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Published in:
Physica D: Nonlinear Phenomena

Publication date:
2001

ELSEVIER

Physica D 155 (2001) 159–183

PHYSICA D

www.elsevier.com/locate/physd

Actions of the Neumann systems via Picard–Fuchs equations

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Received 7 July 2000; accepted 27 February 2001
Communicated by I. Gabitov

Abstract

The Neumann system describing the motion of a particle on an n-dimensional sphere with an anisotropic harmonic potential has been celebrated as one of the best understood integrable systems of classical mechanics. The present paper adds a detailed discussion and the determination of its action integrals, using differential equations rather than standard integral formulas. We show that the actions of the Neumann system satisfy a Picard–Fuchs equation which in suitable coordinates has a rather simple form for arbitrary n. We also present an explicit form of the related Gauß–Manin equations. These formulas are used for the numerical calculation of the actions of the Neumann system. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Neumann system; Integrable systems; Action variables; Picard–Fuchs equation

1. Introduction

The classical mechanical problem of a point particle moving on a unit sphere, under the influence of an anisotropic harmonic force, was the first case solved in terms of hyperelliptic theta functions of genus 2 (also called ultraelliptic theta functions). This was done by Neumann [8] in his 1856 doctoral dissertation. An amazing relation between the Neumann system and the spectral theory of Schrödinger operators was discovered by Moser and Trubowitz (see [7,12]). The action variables of the Neumann system from this point of view have been discussed by Novikov and Veselov [9].

We will consider the natural n-dimensional generalization of the Neumann system, which is a Hamiltonian system in $\mathbb{R}^{2n+2}$ with the Hamiltonian

$$H(\boldsymbol{x}, \boldsymbol{y}) = \tfrac{1}{2}(|\boldsymbol{y}|^2|\boldsymbol{x}|^2 - (\boldsymbol{y}, \boldsymbol{x})^2) + \tfrac{1}{2}(\boldsymbol{x}, A\boldsymbol{x}), \tag{1}$$

where $\boldsymbol{x} = (x_0, \dots, x_n)^t$, $\boldsymbol{y}$ is the conjugate momentum, and (,) is the standard Euclidean scalar product. $\boldsymbol{A}$ is a positive definite matrix which may be assumed diagonal with eigenvalues $0 < a_0 < \cdots < a_n$. It leaves the unit

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0167-2789/01/$ – see front matter
PII: S0167-2789(01)00257-3

sphere $S^n \subset \mathbb{R}^{n+1}$-invariant: $|\boldsymbol{x}|^2$ is a geometric constant which is chosen as 1. $H(\boldsymbol{x}, \boldsymbol{y})$ possesses a set of algebraic integrals $F_0, \dots, F_n$ in involution, which, surprisingly, have only recently been found (see [7]):

$$F_\nu(\boldsymbol{x}, \boldsymbol{y}) = x_\nu^2 + \sum_{\mu \neq \nu} \frac{(x_\nu y_\mu - x_\mu y_\nu)^2}{a_\nu - a_\mu}, \qquad \sum_{\nu=0}^{n} F_\nu = 1. \tag{2}$$

These integrals are related to another set of integrals, $\eta_1, \dots, \eta_n$, defined by the formula

$$\sum_{\nu=0}^{n} \frac{F_\nu}{z - a_\nu} = \frac{Q(z)}{A(z)}, \tag{3}$$

where

$$Q(z) = z^n + 2\eta_1 z^{n-1} + 2\eta_2 z^{n-2} + \cdots + 2\eta_n, \tag{4}$$

$$A(z) = \prod_{\nu=0}^{n} (z - a_\nu). \tag{5}$$

The Liouville tori of this system are in general certain coverings of real parts of Jacobi varieties of the hyperelliptic curve

$$w^2 = R(z), \quad R(z) = -A(z)Q(z). \tag{6}$$

The actions of the Neumann system are the periods of the Abelian integral

$$I_j = \frac{1}{2\pi} \oint \frac{Q(z)}{\sqrt{R(z)}}\, \mathrm{d}z, \quad j = 1, \dots, n \tag{7}$$

over certain real cycles γ_j (see Section 3 for details).

Similar representations of the actions in terms of Abelian integrals can be given for almost all known integrable systems; for the Kovalevskaya top this has only recently been achieved in [2,6]. To use the actions, e.g., for the computation of semiclassical approximations to quantum mechanical eigenvalues [6,10,13], they have to be evaluated numerically. Of course, (7) is an explicit formula, but from the computational point of view it is not efficient when the integral (7) must be determined for all values of the constants of motion. We suggest another way for calculating the actions, using the Picard–Fuchs equation they satisfy. The existence of such linear ordinary differential equations follows from general de Rham-type arguments, but their explicit form can be very complicated (see, e.g. [2] where the case of the Kovalevskaya top is discussed). The main goal of this paper is to show that for the Neumann system, the Picard–Fuchs (and the related Gauß–Manin) equations in appropriate variables have a relatively simple form, which makes them very convenient for numerical computations.

We should mention that the simplicity of the Picard–Fuchs/Gauß–Manin equations depends very much on the choice of constants of motion. In particular, if we take F_ν or η_j as independent variables in the differential equation they are quite cumbersome already for $n = 2$. However, with the variables q_i which are the zeros of the polynomial $Q(z)$,

$$Q(z) = \prod_{j=1}^{n} (z - q_j), \tag{8}$$

they become much simpler. Although the Picard–Fuchs equations look more natural and are simpler, the Gauß–Manin equations are more suitable from the computational point of view. The reason is that for each q_i, the corresponding

Picard–Fuchs equation has a different set of dependent variables, while the corresponding n Gauß–Manin equations give commuting flows for the same set of dependent variables.

The outline of this paper is as follows. In Section 2, we recall the basic properties of the Neumann system. Separation of variables is achieved in a standard way [7] using elliptical spherical coordinates. In Section 3, we discuss the explicit formulas for the actions and introduce reduced actions using the obvious reflection symmetry group $\mathbb{Z}_2^{n+1}$. These two actions differ only by some constant coefficients (and therefore satisfy the same linear equations), but the reduced actions are continuous functions of the constants of motion while the usual actions are not. Sections 4 and 5 contain the main results. We find the explicit form of the corresponding Picard–Fuchs and Gauß–Manin equations in the variables q_i. In Section 6, we use these results to compute the actions for the Neumann system.

2. Separation of variables

In the first part of this section, we follow closely the presentation of Section 7 of Moser's paper [7].

Consider a particle moving on a unit sphere $S^n \subset \mathbb{R}^{n+1}(\boldsymbol{x})$, and let it be attracted towards the center of the sphere by a non-isotropic harmonic potential

$$V(\boldsymbol{x}) = \tfrac{1}{2}(\boldsymbol{x}, Ax), \tag{9}$$

where the matrix $\boldsymbol{A}$ is positive symmetric. Without loss of generality, we may assume $\boldsymbol{A} = \mathrm{diag}(a_\nu)$ with spring constants $0 < a_0 < \cdots < a_n$. The constraint $|\boldsymbol{x}|^2 = 1$ may be accounted for in different ways. One possibility is in terms of the Hamiltonian (1), where the kinetic energy has an appropriate dependence on the point $\boldsymbol{x}$. A more familiar starting point is to include a force $\lambda\boldsymbol{x}$ perpendicular to the sphere,

$$\ddot{\boldsymbol{x}} + Ax = \lambda\boldsymbol{x}, \tag{10}$$

and to determine its strength λ from the constraint. Differentiating $|\boldsymbol{x}|^2 = 1$ twice with respect to time t, and taking the scalar product of (10) with $\boldsymbol{x}$, we find that λ is

$$\lambda = (\boldsymbol{x}, Ax) - |\dot{\boldsymbol{x}}|^2. \tag{11}$$

It may be checked by direct computation that Eq. (10) is then equivalent to the canonical equations derived from (1).

In the neighborhood of the stable equilibrium points $\boldsymbol{x} = \pm\boldsymbol{e}_0 = (\pm 1, 0, \dots, 0)$, we have, to lowest order, $\lambda = a_0$ and therefore $\ddot{x}_j \approx -(a_j - a_0)x_j$. Hence the frequencies of small oscillations at energies close to $h = V_{\min} = \frac{1}{2}a_0$ are

$$\omega_{j0} = \sqrt{a_j - a_0} \quad (j = 1, \dots, n). \tag{12}$$

The n eigendirections are tangent to the great circles through $\boldsymbol{e}_0$ and the other equilibrium points $\boldsymbol{e}_j$. The equilibria at $\boldsymbol{x} = \pm\boldsymbol{e}_j$ have energies $\frac{1}{2}a_j$ and $n - j$ stable eigendirections (tangent to the great circles through $\boldsymbol{e}_j$ and $\boldsymbol{e}_k$, $k = j+1, \dots, n$) with eigenfrequencies $\omega_{kj} = \sqrt{a_k - a_j}$. The j great circles connecting to $\boldsymbol{e}_0, \dots, \boldsymbol{e}_{j-1}$ give unstable eigendirections with eigenvalues $\sqrt{a_j - a_i}$, $j > i$.

The $\frac{1}{2}n(n+1)$ great circles connecting all pairs of equilibrium points, plus the corresponding momenta, are invariant sets for the equations of motion. This implies the existence of periodic motion along them, of oscillatory or rotational type, depending on the available energy. Parametrizing the circle through $\boldsymbol{e}_\mu$ and $\boldsymbol{e}_\nu$ ($\mu < \nu$) with the angle φ via $(x_\mu, x_\nu) = (\cos\varphi, \sin\varphi)$, it is straightforward to describe this periodic motion with the equation

$$2\ddot{\varphi} = -(a_\nu - a_\mu)\sin 2\varphi, \tag{13}$$

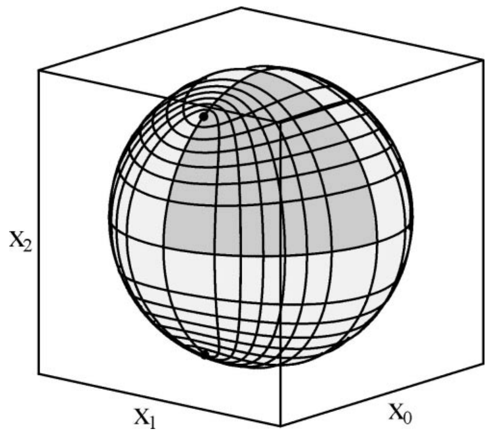

Fig. 1. Jacobi's elliptical spherical coordinates u_1, u_2 on S^2, for $(a_0, a_1, a_2) = (1, 2, 4)$. Coordinate lines $u_1 =$ const are more or less deformed circles with centers on the x_0-axis; $u_1 = a_0$ is the great circle $x_0 = 0$. Coordinate lines $u_2 =$ const surround the x_2-axis; $u_2 = a_2$ is the great circle $x_2 = 0$. The great circle $x_1 = 0$ is composed from pieces with $u_1 = a_1$ or $u_2 = a_1$. One full sheet of coordinates u_1, u_2 is needed for each octant of S^2; one octant is shaded. The lines are drawn at equidistant values of u_1 and u_2.

which, for each half of the great circle, is nothing but the pendulum equation. It admits solutions for energies $h \geq \frac{1}{2}a_\mu$. For $n \geq 2$ and intermediate values of the energy, there also exist isolated periodic orbits not related to great circles on S^n.

To solve the equations in general, one uses the fact that they separate in Jacobi's "elliptical spherical coordinates" $(u_1, \dots, u_n)$ on S^n. These are implicitly given as the zeros $u_j \equiv z_j(\boldsymbol{x})$ of

$$f(z) := \sum_{\nu=0}^{n} \frac{x_\nu^2}{z - a_\nu} = 0. \tag{14}$$

(The coordinates u_j are not to be confused with the $n+1$ elliptic coordinates in $\mathbb{R}^{n+1}$, also due to Jacobi, but defined as the zeros of $f(z) - 1 = 0$.) The u_j are labeled according to the scheme

$$a_0 \leq u_1 \leq a_1 \leq \cdots \leq a_{n-1} \leq u_n \leq a_n. \tag{15}$$

Note that 2^{n+1} copies of the set $\{u_1, \dots, u_n\}$ are needed to cover the entire sphere S^n. On the circle $n = 1$, the coordinate $u = a_1 x_0^2 + a_0 x_1^2$ runs from $u = a_0$ at $(x_0, x_1) = (0, \pm 1)$ to $u = a_1$ at $(x_0, x_1) = (\pm 1, 0)$. On the 2-sphere S^2, the coordinate lines for u_1 and u_2 are shown in Fig. 1. Their orthogonality is demonstrated below (see (19)). The circle $x_1 = 0$ is a singular line made up of segments $u_1 = a_1$ and $u_2 = a_1$ which meet at the four points $x_0^2 = (a_1 - a_0)/(a_2 - a_0)$.

We shall use Latin dummies for indices that run from 1 to n, and Greek for those that run from 0 to n. The inverse transformation $\{u_j\} \to \{x_\nu\}$ is obtained from writing (14) as

$$\sum_{\nu=0}^{n} \frac{x_\nu^2}{z - a_\nu} = \frac{\prod_j (z - u_j)}{\prod_\nu (z - a_\nu)} =: \frac{U(z)}{A(z)}. \tag{16}$$

This is easily checked by comparing the zeros and poles of both sides. The relation shows that the x_ν^2 are given as the residues of $f(z)\,\mathrm{d}z$,

$$x_\nu^2 = \frac{U(a_\nu)}{A'(a_\nu)} = \frac{\prod_j (a_\nu - u_j)}{\prod_{\mu \neq \nu} (a_\nu - a_\mu)}. \tag{17}$$

Let us now prove that the u_j are an orthogonal system on S^n. To this end, we calculate the metric

$$\mathrm{d}s^2 = \sum_{\nu=0}^{n} \mathrm{d}x_\nu^2 \tag{18}$$

in terms of u_j. Taking the derivative of the logarithm of Eq. (17), we get

$$\frac{\mathrm{d}x_\nu}{x_\nu} = \frac{1}{2}\sum_j \frac{\mathrm{d}u_j}{u_j - a_\nu},$$

and hence

$$4\,\mathrm{d}s^2 = \sum_\nu x_\nu^2 \left(\sum_j \frac{\mathrm{d}u_j}{u_j - a_\nu}\right)^2 = \sum_\nu \sum_j \frac{x_\nu^2\,\mathrm{d}u_j^2}{(u_j - a_\nu)^2} + \sum_\nu \sum_{j\neq k} \frac{x_\nu^2\,\mathrm{d}u_j\,\mathrm{d}u_k}{(u_j - a_\nu)(u_k - a_\nu)}.$$

Rewriting the second term as

$$\sum_\nu \sum_{j\neq k} \frac{x_\nu^2\,\mathrm{d}u_j\,\mathrm{d}u_k}{u_k - u_j}\left(\frac{1}{u_j - a_\nu} - \frac{1}{u_k - a_\nu}\right),$$

we see that the off-diagonal elements of the metric vanish, and obtain

$$\mathrm{d}s^2 = \sum_{j=1}^{n} g_j(\boldsymbol{u})\,\mathrm{d}u_j^2, \tag{19}$$

where

$$g_j(\boldsymbol{u}) = \frac{1}{4}\sum_\nu \frac{x_\nu^2}{(u_j - a_\nu)^2} = -\frac{1}{4}\,\frac{\mathrm{d}}{\mathrm{d}z}\sum_\nu \frac{x_\nu^2}{z - a_\nu}\bigg|_{z=u_j} = -\frac{1}{4}\,\frac{U'(u_j)}{A(u_j)}. \tag{20}$$

The kinetic and potential energies can now be expressed in terms of the u_j as

$$T = \frac{1}{2}|\dot{\boldsymbol{x}}|^2 = \frac{1}{2}\sum_{j=1}^{n} g_j(\boldsymbol{u})\dot{u}_j^2, \tag{21}$$

$$V = \frac{1}{2}(\boldsymbol{x}, A\boldsymbol{x}) = \frac{1}{2}\sum_{\nu=0}^{n} a_\nu - \frac{1}{2}\sum_{j=1}^{n} u_j. \tag{22}$$

To see this, we expand both sides of Eq. (16) in $1/z$. The left-hand side is

$$\sum_\nu \frac{x_\nu^2}{z(1 - a_\nu/z)} = \frac{1}{z}\sum_\nu x_\nu^2\left(1 + \frac{a_\nu}{z} + \frac{a_\nu^2}{z^2} + \cdots\right) = \frac{1}{z} + \frac{1}{z^2}\sum_\nu\left(a_\nu x_\nu^2 + \frac{a_\nu^2 x_\nu^2}{z} + \cdots\right),$$

and the right-hand side is

$$\begin{aligned}\frac{\prod_j (z - u_j)}{\prod_\nu z(1 - a_\nu/z)} &= \frac{1}{z^{n+1}}\prod_j (z - u_j)\prod_\nu\left(1 + \frac{a_\nu}{z} + \frac{a_\nu^2}{z^2} + \cdots\right)\\ &= \frac{1}{z^{n+1}}\left(z^n - \left(\sum_j u_j\right)z^{n-1} + \cdots\right)\left(1 + \frac{1}{z}\sum_\nu a_\nu + \cdots\right).\end{aligned}$$

Comparison of the coefficients of $1/z^2$ on both sides leads to the equality (22).

We introduce the canonical momenta

$$p_j = \frac{\partial T}{\partial \dot{u}_j} = g_j(\boldsymbol{u})\dot{u}_j, \tag{23}$$

and obtain the Hamiltonian

$$H(\boldsymbol{u}, \boldsymbol{p}) = T + V = \frac{1}{2}\sum_{j=1}^{n}\left(\frac{p_j^2}{g_j(\boldsymbol{u})} - u_j\right) + c, \tag{24}$$

where $c = \frac{1}{2}\sum_\nu a_\nu$ is a constant of no interest. The Hamilton–Jacobi equation $H(u, \partial S/\partial u) = \text{const}$ in this case has the form

$$\sum_j \left(-4\frac{A(u_j)}{U'(u_j)}\left(\frac{\partial S}{\partial u_j}\right)^2 - u_j\right) - 2\eta_1 = 0. \tag{25}$$

We are looking for the solution of this equation $S(u, \eta)$ which depends on n additional parameters $\eta_1, \dots, \eta_n$. To separate the variables, we need the following two identities found by Jacobi. For arbitrary distinct u_j, the corresponding polynomial $U(z) = \prod_j (z - u_j)$ as in (16), and an arbitrary polynomial P of the form

$$P(z) = \eta_1 z^{n-1} + \eta_2 z^{n-2} + \cdots + \eta_n, \tag{26}$$

it holds that

$$\eta_1 = \sum_{j=1}^{n}\frac{P(u_j)}{U'(u_j)}, \qquad \sum_j u_j = \sum_j \frac{u_j^n}{U'(u_j)}. \tag{27}$$

To prove these identities, consider the integrals

$$\frac{1}{2\pi i}\oint \frac{P(z)}{U(z)}\,\mathrm{d}z, \qquad \frac{1}{2\pi i}\oint \frac{z^n}{U(z)}\,\mathrm{d}z$$

with a contour that encloses all u_j, and calculate them using the theorem of residues.

Replacing $-2\eta_1$ and $\sum u_j$ in (25) by means of Eqs. (27), the Hamilton–Jacobi equation is transformed into

$$\sum_j \frac{1}{U'(u_j)}\left(-4A(u_j)\left(\frac{\partial S}{\partial u_j}\right)^2 - Q(u_j)\right) = 0, \tag{28}$$

where $Q(z) = z^n + 2P(z) = z^n + 2\eta_1 z^{n-1} + \cdots + 2\eta_n$. Setting each term in the sum equal to zero, we obtain

$$\left(\frac{\partial S}{\partial u_j}\right)^2 = -\frac{Q(u_j)}{4A(u_j)}, \tag{29}$$

so that

$$S(u, \eta) = \frac{1}{2}\sum_{j=1}^{n}\int^{u_j}\sqrt{\frac{Q(z)}{-A(z)}}\,\mathrm{d}z$$

is the solution we are looking for. The parameters $\eta_1, \dots, \eta_n$ are integrals of the motion which are related to the involutive algebraic integrals (2) by the formula (3) [7].

3. Actions and reduced actions

According to the Liouville–Arnold theorem [1], the generic level set corresponding to fixed values of these integrals must be a union of n-dimensional tori. In the neighborhood of these generic tori, the actions are defined by the formula

$$I_j = \frac{1}{2\pi}\int_{\gamma_j} \boldsymbol{y}\,\mathrm{d}\boldsymbol{x}, \quad j = 1, \dots, n,$$

where $\gamma_1, \dots, \gamma_n$ is a basis of cycles on these tori. In the coordinates $u_1, \dots, u_n$, the tori are given by fixing $\eta_1, \dots, \eta_n$, and the canonical one-form is

$$\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} = \sum_{i=1}^{n} p_i\,\mathrm{d}u_i = \frac{1}{2}\sum_{j=1}^{n}\sqrt{\frac{Q(u_j)}{-A(u_j)}}\,\mathrm{d}u_j, \tag{30}$$

since

$$p_j = \frac{\partial S}{\partial u_j} = \frac{1}{2}\sqrt{\frac{Q(u_j)}{-A(u_j)}}. \tag{31}$$

The last two equations can be rewritten in the form

$$w_j^2 = R(u_j), \quad w_j = 2A(u_j)p_j, \quad R(z) = -A(z)Q(z). \tag{32}$$

The zeros of $R(z)$ are the constants $a_0, \dots, a_n$ plus the n zeros $q_1, \dots, q_n$ of the polynomial $Q(z)$ which are known to be real [7]. We label them in increasing order: $z_0 \le z_1 \le \cdots \le z_{2n}$. Eqs. (32) shows that the points $P_j = (w_j, u_j)$ for any j belong to the real part of the hyperelliptic curve Γ, of genus n,

$$\Gamma = \{(w, u) \in \mathbb{C}^2 : w^2 = R(u)\}. \tag{33}$$

The real part of Γ consists of n ovals and one unbounded part; all P_i belong to the ovals since by definition the u_j are bounded. Let us denote by c_j the oval projecting into the segments $[z_{2j-1}, z_{2j}]$ under the map $\pi : (w, u) \to u$ (see Fig. 2).

The elliptical spherical coordinates $u_1, \dots, u_n$ determine uniquely only $x_0^2, \dots, x_n^2$ (see (17)), but not the signs of $x_0, \dots, x_n$. Now, $x_\nu = 0$ when one of the coordinates satisfies $u_j = a_\nu$. Consequently, when one of the ends of the segment $[z_{2j-1}, z_{2j}]$ is equal to a_ν, then the coordinate x_ν changes sign along the cycle c_j, i.e., it crosses over to another "2^{n+1}-tant" of $\mathbb{R}^{n+1}$. This means that the cycle γ_j on the Liouville torus corresponds to the oval c_j taken

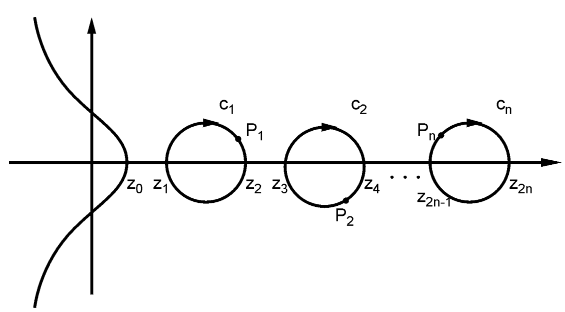

Fig. 2. The real part of the hyperelliptic curve $w^2 = R(u)$.

twice. The same is true when both ends of the segment $[z_{2j-1}, z_{2j}]$ are values a_ν and $a_{\nu+1}$, respectively; then the oval c_j involves two crossovers, at $x_\nu = 0$ and $x_{\nu+1} = 0$, while the cycle γ_j involves four. This observation leads to the following definition of cycles in the homology group of the curve Γ:

$$\gamma_j = c_j \quad \text{if the segment } [z_{2j-1}, z_{2j}] \text{ contains none of } a_0, \dots, a_n, \qquad \gamma_j = 2c_j \quad \text{otherwise.}$$

We can summarize these results in the following theorem.

Theorem 1. *The actions of the Neumann system are the complete hyperelliptic integrals*

$$I_j = \frac{1}{2\pi} \oint_{\gamma_j} \frac{Q(u)}{2w}\, \mathrm{d}u \tag{34}$$

over the cycles γ_j on the curve Γ defined above.

This actually means that the Liouville tori for the Neumann system are coverings of the real Jacobian varieties of the curves Γ, which are isomorphic to the direct product of the cycles c_j. We now illustrate this result for the simplest case $n = 1$. With the natural angular coordinate ϕ on the unit circle S^1, we have the energy

$$h = \tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}(a_0 \cos^2\phi + a_1 \sin^2\phi).$$

The coordinate u is defined by the equation

$$\frac{\cos^2\phi}{a_0 - u} + \frac{\sin^2\phi}{a_1 - u} = 0,$$

which gives $u = a_1 \cos^2\phi + a_0 \sin^2\phi$. In the coordinate ϕ, the action is obviously

$$I = \frac{1}{2\pi} \oint \sqrt{2h - (a_0 \cos^2\phi + a_1 \sin^2\phi)}\, \mathrm{d}\phi,$$

which in the coordinate u gives

$$I = \frac{1}{2\pi} \oint \sqrt{\frac{u - q}{(a_1 - u)(u - a_0)}}\, \mathrm{d}u, \qquad q = a_0 + a_1 - 2h,$$

where the integral is taken over the cycles c_1 in the previous notation. Comparing this formula with (34), we see the factor 2 which is compensated by the choice $\gamma_1 = 2c_1$. The oval c_1 projects to the interval $[q, a_1]$ in the case of oscillatory motion ($q > a_0$), and to the interval $[a_0, a_1]$ for rotations ($q < a_0$). Hence, for $n = 1$, the possibility $\gamma_j = c_j$ does not occur.

Another interpretation of the factor 2 derives from the following observation. Writing the energy as

$$h = \tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{4}(a_0 + a_1) + \tfrac{1}{4}(a_0 - a_1)\cos 2\phi,$$

we see that up to a constant it is the energy of a pendulum in the coordinate $\psi = 2\phi$. Thus, the Neumann system for $n = 1$ corresponds to *two* pendulums on the ϕ-circle.

The first case where it is possible that a u_j interval $[z_{2j-1}, z_{2j}]$ contains no a_ν is $n = 2$ (see Section 6). It is obvious that at the transition from $\gamma_j = 2c_j$ to c_j, the action I_j must be a *discontinuous* function on phase space. This defect can be removed if we consider the *reduced actions* J_j, which are the actions of the Neumann system after reduction by the obvious symmetry group $G = \mathbb{Z}_2^{n+1}$ generated by the $n+1$ reflections $(x_i, y_i) \to (-x_i, -y_i)$. The reduced space is smooth outside the fixed hyperplanes $x_i = y_i = 0$ which correspond to Neumann systems of dimension $n - 1$. Since the actions are defined only in the vicinity of a regular Liouville torus, we should exclude these motions anyway; the corresponding tori have dimension $n - 1$ and therefore are singular.

The coordinates $(u_1, \dots, u_n)$ are well defined on the reduced space because by definition they depend only on $x_0^2, \dots, x_n^2$. Eq. (31) gives the equation of the Liouville tori in the reduced space, which is isomorphic to the product of the ovals (or equivalently to the bounded real components of the Jacobian variety of Γ). Eq. (30) gives the canonical 1-form in the reduced coordinates, so the reduced actions are simply the complete Abelian integrals

$$J_j = \frac{1}{2\pi} \oint_{c_j} \frac{Q(u)}{w}\, \mathrm{d}u. \tag{35}$$

Notice that the J_j are obviously continuous functions of the constants of motion. The "explanation" of this fact is that the reduced system has only one Liouville torus at each level set obtained by fixing the constants of motion. One can show that the original Neumann system has $N = 2^{m+1}$ disjoint Liouville tori, where m is the number of cycles with $\gamma_j = c_j$ (see Appendix A). Therefore, the volume forms $\mathrm{d}I_1, \dots, \mathrm{d}I_n$ and $\mathrm{d}J_1, \dots, \mathrm{d}J_n$ are related by

$$|G| \mathrm{d}J_1 \cdots \mathrm{d}J_n = N\, \mathrm{d}I_1 \cdots \mathrm{d}I_n, \quad |G| = 2^{n+1}. \tag{36}$$

The existence of such a modified version of action variables for integrable systems with several Liouville tori at a given level of the integrals seems to be an interesting phenomenon which deserves further investigation.

In the rest of this paper, we will consider the reduced actions J_j. Since the definitions for J_j and I_j only differ by a constant, the results can be easily reformulated for the usual action I_j.

4. Picard–Fuchs equations

Using ideas from de Rham theory, the task of calculating the actions can be formulated in terms of ordinary linear differential equations. It is numerically much less expensive to integrate these equations than to compute the integrals. More importantly, this method provides insight into the analytic nature of actions and frequencies of the system, in particular their behavior near critical tori. It also provides a way to use analytic continuation in the complex in order to define global actions.

The basic observation is that all actions J_j in (35), as well as their derivatives with respect to the integration constants, are hyperelliptic integrals of the second kind, and that the space of Abelian differentials of the second kind, modulo total differentials, has dimension $2g = 2n$. As a possible basis, we can take the action differential

$$\beta = \frac{Q(z)}{w}\, \mathrm{d}z, \tag{37}$$

$2\pi J = \oint \beta$, and its successive derivatives with respect to one root, e.g., q_1. At this point, it is important that the integration paths are actually the cycles c_j and not the cycles γ_j related to the local actions. Denoting the partial derivatives of β with respect to q_1 by $\beta', \beta'', \dots, \beta^{(k)}, \dots, \beta^{(2n)}$, and using the abbreviations $X = z - q_1$, $Y = (z - q_2) \cdots (z - q_n)$, we have

$$\begin{aligned}
\beta &= Q(z) \frac{\mathrm{d}z}{w} = XY \frac{\mathrm{d}z}{w}, \\
\beta' &= -\frac{1}{2} Y \frac{\mathrm{d}z}{w}, \\
\beta'' &= -\frac{1}{4} \frac{Y}{X} \frac{\mathrm{d}z}{w}, \\
&\vdots \\
\beta^{(k)} &= c_k \frac{Y}{X^{k-1}} \frac{\mathrm{d}z}{w}, \\
&\vdots \\
\beta^{(2n)} &= c_{2n} \frac{Y}{X^{2n-1}} \frac{\mathrm{d}z}{w},
\end{aligned} \tag{38}$$

where

$$c_k = -\frac{1}{2}\left(+\frac{1}{2}\right)\left(+\frac{3}{2}\right)\cdots\left(\frac{2k-3}{2}\right) = -\frac{(2k-3)!!}{2^k}, \qquad c_0 = 1. \tag{39}$$

From de Rham theory, it follows that any $2n+1$ Abelian differentials of the second kind on the curve Γ of genus n are linearly dependent modulo total derivatives since the first cohomology of Γ has dimension $2n$. Thus, there exist coefficients $b_0, \dots, b_{2n}$ depending on q_j (and a_ν) such that

$$\sum_{s=0}^{2n} b_s \beta^{(s)} = \mathrm{d}F \tag{40}$$

for some meromorphic function F on Γ. Using the fact that the actions are periods of β along closed loops c on Γ, $2\pi J = \oint_c \beta$, we integrate this relation along c and obtain a linear differential equation of order $2n$ for J as a function of q_1,

$$\sum_{s=0}^{2n} b_s J^{(s)} = 0. \tag{41}$$

Such equations are called Picard–Fuchs equations.

The main problem is now to find some F and coefficients $b_0, \dots, b_{2n}$ that satisfy (40). The form of the derivatives $\beta^{(k)}$, Eq. (38), suggests the following form for F:

$$F = \frac{Yw}{X^{2n-1}}.$$

Indeed, with this choice

$$\mathrm{d}F = \frac{w\,\mathrm{d}Y}{X^{2n-1}} + \frac{Y\,\mathrm{d}w^2}{2wX^{2n-1}} - (2n-1)\frac{Yw\,\mathrm{d}z}{X^{2n}}.$$

Rewriting the last term with

$$\frac{Yw}{X^{2n}} = \frac{Yw^2}{X^{2n}w} = -\frac{XY^2A(z)}{X^{2n}w} = -\frac{Y^2A(z)}{X^{2n-1}w},$$

we have

$$\mathrm{d}F = Y\frac{-XA(z)Y' + \frac{1}{2}R' + (2n-1)YA(z)}{X^{2n-1}w}\,\mathrm{d}z,$$

where Y' and R' denote derivatives with respect to z. In order to solve (40), we rewrite this as a sum of powers of X similar to those in (38). Thus, we consider the polynomial of degree $2n$ in the numerator,

$$P(z) = -XA(z)Y' + \tfrac{1}{2}R' + (2n-1)YA(z), \tag{42}$$

and its Taylor expansion at $z = q_1$,

$$P(z) = \sum_{s=0}^{2n} p_s (z-q_1)^s = \sum_{s=0}^{2n} p_s X^s = P_1(X), \tag{43}$$

where $P_1(X) := P(X + q_1)$ is the polynomial P shifted by q_1. Now we have

$$\begin{aligned} \mathrm{d}F &= \frac{Y\sum_{s=0}^{2n} p_s X^s}{wX^{2n-1}}\,\mathrm{d}z = p_0\frac{Y\,\mathrm{d}z}{wX^{2n-1}} + p_1\frac{Y\,\mathrm{d}z}{wX^{2n-2}} + \cdots + p_{2n-1}\frac{Y\,\mathrm{d}z}{w} + p_{2n}\frac{XY\,\mathrm{d}z}{w} \\ &= b_{2n}\beta^{(2n)} + b_{2n-1}\beta^{(2n-1)} + \cdots + b_1\beta' + b_0\beta, \end{aligned}$$

so that

$$b_{2n-s} = \frac{p_s}{c_{2n-s}} = \frac{1}{s!c_{2n-s}}\left.\frac{\mathrm{d}^s P(z)}{\mathrm{d}z^s}\right|_{z=q_1}. \tag{44}$$

In particular,

$$b_{2n} = \frac{p_0}{c_{2n}} = \frac{P_1(0)}{c_{2n}} = \frac{P(q_1)}{c_{2n}} = \frac{(-\frac{1}{2} + (2n-1))\prod_{s=2}^{n}(q_1 - q_s)\prod_{\nu=0}^{n}(q_1 - a_\nu)}{c_{2n}}. \tag{45}$$

Dividing by b_{2n}, we get the normalized form of the Picard–Fuchs equation

$$J^{(2n)} + \sum_{s=0}^{2n-1}\frac{b_s}{b_{2n}}J^{(s)} = 0. \tag{46}$$

Notice that the singularities of this equation correspond to the cases when either $q_1 = q_j$ $(j = 2, \ldots, n)$ or $q_1 = a_\nu$, i.e., when the curve Γ becomes singular. Indeed, as one can see from the previous formulas all the coefficients $p_0, \ldots, p_{2n}$ and therefore $b_0, \ldots, b_{2n}$ are polynomials in $q_1, \ldots, q_n$. It is obvious what the corresponding equations in the variables $q_2, \ldots, q_n$ are.

Coefficients for the Picard–Fuchs equation of the differential $\mathrm{d}z/w$ have been given by Schlesinger in his classical book on linear differential equations [11]. Our formula can be considered as the generalization of those results to the differentials which determine the action in the Neumann system.

An important benefit of the linear differential Picard–Fuchs equations is that they give information on the nature of singularities of their solutions, i.e., of the actions and their derivatives. The singular points of the equation in the variable q_j are at $q_j = a_\nu$, $q_j = q_k$ $(k \neq j)$ and $q_j = \infty$; they are all found to be regular singular points. A linear differential equation with this property is called Fuchsian; from the general theory, it follows that Picard–Fuchs equations are always Fuchsian. The indicial equations can be solved for the leading indices of independent fundamental solutions.

To summarize the results of this section, we formulate the following theorem.

Theorem 2. *The Picard–Fuchs equation for the actions of the Neumann system with respect to the independent variable q_i has the form*

$$\sum_{s=0}^{2n}\frac{b_s}{b_{2n}}J^{(s)} = 0,$$

where the coefficients b_s are given by formula (44) *with q_1 replaced by q_i*, (42) *and* (39), *where $X = z - q_i$ and $Y = Q/X,\ R = -QA$.*

4.1. $n = 1$

Consider the Picard–Fuchs equation for the elliptic case $n = 1$. In this case, $b_1 = 0$ and $A(q) = (q - a_0)(q - a_1)$ so that

$$4(q - a_0)(q - a_1)J'' + J = 0, \tag{47}$$

where the prime denotes differentiation with respect to q. This second-order ODE has singularities at the three points $q = s_i$, $(s_0, s_1, s_2) = (a_0, a_1, \infty)$. The singularity at ∞ becomes obvious with the substitution $q \to 1/t$, which turns (47) into

$$\frac{\mathrm{d}^2 J}{\mathrm{d}t^2} + \frac{2}{t}\frac{\mathrm{d}J}{\mathrm{d}t} + \frac{1}{4t^2(1-a_0 t)(1-a_1 t)} J = 0. \tag{48}$$

This is a particular case of the classical hypergeometric equations (see, e.g. [14]). The indicial equation is $\alpha(\alpha - 1) = 0$ at both finite singular points, and $(\alpha + \frac{1}{2})^2 = 0$ at ∞. In terms of the Riemann P symbol, we thus have

$$J(q) = P\left\{\begin{matrix} a_0 & a_1 & \infty & \\ 0 & 0 & -\frac{1}{2} & q \\ 1 & 1 & -\frac{1}{2} & \end{matrix}\right\} = F\left(-\frac{1}{2}, -\frac{1}{2}; 0; \frac{q - a_0}{a_1 - a_0}\right), \tag{49}$$

where F is the hypergeometric function [14]. The indices tell us [4] that one regular solution in the neighborhood of each finite singular point $s_i = a_0$ or a_1 is analytic with a linear leading term,

$$f_1(q; s_i) = q - s_i + \mathrm{O}((q - s_i)^2), \tag{50}$$

and that an independent solution can be given in the form

$$f_2(q; s_i) = 1 + \mathrm{O}(q - s_i) + c_i f_1(q; s_i) \ln(q - s_i), \tag{51}$$

where the constant c_i in this case turns out to be $\pm\frac{1}{4}$ for $s_i = a_{0,1}$.

4.2. $n = 2$

Evaluating the general formulas for the case $n = 2$, we obtain the Picard–Fuchs equation

$$\begin{aligned} 0 &= 4A(q_1)Y(q_1)J'''' + (4q_1^2(7q_1 - 6q_2) - 4q_1(5q_1 - 4q_2)T + 4(3q_1 - 2q_2)S - 4P)J''' \\ &\quad + (9q_1(4q_1 - 3q_2) - 3(5q_1 - 3q_2)T + 3S)J'' + 3(q_1 - q_2)J' + \tfrac{3}{4}J = 0, \end{aligned} \tag{52}$$

where the prime denotes differentiation with respect to q_1, and T, S, P are the symmetric functions of the fixed zeros a_ν,

$$T = a_0 + a_1 + a_2, \qquad S = a_0 a_1 + a_1 a_2 + a_2 a_0, \qquad P = a_0 a_1 a_2. \tag{53}$$

The corresponding equation for the derivatives with respect to q_2 is obtained by exchanging q_1 and q_2.

The coefficient of the second highest derivative in the normalized Picard–Fuchs equation has the following simple partial fraction decomposition:

$$\frac{b_{2n-1}}{b_{2n}} = \frac{2}{q_1 - a_0} + \frac{2}{q_1 - a_1} + \frac{2}{q_1 - a_2} + \frac{1}{q_1 - q_2}. \tag{54}$$

This can be used to read off the indices of the finite regular singular points as long as q_2 is not equal to one of the a_ν. They are 0, 1, 1, 2 for $q_1 = a_\nu$, 0, 1, 2, 2 for $q_1 = q_2$, and $-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$ at infinity. When q_2 is confluent with one of the a_ν, the indices at $a_\nu = q_2$ are 0, $\frac{1}{2}$, 1, $\frac{3}{2}$.

4.3. General n

In the general case, it is not obvious whether it is possible (or useful) to obtain more explicit expressions for the b_s than (44). b_{2n-1}/b_{2n} is interesting because it determines the indices at the regular singular points when there is just one double root in $R(z)$. Using $b_{2n-1}/b_{2n} = (2n - \frac{3}{2})(\log P)'|_{z=q_1}$, a simple calculation shows that

$$\frac{b_{2n-1}}{b_{2n}} = (2n-3)\sum_{j=2}^{n}\frac{1}{q_1 - q_j} + (2n-2)\sum_{r=0}^{n}\frac{1}{q_1 - a_\nu}.$$

The fact that the coefficients in the partial fraction decomposition of b_{2n-1}/b_{2n} are either $p_c = 2n-2$ or $p_c = 2n-3$ determines the indices at the corresponding regular singular points as the solutions of the equation $\alpha(\alpha-1)\cdots(\alpha - 2n+2)(\alpha - 2n + 1 + p_c)$; hence they are $0, 1, \ldots, 2n-2$ and 1 or 2, respectively. At infinity, the indices are $-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, \frac{1}{2}(2n+1)$.

One can also verify the following expressions for the coefficients b_0 and b_1:

$$b_1 = -2nq_1 + 2\sum_{j=1}^{n} q_j, \tag{55}$$

$$b_0 = -\tfrac{1}{2}. \tag{56}$$

For nondegenerate potentials with $a_\mu \neq a_\nu$, $n > 3$, the discriminant b_{2n} can at most have a triple zero for real motions when $a_j = q_{j-1} = q_j = q_{j+1}$. The indices are then $0, 1, \ldots, 2n-4$ and $-\frac{1}{2}, \frac{1}{2}, 2$. There are two ways in which double zeros can occur in b_{2n}. When $q_j = q_{j+1} = a_\nu$, the indices become $0, 1, \ldots, 2n-3$ and $\frac{1}{2}, \frac{3}{2}$. In case of a double zero of the type $q_{j-1} = q_j = q_{j+1}$, the indices are $0, 1, \ldots, 2n-3$ and $\frac{1}{2}, \frac{5}{2}$.

5. Gauß–Manin equations

Instead of constructing a basis from successive derivatives of the action differential β, a standard basis of Abelian differentials such as

$$(\alpha_0, \alpha_1, \ldots, \alpha_{2n-1}) = \left(\frac{\mathrm{d}z}{w}, \frac{z\,\mathrm{d}z}{w}, \ldots, \frac{z^{2n-1}\,\mathrm{d}z}{w}\right) \tag{57}$$

may be taken to express β and any of its derivatives, up to total derivatives $\mathrm{d}F$, as linear combinations thereof.

The dependence of the integrals $\boldsymbol{K}_i := \oint \alpha_i$ on $\boldsymbol{q} = (q_1, \ldots, q_n)^t$ is governed by equations of the type

$$\frac{\mathrm{d}\boldsymbol{K}}{\mathrm{d}q_i} = \boldsymbol{M}^{(i)}\boldsymbol{K}, \quad i = 1, \ldots, n, \tag{58}$$

where $\boldsymbol{K} = (\boldsymbol{K}_0, \boldsymbol{K}_1, \ldots, \boldsymbol{K}_{2n-1})^t$ and the $2n \times 2n$ matrices $\boldsymbol{M}^{(i)}$ are found by decomposition of $\partial\alpha_k/\partial q_i$ in terms of α_l and suitable total derivatives. Such equations are called Gauß–Manin equations. From a knowledge of $\boldsymbol{K}_i(q_1, \ldots, q_n)$ the action is obtained with the linear relation $\beta = \alpha_n + 2\sum_{i=0}^{n-1}\eta_i\alpha_i$. The simplicity of this relation is the main advantage of this basis.

Notice one important difference between Picard–Fuchs and Gauß–Manin equations. The latter define n flows with respect to the "times" q_i on the $2n$-dimensional space of Abelian integrals. In the α-basis, each flow advances the same $2n$ quantities $K_i(\boldsymbol{q})$. The individual Picard–Fuchs equations may of course also be written as first-order systems with $2n$ components, namely, J and its first $2n-1$ derivatives with respect to q_i; but these components are different for each flow. This has implications for the numerical evaluation of the equations. If we want to integrate, say, from $\boldsymbol{q} = \boldsymbol{q}^i$ to $\boldsymbol{q}^f$, the Gauß–Manin equations for the directions q_i and q_j may be combined to follow a direct route. In the case of Picard–Fuchs equations, on the other hand, integration along the q_i direction must be followed by a linear but $\boldsymbol{q}$-dependent transformation between the respective sets of I derivatives before the integration can proceed in the q_j direction. We find that this is a severe limitation for their practical use. For example, in order to obtain the actions for an energy surface $\mathrm{d}\eta_1 = 0$ in the case $n = 2$, it is by far simpler to use the Gauß–Manin equations, in a combination $\mathrm{d}q_1 = -\mathrm{d}q_2$.

The n flows given by the solution of the linear $\boldsymbol{q}$-dependent systems (58) must commute, otherwise the solution $\boldsymbol{K}(\boldsymbol{q})$ would not exist. Therefore, the compatibility relations

$$\frac{\partial \boldsymbol{M}^{(i)}}{\partial q_j} - \frac{\partial \boldsymbol{M}^{(j)}}{\partial q_i} + [\boldsymbol{M}^{(i)}, \boldsymbol{M}^{(j)}] = 0 \tag{59}$$

must be satisfied.

The Gauß–Manin equation for Neumann systems of arbitrary n can be given explicitly once the following general result on Abelian integrals of the second kind is established.

Theorem 3. *Let $P(z) = \prod_{j=1}^{2n}(z - z_j) = \sum_{i=0}^{2n}(-1)^i \sigma_i z^{2n-i}$ be a polynomial of degree $2n$, all zeros z_j different, and consider the Riemann surface $w^2 = (q - z)P(z)$. Let $(K_0, \dots, K_{2n-1}) = \boldsymbol{K} = \boldsymbol{K}(q)$ be integrals over the same closed loop, of the standard basis of Abelian differentials of the second kind, $K_i = \oint \alpha_i = \oint z^i\, \mathrm{d}z/w$. Then the following ordinary differential equation holds*:

$$\frac{\mathrm{d}\boldsymbol{K}}{\mathrm{d}q} = MK, \quad \boldsymbol{M} = \sum_{k=1}^{2n} \frac{\boldsymbol{M}_k}{q - z_k}, \quad \boldsymbol{M}_k = -\frac{\boldsymbol{l}_k \otimes \boldsymbol{r}_k}{2D_k}, \tag{60}$$

where the partial discriminant D_k is

$$D_k = \left.\frac{P(z)}{z - z_k}\right|_{z=z_k} = \prod_{j\neq k}(z_k - z_j) =: \sum_{i=0}^{2n-1} (-1)^i \sigma_i^{(k)} z_k^{2n-1-i}, \tag{61}$$

and the matrix $\boldsymbol{M}_k$ derives from the tensor product of the two vectors $\boldsymbol{l}_k$ and $\boldsymbol{r}_k$, with components $(m = 1, \dots, 2n)$,

$$\boldsymbol{l}_k = (1, z_k, z_k^2, \dots, z_k^{2n-1})^t \equiv (l_{k1}, \dots, l_{k,2n})^t, \tag{62}$$

$$\boldsymbol{r}_k = (r_{k1}, \dots, r_{k,2n})^t, \qquad r_{km} = (-1)^{m-1}[m\sigma_{2n-1-m}^{(k)} z_k + (m-1)\sigma_{2n-m}^{(k)}]. \tag{63}$$

The $\sigma_i^{(k)}$ are the symmetric functions of the zeros of $P(z)/(z - z_k)$, with $\sigma_{-1}^{(k)} := 0$. The matrices $\boldsymbol{M}_k$ satisfy

$$\boldsymbol{M}_k^2 = 0. \tag{64}$$

Proof. The task is to find a decomposition of $\mathrm{d}\alpha_i/\mathrm{d}q$ into the basic differentials α_j and a total derivative $\mathrm{d}F$ (the same for all i),

$$\frac{\mathrm{d}\alpha_i}{\mathrm{d}q} = \sum_{j=0}^{2n-1} b_{ij}\alpha_j + b_{i,2n}\, \mathrm{d}F \quad (i = 0, \dots, 2n-1). \tag{65}$$

Once the b_{ij} have been determined, they constitute the elements of the matrix $\boldsymbol{M}$. Now, the decomposition (65) can be achieved with $F = w/(q-z)$ and

$$\mathrm{d}F = \frac{\mathrm{d}F}{\mathrm{d}z}\,\mathrm{d}z = \left(\frac{1}{2}\frac{w}{(q-z)^2} + \frac{P'(z)}{2w}\right)\mathrm{d}z = \frac{P + (q-z)P'}{2(q-z)w}\,\mathrm{d}z. \tag{66}$$

Differentiating $\alpha_i = z^i\,\mathrm{d}z/w$ with respect to q, we obtain $-z^i\,\mathrm{d}z/(2(q-z)w)$; hence (65) becomes

$$-z^i = \sum_{j=0}^{2n-1} 2b_{ij}(q-z)z^j + b_{i,2n}(P + (q-z)P'). \tag{67}$$

The coefficients b_{ij} for fixed i are determined by comparison of like powers z^j:

$$\begin{pmatrix} q & 0 & & & \cdots & -(q\cdot\sigma_{2n-1} - 1\cdot\sigma_{2n}) \\ -1 & q & 0 & & \cdots & +(2q\cdot\sigma_{2n-2} + 0\cdot\sigma_{2n-1}) \\ 0 & -1 & q & 0 & \cdots & -(3q\cdot\sigma_{2n-3} + 1\cdot\sigma_{2n-2}) \\ \vdots & & & & & \vdots \\ 0 & \cdots & & -1 & q & +(2nq\cdot\sigma_0 + (2n-2)\cdot\sigma_1) \\ 0 & 0 & \cdots & 0 & -1 & -(2n-1)\cdot\sigma_0 \end{pmatrix}\begin{pmatrix} 2b_{i0} \\ 2b_{i1} \\ 2b_{i2} \\ \vdots \\ 2b_{i,2n-1} \\ b_{i,2n} \end{pmatrix} = -\boldsymbol{e}_i, \tag{68}$$

where $\boldsymbol{e}_i$ has components $e_{ij} = \delta_{ij}$. Let us denote the $(2n+1)\times(2n+1)$ matrix in this equation by $\boldsymbol{N}$; it is the same for all i. Multiplying with $\boldsymbol{N}^{-1}$ and taking the transposed equations for $i = 0, \ldots, 2n-1$, we see that the matrix $2b_{ij}$, hence $2\boldsymbol{M}$, is the upper left $(2n\times 2n)$ minor of $-\boldsymbol{N}^{-1t}$.

Developing with respect to the last column and sorting terms, the determinant is found to be $\det(\boldsymbol{N}) = P(q) = \prod_{j=1}^{2n}(q-z_j)$. This shows that $\boldsymbol{N}$ is singular at $q = z_j$ $(j = 1, \ldots, 2n)$, regular else. The inverse matrix $\boldsymbol{N}^{-1}$ must therefore have corresponding first-order poles. To obtain $\boldsymbol{N}^{-1t}$, we decompose $\boldsymbol{N}$ as

$$\boldsymbol{N} = \boldsymbol{A} + q\boldsymbol{B} \tag{69}$$

with $\boldsymbol{A}$ and $\boldsymbol{B}$ independent of q. The matrix $\boldsymbol{A}$ has full rank $2n+1$ whereas $\boldsymbol{B}$ has corank 1, its last row being identically zero. The matrices $\boldsymbol{N}_k := \boldsymbol{A} + z_k\boldsymbol{B}$ are singular, with left kernels $\tilde{\boldsymbol{l}}_k = (1, z_k, z_k^2, \ldots, z_k^{2n})^t$ and right kernels $\tilde{\boldsymbol{r}}_k = (r_{k1}, r_{k2}, \ldots, r_{k,2n}, 1)^t$, where the r_{km} are given in (63). We also need the left and right null eigenvectors of $\boldsymbol{B}$; they are $\tilde{\boldsymbol{l}}_{2n+1} = (0, \ldots, 0, 1)^t$ and $\tilde{\boldsymbol{r}}_{2n+1} = (\sigma_{2n-1}, -2\sigma_{2n-2}, 3\sigma_{2n-3}, \ldots, (2n-1)\sigma_1, -2n\sigma_0, 1)^t$, respectively.

Let us now form the two $(2n+1)\times(2n+1)$ matrices $\boldsymbol{L}$ and $\boldsymbol{R}$ with elements $L_{kj} = \tilde{l}_{jk}$ and $R_{kj} = \tilde{r}_{jk}$. We assert that

$$\boldsymbol{L}^t NR = \boldsymbol{L}^t(\boldsymbol{A} + q\boldsymbol{B})\boldsymbol{R} = \mathrm{diag}((q-z_1)D_1, \ldots, (q-z_{2n})D_{2n}, 1). \tag{70}$$

To see this, consider first the elements $(\boldsymbol{L}^t NR)_{jk} = \tilde{\boldsymbol{l}}_j^t\boldsymbol{N}\tilde{\boldsymbol{r}}_k$ for $j, k \le 2n$. We decompose

$$\tilde{\boldsymbol{l}}_j^t\boldsymbol{N}\tilde{\boldsymbol{r}}_k = \tilde{\boldsymbol{l}}_j^t(\boldsymbol{N} - \boldsymbol{N}_j)\tilde{\boldsymbol{r}}_k + \tilde{\boldsymbol{l}}_j^t\boldsymbol{N}_j\tilde{\boldsymbol{r}}_k = (q-z_j)\tilde{\boldsymbol{l}}_j^t\boldsymbol{B}\tilde{\boldsymbol{r}}_k, \tag{71}$$

where the last equation follows because $\tilde{\boldsymbol{l}}_j$ is in the left kernel of $\boldsymbol{N}_j$. With a similar argument, we find that $\tilde{\boldsymbol{l}}_j^t\boldsymbol{N}\tilde{\boldsymbol{r}}_k = (q-z_k)\tilde{\boldsymbol{l}}_j^t\boldsymbol{B}\tilde{\boldsymbol{r}}_k$, and since $z_j \ne z_k$, we conclude that the nondiagonal elements vanish. As for the diagonal elements, it is not difficult to check that

$$\tilde{\boldsymbol{l}}_k^t\boldsymbol{B}\tilde{\boldsymbol{r}}_k = -\sigma_{2n-1} + 2\sigma_{2n-2}z_k - 3\sigma_{2n-3}z_k^2 + \cdots = \left.\frac{\mathrm{d}P}{\mathrm{d}z}\right|_{z=z_k} = D_k. \tag{72}$$

Similarly, we have $(\boldsymbol{L}^t NR)_{2n+1,k} = (q - z_k)\tilde{\boldsymbol{l}}^t_{2n+1}\boldsymbol{B}\tilde{\boldsymbol{r}}_k = 0$ for $k \le 2n$, because $\tilde{\boldsymbol{l}}^t_{2n+1}\boldsymbol{B} = 0$, and $(\boldsymbol{L}^t NR)_{j,2n+1} = 0$ for $j \le 2n$ with analogous arguments. Finally, $(\boldsymbol{L}^t NR)_{2n+1,2n+1} = \tilde{\boldsymbol{l}}^t_{2n+1}\boldsymbol{A}\tilde{\boldsymbol{r}}_{2n+1} = 1$.

Having established the diagonalization (70), we invert it to obtain

$$\boldsymbol{N}^{-1t} = \boldsymbol{L}\,\mathrm{diag}\left(\frac{1}{D_1(q - z_1)}, \dots, \frac{1}{D_{2n}(q - z_{2n})}, 1\right)\boldsymbol{R}^t, \tag{73}$$

and using the fact that the last column of $\boldsymbol{L}$ is $(0, \dots, 0, 1)^t$, we conclude that neither the $(2n+1)$th components of $\tilde{\boldsymbol{l}}_k$ and $\tilde{\boldsymbol{r}}_k$ nor the vectors $\tilde{\boldsymbol{l}}_{2n+1}$ and $\tilde{\boldsymbol{r}}_{2n+1}$ contribute to the upper left $(2n \times 2n)$ block of $\boldsymbol{N}^{-1t}$. This establishes Eq. (60). The properties (64) of $\boldsymbol{M}_k$ follow from the orthogonality $\boldsymbol{r}^t_k\boldsymbol{l}_k = 0$ which is easily checked with (62) and (63). □

Application of this theorem to the Neumann system is straightforward. The n Eqs. (58) are obtained with $w^2 = -A(z)Q(z) =: (q_m - z)P_m(z),\ m = 1, \dots, n$, where the set $\{z^{(m)}_k : k = 1, \dots, 2n\}$ of zeros of $P_m(z)$ consists of the a_ν and all q_i except q_m. This completes the derivation of the Gauß–Manin equations. One general statement can be made in view of the nature of the matrices $-2D^{(m)}_k\boldsymbol{M}^{(m)}_k = \boldsymbol{l}^{(m)}_k \otimes \boldsymbol{r}^{(m)}_k$. The rank of $\boldsymbol{M}^{(m)}_k$ is at most 1, and the $(2n - 1)$-dimensional orthogonal complement of the vector $\boldsymbol{r}^{(m)}_k$ is its eigenspace with eigenvalue 0, i.e., if initial conditions are chosen from that space, the solutions are regular at $q_m = z^{(m)}_k$. From $(\boldsymbol{l}^{(m)}_k, \boldsymbol{r}^{(m)}_k) = 0$ for all k, it follows that the characteristic polynomial of each matrix $\boldsymbol{M}^{(m)}_k$ is λ^{2n}, i.e., all $2n$ eigenvalues are zero. Nevertheless, there exist only $2n - 1$ independent eigenvectors; the Jordan normal form contains a non-trivial block which means that one of the solutions has a logarithmic divergence at $q_m = z^{(m)}_k$.

5.1. $n = 1$

The Gauß–Manin equation for the differentials $\alpha_0 = \mathrm{d}z/w$ and $\alpha_1 = z\,\mathrm{d}z/w$ are easily derived with the help of

$$\begin{pmatrix} \beta^{(0)} \\ \beta^{(1)} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -2q & 2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \tag{74}$$

compare with (38). For $n = 1$, this gives no particular advantage over the Picard–Fuchs equation, but as an illustration of Theorem 3, it is interesting to note that the equations take of the form

$$2(a_1 - a_0)\frac{\mathrm{d}}{\mathrm{d}q}\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \left[\frac{1}{q - a_0}\begin{pmatrix} 1 \\ a_0 \end{pmatrix} \otimes \begin{pmatrix} a_0 \\ -1 \end{pmatrix} - \frac{1}{q - a_1}\begin{pmatrix} 1 \\ a_1 \end{pmatrix} \otimes \begin{pmatrix} a_1 \\ -1 \end{pmatrix}\right]\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \tag{75}$$

5.2. $n = 2$

The Gauß–Manin equations are obtained from Theorem 3. The change of $\boldsymbol{K}_i = \oint \alpha_i$ with q_1 is

$$\frac{\mathrm{d}\boldsymbol{K}_i}{\mathrm{d}q_1} = M^{(1)}_{ij}\boldsymbol{K}_j \tag{76}$$

with

$$\boldsymbol{M}^{(1)} = -\sum_{k=1}^{4}\frac{1}{2D^{(1)}_k}\frac{\boldsymbol{l}^{(1)}_k \otimes \boldsymbol{r}^{(1)}_k}{q_1 - z^{(1)}_k}. \tag{77}$$

The sum is over the $2n = 4$ singular points $z^{(1)}_k$ of the equation for q_1, $(z^{(1)}_1, z^{(1)}_2, z^{(1)}_3, z^{(1)}_4) = (a_0, a_1, a_2, q_2)$, and the constants $D_k^{(j)}$ are defined via

$$\frac{1}{D_k^{(j)}} = \operatorname*{Res}_{z=z_k} \frac{z-q_j}{w^2}. \tag{78}$$

(The constants $D_k^{(2)}$ appear in the equation for q_2.) The vectors $\boldsymbol{l}_k^{(1)}$ and $\boldsymbol{r}_k^{(1)}$ are

$$\boldsymbol{l}_k^{(1)t} = (1, z_k^{(1)}, z_k^{(1)2}, z_k^{(1)3}), \qquad \boldsymbol{r}_k^{(1)t} = (z_k^{(1)}\sigma_2^{(1k)}, -(2z_k^{(1)}\sigma_1^{(1k)} + \sigma_2^{(1k)}), 3z_k^{(1)} + 2\sigma_1^{(1k)}, -3) \tag{79}$$

with

$$\sigma_1^{(jk)} = \sum_{l\neq k} z_l^{(j)}, \qquad \sigma_2^{(jk)} = \sum_{l<m} z_l^{(j)} z_m^{(j)} \quad (l \neq k, m \neq k). \tag{80}$$

Performing the sum in (77) explicitly, leads to $M_{ij}^{(1)} = -(1/2D_1)m_{ij}^{(1)}$ with

$$m_{ij}^{(1)} = \begin{pmatrix} q_1^2 q_{12} - q_1 q_{12} T + q_1 S & q_1^2 - q_1 A - B & q_1 + 2A & -3 \\ q_1 q_2 S + q_{12} P & q_1 m_{12}^{(1)} & q_1 m_{13}^{(1)} & -3q_1 \\ q_1 m_{21}^{(1)} & m_{21}^{(1)} - 2q_1^2 B & q_1^2 m_{13}^{(1)} & -3q_1^2 \\ q_1^2 m_{21}^{(1)} & q_1 m_{32}^{(1)} & m_{21}^{(1)} + 3q_1^3 A - q_1^2 B & -3q_1^3 \end{pmatrix}, \tag{81}$$

where in addition to (53) we used the abbreviations $q_{12} := q_1 - q_2,\ A = q_2 + T,\ B = q_2 T + S$. The corresponding expressions in the equation for $\mathrm{d}K_i/\mathrm{d}q_2$ are obtained with $q_1 \leftrightarrow q_2$.

Remark. We should mention that there is another natural set of differentials

$$\varphi_\nu = \frac{\mathrm{d}z}{(z-z_\nu)\sqrt{R(z)}}, \quad \nu = 0, 1, \ldots, 2n,$$

where z_ν are the zeros of $R(z) : z_k = a_k$ for $k = 0, \ldots, n,\ z_{k+n} = q_k$ for $k = 1, \ldots, n$. Their sum is cohomological to zero,

$$\sum_{\nu=0}^{2n} \varphi_\nu = \mathrm{d}F, \quad F = -2R(z)^{-1/2},$$

but any $2n$ of them form a basis in the first cohomology of the corresponding hyperelliptic curve. These differentials satisfy the following simple system:

$$\partial_\mu \varphi_\nu = \frac{1}{2}\frac{\varphi_\nu - \varphi_\mu}{z_\nu - z_\mu}, \quad \mu \neq \nu.$$

So, the corresponding Gauß–Manin equations for

$$L_\sigma = \oint_\gamma \varphi_\sigma$$

have a very simple form (which is actually a particular case of the Knizhnik–Zamolodchikov equation [5] [1])

$$\partial_\mu L_\sigma = \frac{1}{2}\frac{L_\sigma - L_\mu}{z_\sigma - z_\mu}, \quad \mu \neq \sigma, \tag{82}$$

[1] We are grateful to Vitaly Tarasov, who explained this to us.

$$\partial_\sigma L_\sigma = \partial_\sigma \left(-\sum_{\nu\neq\sigma} L_\nu \right) = -\frac{1}{2}\sum_{\nu\neq\sigma}\frac{L_\sigma - L_\nu}{z_\sigma - z_\nu}. \tag{83}$$

Unfortunately in this basis it seems to be impossible to find simple explicit formulae for the action of the Neumann system, so the basis we used in the present paper looks more suitable from this point of view.

6. Calculation of energy surfaces in action variables

Let us introduce the *moment map* $F : T^*S^n \to \mathbb{R}^n$ given by the integrals $F_1, \dots, F_n$ of the Neumann system. We call the image of this map in $\mathbb{R}^n$ the *moment space*. The Picard–Fuchs and Gauß–Manin equations are equations on moment space. For each dimension in moment space there is one independent flow. For the Gauß–Manin equations, the dependent variables are all the same for the different commuting flows.

In the Neumann system, the moment space has the trivial topology of $\mathbb{R}^n$; however, the differential equations have singularities at the critical values of the moment map. From the explicit formula (35), we know that there exists a global continuous solution. To obtain this particular solution from integration of the differential equation, we have to (1) understand the structure of the singularities, (2) find appropriate initial conditions, and (3) determine how to continue the solution across the singularities.

The bifurcation diagram is the set of critical values of the moment map. We have already seen that the singularities of the differential equations occur when the curve Γ becomes singular, i.e., when there is a collision between the roots of w^2. This set is called the *discriminant surface*. The bifurcation diagram is contained in the discriminant surface; for the determination of actions we are only interested in that part of the discriminant surface whose pre-image contains real tori.

The bifurcation diagram divides the moment space into regions of regularity which we call m-phases. Throughout a given m-phase, all pre-images of the moment map are topologically equivalent. In particular, the numbers (ρ, o, m) defined in Appendix A are constant within an m-phase, where ρ is the number of rotational degrees of freedom, o the number of oscillatory modes that cross a hyperplane $x_\nu = 0$, and m the number of oscillatory modes without hyperplane crossing. $N = 2^{m+1}$ is the number of disjoint tori in phase space.

The m-phases are most conveniently determined from the nature of the u_j-intervals $[z_{2j-1}, z_{2j}]$. Consider Fig. 2 and focus on the forbidden real intervals between the ovals. From each a_j there extends exactly one such region, either to the left (l) or to the right (r). Moreover, the rightmost forbidden interval always extends to the right, otherwise there could not be one oval in each interval $[a_{j-1}, a_j]$, and hence no real motion. The sequence of forbidden intervals can therefore be encoded as a string of $n+1$ letters l and r, starting with the letter for the interval $[z_0, z_1]$, and ending with the letter r. The total number of such strings is 2^n.

For $n = 1$, the only two possibilities are rr (oscillation) and lr (rotation). In general, there are four different types of motion for the variable u_j, depending on the letters for the neighboring forbidden intervals:

lr =: R	u_j is free to move from a_{j-1} to a_j; this is a rotation (when u_j reaches an endpoint of its interval, it changes to a different leaf of the covering of S^n by the coordinate system).
rr =: $\mathrm{O_r}$	u_j oscillates in the right part of its interval, between q_j and a_j.
ll =: $\mathrm{O_l}$	u_j oscillates in the left part of its interval, between a_{j-1} and q_{j+1}.
rl =: $\mathrm{O_m}$	u_j oscillates in the interior part of its interval, between q_j and q_{j+1}.

In the case $n = 2$, there are four types of tori encoded by llr, lrr, rlr, and rrr. The projections of Liouville–Arnold tori onto the sphere S^2 are shown as dark patches in Fig. 3 where again $(a_0, a_1, a_2) = (1, 2, 4)$. In detail, the four types of tori are:

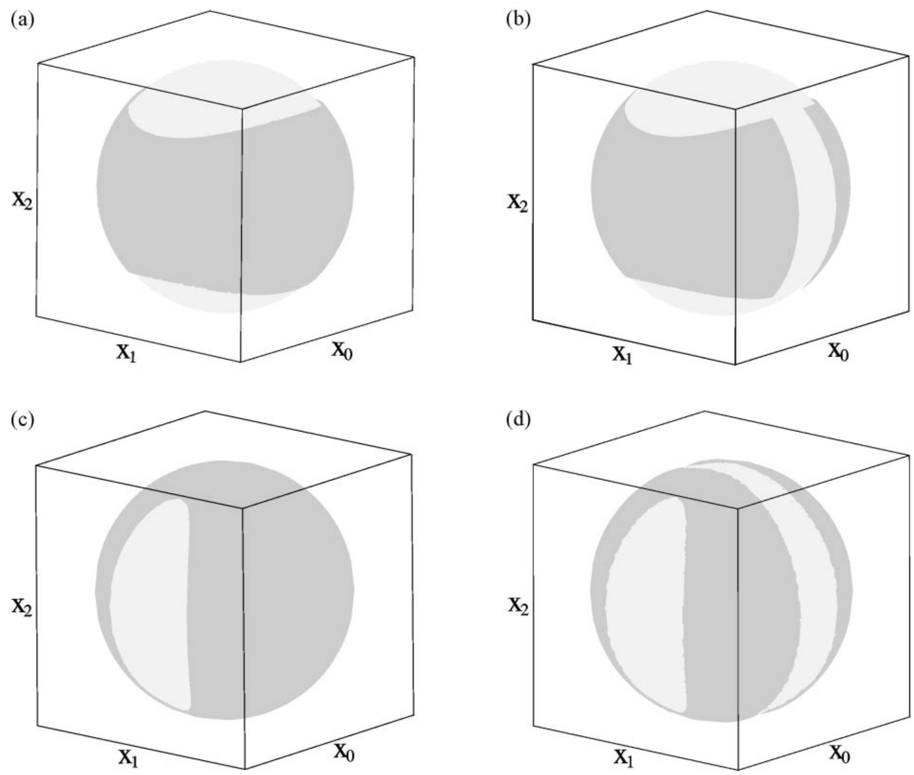

Fig. 3. Projections of invariant tori (dark) from the phase space T^*S^2 onto the sphere S^2. The boundaries (caustics) are given by coordinate lines of the elliptical spherical coordinate system: (a) type lrr $=$ $\mathrm{RO_r}$; (b) type rrr $=$ $\mathrm{O_rO_r}$; (c) type llr $=$ $\mathrm{O}_l\mathrm{R}$; (d) type rlr $=$ $O_\mathrm{m}\mathrm{R}$.

lrr $=$ $\mathrm{RO_r}$ — u_1 varies in its full interval $[a_0, a_1]$, and with two sweeps back and forth completes one rotation about the x_2-axis; u_2 oscillates in $[q_2, a_2]$, but two of its periods are needed to complete one period of x_2 oscillations. The same projection on S^2 corresponds to two tori in T^*S^2 which differ in the sense of rotation. $(\rho, o, m) = (1, 1, 0),\ N = 2$.

rrr $=$ $\mathrm{O_rO_r}$ — Both variables oscillate, u_1 in $[q_1, a_1]$ and u_2 in $[q_2, a_2]$. Each of the two patches in the projection corresponds to one torus in T^*S^2. $(\rho, o, m) = (2, 2, 0),\ N = 2$.

llr $=$ $\mathrm{O_lR}$ — The motion type is similar to $\mathrm{RO_r}$, but with the behavior of u_1 and u_2 interchanged. The rotation is now about the x_0-axis, and the oscillation in the x_0-coordinate. The projection corresponds to two tori in T^*S^2. $(\rho, o, m) = (1, 1, 0),\ N = 2$.

rlr $=$ $\mathrm{O_mR}$ — The oscillations of u_1 are restricted to the interval $[q_1, q_2]$ so that there are two patches in the projection, each corresponding to two tori in T^*S^2 which differ in the sense of rotation about the x_0-axis. $(\rho, o, m) = (1, 0, 1),\ N = 4$.

Fig. 4 contains four different versions of m-*phase diagrams* which show how the type of motion depends on the values of the constants of motion. The plane of zeros (q_1, q_2) gives the simplest picture (see Fig. 4a). The disadvantage of this set of constants of motion is that they are not smooth function on phase space. Only η_j and F_ν are smooth functions on phase space, and therefore qualify as good coordinates for a *bifurcation diagram*. The

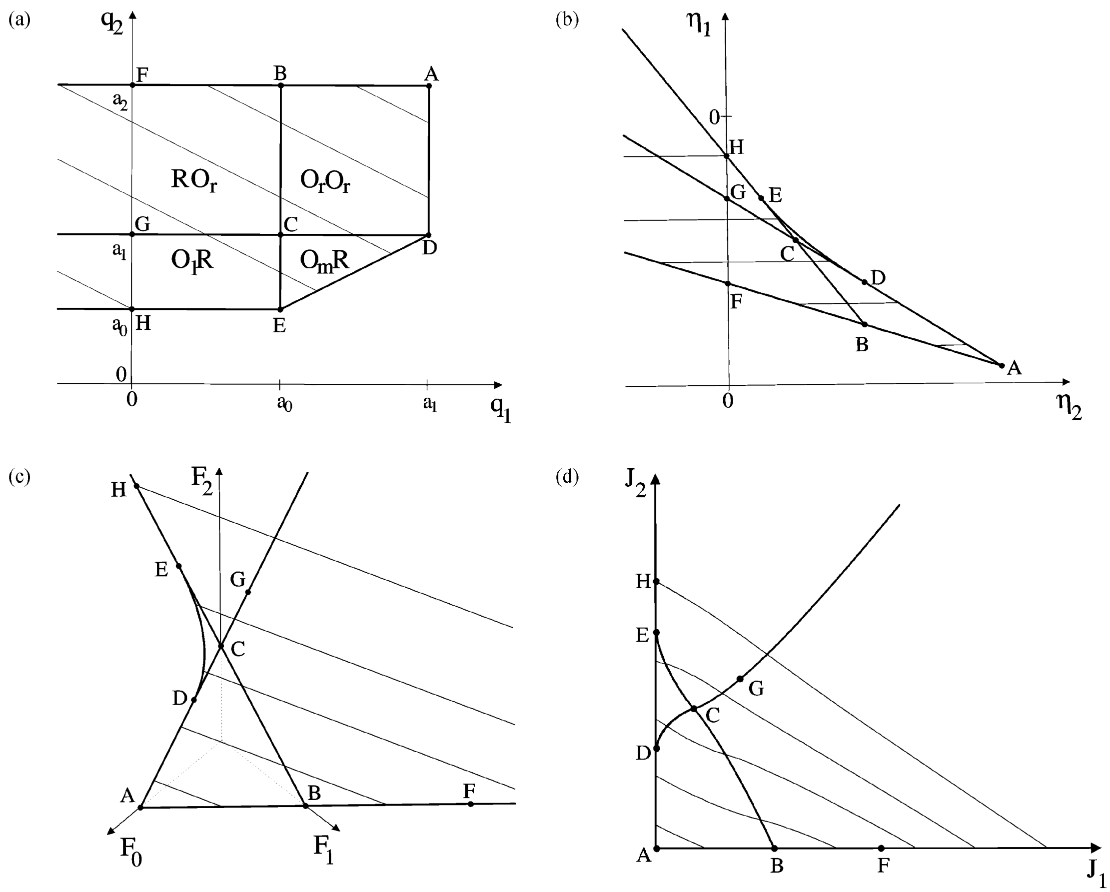

Fig. 4. Phase diagrams with integrals: (a) (q_1, q_2); (b) (η_1, η_2); (c) (F_0, F_1, F_2); (d) (J_1, J_2). The actions are given for the symmetry-reduced system. The correspondence between the four figures can be read off the points A, ..., F. The thin lines are isoenergy lines.

transformation from these smooth constants to (q_1, q_2) is singular when $q_1 = q_2$, while the transformation to (J_1, J_2) is singular, in addition, when $q_j = a_\nu$. In spite of these singularities, each set has its advantages. The q_j are best suited to determine the Picard–Fuchs and Gauß–Manin equations; singularities in the real transformation are then less important because the equations are considered in the complex domain.

The Neumann system on the sphere S^2 is related to a hyperelliptic curve Γ of genus $g = 2$ given by the equation $w^2 = R(z) = -A(z)Q(z)$ with polynomials

$$Q(z) = (z - q_1)(z - q_2) = z^2 + 2\eta_1 z + 2\eta_2, \qquad A(z) = (z - a_0)(z - a_1)(z - a_2), \tag{84}$$

where

$$2\eta_1 = 2h - (a_0 + a_1 + a_2) = -(q_1 + q_2) \quad \text{and} \quad 2\eta_2 = q_1 q_2, \tag{85}$$

η_1 represents the energy constant h. Lines of constant energy h are given as $q_1 + q_2 = \text{const}$ in Fig. 4a. Fig. 4b displays the same information in the plane of separation constants (η_1, η_2); the correspondence is given by (85). Isoenergy lines are $\eta_1 = \text{const}$. By the linear transformation $2\eta_1 = F_0(a_1 + a_2) + F_1(a_2 + a_0) + F_2(a_0 + a_1)$, $2\eta_2 =$

$F_0a_1a_2 + F_1a_2a_0 + F_2a_0a_1$, the phase diagram is taken to the plane of the algebraic integrals (F_0, F_1, F_2) (see Fig. 4c).

The eight points A, ..., H help to see the connection between the different phase diagrams. Three of these points represent equilibria: A corresponds to the stable equilibria $(x_0, x_1, x_2) = (\pm 1, 0, 0)$ at minimum energy $h(\mathrm{A}) = \frac{1}{2}a_0$. B corresponds to the unstable elliptic–hyperbolic equilibria $(x_0, x_1, x_2) = (0, \pm 1, 0)$ at energy $h(\mathrm{B}) = \frac{1}{2}a_1$, and includes motion along the circle $x_2 = 0$ which is asymptotic to the equilibrium points. C corresponds to the unstable equilibria $(x_0, x_1, x_2) = (0, 0, \pm 1)$ at energy $h(\mathrm{C}) = \frac{1}{2}a_2$. The pre-image in phase space is a separatrix which projects onto the entire configuration sphere.

Equilibria mark transitions in the topology of isoenergy surfaces in phase space. Increasing h through $h(\mathrm{A})$, the transition is from an empty set to two 3-spheres S^3, corresponding to the two accessible disks on the configuration sphere S^2. At energy $h(\mathrm{B})$, the accessible region on S^2 becomes an annulus, and the two S^3 merge to form an isoenergy surface of type $S^1 \times S^2$. Finally, at energies above $h(\mathrm{C})$, the entire configuration sphere is accessible, and the isoenergy surface becomes the real projective space $\mathbb{R}P^3$.

The phase diagrams show that these transitions at A, B, and C involve bifurcations of invariant tori: new types of tori appear as energy is increased. In addition, there occur bifurcations of critical tori at D and E. Increasing h through $h(\mathrm{D}) = a_0 + a_2 - a_1$, oscillation along the circle $x_1 = 0$ about the stable equilibrium points becomes unstable, and two stable rotations $u_1 = q_1 = q_2$ emerge which are images of each other under time reversal (pitchfork bifurcation). An inverse pitchfork bifurcation takes place when h increases through $h(\mathrm{E}) = a_1 + a_2 - a_0$: the two stable rotations $u_1 = q_1 = q_2$ and the unstable rotation along the circle $x_0 = 0$ merge into one stable rotation. For the Picard–Fuchs equation the points D and E are also special because they do not have the generic indices as a result of a coalescence of singularities.

From the point of view of foliation of phase space by isoenergy surfaces and invariant tori, the points A, ..., E mark five topological transitions. The points F, G, and H play no particular role in the organization of phase space; they merely serve for orientation in the different phase diagrams.

For arbitrary n, the singularity structure of moment space is quite similar. The general phase diagram for the Liouville–Arnold tori of the Neumann system is easily discussed in terms of the constants q_j. Its outer boundary in $\mathbb{R}^n(q_1, \dots, q_n)$ is given by the $3n - 2$ planes $q_j = a_j$ $(j = 1, \dots, n)$, $q_j = a_{j-2}$, $(j = 2, \dots, n)$, and $q_j = q_{j+1}(j = 1, \dots, n-1)$. They correspond to critical tori of elliptic type. The inner planes $q_j = a_{j-1}(j = 1, \dots, n)$ correspond to hyperbolic critical tori and their associated separatrices. In the following discussion of the change of tori at separatrices, we assume that all the q_j except the one under consideration are fixed at non-critical values.

1. When $q_j = a_j$ for any j, the variable u_j cannot vary, $u_j = a_j$ and $p_j = 0$. For q_j slightly lower, u_j oscillates between q_j and a_j. This shows that the bifurcation involves an elliptic critical torus.
2. Similarly, when $q_j = a_{j-2}$ for any j, the coordinate u_{j-1} is fixed to a_{j-2}, and $p_{j-1} = 0$. For q_j slightly larger, u_{j-1} oscillates between q_j and a_{j-2}. Again, the critical torus is elliptic.
3. When $q_j = q_{j+1}$ (which may only happen between a_{j-1} and a_j), then $u_j = q_j$ is fixed, $p_j = 0$. For q_j slightly lower, u_j oscillates between q_j and q_{j+1}. Also here, the critical torus is elliptic.
4. When $q_j = a_{j-1}$, the situation is different. For q_j slightly above a_{j-1}, the variable u_j has q_j as its left endpoint and varies from there to either a_j or q_{j+1}, whichever is smaller. For q_j slightly below a_{j-1}, the variable u_{j-1} has q_j as its right endpoint and varies from there to either a_{j-2} or q_{j-1}, whichever is larger. Thus there is a transition between different types of tori; at the transition point the tori coalesce and form a separatrix.

The point $(q_1, q_2, \dots, q_n) = (a_0, a_1, \dots, a_{n-1})$ may be viewed as the organizing center of the bifurcation diagram. It is the intersection point of all inner bifurcation planes, hence it is corner point to 2^n regions with different types of tori. Each region is locally bounded by the n planes $q_j = a_{j-1}$, and extends without further internal boundaries to the outer boundary (or to $-\infty$ in the variable q_1). Again this shows that there are exactly 2^n

topologically different types of tori. The codimension n corners of the regions that are not contained in any of the hyperplanes $q_j = q_{j+1}$ correspond to stable isolated periodic orbits. All other isolated periodic orbits are unstable.

Let us now comment on the computation of actions based on the Gauß–Manin equations. We start at A, the simplest point in the bifurcation diagram. For arbitrary n, it is given by $q_i = a_i$. The corresponding motion is the stable equilibrium point at the minimum of the potential. Accordingly all actions are zero at A. To find initial conditions for the Gauß–Manin equations, we have to find the value of the integrals K_i^j. Since at A, the curve has n double roots the genus of the curve drops to 0, and the integrals can be evaluated using residue calculus. To calculate the action J_j we have to chose the cycle c_j which can be contracted to a point at A such that

$$K_l^j\Big|_{\mathrm{A}} = \oint_{c_j} \frac{z^l}{w}\Big|_{\mathrm{A}}\,\mathrm{d}z = 2\pi i \operatorname*{Res}_{z=a_j} \frac{z^l\sqrt{z-a_0}}{A(z)} = 2\pi \frac{a_j^l}{\sqrt{a_j-a_0}\prod_s{}'(a_j-a_s)}, \tag{86}$$

where $\prod'$ runs from $s=1$ to n excluding $s=j$.

For each j, this initial condition corresponds to a solution which is regular at the singular point A. The singularity in $\mathbf{M}$ at A is canceled by $\mathbf{K}^j$; yet, the first step of the integration cannot be done numerically. Since it is not obvious that our solution must be analytic, we cannot use the differential equation to find the Taylor expansion of the solution at A. However, we can again use the integral solution and calculate its series at A. We choose to step away from A in the direction $\Delta\mathbf{q} = (-1, \dots, -1)$, i.e., we want to calculate $\mathbf{K}^j(\mathbf{q}_{\mathrm{A}} - \epsilon\Delta\mathbf{q})$. The first term in the expansion is already known, the second term is given by

$$\sum_k \frac{\partial}{\partial q_k} K_l^j\Bigg|_{\mathrm{A}} = 2\pi i \operatorname*{Res}_{z=a_j} - \frac{z^l}{2\sqrt{a_0-z}\prod_s(z-a_s)} \sum_k \frac{1}{z-a_k} = \frac{\pi}{2} a_j^{l-1} \frac{2la_0-(2l-1)a_j}{\sqrt{a_j-a_0}^3\prod_s{}'(a_j-a_s)}. \tag{87}$$

Therefore, we find to first-order in ϵ,

$$K_l^j(\mathbf{q}_{\mathrm{A}} + \epsilon\Delta\mathbf{q}) = K_l^j\Big|_{\mathrm{A}} \left(1 - \frac{\epsilon}{4}\left(\frac{2l}{a_j} - \frac{1}{a_j-a_0}\right)\right) + \mathrm{O}(\epsilon^2). \tag{88}$$

From there it is straightforward to integrate the Gauß–Manin equations in the entire m-phase rr, ..., r. The procedure runs into numerical problems when it meets a bifurcation point whose pre-image contains a hyperbolic object. The integrals K_i^j then typically possess logarithmic singularities.

In principle, we could choose a special initial point in every m-phase, compute the integrals $K_i^j = \oint_{c_j}\alpha_i$ for that point, and start the integration from there. However, it is possible to cross from one m-phase to a neighboring one by encircling the singularity in the complex. This will give the analytic continuation of the solution chosen at A. We now show that the real part of this analytic continuation is unique and continuous. Therefore, it is exactly the solution we are looking for when calculating the reduced actions J_j.

Consider the curve $w^2 = R(u)$ in Fig. 2. We have the real cycles c_j which map onto the intervals $[z_{2j-1}, z_{2j}]$ under the projection $(w, u) \to u$. To obtain a (non-canonical) basis of cycles we add the imaginary cycles d_j $(j = 1, \dots, n)$, which map onto the intervals $[z_{2j-2}, z_{2j-1}]$. We found that crossing an interior plane in the bifurcation diagram implies $q_j = a_{j-1}$. That means we take the two "endpoints" of a d-cycle and deform the adjacent cycles so as to exchange the endpoints. The resulting new cycles $\tilde{c}$, $\tilde{d}$, are then expressed in terms of the old. For example, for $j=2$ we find from elementary topological arguments that

$$\tilde{c}_1 = c_1 + d_2, \qquad \tilde{d}_2 = d_2, \qquad \tilde{c}_2 = c_2 + d_2.$$

This shows that the real c-cycles are only changed by imaginary cycles. Since the periods of the forms over imaginary cycles are purely imaginary, the real part of the analytic continuation around an interior plane of the bifurcation diagram is unique and well defined. In particular, this means that when integrating a fixed linear combination of the Gauß–Manin equations, corresponding to a direction $\Delta\mathbf{q}$ in moment space transversely crossing an interior plane, we can use the following procedure. Integrate towards the plane and stop a distance ϵ before reaching it. The solutions blow up logarithmically with ϵ. Denote the current point by $\mathbf{q}_s - \epsilon\Delta\mathbf{q}$, where $\mathbf{q}_s$ is the point of intersection with the singular hyperplane. The solution at that point is $\mathbf{K}(\mathbf{q}_s - \epsilon\Delta\mathbf{q})$. Then restart the integration on the other side of the singularity at $\mathbf{q}_s + \epsilon\Delta\mathbf{q}$ with the initial condition $\mathbf{K}(\mathbf{q}_s - \epsilon\Delta\mathbf{q})$. Of course this introduces an additional error in the numerical solution, but it can be well controlled by ϵ. To obtain a better initial value on the other side, one would have to perform analytical continuation.

The energy surfaces in action space shown in Fig. 4d can therefore be obtained by the following procedure. For each action J_j, calculate the initial conditions $K^i_j(\mathbf{q}_\mathrm{A} + \epsilon\Delta\mathbf{q})$ given by (86) and (88). To find a particular value of the energy h, integrate the flow corresponding to $-\sum_{s=1}^{j}\mathbf{M}^j$ with this initial condition. When the desired value of h is reached, use $n-1$ flows that preserve the value of h to generate the energy surface in action space. For $n=2$, only one such flow is needed, corresponding to $\mathbf{M}^1$–$\mathbf{M}^2$. The new initial condition is obtained from the previous integration. Whenever a codimension one singularity of the bifurcation diagram is approached, with any flow, use the above procedure to get across it.

7. Concluding remarks

Ideally, one would like to consider the dependence of the actions on any set of n physical integration constants $k_1, \ldots, k_n$, and to have Picard–Fuchs equations with the derivatives $\mathrm{d}^i J/\mathrm{d}k_j^i$, or Gauß–Manin equations for $\mathrm{d}\boldsymbol{K}/\mathrm{d}k_j$. In principle, this should be possible whenever the actions are complete Abelian integrals, and the derivatives of first or second kind. In practice, however, there tend to be complications when more than n zeros of the polynomial $P(z)$ in Theorem 3 depend on the parameters k_j. The Neumann system is special in that only n branching points $q_1, \ldots, q_n$ of the Riemann surface $w^2 = -A(z)Q(z)$ depend on the physical constants η_j, or F_ν. The values q_j can then be taken as integration constants, and only then do the Picard–Fuchs or Gauss–Manin equations exhibit the simplicity that we found.

However, if the constants η_j are taken as variables of moment space, instead of the q_j, it becomes highly cumbersome to write down Gauß–Manin or Picard–Fuchs equations even for the Neumann system. The reason is the nonlinear relationship between the two sets (except for η_1). In other integrable systems with Riemann surfaces $w^2 = P(z)$, it is usually not the case that the n constants of integration are mapped to just n zeros of the polynomial $P(z)$, and then things become much more complicated. The Kovalevskaya system, for instance, has a Riemann surface of genus 2, but all five zeros change with the two integration constants. The complexity of the Gauß–Manin equations in terms of energy and separation constant as given in [2] is so enormous that their practical usefulness may be questioned. For the Neumann system, however, we have shown that simple and effective formulas for arbitrary n exist.

Finally, we want to stress that there is more to this approach than its effectiveness for numerical computation: the Gauss–Manin equations contain all information about the derivatives of actions, in particular the frequencies. In this connection, we refer to an interesting paper by Horozov [3] where he applied Picard–Fuchs equations to the proof of nondegeneracy (for *all* non-critical tori) of the spherical pendulum. It would be interesting to use our results to investigate the nondegeneracy of the Neumann system.

Acknowledgements

APV is grateful to Vitaly Tarasov for the useful discussions.

Appendix A

Proposition A.1. *For the Neumann system, the number N of disjoint tori in the pre-image of the moment map is $N = 2^{m+1}$, where m is the number of real ovals of the corresponding curve $w^2 = R(u)$ which do not contain any a_j.*

Proof. The reduced phase space has one "2^{n+1}-tant" of $\mathbb{R}^{n+1}$ as configuration space, with the identification induced by G on its boundaries. The elliptical coordinates are not a smooth coordinate system for the reduced phase space. The transformation to elliptical spherical coordinates smoothly maps each "2^{n+1}-tant" of S^n with constant signs of x_ν onto the n-cube I^n with sides $a_{j-1} \le u_j \le a_j$. The transformation is singular on the faces $u_j = a_\nu$ of the cube, which are the images of $x_\nu = 0$. Except for $x_0 = 0$ and $x_n = 0$ all other sub-spheres are mapped to two faces.

The coordinates ϕ_j defined by

$$\int_0^{\phi_j} \mathrm{d}\phi = \int_{a_{j-1}}^{u_j} \frac{\mathrm{d}u}{2\sqrt{-A(u)}}$$

remove the corresponding pole singularities in the metric $g_j(\boldsymbol{u})$. The ϕ_j are coordinates on a torus T^n that covers the sphere S^n. The codimension two branch points of the covering are given by $u_j = u_{j-1} = a_j$, the poles of g_j^{-1}. Locally the transformation from u_j to ϕ_j is just a reparametrization of coordinate lines. Globally each coordinate line ϕ_j on T^n covers its image in the cube I^n four times: crossing each of the two faces $u_j = a_\nu$ of the cube in the u_j-direction restores the original signs of x_ν. We can picture the T^n covering the configuration space S^n as 4^n cubes with periodic boundary conditions (see Fig. 5). Since the map from S^n to I^n is 2^{n+1}-fold, the covering from T^n to S^n is $4^n/2^{n+1} = 2^{n-1}$-fold.

The projection onto configuration space of the set of n-dimensional invariant tori in phase space, corresponding to fixed constants of motion, is a union of disjoint rectangular boxes (with periodic boundary conditions) in the ϕ_j coordinates. The faces of a box correspond to the ovals $z_{j-1} \le u_j \le z_j$. Denote by μ_j the number of endpoints of

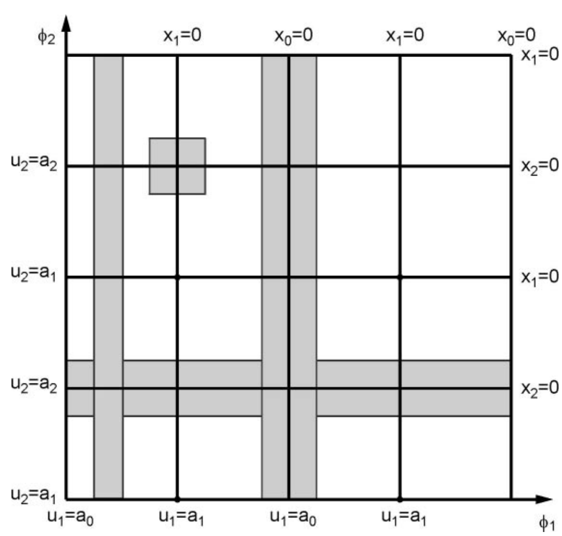

Fig. 5. The T^2 covering S^2 twice, and the four types of tori (gray) for $n = 2$: (a) horizontal strip: type lrr $= \mathrm{RO_r}$; (b) box in the upper left: type rrr $= \mathrm{O_rO_r}$; (c) vertical strip at the center: type llr $= \mathrm{O_lR}$; (d) left vertical strip: type rlr $= \mathrm{O_m}R$. Compare with Fig. 3.

the jth oval that are not among $a_0, \ldots, a_n$. The number of fundamental pieces I^n needed to cover this box in T^n is given by the product of the corresponding numbers 2^{μ_j} for each degree of freedom. The cases $\mu = 2$ (rotation) and $\mu = 1$ (oscillation) already appear for $n = 1$. Another oscillatory type of motion, with $\mu = 0$, occurs for $n > 1$. Denote by ρ, o, m, the number of intervals with $\mu_j = 2, 1, 0$, respectively; $\rho + o + m = n$. The number of fundamental pieces needed to cover an invariant torus is then $4^\rho 2^o 1^m = 2^{2\rho+o} = 4^n/2^{2m+o}$. As the total number of fundamental cubes is 4^n, there are 2^{2m+o} copies of the torus in T^n. Dividing this by the covering number 2^{n-1} we find that for this type of torus there are $2^{m+1-\rho}$ disjoint projections onto configuration space S^n. But notice that for each rotational degree of freedom, there are two different possibilities for the momentum, so that the total number of disjoint tori in phase space is $N = 2^{m+1}$. □

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