subset of a (non-unique) codimension-$6k$ manifold (see Fig. 3).
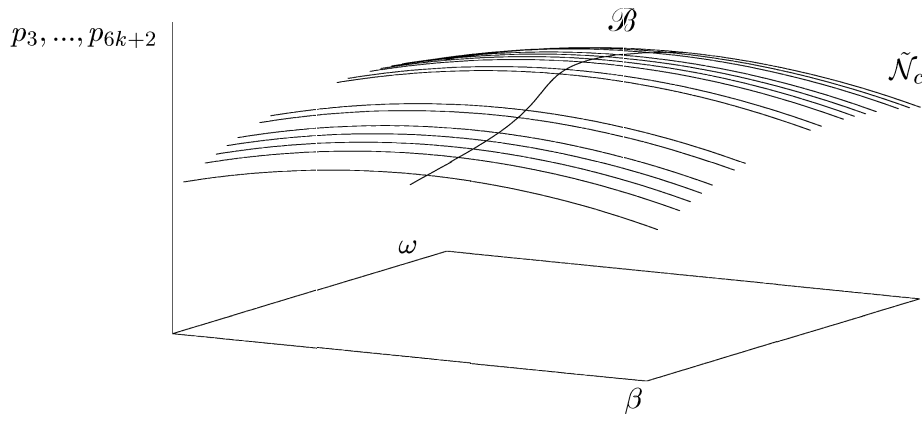
Fig. 3. The parameter space of the unfolding, which is (at least) $6k + 2$-dimensional. Indicated is the large measure subset $\tilde{\mathcal{N}}_c$ of the codimension-$6k$ manifold $\tilde{\mathcal{N}}$ in the space of parameters. For $p \in \tilde{\mathcal{N}}_c$ the family of diffeomorphisms ϕ_p has an invariant quasi-periodic circle. Also indicated is the "bifurcation curve" $\mathscr{B}$: for $p \in \mathscr{B} \cap \tilde{\mathcal{N}}_c$, invariant circles are not normally hyperbolic. Note that this picture also holds for the non-skew case, where $k = 0$.

6.2. Normal form

This subsection presents a normal form for a generically perturbed integrable skew Hopf bifurcation family, and gives a sufficient condition for the existence of an bifurcating invariant 2-torus.

It is convenient to formulate the result in the complex representation, where the phase space M is identified with $S^1 \times \mathbb{C}$ and (x, y) with $(x, z, \bar{z})$, where $z = y_1 + iy_2$. At this point the parameter μ, dropped in the above, is incorporated again. Note that statements are made either on the vertical diffeomorphism ϕ or the parameterized family ϕ_p, which are related by: $\phi(x, y, p) = (\phi_p(x, y), p)$.

If in the notation of the previous subsection $\phi \in \mathcal{V}$ and $p_* \in \tilde{\mathcal{N}}_c$, then there is in a neighbourhood of p_* a coordinate system $(x, y, \omega, \mu, \lambda)$ with the following properties:

1. $p_* = (\omega_*, 0, 0)$ with $\omega_* \in \mathbb{R}_c$.

2. The manifold $\tilde{\mathcal{N}} = \Pi(\mathcal{N})$ is given by:
$$\lambda = 0\,,$$
where $\lambda \in \mathbb{R}^{6k}$.

3. If $\lambda = 0$ and $\omega_* \in \mathbb{R}_c$, the diffeomorphism $\phi_{\omega_*, \mu, 0}$ has an invariant circle:
$$S = \{(x, y) \in M : y = 0\}\,.$$

   On the normal bundle $N(S) \cong S^1 \times \mathbb{C}$, the dynamics of $N(\phi_{\omega_*, \mu, 0}$ are given by:
$$(x, z) \mapsto \left(x + \omega, \beta(\mu)e^{ikx}z\right)\,.$$

Hence in these coordinates, for $\omega_* \in \mathbb{R}_c$ fixed, the diffeomorphism $\phi(x, y, \omega_*, \mu, 0)$ takes the following form in the complex representation:
$$\phi(x, z, \bar{z}, \mu) = \left(x + \omega + \chi(x, z, \bar{z}, \mu), \beta e^{ikx}z + \psi(x, z, \bar{z}, \mu), c.c.\right)\,.$$

Here $c.c.$ stands for $c$omplex $c$onjugate (of the previous component); the maps χ and ψ are $\mathcal{O}(|z|)$ and $\mathcal{O}(|z|^2)$ respectively. For this diffeomorphism a normal form is found by the following theorem.

Normal form theorem [61] *For every $N \geqslant 2$ there is a neighbourhood U_N of $S = S^1 \times \{0\} \subset M$, and a smooth coordinate transformation Φ^N defined on U_N, such that $\phi^N = \Phi^N \circ \phi \circ \left(\Phi^N \right)^{-1}$ is of the form:*

$$\phi^N(x, z, \bar{z}, \mu) = \left(x + \omega + F + R_1, \beta e^{ikx} z + G + R_2, c.c. \right) .$$

where:

$$F(x, z, \bar{z}, \mu) = \sum_{n=1}^{N} \sum_{|m|=n} c_m(x, \mu) z^{m_1} \bar{z}^{m_2} , \qquad G(x, z, \bar{z}, \mu) = \sum_{n=2}^{N} \sum_{|m|=n} d_m(x, \mu) z^{m_1} \bar{z}^{m_2} .$$

Let $A = |k(m_1 - m_2)|$ and $B = |k(m_1 - m_2 - 1)|$. The coefficient functions c_m and d_m are of the form:

$$c_m(x, \mu) = c_m^0(\mu) \quad \text{if } A = 0 ; \qquad d_m(x, \mu) = d_m^0(\mu) \quad \text{if } B = 0 ;$$

and:

$$c_m(x, \mu) = \sum_{\ell=1}^{A} c_m^\ell(\mu) e^{i\ell x} \quad \text{if } A \neq 0 ; \qquad d_m(x, \mu) = \sum_{\ell=1}^{B} c_m^\ell(\mu) e^{i\ell x} \quad \text{if } B \neq 0 .$$

Moreover, the remainder terms R_j satisfy:

$$|R_j(x, z, \bar{z}, \mu)| \leqslant C \left(|z|^{N+1} + |\mu|^{N+1} \right) .$$

Note that if the R_j are identically zero, and if also the c_m^ℓ, d_m^ℓ vanish for $\ell \neq 0$, the diffeomorphism ϕ^N is an integrable skew Hopf family. Moreover, the parameter β can be assumed to be real-valued, if necessary after applying a parameter transformation:

$$x = \tilde{x} - \frac{\arg \beta}{k} .$$

6.3. Bifurcation analysis

The results of the previous two subsections are applied to a small generic perturbation of an integrable skew Hopf family, nondegenerate at $p = 0$:

$$\phi_p(x, y) = \left(x + \omega(p) + f(|y|^2, p) + \tilde{f}(x, y, p), \left(\beta(p) + g(|y|^2, p) \right) E_k(x) + \tilde{g}(x, y, p) \right) ,$$

where f, g, $\tilde{f}$ and $\tilde{g}$ are real analytic, such that $|\tilde{f}|_r$, $|\tilde{g}|_r \leqslant \delta$.

Let:

$$\tilde{\phi}_\sigma(x, y) = \phi_p(x, y) + (0, M(x, m) + L(x, \ell)y) ,$$

be the unfolding of ϕ_p introduced in Subsection 6.1. If δ is small enough, and if ϕ_σ is non-degenerate, on some neighbourhood $U \times P$ of $S \times \{\sigma = 0\} \subset M \times P$, the diffeomorphism $\tilde{\phi}_\sigma$ can be conjugated to:

$$\phi_\sigma^N(x, y) = \left(x + \omega + f^N(|y|^2, \sigma) + \tilde{F}(x, y, \sigma) + R_1(x, y, \sigma), \right.$$
$$\left. (\beta + g^N(|y|^2, \sigma)) E_k(x) y + \tilde{G}(x, y, \sigma) + R_2(x, y, \sigma) \right) ,$$

The normal form ϕ_σ^N has the following properties:

1. There is a codimension-$6k$ manifold $\tilde{\mathcal{N}}$ in the parameter space of ϕ_σ^N, and a positive measure subset $\tilde{\mathcal{N}}_c$ on $\tilde{\mathcal{N}}$, such that for $\sigma_* \in \tilde{N}_c$, the map $\phi_{\sigma_*}^N$ has an invariant circle.

2. If $\sigma = (\omega, \mu, \lambda)$ are local coordinates around σ_* such that $\sigma_* = (\omega_*, 0, 0)$, $\mu = (\beta - 1, \mu_2, \dots)$ and such that points satisfying $\lambda = 0$ are in $\tilde{\mathcal{N}}$, then $\omega_* \in \mathbb{R}_c$, and:

$$|R_j(x, y, \omega_*, \mu, 0)| \leqslant C \left(|y|^{N+1} + |\mu|^{N+1} \right) .$$

3. In the local parameters of the previous number:

$$\tilde{F}(x, y, \omega_*, \mu, \lambda) = F(x, y, \mu), \qquad \tilde{G}(x, y, \omega_*, \mu, \lambda) = G(x, y, \mu),$$

where F and G are polynomials in y and trigonometric polynomials in x: they are the real valued analogs of the F and G in the normal form theorem above, with all c_m^0, d_m^0 identically zero.

4. $|f^N - f|, |g^N - g| < C\delta$.

It follows from these properties that on $\tilde{\mathcal{N}}_c$ there is a bifurcation set $\mathcal{B}_c$ given by:

$$\mathcal{B}_c = \{\sigma \in \tilde{\mathcal{N}}_c \colon \beta = 1\}.$$

This set can be extended to be the intersection of a smooth submanifold of codimension 1 in $\mathcal{N}$ with $\tilde{\mathcal{N}}_c$.

To find a sufficient condition for the existence of a bifurcating torus, consider the map:

$$D(\sigma) = \left(\tilde{d}_{20}^1(\sigma), \ldots, \tilde{d}_{02}^{6k}(\sigma)\right),$$

having as components the $10k$ real valued coefficient functions $\tilde{d}_m^\ell$ of the terms in $G(x, y, \sigma)$ which are of second order in y_1, y_2.

If $D(\sigma) = 0$, all terms in $G(x, y, \sigma)$ of second order in y are zero. The condition:

$$\text{rank } \frac{\partial D}{\partial \sigma} \text{ is maximal}, \qquad (6.3)$$

is generic. Assuming (6.3), a submanifold $\mathcal{T}$ of codimension $10k$ of P is defined by:

$$\mathcal{T} = \{\sigma \in P \colon D(\sigma) = 0\}.$$

It is shown in [61] that if:

$$\sigma \in \mathcal{N}_c \cap \mathcal{T},$$

there is an invariant 2-torus bifurcating off the invariant circle S. For parameters σ such that the invariant torus exists, it is normally hyperbolic; since normal hyperbolicity is an open property [34], there is an open set V such that for $\sigma \in V$ there is an invariant 2-torus. Note, however, that V may have the form of a narrow wedge with vertex at $\mathcal{B}_c$.

Dynamics on the torus have now the general form:
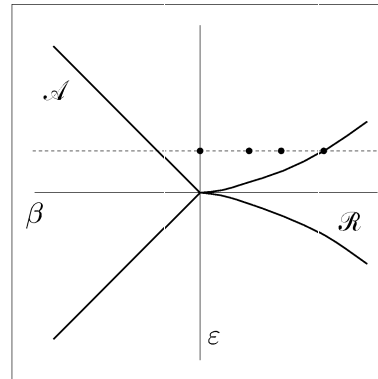
Fig. 4. The (β, ε)-parameter space for the model diffeomorphism $\phi_{\beta,\varepsilon}$ given by (6.4). The form of the regions $\mathcal{A}$ of attracting and $\mathcal{R}$ of repelling invariant circles is indicated, as well as the line $\varepsilon = 0.1$. On that line, in the complement of $\mathcal{A} \cup \mathcal{R}$, four values of β are given by black dots: in Fig. 6 the forward orbit of an arbitrary point under iteration of $\phi_{\beta,\varepsilon}$ is shown for these values.

$$\psi_\sigma(x, s) = (x + \omega + f(x, s, \sigma), s + kx + g(x, s, \sigma)),$$

where f and g are small for $\sigma \in V$ close to B_c. It is not clear a $priori$ whether skew dynamics, or indeed mixed spectrum, persist in some sense. We conjecture that they do not in general.

6.4. Fattening

The previous subsection inferred from the normal hyperbolicity of the bifurcating torus on a subset of P the existence of normal hyperbolic tori on an *open* subset of the parameter space. This illustrates a general principle, which is called *fattening*. However, the open sets obtained in this way may be very small.

First, remark that the remainder terms R_1 and R_2 of $\phi_\sigma^N = \phi_{\omega,\mu,\lambda}^N$ vanish if $\omega \in \mathbb{R}_c$. As a closed set, $\mathbb{R}_c$ is the union of a countable set and a perfect set. Hence, almost every point of $\mathbb{R}_c$ is an accumulation point of the set. It follows that the first derivative of R_1 and R_2 with respect to ω vanishes for these points, and by induction the higher derivatives vanish as well. Restricted to $\lambda = 0$, it can be concluded that the remainders R_j are *flat* in ω in the neighbourhood of almost every $\omega \in \mathbb{R}_c$. Let $\mathscr{A}$ and $\mathscr{R}$ be subsets of parameter space such that ϕ_σ^N has, respectively, an attracting or a repelling invariant circle, As in [11] it follows that, restricted to $\lambda = 0$, $\mathscr{A}$ and $\mathscr{R}$ have infinite order of contact at the bifurcation set $\mathscr{B}_c$. See Fig. 5, lower right hand corner.

The situation in the normal direction ($\lambda \neq 0$) is quite different. To give an idea of the shape of the set of parameters for which an invariant circle exists for a perturbed skew Hopf bifurcation, in [61] the following model system has been considered:

$$\phi_{\beta,\varepsilon} = \left(x + \omega, \left(\beta - |y|^2\right) E_k(x)y + \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix} y + \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix} \right) . \tag{6.4}$$

It has been shown there that:

1. For $|\varepsilon| \leqslant \beta - 1$ (region $\mathscr{A}$), the map $\phi_{\beta,\varepsilon}$ has an invariant attracting circle.

2. For $|\varepsilon| \leqslant f(\beta)$ (region $\mathscr{R}$), where $f^{-1}(\beta) = 1 + 3\left(\dfrac{\beta}{2}\right)^{\frac{2}{3}} + \beta$, the map $\phi_{\beta,\varepsilon}$ has a repelling invariant circle.

These regions are indicated in Fig. 4. Note that the Conditions 1 and 2 are sufficient to have an invariant hyperbolic circle. The actual region where hyperbolic circles exist may be larger, but compare Appendix A.

In order to obtain an idea of the dynamics outside these regions, the attracting set of $\phi_{\beta,\varepsilon}$ has been simulated by forward numerical iteration of a random initial point, discarding the initial $5 \cdot 10^4$ iterates. This has been done for $\omega = \sqrt{2} - 1$, $\varepsilon = 0.1$ and $\beta = 1$, 1.1, 1.15 and 1.24 respectively (see Fig. 6). The last value of β is chosen such that it is inside a region where a repelling invariant circle is known to exist — hence the simulated set, which might have a resemblance to the true invariant set, cannot be homeomorphic to a circle.

6.5. Resonances

This subsection presents conclusions reached in [60] on the following model skew Hopf bifurcation system at lowest order resonance (that is, with (ω,μ) close to $(0,0)$):

$$\phi_p(x,y) = \left(x + \omega + a\sin x + c|y|^2 + f(x,y,p), \left(1 + \mu + b\cos x + d|y|^2\right) E_k(x)y + g(x,y,p) \right) .$$

Here f, g are small generic perturbations of the model system considered in Subsection 5.2.

The organizing bifurcations in the non-symmetric system are the following well-known codimension two bifurcations of diffeomorphisms which occur generically: the saddle-Hopf bifurcation (see [25], [30], [58]) and the codimension two degenerate Hopf bifurcations for diffeomorphisms (see [20], [21], [22]). The Bogdanov–Takens bifurcation (for diffeomorphisms!) of invariant circles is also an organizing bifurcation. However, the theory of this bifurcation is not fully developed yet, and the
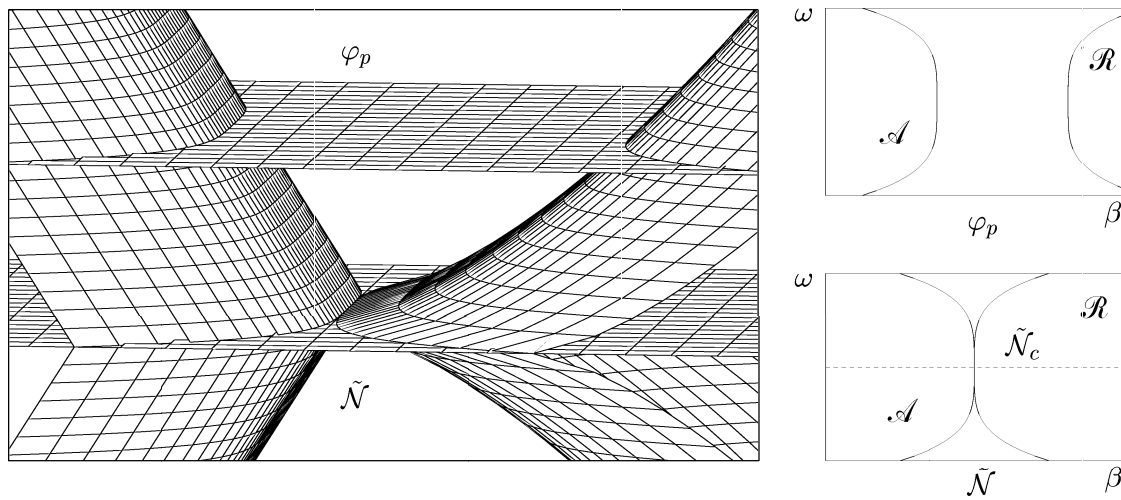
Fig. 5. A sketch of the parameter space of the unfolding $\tilde{\phi}_\sigma$ of ϕ_p (notations as in Subsection 6.3). The axes of the left picture are the same as in Fig. 3. The manifold $\tilde{\mathcal{N}}$ is indicated. The curved surfaces left and right bound a pair of blunt cusps of the "fattened" set $\mathscr{A} \cup \mathscr{R}$ of hyperbolic invariant circles. The original family ϕ_p (here assumed to be two-dimensional), which is a subfamily of the unfolding, is indicated as well. On the right, the intersections of $\tilde{\mathcal{N}}$ and the parameter space of ϕ_p with $\mathscr{A} \cup \mathscr{R}$ are sketched. The dashed line is in the set $\tilde{\mathcal{N}}_c$ for which quasi-periodic invariant circles exist.

reader is referred to papers on the Bogdanov–Takens bifurcation of invariant points in vector fields and diffeomorphisms ([6], [25], [30], [51], [57]).

All the dynamical features of these bifurcations are present here, homoclinic and heteroclinic connections, sensitive dynamics, chaos and Aubry–Mather invariant sets (Cantori).

The curves of Hopf bifurcations of diffeomorphisms (Naimark–Sacker bifurcations) persist: every normal rotation number occurs exactly k times, including the weakly and strongly resonating cases, because of the essential dependence of the normal rotation on the torus coordinate x. Hence at a dense set of points (the weakly resonating cases) on the Naimark–Sacker bifurcation curve, two saddle-node bifurcation curves emanate, bounding a so-called *resonance tongue*. Also, all strongly resonating cases occur k times (see [30], [37], [57]).

A. Conjecture

This appendix conjectures a more general perturbation theorem than the one in Subsection 6.1. The relevance of the result, if it were true, is illustrated.

A.1. A conjectured perturbation theorem

Recall the definitions of section 6.1, the definitions of normal bundle and normal conjugacy from Subsection 4.2, and the definition of the set $\mathbb{R}_c = \mathbb{R}_c(\gamma, \tau)$ from Subsection 3.2.

Conjecture. *Let $\phi^0 \in X_r$ be given by:*

$$\phi^0(x, y, p) = (x + a + \chi, E(x, b)y + M(x, m) + L(x, \ell)y + \psi, p),$$

where $\chi(x, y, p)$ and $\psi(x, y, p)$ are real analytic functions having complex extension to $(M \times p) + r$, which are $\mathcal{O}(|y|)$ and $\mathcal{O}(|y|^2)$ respectively as $|y| \to 0$.

Let $p_0 = (a_0, b_0, m_0, \ell_0)$ be such that:

$$b_0 = (\cos\theta, \sin\theta).$$

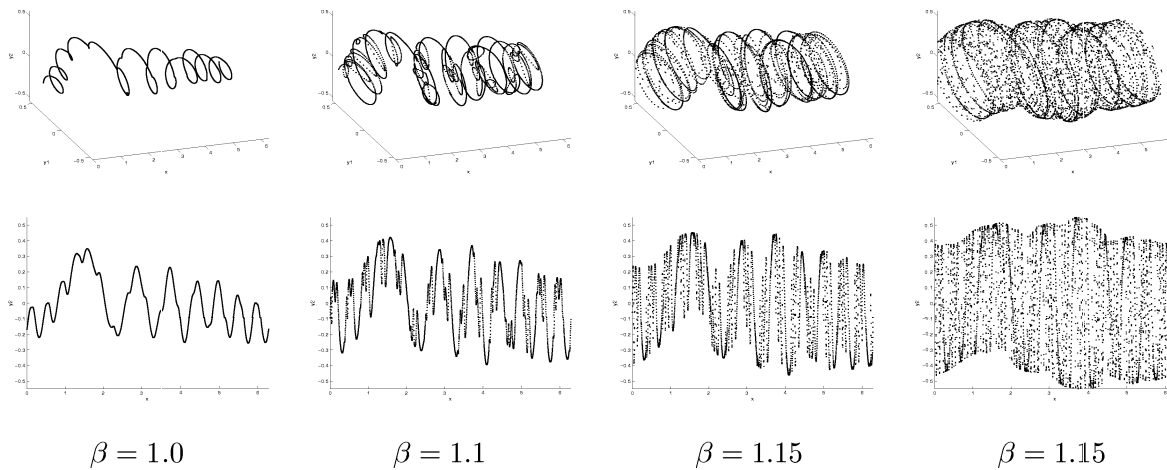$\beta = 1.0$ $\beta = 1.1$ $\beta = 1.15$ $\beta = 1.15$

Fig. 6. Simulation of the dynamics of $\phi_{\beta,\varepsilon}$ outside the region $\mathscr{A} \cup \mathscr{R}$. The parameters have been chosen as follows: $\omega = \sqrt{2} - 1$ and $\varepsilon = 0.1$. For the values of β indicated, the point $(x, y) = (0, 0)$ has been iterated $5 \cdot 10^4$ times, and then $6 \cdot 10^3$ points of the trajectory have been plotted. The top row gives a three-dimensional view of these trajectories, the bottom row shows a side view.

for some $\theta \in [0, 2\pi)$.

Let $\gamma, \tau > 0$ be fixed. Then there is a neighbourhood $\mathscr{N}$ of p_0, and a neighbourhood $\mathscr{V}$ of ϕ^0 in X_r, such that for all $\phi \in \mathscr{V}$, there is a map $\Phi \colon M \times \mathscr{N} \to M \times P$ with the following properties:

1. *Φ is a diffeomorphism onto its image, C^∞-close to the identity map.*

2. *Φ is affine (equal to its normal linear part) in y, and real analytic in x.*

3. *Φ preserves the projection to P, that is, Φ is of the form:*

$$\Phi(x, y, p) = (\Phi_p(x, y), \Pi(p)) \ ,$$

   where $\Phi_p \colon M \times \mathscr{N} \to M$ and $\Pi \colon \mathscr{N} \to P$.

4. *For every $p \in \mathscr{N}_c = \{p \in \mathscr{N} : a \in \mathbb{R}_c\}$, the diffeomorphisms Φ_p normally conjugates ϕ_p^0 to $\phi_{\Pi(p)}$ at the invariant circle $S = S^1 \times \{0\} \subset M$.*

A.2. Lyapunov exponents

The previous subsection conjectured that skew Hopf families of the following type may be persistent on a subset of large measure in the space of parameters:

$$\phi_\sigma(x, y) = (x + \omega, \beta E_k(x)y + M(x, m) + L(x, \ell)y) \ . \tag{A.1}$$

Here $M(x, m)$ and $L(x, \ell)$ are trigonometric polynomials in x, and linear in their respective parameters m and ℓ.

If $m = 0$, the circle $S = S^1 \times \{0\}$ is invariant, and the map ϕ_σ can be seen as a cocycle for which Lyapunov exponents can be defined (an *ad hoc* definition of Lyapunov exponents in the present case follows below). The rest of this subsection will show that if $m = 0$ and $\ell \neq 0$, the two Lyapunov exponents of ϕ_σ are not equal.

The map $L(x, \ell)$ is written as:

$$L(x, \ell) = \begin{pmatrix} L_1(x) & L_2(x) \\ L_2(x) & L_1(x) \end{pmatrix} \ .$$

Introduce:
$$A(x) = \beta E_k(x) + L(x, \ell) \,,$$

and let $A_n(x)$ denote:

$$A_n(x) = A(x + (n-1)\omega)A(x + (n-2)\omega) \dots A(x) \,.$$

Note that with this definition, the n'th iterate of ϕ takes the form:

$$\phi^n(x, y) = (x + n\omega, A_n(x)) \,.$$

The *upper Lyapunov exponent* $\lambda_+(x)$ of the normally linear diffeomorphism ϕ is given by:

$$\lambda_+(x) = \limsup_{n \to \infty} \frac{1}{n} \log \|A_n(x)\| \,,$$

where $\|A\|$ is the spectral radius matrix norm (the square root of the largest eigenvalue in modulus of A^*A).

The *lower Lyapunov exponent* is defined implicitly by:

$$\lambda_+(x) + \lambda_-(x) = \limsup_{n \to \infty} \frac{1}{n} \log |\det A_n(x)| \,.$$

If ω is non-resonant λ_+ and λ_- do not depend on x.

A result of Herman [33] implies that the upper Lyapunov exponent can be estimated from below by:

$$\lambda_+ \geqslant \log \sqrt{\beta^2 + (\ell_{-k}^1)^2 + (\ell_{-k}^2)^2} \geqslant \log \beta \,.$$

On the other hand, the ergodic theorem yields that:

$$\lambda_+ + \lambda_- = \frac{1}{2\pi} \int\limits_0^{2\pi} \log |\det A(x)| \partial x = \frac{1}{2\pi} \int\limits_0^{2\pi} \log \left| \beta^2 - L_1^2(x) - L_2^2(x) \right| \partial x < 2 \log \beta \,.$$

Note that the last inequality holds only if $\ell \neq 0$. Hence, if $\ell \neq 0$, then:

$$\lambda_+ \geqslant \log \beta > \lambda_- \,.$$

This inequality of Lyapunov exponents has interesting consequences: by Oseledec's theorem, it is known that there are two measurable invariant normal bundles of S, spanning $S \times \mathbb{R}^2$, associated to λ_+ and λ_- respectively. However, because of essential non-reducibility, these bundles cannot be continuous (basically because to a continuous normal bundle a linking number with respect to S can be associated, which increases under iteration of the diffeomorphism). It is not known whether quasi-periodic invariant circles with this kind of normal dynamics persist under small perturbations. If the conjecture above is true they do persist, and this would lead to a new type of bifurcations.

Acknowledgments

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