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Maes, Christian; Redig, Frank; Takens, Floris; Moffaert, Annelies van; Verbitski, Evgeny

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Intermittency and weak Gibbs states

Christian Maes†, Frank Redig†, Floris Takens‡, Annelies van Moffaert† and
Evgeny Verbitski‡

† Instituut voor Theoretische Fysica K U Leuven, Celestijnenlaan 200D, B-3001 Leuven, Belgium
‡ Department of Mathematics, University of Groningen, PO Box 800, 9700 AV, Groningen,
The Netherlands

E-mail: `christian.maes@fys.kuleuven.ac.be`, `frank.redig@fys.kuleuven.ac.be`,
`f.takens@math.rug.nl`, `annelies.vanmoffaert@fys.kuleuven.ac.be` and
`e.verbitski@math.rug.nl`

Abstract. We show that the natural invariant state for Manneville–Pomeau maps can be characterized as a weakly Gibbsian state. In this way we make a connection between the study of intermittency via non-uniformly expanding maps and the thermodynamic formalism for non-uniformly convergent interactions.

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1. Introduction

In this paper we connect the notion of a weakly Gibbsian state, as has recently emerged from the statistical mechanical study of certain lattice spin systems, with the concept of intermittency, as modelled by Manneville–Pomeau maps.

Weakly Gibbsian states were introduced by Dobrushin in his last conference talk in Renkum [6]. What was sought was a Gibbsian restoration of certain physically relevant examples of non-Gibbsian states. A first part of the Dobrushin programme has been recently completed in [22] where it is shown that essentially all restrictions to a sublattice of the low-temperature phases in the realm of the Pirogov–Sinai theory for lattice spin systems are weakly Gibbsian. The typical scenario is the occurrence of a 'configuration-dependent range of the interaction'. This implies that the relative energies are no longer uniformly bounded (as is the case for the usual Gibbsian set-up) but can be unbounded as dictated by configuration-dependent length scales. This divides the set of lattice spin configurations into two disjoint sets: the 'good' ones for which the effective interaction is short range, and the 'bad' ones, for which the total interaction is diverging. Instead of introducing the somewhat abstract formalism defining weakly Gibbsian states, we refer to [23] for general definitions and properties and we only underline the above via a concrete and, for our purposes, illustrative example.

1.1. Example of a weakly Gibbsian state

Consider the standard ferromagnetic Ising model on the square lattice $\mathbb{Z}^2$ with the usual nearest-neighbour interactions. The finite-volume Gibbs measure $\mu_{\beta,n}$ on a box Λ_n with plus boundary

conditions is defined on the infinite-volume Ising configuration space $\Omega = \{+1, -1\}^{\mathbb{Z}^2}$ via

$$\mu_{\beta,n}(\sigma) = \frac{I[\sigma = \bar{1} \text{ on } \Lambda_n^c]}{Z_n(\beta)} \exp\left[\beta \sum_{\langle xy \rangle \cap \Lambda_n \neq \emptyset} \sigma_x \sigma_y\right]$$

where $\Lambda_n = [-n, n]^2 \cap \mathbb{Z}^2$ is a finite box, $I[\sigma = \bar{1} \text{ on } \Lambda_n^c]$ is the indicator of the event that $\sigma_z = +1$ for all $z \in \Lambda_n^c = \mathbb{Z}^2 \setminus \Lambda_n$, $\beta > 0$ denotes the inverse temperature, $Z_n(\beta)$ is the normalizing partition function and the sum in the exponent is over all nearest-neighbour pairs $\langle xy \rangle$ at least one of which is in the box Λ_n. It is well known that the weak limit $\lim_n \mu_{\beta,n} = \mu_\beta$ exists for all β. This limit is a translation-invariant Gibbs measure at inverse temperature β for the formal Hamiltonian

$$H(\sigma) = -\sum_{\langle xy \rangle} \sigma_x \sigma_y.$$

Of course, this sum is only well defined in terms of the corresponding relative energies

$$H(\sigma) - H(\eta) = H_\Lambda(\sigma) = -\sum_{\langle xy \rangle \cap \Lambda \neq \emptyset} (\sigma_x \sigma_y - \eta_x \eta_y)$$

defined for a finite region $\Lambda \subset \mathbb{Z}^2$ and $\eta = \sigma$ on Λ^c (η coincides with σ outside a finite volume). That μ_β is a Gibbs measure for H means that its conditional probabilities are described via these relative energies as, for example, in

$$\mu_\beta(\sigma_0 | \sigma_x, x \neq 0) = \frac{1}{1 + \exp\left[-2\beta \sum_{\langle x0 \rangle} \sigma_0 \sigma_x\right]}.$$

In particular, the measure μ_β admits a continuous version of its conditional probabilities.

We are interested in the restriction ν_β of this infinite-volume probability measure μ_β to a lattice line (to be identified with $\mathbb{Z}$), say one which contains the origin. It was proven in [9, 30] that at low temperatures (β sufficiently large) ν_β is not Gibbsian, i.e. does not admit a continuous version of its conditional probabilities. It was, however, realized by Dobrushin that ν_β remains weakly Gibbsian. This means the following. There exists a translation-invariant tail-set $K \subset \{+1, -1\}^{\mathbb{Z}}$ of 'good' one-dimensional lattice spin configurations which has full measure ($\nu_\beta(K) = 1$) and for which one can find a translation-invariant interaction potential (U_A), which is absolutely summable on K and is compatible with ν_β, i.e. the interaction potential is a collection of functions $U_A : \{+1, -1\}^A \to \mathbb{R}$ parametrized by the finite subsets A of $\mathbb{Z}$, for which

$$\sum_{A \ni 0} |U_A(\xi)| < \infty \qquad \xi \in K$$

(absolute convergence) and for which the Dobrushin–Lanford–Ruelle (DLR) equations with respect to ν_β are satisfied:

$$\int f(\xi)\, \mathrm{d}\nu_\beta(\xi) = \int \sum_{\omega \in \{+1, -1\}^V} \frac{1}{Z_V^U(\xi^c)} f(\omega_V \xi_{V^c}) \exp\left[-\sum_{A \cap V \neq \emptyset} U_A(\omega_V \xi_{V^c})\right] \mathrm{d}\nu_\beta(\xi) \quad (1.1)$$

for all continuous functions f on $\{+1, -1\}^{\mathbb{Z}}$ for all finite $V \subset \mathbb{Z}$, and where $\omega_V \xi_{V^c}$ is a configuration, which coincides with ω on V, and with ξ on V^c.

In this sense, ν_β is weakly Gibbsian for the formal Hamiltonian

$$H(\xi) = \sum_A U_A(\xi)$$

but now, even the local Hamiltonians

$$H_V^U(\xi) = \sum_{A \cap V \neq \emptyset} U_A(\xi)$$

are only well defined for $\xi \in K$.

The proof of this result (i.e. the existence of a tail-set K) was given in [6, 7, 24, 25] with a more general version in [22]. It turns out that one can choose the potential (U_A) so that it is non-vanishing only for A a lattice interval. In particular, one shows that for every $\xi \in K$ there is a (configuration-dependent length) $\ell(\xi) < +\infty$ for which

$$|U_{[0,k]}(\xi)| \leqslant c_1 I[k \leqslant \ell(\xi)] + c_2 \exp[-c_3 k]$$

for all $k > 0$ and where the finite constants c_1, c_2, c_3 depend on β. In other words, the potential starts decaying only after a 'random' distance which is itself a function of the configuration. That is the meaning of saying that the interaction is effectively short ranged with a 'configuration-dependent interaction range'. In the model, this range $\ell(\xi)$ measures the distance to the right of the origin after which the proportion of +1-spins to the right of the origin becomes forever larger than a given (large) amount. It is this structure of the interaction that reminds us of the phenomenon of intermittency in the theory of dynamical systems.

1.2. Intermittency

Since the beginning of the 1980s intermittency has been widely studied as a common phenomenon in the transition to turbulence [4]. While it is difficult to give a good definition, its simplest manifestation is probably the occurrence of randomly spread bursts or fluctuations happening between periods where the system undergoes a limit cycle or periodic motion. While varying some control parameter, the average frequency of these fluctuations becomes larger and larger. Here we will not discuss the nature of this intermittent regime except to investigate some Gibbsian aspects of the steady state for some model systems.

To see what we have in mind, it is best to start from so-called (uniformly) expanding interval maps. Under some additional smoothness conditions, there is a unique ergodic absolutely continuous time-invariant measure. Its density is a continuous function bounded away from zero. The standard Gibbs formalism can be applied and an exponentially decaying interaction can be identified with which this invariant measure is compatible. Imagine now what happens if an indifferent fixed point appears. In the neighbourhood of this point the expansion of the map shrinks to zero, because the derivative in the indifferent fixed point is equal to one. This non-uniformity in the expansion has as a consequence that the system can stay for longer times in the neighbourhood of this fixed point before it is expelled to a region where the map is again truly expanding. These fluctuations are rare but are nevertheless responsible for breaking the uniform convergence of an associated interaction potential. It is this feature that we study here.

We start in the next section with the introduction of the simplest models. Section 3 is devoted to the presentation of our main result: the weakly Gibbsian character of the absolutely continuous invariant measure.

2. Model. Interval maps with indifferent fixed points

2.1. Model

We study the following class of non-uniformly expanding interval maps.

Definition 2.1. *We say that $T : [0, 1] \to [0, 1]$ is a Manneville–Pomeau-type map (an MP map) if:*
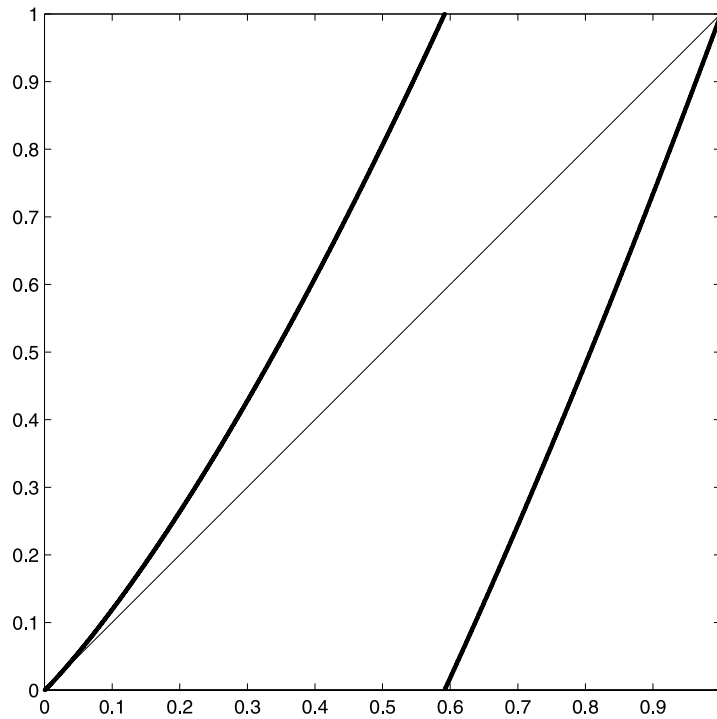
Figure 1. A Manneville–Pomeau-type map.

(a) *T is piecewise monotonic with two full branches, i.e. there exists a p > 0 such that $T|_{(0,p)}$ and $T|_{(p,1)}$ are strictly monotonic, continuous and $T(0, p) = T(p, 1) = (0, 1)$;*

(b) *the branches $T|_{(0,p)}$ and $T|_{(p,1]}$ are C^2;*

(c) *$T'(x) > 1$ for all $x > 0$ and $T'(x) \geqslant \lambda > 1$ for $x \in (p, 1)$.*

(d) *T has the following asymptotic behaviour when $x \to 0_+$:*

$$T(x) = x + Cx^{1+\alpha}(1 + u(x))$$

for some constants $C > 0$, $\alpha \in (0, 1)$, and u is a C^2 function such that

$$\lim_{x \to 0_+} u'(x) = \lim_{x \to 0_+} u''(x) = 0.$$

As an example we can consider the original Manneville–Pomeau map itself (see figure 1), defined as follows:

$$T(x) = x + x^{1+\alpha} \quad \text{mod} \, 1.$$

It is easy to see that (c) and (d) imply that 0 is a unique indifferent fixed point.

2.2. Absolutely continuous invariant measure, ergodic properties

Pianigiani [26], using the first return map, established the existence of absolutely continuous invariant measures for MP maps.

The constructed absolutely continuous T-invariant measure μ for the MP maps is a Sinai–Ruelle–Bowen measure: for almost every $x \in [0, 1]$ with respect to the Lebesgue measure one has the weak convergence

$$\frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i(x)} \to \mu$$

where δ_y is the Dirac measure at y.

Thaler [31, 32] has proven the following estimates on the density $h(x) = \mathrm{d}\mu/\mathrm{d}x$ for MP maps: there exist constants $C_*, C^* \in (0, \infty)$ such that

$$\frac{C_*}{x^\alpha} < h(x) < \frac{C^*}{x^\alpha} \qquad \text{for all} \quad x > 0. \tag{2.1}$$

It is also not very difficult to see that the dynamical system $([0, 1], \mu, T)$ is exact:

$$\lim_{n \to \infty} \mu(T^n(A)) = 1$$

for all measurable sets A with $\mu(A) > 0$, implying ergodicity and mixing.

Determining a rate of mixing (or decay of correlations) for the MP-type maps attracted a lot of attention. This problem has been studied in [14, 15, 21, 36]. It turns out that for the Manneville–Pomeau-type maps one has a polynomial decay of correlations: for sufficiently smooth f, g (say, Hölder continuous)

$$|\rho(n)| = \left| \int f(x) g(T^n(x)) \, \mathrm{d}\mu - \int f(x) \, \mathrm{d}\mu \int g(x) \, \mathrm{d}\mu \right| = \mathrm{O}(n^{-1/\alpha+1}).$$

It should be mentioned that there are several other possibilities for piecewise-monotonic interval maps with indifferent fixed points to have a finite absolutely continuous invariant measure even with bounded density, e.g. [17, 38]. What is important is that the presence of an indifferent periodic point (i.e. $T^p(x) = x$ and $|(T^p)'(x)| = 1$ for some $x \in [0, 1]$ and $p \geqslant 1$) should be compensated by some singularities of the first or second derivative of T, because if the map is C^2, only infinite absolutely continuous T-invariant measures exist, see [3].

2.3. Thermodynamic formalism

Formally, an MP map T is not a continuous transformation of a compact metric space (the interval $[0, 1]$). However, we can make one from T. This is done by doubling the point of discontinuity p, i.e. substituting it by two points p_- and p_+, such that $p_- < p_+$, and putting $T(p_-) = \lim_{x \uparrow p} T(x)$ and $T(p_+) = \lim_{x \downarrow p} T(x)$. We repeat the procedure with all the preimages of p_- and p_+. In this way we obtain an 'enlarged' space X, which is totally ordered and order complete. Moreover, X is a compact space. In this new space X, the intervals $\bar{I}_i$ form a partition. X has points which are isolated from one side, but there are no completely isolated points. Since at most a countable number of points are affected by this operation, and since we are studying measures that are absolutely continuous with respect to the Lebesgue measure, the described modifications take place on a set of measure 0, and are therefore irrelevant from a measure-theoretic point of view. Note also, that this operation makes the coding map $\pi : \{0, 1\}^{\mathbb{Z}_+} \to X$, given by

$$\pi(\omega_0 \omega_1 \omega_2 \ldots) = \bar{I}_{\omega_0} \cap T^{-1} \bar{I}_{\omega_1} \cap T^{-2} \bar{I}_{\omega_2} \cap \ldots$$

a homeomorphism. The coding π conjugates T with the left shift σ on $\Sigma = \{0, 1\}^{\mathbb{Z}_+}$, i.e. $T \circ \pi = \pi \circ \sigma$. For details see [13] and [17, appendix A.5].

Consider the function $\varphi = -\log|T'|$, where $T'(x)$ is the left or right derivative of T at x if x is isolated from the right or left, respectively). The topological pressure of φ is

$$P(\varphi) = \sup_{\nu}\left(h_\nu(T) + \int \varphi \, \mathrm{d}\nu\right),$$

where the supremum is taken over all T-invariant measures and $h_\nu(T)$ is the measure-theoretic entropy (Kolmogorov–Sinai entropy) (see, for example, the variational principle in [34]).

The absolutely continuous invariant measure μ is an equilibrium state for φ, i.e.

$$P(\varphi) = h_\mu(T) + \int \varphi \, \mathrm{d}\mu. \tag{2.2}$$

Since μ is an absolutely continuous invariant measure, the measure-theoretic entropy (Kolmogorov–Sinai entropy) is given by Rokhlin's formula [20]:

$$h_\mu(T) = \int \log|T'| \, \mathrm{d}\mu = -\int \varphi \, \mathrm{d}\mu$$

and hence, $P(\varphi) = 0$. However, μ is not the only equilibrium state. The Dirac measure at 0, which we denote by δ_0, satisfies (2.2) as well. Hence, every measure from the convex hull of μ and δ_0

$$\mathcal{A} = \left\{t\mu + (1-t)\delta_0 \mid t \in [0, 1]\right\}$$

is an equilibrium state. There are no other equilibrium states for φ.

Non-uniqueness of the equilibrium states for φ results in a singular behaviour of the pressure function $P(q\varphi)$, $q \in \mathbb{R}$. Combining the results from [27, p 511] and [33, theorem 3.6] we obtain the following statement on the type of phase transition.

Theorem 2.2. *Let T be an MP map. The pressure function $P(q\varphi)$ is continuous, convex and non-increasing. Moreover, $P(q\varphi) = 0$ for $q \geqslant 1$, $P(q\varphi) > 0$ for $q < 1$, and $P(q\varphi)$ is a real-analytic function of q for $q < 1$. At the critical point one has the following asymptotics:*

$$\frac{P(q\varphi)}{1-q} \to h_\mu(T) \qquad as \quad q \nearrow 1.$$

3. Main results: Gibbs properties of MP maps

3.1. Unbounded distortion

Consider a piecewise-monotonic map T of the unit interval I. Denote by $\{I_k\}$ the intervals of monotonicity of T. Assume that T can be continued up to a C^2 diffeomorphism T_k on the closure of I_k, and $T_k(\bar{I}_k) = [0, 1]$. Assume also that T is expanding, i.e. there exists $\lambda > 1$ such that $|T'(x)| \geqslant \lambda$ for all $x \in I_k$.

Such a map T admits an absolutely continuous invariant measure μ, whose density h is a continuous function bounded away from 0, see [2]. This measure μ has the following useful property: there exists a constant $C > 1$ such that for all $x \in [0, 1]$ and every $n \geqslant 1$ one has

$$\frac{1}{C} \leqslant \frac{\mu\left(I_{i_1,\dots,i_n}(x)\right)}{\exp\left(\sum_{k=0}^{n-1} \varphi\left(T^k(x)\right)\right)} \leqslant C \tag{3.1}$$

where $\varphi = -\log|T'|$ and $I_{i_1,\dots,i_n}(x) = I_{i_1} \cap T^{-1}I_2 \cap \dots \cap T^{-n+1}I_{i_n}$ is that interval of monotonicity for T^n, which contains x.

This property (3.1), which we call Bowen's boundness property, is often taken as a definition of a Gibbs state in dynamical systems. Indeed, the inequalities in (3.1) can be derived from standard definitions of the Gibbs state (see [17] for details). We can also obtain these inequalities from the properties of expanding maps and absolutely continuous measures directly.

First of all, expanding interval maps have the so-called bounded distortion property: there exists some constant $C > 0$ such that for the Lebesgue measure m

$$\frac{1}{C} \leqslant \frac{|(T^n)'(x)|}{|(T^n)'(y)|} \leqslant C \tag{3.2}$$

for all $x, y \in I_{i_1, \ldots, i_n}$. Secondly, $T^n\big(I_{i_1, \ldots, i_n}\big) = (0, 1)$. Thus, from the mean value theorem we conclude that there exists a point $x^* \in I_{i_1, \ldots, i_n}$ such that

$$1 = m\big(T^n I_{i_1, \ldots, i_n}\big) = |(T^n)'(x^*)| m(I_{i_1, \ldots, i_n}).$$

Now, taking into account (3.2) and the fact that the density h is a continuous function bounded away from 0 we obtain (3.1).

Manneville–Pomeau maps do not have the bounded distortion property. This is most clearly seen on the leftmost interval of monotonicity of T^n. This interval contains zero, therefore $\inf_{x \in I_{0,\ldots,0}} |(T^n)'(x)| = 1$. On the other hand, $\sup_{x \in I_{0,\ldots,0}} |(T^n)'(x)| \geqslant 1/m(I_{0,\ldots,0}) \to \infty$ as $n \to \infty$. As a result the ratio

$$\frac{\mu\big(I_{i_1, \ldots, i_n}(x)\big)}{\exp\big(\sum_{i=0}^{n-1} \varphi\big(T^k(x)\big)\big)}$$

is not uniformly bounded in x and n. However, one can find bounds from above and below, which are polynomial in n and uniform in x. This observation (i.e. the violation of (3.1) for the absolutely continuous measure in the case of MP maps) motivated Yuri [37] to call μ weakly Gibbs. We wish to show that μ is indeed a weakly Gibbs measure in the sense of [22, 23].

3.2. Weakly Gibbsian measures for MP maps

Often the natural invariant measures for dynamical systems (such as SRB measures) can be connected with the Gibbs states as they appear in mathematical statistical mechanics. There are two possible ways to do so. The first approach is to follow the prescription of Capocaccia which requires the existence of a well defined relative energy under 'local' transformations. This approach is quite general; it, for example, allows us to establish a thermodynamic formalism even for some systems without Markov partitions, e.g. expansive homeomorphisms with the specification property [12, 29]. We will follow this approach in section 3.2.1.

The second and more traditional way is to use an appropriate (Markov) partition and the symbolic dynamics, and discuss the Gibbsian aspects of the image measure as is usual for lattice spin systems. This approach, for example, allows us to study the properties of the conditional probabilities on finite sets (boxes), given the configuration outside. This will be done in section 3.2.2. In particular, in theorem 3.6 we will construct a symbolic interaction potential for MP maps and show that it is not absolutely convergent, but is convergent on a set of measure 1. Moreover, similar to the projection of the Ising model discussed in the introduction, we will establish that for almost every configuration ω one can find a configuration-dependent length $l(\omega)$ after which the interaction potential decays exponentially. Finally, in theorem 3.7 we will relate the distribution of this length $l(\omega)$ with the decay of correlations for the MP maps.

3.2.1. Gibbs property and multipliers.   We will use a definition of a Gibbs state introduced by Capocaccia in [5], see also [10, 11, 17, 29]. According to this definition, a measure μ is called Gibbs if the result $\tau_* \mu$ of an action by an arbitrary conjugating homeomorphism (defined below) is absolutely continuous with respect to μ and the corresponding Radon–Nykodim derivative, which depends on τ and is called a multiplier for τ, has certain properties. The DLR equations (1.1) can then be rewritten in terms of these multipliers. We recall some definitions from [5, 12, 29].

Definition 3.1. *A continuous transformation T of a compact metric space (X, d) is called expansive if there exists $\gamma > 0$ such that if $d(T^k(x), T^k(y)) < \gamma$ for all $k \geqslant 0$, then $x = y$. Two points $x, y \in X$ are called conjugated if $d(T^k(x), T^k(y)) \to 0$ as $k \to \infty$. Two points $x, y \in X$ are called n-conjugated if $T^n(x) = T^n(y)$.*

Clearly, MP maps are expansive. Also, it is easy to see that if T is an expansive endomorphism, and two points $x, y \in X$ are conjugated, then they are n-conjugated for some $n \in \mathbb{N}$. Therefore, points x and y are conjugated if and only if their symbolic representations $\omega = (\omega_0, \omega_1, \ldots)$, $\omega' = (\omega'_0, \omega'_1, \ldots)$, coincide starting from a certain place, i.e. there exists $n \in N$ such that

$$\omega_k = \omega'_k \qquad \text{for} \quad k \geqslant n.$$

Definition 3.2. *A homeomorphism $\tau : U \to X$, defined on a closed set U, $U \subseteq X$, is called* conjugating, *if x and $\tau(x)$ are conjugated for every $x \in U$.*

Remark. Generally conjugating homeomorphisms do not form a group, but a *pseudogroup*: composition of two conjugating homomorphisms τ', τ'', defined on U' and U'', respectively, can be defined provided $U = (\tau')^{-1}(U'' \cap \tau'(U'))$ is not empty. In this case $\tau = \tau'' \circ \tau'$ is a conjugating homeomorphism defined on U. We will use this observation later, when we discuss the cocycle property of multipliers.

If two points x and y are conjugated, then there is a unique germ of a conjugating homeomorphism mapping a neighbourhood of x into a neighbourhood of y [5, 11]. Those germs form a groupoid [35].

We are going to describe a set of conjugating homeomorphisms $\mathcal{E}$ [17], for MP maps T, using the fact that $T : X \to X$ is topologically conjugated to a one-sided shift $\sigma : \Sigma \to \Sigma$, $\Sigma = \{0, 1\}^{\mathbb{Z}_+}$, by a coding map $\pi : \Sigma \to X$. By definition $\mathcal{E} = \cup_n \mathcal{E}_n$, where $\mathcal{E}_n$ is defined as follows. We say that $\tau \in \mathcal{E}_n$, $n \geqslant 1$, if and only if

(a) there exist $(i_0, \ldots, i_{n-1})$, $(j_0, \ldots, j_{n-1}) \in \{0, 1\}^n$ such that

$$\tau : \bar{I}_{i_0, \ldots, i_{n-1}} \to \bar{I}_{j_0, \ldots, j_{n-1}}$$

(b) for every point $x \in \bar{I}_{i_0, \ldots, i_{n-1}}$ with the symbolic representation

$$\omega = \pi^{-1}(x) = (i_0, \ldots, i_{n-1}, \omega_n, \omega_{n+1}, \ldots)$$

the image $\tau(x) = y$ has a symbolic representation

$$\omega' = \pi^{-1}(y) = (j_0, \ldots, j_{n-1}, \omega_n, \omega_{n+1}, \ldots).$$

This means, that τ alters the first n symbols in the symbolic representation of x. In a compact form τ can be written as follows:

$$\tau(x) = \pi\big((j_0, \ldots, j_{n-1}) \vee \sigma^n(\pi^{-1}(x))\big)$$

where $(j_0, \ldots, j_{n-1}) \vee \sigma^n(\pi^{-1}(x))$ is the concatenation of strings $(j_0, \ldots, j_{n-1})$ and $\sigma^n(\pi^{-1}(x))$.

Now, we give the definition of (weakly) Gibbs states.

Definition 3.3. *Suppose (X, d) is a compact metric space, $T : X \to X$ is a continuous expansive transformation.*

(a) *A family of non-negative functions $\{R_\tau\}$, indexed by all conjugating homeomorphisms $\{\tau \mid \tau : U \to \tau(U)\}$, is called a* family of multipliers *if the following cocycle relation*

$$R_{\tau''}\left(R_{\tau'} \circ \tau''^{-1}\right) = R_{\tau'' \circ \tau'} \tag{3.3}$$

holds on $U = (\tau')^{-1}(\tau'(U') \cap U'')$, whenever U is not empty.

(b) *A measure μ is called* weakly Gibbs *for the family of multipliers $\{R_\tau\}$ if for every conjugating homeomorphism $\tau : U \to \tau(U)$, the push-forward $\tau_*(\mu|_U)$ is absolutely continuous with respect to $\mu|_{\tau(U)}$ and*

$$\frac{d\tau_*(\mu|_U)}{d\mu|_{\tau(U)}} = R_\tau. \tag{3.4}$$

(c) *A measure μ is called* Gibbs *if it is weakly Gibbs for some family of positive and continuous multipliers $\{R_\tau\}$.*

(d) *A function $\varphi : X \to \mathbb{R}$ is called a* dynamical potential *for the measure μ if μ is (weakly) Gibbs and*

$$R_\tau = \exp\left(\sum_{k=0}^{\infty} \varphi(T^k \circ \tau^{-1}(x)) - \varphi(T^k(x))\right). \tag{3.5}$$

Remark.

(a) Condition (3.4) is equivalent to the requirement

$$\int f \circ \tau \, d\mu = \int f R_\tau \, d\mu$$

for all continuous f supported on $\tau(U)$.

(b) If τ is a conjugating homeomorphism, so is τ^{-1}. It is easy to see that due to expansiveness of T two conjugated points x and y are n-conjugated for some $n \geqslant 0$, i.e. $T^n(x) = T^n(y)$. Therefore, the sum in (3.5) is actually finite.

(c) It is also easy to check that any family of functions R_τ obtained from (3.5) is a family of multipliers in the sense of (3.3).

Therefore, in order to decide if a given measure ν is Gibbs or not, we have to understand what happens to ν under the action of all possible conjugating homeomorphisms from $\mathcal{E}$. This seems to be an enormous task. Nevertheless, the problem becomes much easier, if we can relate the measure ν to some transfer operator.

Suppose we are given some Borel bounded and non-negative function $\psi : X \to \mathbb{R}$. Define the corresponding (Ruelle's) transfer operator $\mathcal{L}_\psi$, acting on bounded Borel measurable functions, as follows:

$$\mathcal{L}_\psi f(x) = \sum_{y:T(y)=x} \psi(y) f(y).$$

By induction,

$$\mathcal{L}_\psi^n f(x) = \sum_{y:T^n(y)=x} \psi(y) \psi(T(y)) \ldots \psi(T^{n-1}(y)) f(y).$$

Also define an adjoint operator $\mathcal{L}_\psi^*$, acting on Borel measures, by requiring that the equality

$$\int \mathcal{L}_\psi f \, d\nu = \int f \, d\mathcal{L}_\psi^* \nu$$

holds for all f. The following is a corollary of a theorem by Ruelle [29].

Theorem 3.4. *Suppose $T : X \to X$ is a Manneville–Pomeau map, and ν is a Borel measure (not necessarily T-invariant) such that*

$$\mathcal{L}_\psi^* \nu = \nu$$

and ψ is positive ν-a.s. Then for every $\tau \in \mathcal{E}$ the following holds:

$$\tau_*(\nu|_U) \ll \nu|_{\tau(U)} \qquad and \qquad R_\tau := \frac{\mathrm{d}\tau_* \nu}{\mathrm{d}\nu} = \frac{\prod_{k=0}^{\infty} \psi\left(T^k \circ \tau^{-1}(x)\right)}{\prod_{k=0}^{\infty} \psi\left(T^k(x)\right)}. \tag{3.6}$$

Proof. Let $\tau \in \mathcal{E}$. Then there exists $n \geqslant 1$ and $(i_0, \ldots, i_{n-1})$, $(j_0, \ldots, j_{n-1})$ such that

$$\tau : \bar{I}_{i_0, \ldots, i_{n-1}} \to \bar{I}_{j_0, \ldots, j_{n-1}}.$$

Consider an arbitrary bounded Borel function f, vanishing outside $\bar{I}_{j_0, \ldots, j_{n-1}}$. Then

$$\mathcal{L}_\psi^n (f \circ \tau)(x) = \prod_{k=0}^{n-1} \psi\left(T^k(y)\right) f(\tau(y))$$

where $y \in \bar{I}_{i_0, \ldots, i_{n-1}}$ is such that $T^n(y) = x$. Or, equivalently,

$$\mathcal{L}_\psi^n (f \circ \tau)(x) = \prod_{k=0}^{n-1} \psi\left(T^k(\tau^{-1}(z))\right) f(z)$$

where $z \in \bar{I}_{j_0, \ldots, j_{n-1}}$ is such that $T^n(z) = x$. Therefore,

$$\mathcal{L}_\psi^n \left(f \circ \tau\right)(x) = \prod_{k=0}^{n-1} \psi\left(T^k(z)\right) \frac{\prod_{k=0}^{n-1} \psi\left(T^k \circ \tau^{-1}(z)\right)}{\prod_{k=0}^{n-1} \psi\left(T^k(z)\right)} f(z)$$

$$= \left(\mathcal{L}_\psi^n \left(\frac{\prod_{k=0}^{n-1} \psi \circ T^k \circ \tau^{-1}}{\prod_{k=0}^{n-1} \psi \circ T^k} f\right)\right)(x).$$

Since $\mathcal{L}_\psi^* \nu = \nu$, one finds that

$$\int f \circ \tau \, \mathrm{d}\nu = \int \frac{\prod_{k=0}^{n-1} \psi \circ T^k \circ \tau^{-1}}{\prod_{k=0}^{n-1} \psi \circ T^k} f \, \mathrm{d}\nu.$$

Moreover, since f is arbitrary, we conclude that

$$\tau_* \nu|_{\tau(U)} \ll \nu|_{\tau(U)}$$

and

$$\frac{\mathrm{d}\tau_* \nu|_{\tau(U)}}{\mathrm{d}\nu|_{\tau(U)}} = \frac{\prod_{k=0}^{n-1} \psi \circ T^k \circ \tau^{-1}}{\prod_{k=0}^{n-1} \psi \circ T^k} = \frac{\prod_{k=0}^{\infty} \psi \circ T^k \circ \tau^{-1}}{\prod_{k=0}^{\infty} \psi \circ T^k}$$

since $T^k \tau^{-1}(x) = T^k(x)$ for all $k \geqslant n$. $\square$

Remark. The above theorem immediately checks the requirements in definition 3.3 of weakly Gibbs states: it establishes the absolute continuity of $\tau_* \mu$ with respect to μ and gives a corresponding family of multipliers

$$R_\tau = \frac{\prod_{k=0}^{\infty} \psi\left(T^k \circ \tau^{-1}(x)\right)}{\prod_{k=0}^{\infty} \psi\left(T^k(x)\right)}. \tag{3.7}$$

Thus our problem consists in establishing the properties of the product in the right-hand side of (3.7).

Theorem 3.5. *Let T be a Manneville–Pomeau-type map. Let μ be the absolutely continuous invariant measure for T. Then μ is not a Gibbs, but is a weakly Gibbsian measure: the multipliers R_τ, given by (3.4), are well defined non-negative integrable functions, but not all of them are positive and continuous.*

Proof. Let h be the density of μ, and let us introduce a normalized transfer operator $\mathcal{L}_0$, corresponding to ψ_0, which is given by

$$\psi_0(x) = \begin{cases} \dfrac{h(x)}{h(T(x))} \dfrac{1}{|T'(x)|} & \text{for} \quad x > 0 \\ 1 & \text{for} \quad x = 0. \end{cases} \tag{3.8}$$

Hu in [14] showed that ψ_0 is a continuous function on X, satisfying $0 \leqslant \psi_0(x) \leqslant 1$, $\psi_0(x) = 1$ iff $x = 0$, and the only zero of ψ_0 at p_+, which was obtained by doubling the point p. Note, that $T(p_+) = 0$.

Consider also the transfer operator $\mathcal{L}$ corresponding to $\psi = 1/|T'|$. It is well known [19] that the transfer operator maps $L^1(m)$ to itself (m denotes the Lebesgue measure), the density $h = d\mu/dm$ satisfies $\mathcal{L}h = h$, and

$$\int \mathcal{L}f \, dm = \int f \, dm \tag{3.9}$$

for all $f \in L^1(m)$ and thus, $\mathcal{L}^* m = m$.

The normalized transfer operator $\mathcal{L}_0$ has the following properties:

(a) $\mathcal{L}_0 \mathbb{I} = \mathbb{I}$, where $\mathbb{I}(x) = 1$ for all x;
(b) operators $\mathcal{L}$ and $\mathcal{L}_0$ are related by the following formula:

$$\mathcal{L}_0 f = \frac{1}{h} \mathcal{L}(hf) \qquad \text{for all} \quad f$$

(c) for every $f \in L^1(\mu)$

$$\int \mathcal{L}_0 f \, d\mu = \int f \, d\mu$$

and hence $\mathcal{L}_0^* \mu = \mu$.

The last property of $\mathcal{L}_0$ follows easily from the corresponding property of $\mathcal{L}$. Indeed,

$$\int \mathcal{L}_0(f) \, d\mu = \int \frac{1}{h} \mathcal{L}(hf)h \, dm = \int \mathcal{L}(hf) \, dm = \int hf \, d\mathcal{L}^* m = \int fh \, dm = \int f \, d\mu.$$

Therefore, we can apply theorem 3.4 to μ: substitute the expression (3.8) for ψ_0 into the corresponding expression (3.6) for R_τ. Let $\tau \in \mathcal{E}_n$ and assume that none of the points $\{T^k \circ \tau^i(x) | k = 0, \ldots, n-1, i = -1, 0\}$ is equal to 0. Then, taking into account that $T^n \tau^{-1}(x) = T^n(x)$, we obtain

$$R_\tau(x) = \frac{h(\tau^{-1}(x))}{h(x)} \frac{|(T^n)'(x)|}{|(T^n)'(\tau^{-1}(x))|}.$$

The part of the previous formula involving the derivative of T^n depends continuously on x and is positive. The ratio $h(\tau^{-1}(x))/h(x)$ can be arbitrary large (small). Indeed, suppose $\tau \in \mathcal{E}_n$ and $\tau : \bar{I}_{0,\ldots,0} \to I_{i_0,\ldots,i_{n-1}}$, where $i_0 = 1$. The density $h(x)$ is bounded on $I_{i_0,\ldots,i_{n-1}}$, on the other hand, since $h(t)$ is singular at $t = 0$ one has that R_τ is singular as well.

It is also clear that any other family of multipliers $\{\tilde{R}_\tau\}$, satisfying (3.4) of definition 3.3, can differ from the obtained densities only on a set of measure 0. Thus, they cannot be positive and continuous. Hence μ is not a Gibbs state, but is weakly Gibbsian. $\qquad\square$

In the following section we study the absolutely continuous invariant measure μ in the symbolic representation. We prove that there exists an almost surely absolutely convergent potential U such that the conditional probabilities of μ can be represented in a standard Gibbsian way. This strengthens the result of theorem 3.5, and establishes a relation between the notions of weakly Gibbsian measures commonly used in statistical mechanics and definition 3.3. Moreover, certain properties of the measure μ can be understood from the decay of the potential U.

3.2.2. Symbolic dynamics: the potential. Consider again the coding $\pi : \{0, 1\}^{\mathbb{Z}_+} \to X$. The question we want to deal with here is to see which kind of a potential (in the sense of equilibrium statistical mechanics) is associated with $\nu = \pi^*\mu$ and how the properties of this potential can be related to the decay of correlations.

Let us introduce some notation: put $\Omega = \{0, 1\}^{\mathbb{Z}_+} = \{\omega = (\omega_i) : \omega_i \in \{0, 1\}, i \in \mathbb{Z}_+\}$, $[\omega_0, \ldots, \omega_{n-1}]$ is the cylinder with first coordinates $\omega_0, \ldots, \omega_{n-1}$. If $\omega \in \Omega$ and $\Lambda \subset \mathbb{Z}_+$ then ω_Λ is a projection of ω to $\{0, 1\}^\Lambda$, so $\omega_{\{i\}} = \omega_i$. For $\Lambda, \Lambda' \subseteq \mathbb{Z}_+$, $\Lambda \cap \Lambda' = \emptyset$, and $\xi \in \{0, 1\}^\Lambda$, $\eta \in \{0, 1\}^{\Lambda'}$, we let $\zeta = \xi\eta \in \{0, 1\}^{\Lambda \cup \Lambda'}$ be such that $\zeta|_\Lambda = \xi$ and $\zeta|_{\Lambda'} = \eta$. For any $\Lambda \subseteq \mathbb{Z}_+$ denote by Λ^c the complement of Λ in $\mathbb{Z}_+$. For any $\xi \in \{0, 1\}^\Lambda$, $\eta \in \{0, 1\}^{\Lambda^c}$, the conditional probability of observing ξ on Λ given η on the complement will be denoted by $\nu\left(\omega|_\Lambda = \xi \,\middle|\, \omega|_{\Lambda^c} = \eta\right)$, or, shortly, $\nu(\xi|\eta)$. Finally, $\bar{0}$ and $\bar{1}$ are the configurations consisting entirely of 0's and 1's.

We start by observing that ν is certainly not Gibbsian in the usual sense. The reason is that the ν-probability of the cylinder $\{\omega : \omega_0 = \omega_1 = \cdots = \omega_n = 0\}$ only decays polynomially, see (3.28). As a result the relative entropy density $i(\delta_{\bar{0}}|\nu)$ between the Dirac measure on the configuration of all zeros $\delta_{\bar{0}}$ and ν vanishes:

$$i(\delta_{\bar{0}}|\nu) = \lim_{n\to\infty} \frac{1}{n} I_n(\delta_{\bar{0}}|\nu) = 0$$

where

$$I_n(\lambda|\rho) = I(\lambda_n|\rho_n) = \int \log \frac{\mathrm{d}\lambda_n}{\mathrm{d}\rho_n} \, \mathrm{d}\lambda_n = \int \frac{\mathrm{d}\lambda_n}{\mathrm{d}\rho_n} \log \frac{\mathrm{d}\lambda_n}{\mathrm{d}\rho_n} \, \mathrm{d}\rho_n$$

is the Kullback–Liebler information between the projections λ_n, ρ_n of the measures λ, ρ onto cylinders of length $n+1$, i.e. $\Omega_n = \{0, 1\}^{n+1}$. However, by the variational principle of statistical mechanics, see, e.g., [9], this means that ν cannot be Gibbsian (since then $\delta_{\bar{0}}$ would be Gibbsian with the same potential which is absurd).

Using (3.4), the conditional probabilities for ν can be written [17, 28] as

$$\nu(\omega_0, \ldots, \omega_n | \omega_{n+1}, \omega_{n+2}, \ldots) = \frac{R_{\tau_{[\omega_0,\ldots,\omega_n]}}\left(\pi^{-1}(\omega_0 \ldots \omega_n\omega_{n+1} \ldots)\right)}{\sum_{\omega'_0,\ldots,\omega'_n} R_{\tau_{[\omega'_0,\ldots,\omega'_n]}}\left(\pi^{-1}(\omega'_0 \ldots \omega'_n\omega_{n+1} \ldots)\right)} \tag{3.10}$$

where $\tau_{[\omega_0,\ldots,\omega_n]} \in \mathcal{E}_{n+1}$ is a conjugating homeomorphism, mapping $\bar{I}_{\omega_0,\ldots,\omega_n}$ to $\bar{I}_{1,\ldots,1}$. Note, that here we have chosen $\bar{1}$ as a reference state. It is easy to see that actually (3.10) does not depend on the choice of the reference state.

In particular, for $n = 0$ we find that the conditional probability to find $\omega_0 \in \{0, 1\}$ at the origin while the rest of the configuration on $\{1, 2, \ldots\}$ is $\omega_{0^c} = (\omega_1, \omega_2, \ldots)$, is simply given by

$$\nu(\omega_0|\omega_1, \omega_2, \ldots) = \frac{\psi_0(\pi^{-1}(\omega))}{\psi_0(\pi^{-1}(\omega^0)) + \psi_0(\pi^{-1}(\omega))} \tag{3.11}$$

where $\omega = (\omega_0, \omega_1, \ldots)$ and ω^0 is ω 'flipped' at the origin, i.e. $\omega_0^0 = 1 - \omega_0$ and $\omega_i^0 = \omega_i$ for $i \neq 0$. Since ψ_0 is continuous, we immediately conclude that the non-Gibbsian character of ν is not related to the presence of essential discontinuities in the conditional probabilities (as in the case of the restricted Ising model of the introduction). However, $\nu(\omega_0|\omega_{0^c})$ is not uniformly non-null: it is easily seen that for the continuous version of the conditional probabilities one has

$$\nu(\omega_0 = 1|\omega_{0^c} = \bar{0}) = 0$$
$$\nu(\omega_0 = 0|\omega_{0^c} = \bar{0}) = 1 \tag{3.12}$$

where $\bar{0}$ denotes the configuration of all zeros. Therefore, we expect to find a potential $U(\Lambda, \omega)$ for which the sums that form the local Hamiltonian

$$H_\Lambda^U(\omega) = \sum_{A \cap \Lambda \neq \emptyset} U(A, \omega) \tag{3.13}$$

will diverge at $\omega = \bar{0}$. More precisely we have the following.

Theorem 3.6. *There exists a translation-invariant potential $U(\Lambda, \omega)$ with the following properties:*

(a) $U(\Lambda, \omega) = 0$ unless $\Lambda = [i, j]$;
(b) $\exists \delta > 0$, $\exists l : \Omega \to \mathbb{R}_+ \cup \{\infty\}$ such that on the set $K := \{\omega : l(\omega) < \infty\}$ we have the estimate

$$|U([0, n], \omega)| \leqslant \begin{cases} C_2(\omega) & n < l(\omega) \\ C_1(\omega) \exp(-\delta n) & n \geqslant l(\omega) \end{cases}$$

for some $C_i(\omega) < \infty$, $\omega \in K$.
(c) ν is weakly Gibbsian with potential U, see (3.27).

Proof. We consider the Kozlov potential [18] with reference state $\omega = \bar{1}$:

$$U([i, j], \omega) = \log \frac{\nu(1_{\{i\}}\omega_{(i,j]}|1_{[i,j]^c})\nu(\omega_{[i,j)}1_{\{j\}}|1_{[i,j]^c})}{\nu(\omega_{[i,j]}|1_{[i,j]^c})\nu(1_{\{i\}}\omega_{(i,j)}1_{\{j\}}|1_{[i,j]^c})}. \tag{3.14}$$

From the inequality $|\log a - \log b| \leqslant |a - b|/\min\{a, b\}$, we have the estimate

$$|U([0, n], \omega)| \leqslant c(\omega)\varphi_n(\omega) \tag{3.15}$$

where

$$c(\omega)^{-1} = \tfrac{1}{2} \min_n \min_{a,b} \nu(a|\omega_{(0,n-1)}b_n 1_{(n,\infty)}) \geqslant 0 \tag{3.16}$$

with equality only for $\omega = \bar{0}$, and where

$$\varphi_n(\omega) = \sup_{\xi, \xi'} \left| \nu(\omega_0|\omega_{[1,n-1]}\xi_{[n,\infty)}) - \nu(\omega_0|\omega_{[1,n-1]}\xi'_{[n,\infty)}) \right|. \tag{3.17}$$

Using expression (3.11) for the conditional probabilities, we obtain

$$\varphi_n(\omega) = \sup_{\xi,\xi'} \left| \frac{\psi_0 \circ \pi(\omega_{[0,n-1]}\xi_{[n,\infty)})}{\psi_0 \circ \pi(\omega^0_{[0,n-1]}\xi_{[n,\infty)}) + \psi_0 \circ \pi(\omega_{[0,n-1]}\xi_{[n,\infty)})} \right.$$

$$\left. - \frac{\psi_0 \circ \pi(\omega_{[0,n-1]}\xi'_{[n,\infty)})}{\psi_0 \circ \pi(\omega^0_{[0,n-1]}\xi'_{[n,\infty)}) + \psi_0 \circ \pi(\omega_{[0,n-1]}\xi'_{[n,\infty)})} \right|$$

$$\leqslant C_3 \left[\sup_{x,y \in I_{\omega_0\omega_1\ldots\omega_{n-1}}} |\psi_0(x) - \psi(y)| + \sup_{x,y \in I_{\omega^0_0\omega_1\ldots\omega_{n-1}}} |\psi_0(x) - \psi_0(y)| \right] \qquad (3.18)$$

with

$$C_3 = 2 \sup_{\xi} \frac{1}{\psi_0 \circ \pi(\xi) + \psi_0 \circ \pi(\xi^0)} < \infty.$$

Estimating (3.18) is only problematic in the case where x or y are very close to zero. Since h is bounded and Lipschitz on $(\epsilon, 1]$ for any $\epsilon > 0$ (see [14]), we know that if $\epsilon < x \leqslant y$ for some $\epsilon > 0$, then there exists a constant $C_4 = C_4(\epsilon)$ such that

$$|\psi_0(x) - \psi_0(y)| \leqslant C_4|x - y|. \qquad (3.19)$$

Of course, things remain bad for $x = 0$ and we must therefore restrict ourselves to 'good' configurations. We first define what we mean by this. Put $\beta = \int \nu(d\omega)\omega_0 = \mu(I_1) > 0$ and define

$$\ell(\omega) = \inf\left\{ n \in \mathbb{N}_0 : \frac{1}{k}\sum_{i=0}^{k-1} \omega_i > \tfrac{1}{2}\beta \text{ for all } k \geqslant n \right\}. \qquad (3.20)$$

We say that ω is 'good' if $\ell(\omega) < \infty$ and we collect them in the set

$$K = \{\omega : \ell(\omega) < \infty\}. \qquad (3.21)$$

Note that K is a set in the tail field. Indeed, if two configurations ω and ω' are such that

$$\left\{ i \in \mathbb{N}_0 : \omega_i \neq \omega'_i \right\}$$

is a finite set, then $\ell(\omega) < \infty$ if and only if $\ell(\omega') < \infty$. Thus, if $\omega \in K$, then $\omega^0 \in K$ as well. For $\omega \in K$, we define

$$\epsilon = \epsilon(\omega) = \tfrac{1}{2} \inf\left\{ x \in I_{\omega_0\ldots\omega_{\ell(\omega)-1}} \cup I_{\omega^0_0\ldots\omega_{\ell(\omega^0)-1}} \right\}. \qquad (3.22)$$

By the definition of $\ell(\omega)$ and K, $\epsilon(\omega) > 0$ for every $\omega \in K$. Moreover, for every $n \geqslant \ell(\omega)$, if $x \in I_{\omega_0\ldots\omega_{n-1}} \cup I_{\omega^0_0\ldots\omega_{n-1}}$ then $x > \epsilon(\omega)$.

Combining (3.18) and (3.19), we have that

$$\varphi_n(\omega) \leqslant C(\omega)\left(|I_{\omega_0,\ldots,\omega_{n-1}}| + |I_{\omega^0_0\ldots\omega_{n-1}}| \right). \qquad (3.23)$$

Now use that $1/|T'(x)| \leqslant e^{-\delta'}$ for $x \in [p, 1]$ and $\delta' = \log\lambda > 0$; this gives the estimate

$$m(I_{\omega_0\omega_1\ldots\omega_{n-1}}) \leqslant \exp\left(-\delta' \sum_{i=0}^{n-1} \omega_i \right). \qquad (3.24)$$

Therefore, for $\omega \in K$ and $n \geqslant \ell(\omega)$ we have

$$\varphi_n(\omega) \leqslant C(\omega)\,e^{-\delta n} \qquad (3.25)$$

with $\delta = \frac{1}{2}\beta\delta'$. For $\omega \in K$ and $n \leqslant l(\omega)$ we have the trivial bound

$$\varphi_n(\omega) \leqslant 2. \tag{3.26}$$

Together with (3.14)–(3.16), this finishes the proof of claims (a) and (b) of the theorem and shows that the potential is absolutely convergent on the set K.

In order to prove that ν is weakly Gibbsian with potential U we still have to establish two facts:

(a) the potential U is absolutely convergent on a set of ν-measure one;
(b) ν is consistent with the potential, i.e.

$$\nu(\omega_0|\omega) = \frac{\exp\left(-H_{\{0\}}(\omega)\right)}{\exp(-H_{\{0\}}(\omega)) + \exp(-H_{\{0\}}(\omega^0))} \qquad \nu - \text{a.s.} \tag{3.27}$$

where $H_{\{0\}}$ is the local Hamiltonian defined in (3.13).

The second point follows from the first one and from the continuity of the conditional probabilities (see, e.g., [23, 24]). The first fact is a simple consequence of the ergodic theorem:

$$\nu(K^c) \leqslant \nu\left(\left\{\omega : \liminf_{n \uparrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \omega_i \leqslant \frac{1}{2}\beta\right\}\right) = 0. \qquad \square$$

Theorem 3.6 states that the potential $U([0, n], \omega)$ decays exponentially for n larger than some configuration-dependent 'correlation length' $\ell(\omega)$. As we have seen above, the correlations for the MP maps decay polynomially: $\rho_n = O(n^{-1/\alpha+1})$, where $\alpha \in (0, 1)$ is the parameter of the MP map. It turns out that the 'distribution' of the correlation length $\ell(\omega)$ is closely related to the above decay of correlations. This is the content of the following proposition.

Theorem 3.7. *One has the following estimates of $\nu(\{\omega : \ell(\omega) \geqslant n\})$ depending on the parameter α:*

(a) for $\alpha \in (\frac{1}{2}, 1)$ there exist constants $C_1, C_2 > 0$ such that

$$C_1 n^{-1/\alpha+1} \leqslant \nu(\{\omega : \ell(\omega) \geqslant n\}) \leqslant C_2 n^{-1/\alpha+1}$$

(b) for $\alpha = \frac{1}{2}$ there exist constants $C_1, C_2 > 0$ such that we have

$$C_1 n^{-1} \leqslant \nu(\{\omega : \ell(\omega) \geqslant n\}) \leqslant C_2 n^{-1} \log n$$

(c) for $\alpha \in (0, \frac{1}{2})$ and any $\delta > 0$ there exist constants $C_1, C_2 > 0$ such that

$$C_1 n^{-1/\alpha+1} \leqslant \nu(\{\omega : \ell(\omega) \geqslant n\}) \leqslant C_2 n^{-1/\alpha+1+\delta}.$$

We are going to use the following result in the proof of the above statement.

Theorem 3.8. *Let $\Sigma = A^{\mathbb{Z}_+}$, where $A = \{1, \ldots, N\}$, and let ν be a shift-invariant probability measure on Σ. For a bounded function $f : A \to \mathbb{R}$ with $\int f \, d\nu = 0$, denote*

$$\rho_k(f) = \left|\int f(\omega_0) f(\omega_k) \, d\nu\right| \qquad k \in \mathbb{N}.$$

Suppose, that for some $m \geqslant 1$ one has

$$\zeta_m := \sum_{k=1}^{\infty} \rho_k^{1/m} < \infty.$$

Then there exists a constant $C > 0$ such that

$$\int \left| \sum_{k=0}^{n-1} f(\omega_k) \right|^{2m} \mathrm{d}\nu \leqslant C\left(\zeta_m\right)^m n^m.$$

for all $n \geqslant 1$.

Proof. The cases $m = 1$ and 2 are, in fact, lemmas 2 and 4 in [1, chapter 4, p 172]. The statement can be easily generalized for any other integer $m > 2$. The remaining case $m = k + \delta$, where k is an integer and $\delta \in (0, 1)$ can be proved along the lines of lemma 7.4 [8, p 225]. One has to stress, that though lemma 7.4 in [8] is proved under very different assumptions, discrete Markov chains with exponential mixing, its proof can be adopted to our purposes with minor modifications. □

Proof of theorem 3.7. The lower bound is easy:

$$\nu(\{\omega : \ell(\omega) \geqslant n\}) \geqslant \mu(I_{00\ldots0}).$$

Since $h(x) \geqslant C_3 x^{-\alpha}$ and $m(I_{00\ldots0}) \geqslant C_4 n^{-1/\alpha}$:

$$\mu(I_{00\ldots0}) \geqslant \int_0^{C_4 n^{-1/\alpha}} C_3 x^{-\alpha} \mathrm{d}x = C_1 n^{1-1/\alpha}. \tag{3.28}$$

The upper bound is more difficult: put $S_n(\omega) := \sum_{i=0}^{n-1}(\omega_i - \beta)$, where we recall that $\beta = \int \omega_0 \, \mathrm{d}\nu > 0$.

We have to estimate $\nu(\{\omega : \ell(\omega) \geqslant n\})$, i.e.

$$\nu\left(\left\{\omega : \exists k \geqslant n \text{ such that } \left|\frac{S_k}{k}\right| > \tfrac{1}{2}\beta\right\}\right) = \nu\left(\left\{\omega : \sup_{k \geqslant n} \left|\frac{S_k}{k}\right| > \tfrac{1}{2}\beta\right\}\right). \tag{3.29}$$

According to theorem 12 in [16] the following conditions are equivalent:

(a) $$\nu\left(\left\{\omega : \sup_{k \geqslant n} \left|\frac{S_k}{k}\right| > \frac{\beta}{2}\right\}\right) = \mathrm{O}(n^{-\gamma}) \qquad \text{as} \quad n \to \infty$$

(b) $$\nu\left(\left\{\omega : \left|\frac{S_n}{n}\right| > \frac{\beta}{2}\right\}\right) = \mathrm{O}(n^{-\gamma}) \qquad \text{as} \quad n \to \infty.$$

Therefore, these two probabilities have similar asymptotic behaviour, however, the second quantity is much easier to deal with.

Let us start with the case $\alpha \in \left[\frac{1}{2}, 1\right)$. By the Chebyshev inequality

$$\nu\left(\left\{\omega : \left|\frac{S_n}{n}\right| > \tfrac{1}{2}\beta\right\}\right) \leqslant \frac{4\int |S_n|^2 \, \mathrm{d}\nu}{n^2\beta^2} \leqslant C\frac{1}{n}\sum_{k=0}^{n-1} \rho_k.$$

Taking into account that $\rho_k = \mathrm{O}(k^{-1/\alpha+1})$ for $k \geqslant 1$, we conclude that for some C_2

$$\nu\left(\left\{\omega : \left|\frac{S_n}{n}\right| > \tfrac{1}{2}\beta\right\}\right) \leqslant C_2 n^{-1/\alpha+1} \qquad \text{for} \quad \alpha \in \left(\tfrac{1}{2}, 1\right)$$

and

$$\nu\left(\left\{\omega : \left|\frac{S_n}{n}\right| > \tfrac{1}{2}\beta\right\}\right) \leqslant C_2 n^{-1} \log n \qquad \text{for} \quad \alpha = \tfrac{1}{2}.$$

The above argument cannot produce an estimate which decays faster than $1/n$. Therefore, for $\alpha \in \left(0, \frac{1}{2}\right)$ we have to use higher moments of S_n in order to obtain better estimates.

Consider $\alpha \in \left(0, \frac{1}{2}\right)$ and take any sufficiently small $\delta \in (0, 1)$ such that $m = \frac{1-\alpha}{\alpha}(1-\delta) \geqslant 1$. Since $\rho_k = \mathrm{O}(k^{-1/\alpha+1})$ one has

$$\zeta_m = \sum_{k=1}^{\infty} \rho_k^{1/m} \leqslant C \sum_{k=1}^{\infty} k^{-1/(1-\delta)} < +\infty.$$

Using the Chebyshev inequality and the estimate from theorem 3.8 we conclude that there exists a constant C_2 such that

$$\nu\left(\left\{\omega : \left|\frac{S_n}{n}\right| > \tfrac{1}{2}\beta\right\}\right) \leqslant C_2 n^{-m} = C_2 n^{-1/\alpha+1+\delta'}$$

where $\delta' = (1/\alpha - 1)\delta$. This finishes the proof of the upper bound. $\qquad\square$

Acknowledgment

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