Hedging with forwards and puts in complete and incomplete markets

Benninga, Simon; Oosterhof, Casper

HEDGING WITH FORWARDS AND PUTS IN COMPLETE AND INCOMPLETE MARKETS

Simon Benninga
Casper Oosterhof

SOM theme E: Financial markets and institutions

ABSTRACT

We derive general conditions under which forward and/or put unbiasedness occurs and show that restrictions on the probability distribution suffice for simultaneous unbiasedness of forwards and puts, even if consumers are assumed to be risk averse. We examine the optimal production and hedging decisions by a risk-averse producer. If the producer's state prices are derived from his marginal rates of substitution an unbiased market forward price is overpriced and an unbiased market put price is underpriced. Even in this case the full hedging and separation theorem still holds and, contrary to previous literature, there is a hedging role for puts.

Also downloadable in electronic version: http://som.rug.nl

JEL classification: D21; D81; G10.

We thank two anonymous referees, Mark Helmantel, Peter Smid, Frans Tempelaar, seminar participants at the European Financial Management Association, the X International Tor Vergata Financial Conference on Banking and Finance, and the University of Groningen for their comments and suggestions. All remaining errors are ours.

HEDGING WITH FORWARDS AND PUTS IN COMPLETE AND INCOMPLETE MARKETS

1. Introduction

Income uncertainty and optimal hedging decisions by a competitive commodity producer have been the object of considerable research. This paper examines two issues which are not addressed or have caused some confusion in the hedging literature. We first derive general conditions under which forward and/or put price unbiasedness occurs. Contrary to the traditional belief that unbiasedness occurs only under risk-neutrality, we show that restrictions on the probability distribution suffice for unbiasedness, even if consumers are assumed to be strictly risk averse. Second, we examine the optimal production and hedging decisions by a risk-averse producer. Hedging is utility-enhancing for this producer only if his private state prices (derived from the marginal rates of substitution) differ from the market state prices. If the producer's state prices are derived from his marginal rates of substitution, he will perceive an unbiased market forward contract to be overpriced and an unbiased market put price to be underpriced. Contrary to the previous literature we show there is a hedging role for put options together with forward contracts.

In a pioneering article, McKinnon (1967) presents a model of a commodity producer who minimizes income volatility in a mean-variance framework. He shows that the correlation between stochastic price and production is crucial in the optimal hedging decision. A missing feature in McKinnon's model is that production cannot be chosen. Baron (1970) and Sandmo (1971) develop a model of optimal production under price uncertainty, which is extended by Danthine (1978), Holthausen (1979), and Feder, Just, and Schmitz (1980) to incorporate optimal hedging decisions as well. They show that, when output is nonrandom, the well-known separation theorem holds. The optimal production decision is independent of the producer's risk preferences and expectations and can be separated from the optimal hedging decision. If the forward price is unbiased, the optimal production decision is to produce until the marginal costs equal the forward price and the optimal forward position is a full hedge.

The results above—extended by Benninga, Eldor, and Zilcha (1983) and Lapan, Moschini, and Hanson (1991)—apply to a competitive producer who faces price risk only. For most commodities, however, a producer faces multiple sources of risk. Lapan and Moschini (1994) consider a producer facing price, production, and basis risk. They derive an exact solution to the optimal hedging problem under the assumption that price, production and basis risk are joint-normally distributed and that the producer maximizes an exponential utility function. An important finding is that the optimal hedge depends on the degree of risk aversion, even if the forward price is assumed to be unbiased.

The use of options as a hedging instrument has been examined much less than the use of futures. Lapan, Moschini, and Hanson (1991) consider a producer facing price and basis risk and compare the use of futures to put options as a hedging device. They show that, when the futures price is unbiased, options are redundant hedging instruments since futures provide a payoff that is linear in price risk. Moschini and Lapan (1995) study the problem of a producer facing price, (non-linear) basis, and production risk. They provide analytical solutions to the use of futures contracts and straddles, assuming an exponential utility function and joint-normal distributions between the risk factors. Under the assumption of unbiased forward and straddle prices, they show that the optimal strategy is to buy straddles along with a short position in futures. Batterman, Braulke, Broll, and Schimmelpfennig (2000) compare the use of forward contracts and put options within a one period utility framework. They show that, in case of unbiased put prices, the optimal hedging strategy is to overhedge and the optimal output decision is to produce up to a point where the marginal costs are less than the forward price (assuming unbiasedness of the forward price). Furthermore, forwards will always be preferred to puts when both instruments are perceived as unbiased predictors of future payoffs.

In all of the above papers hedging is the result of well-specified risks and derivatives markets which allow the complete or partial (through cross-hedging) hedging of these risks. An innovative paper by Franke, Stapleton, and

Subrahmanyam (1998) examines the hedging motive when there are unhedgeable background risks. This hedging behavior has similar origins to that discussed in the current paper: Where individuals disagree strongly with the market prices (whether through greater background risk or for the unspecified reasons in this paper), they will be more strongly motivated to use non-linear derivative instruments to try to complete the market.

This paper has two purposes: First, we examine the conditions under which forward contracts and/or put options are unbiased. It is sometimes argued that unbiasedness of derivative instruments only occurs under risk-neutrality.[1] We show that this is not true, and that restricting the probability distribution is sufficient for unbiasedness of forward and put prices. Second, we examine the impact of unbiasedness on optimal hedging and production decisions. Our model extends previous research by showing that there is a hedging role for put options even if only price is stochastic. We also show that, whereas unbiased forward prices do not affect production decisions, the use of puts reduces production.

The remainder of this paper is organized as follows. Section 2 introduces the general model specification in which the conditions for the forward price and the put price to be unbiased are derived. In section 3 we derive the optimal production and risk management decisions for a risk-averse producer. We examine the possibility of optimal hedging and production under market completeness and under market incompleteness. Section 4 concludes the paper.

2. The model

We consider a two-date framework where today is denoted as time 0 and tomorrow as time 1. Time 1 has N states of the world. We examine an asset S, having a spot price S_0 today and $\tilde{S} = \{S_1 < S_2 < ... < S_N\}$ prices in the states of the world tomorrow. The

[1] Among others, Chiang and Trinidad (1997), Wu and Zhang (1997), and Baillie and Bollerslev (2000) argue that forward unbiasedness occurs under the joint assumptions of efficient markets and risk-neutral consumers.

state probabilities are given by $\tilde{\pi} = \{\pi_1, \pi_2, ..., \pi_N\}$, and the state prices by which financial assets are priced are denoted as $\tilde{q} = \{q_1, q_2, ..., q_N\}$.

Assets are priced by the state prices. For example, the equilibrium risk-free rate of interest r_f is given by $\frac{1}{1+r_f} = \sum_{j=1}^{N} q_j$. In general, given the state prices, any asset having state-dependent payoffs $\tilde{A} = \{A_1, A_2, ..., A_N\}$ will have price today $A_0 = \sum_{j=1}^{N} q_j A_j$. As shown by Beja (1972), we can write the value A_0 as a function of the discounted expected payoff plus a covariance term representing the risk of the asset:

$$A_0 = \sum_{j=1}^{N} q_j A_j = \sum_{j=1}^{N} \pi_j \frac{q_j}{\pi_j} A_j = \frac{E\left[\tilde{A}\right]}{1+r_f} + Cov\left(\frac{\tilde{q}}{\tilde{\pi}}, \tilde{A}\right) \tag{1}$$

For future reference we note that in the case of a single representative consumer with a Von Neumann-Morgenstern time-additive utility function, the state prices are derived from the consumer's marginal rates of substitution $q_j = \delta \pi_j \frac{U'(c_j)}{U'(c_0)}$, where δ is the consumer's pure rate of time preference, π_j is the probability of state j, and c_j is consumption in state j. At this point we leave open the question of whether the market state prices are determined by the individual consumer's marginal rates of substitution (see Section 3). Suppose the size of optimal consumption is correlated to the commodity price so that $c_1 < c_2 < .... < c_N$. Since the utility function is concave, it follows that for time-additive utility:

$$\frac{q_1}{\pi_1} = \delta \frac{U'(c_1)}{U'(c_0)} > \frac{q_2}{\pi_2} = \delta \frac{U'(c_2)}{U'(c_0)} > ... > \frac{q_N}{\pi_N} = \delta \frac{U'(c_N)}{U'(c_0)} \tag{2}$$

Before deriving the optimal production and hedging decisions in the next section, we first examine the conditions under which forward contracts and put options will be unbiased.

Unbiasedness of the forward price

In this subsection we derive conditions for the forward price to be unbiased. Let F denote the forward price at date 0 for the delivery of one unit of the asset at date 1. By definition of the forward price, F is set so that the time 0 cost is zero:

$$\sum_{j=1}^{N} q_j \left(S_j - F\right) = 0 \tag{3}$$

Solving equation (3) for the forward price gives $\frac{F}{1+r_f} = \sum_{j=1}^{N} q_j S_j = S_0 \Rightarrow F = \left(1+r_f\right) S_0$. This forward price is unbiased if $F = E\left[\tilde{S}\right]$.

As shown above, we can write:

$$\frac{F}{1+r_f} = \sum_{j=1}^{N} q_j S_j = \sum_{j=1}^{N} \pi_j \frac{q_j}{\pi_j} S_j = \frac{E\left[\tilde{S}\right]}{1+r_f} + Cov\left(\frac{\tilde{q}}{\tilde{\pi}}, \tilde{S}\right) \tag{4}$$

Solving for F gives:

$$F = E\left[\tilde{S}\right] + \left(1+r_f\right) Cov\left(\frac{\tilde{q}}{\tilde{\pi}}, \tilde{S}\right) \tag{5}$$

Thus, the forward price is unbiased if and only if $Cov\left(\frac{\tilde{q}}{\tilde{\pi}}, \tilde{S}\right) = 0$. As the lemma below shows, the covariance is zero for two cases:

Lemma 1: The forward price is unbiased if and only if one of the following holds:

1. The state prices are derived from a risk-neutral representative consumer.
2. The consumer is risk averse and there is one restriction on the probability distribution.

Proof:

Part 1: In this case the state prices are given by $q_j = \frac{\pi_j}{1+r_f}$ and the covariance term is $Cov\left(\frac{\tilde{q}}{\tilde{\pi}}, \tilde{S}\right) = Cov\left(\frac{1}{1+r_f}, \tilde{S}\right) = 0$.

Part 2: Given the market state prices $\tilde{q}$ and the asset prices $\tilde{S}$, unbiasedness of the forward price imposes restrictions on the state probabilities. When there are N states of the world, forward unbiasedness occurs if $F = \frac{\sum_{j=1}^{N} q_j S_j}{\sum_{j=1}^{N} q_j} = \sum_{j=1}^{N} \pi_j S_j$. Given the state prices and the asset prices, this equation can be solved for any state probability π_k. This means that unbiasedness imposes the following general restriction on π_k:

$$\pi_k = \frac{F - \left[\sum_{j=1:\, j \neq k}^{N-1} \pi_j \left(S_j - S_N \right) + S_N \right]}{\left[S_k - S_N \right]} \tag{6}$$

where state $1 \leq k \leq N$. ||

The first result in Lemma 1 is standard, since under risk-neutrality all asset prices are unbiased. The second part of the lemma shows that risk-neutrality is not a necessary condition for forward unbiasedness. The restriction in equation (6) depends on all the other probabilities and on the state prices, meaning that both the probability distribution and the state prices can have any form. Note that if there are only two states of the world, this restriction implies that $\pi_1 = \frac{q_1}{q_1 + q_2}$ and $\pi_2 = \frac{q_2}{q_1 + q_2}$.[2] For three or more states, this restriction is unrelated to the degree of risk aversion of the consumers.

Unbiasedness of the put price

Let $P(X)$ denote the date 0 put price with and exercise price equal to X The put is unbiased if it equals the discounted expected put payoffs. Since we can write the put price as

---

[2] We thank an anonymous referee for pointing this out.

$$P(X)=\sum_{j=1}^{N}q_j\left[X-S_j\right]^+=\frac{E\left[\left[X-\tilde{S}\right]^+\right]}{1+r_f}+Cov\left(\frac{\tilde{q}}{\tilde{\pi}},\left[X-\tilde{S}\right]^+\right) \quad (7)$$

put price unbiasedness occurs if and only if $Cov\left(\frac{\tilde{q}}{\tilde{\pi}},\left[X-\tilde{S}\right]^+\right)=0$.

Lemma 2: The put price is unbiased if and only if one of the following holds:

1. The state prices for every state in which the put is in the money are risk-neutral prices.
2. The forward price is unbiased and the put is either always in or out of the money.
3. A restriction on the probability distribution similar to that derived in Lemma 1 for forward prices is imposed.

Proof:

Part 1: We start by recalling from equation (7) that[3]

$$Cov\left(\frac{\tilde{q}}{\tilde{\pi}},\left[X-\tilde{S}\right]^+\right)=\sum_{j=1}^{N}q_j\left[X-S_j\right]^+-\frac{\sum_{j=1}^{N}\pi_j\left[X-S_j\right]^+}{1+r_f}$$

Now suppose that the put option is exercised in states 1, ..., k of the world, i.e., that $S_{k+1}>X>S_k>S_{k-1}>...>S_1$. In this case the put price is unbiased if and only if

$$Cov\left(\frac{\tilde{q}}{\tilde{\pi}},\left[X-\tilde{S}\right]^+\right)=\sum_{j=1}^{k}q_j\left(X-S_j\right)-\frac{\sum_{j=1}^{k}\pi_j\left(X-S_j\right)}{1+r_f}=\sum_{j=1}^{k}\left(q_j-\frac{\pi_j}{1+r_f}\right)\left(X-S_j\right)=0$$

[3] Since $E\left[A\cdot B\right]=E\left[A\right]E\left[B\right]+Cov\left(A,B\right)$, the covariance term $Cov\left(A,B\right)=E\left[A\cdot B\right]-E\left[A\right]E\left[B\right]$.

Since the factors $\left(X-S_j\right)>0, j=1,...,k$, it follows that the put is unbiased if and only if $\frac{q_j}{\pi_j}=\frac{1}{1+r_f}$ for every state in which it is exercised.

Part 2: Suppose that $X < S_1$, so that the put is always out of the money. In this case, $P(X)=\sum_{j=1}^{N} q_j\left[X-S_j\right]^{+}=0$, so that the put is unbiased. On the other hand, if the put is always in the money, i.e., $X > S_N$, then

$$P(X)=\sum_{j=1}^{N} q_j\left[X-S_j\right]^{+}=\frac{E\left[\left[X-\tilde{S}\right]^{+}\right]}{1+r_f}+Cov\left(\frac{\tilde{q}}{\tilde{\pi}},\left[X-\tilde{S}\right]^{+}\right)$$
$$=\frac{E\left[X-\tilde{S}\right]}{1+r_f}+Cov\left(\frac{\tilde{q}}{\tilde{\pi}},X-\tilde{S}\right)=\frac{E\left[X-\tilde{S}\right]}{1+r_f}$$

since the forward is unbiased.

Part 3: Similar to deriving the restriction imposed on the probability distribution in case of forward unbiasedness, we can solve for put unbiasedness as well. Given the market state prices and the asset prices, unbiasedness of the put price imposes a restriction on the probability distribution. Again, when there are N states of the world, put unbiasedness occurs if $P(X)\cdot\left(1+r_f\right)=\frac{\sum_{j=1}^{N} q_j\left[X-S_j\right]^{+}}{\sum_{j=1}^{N} q_j}=\sum_{j=1}^{N}\pi_j\left[X-S_j\right]^{+}$. Given the state prices $\tilde{q}$ and the asset prices $\tilde{S}$, this equation can be solved for any state probability π_k. This means that unbiasedness imposes the following general restriction on π_k:

$$\pi_k=\frac{\frac{P(X)}{\sum_{j=1}^{N} q_j}-\left(\sum_{j=1;j\neq k}^{N-1}\pi_j\left(\left[X-S_j\right]^{+}-\left[X-S_N\right]^{+}\right)+\left[X-S_N\right]^{+}\right)}{\left[X-S_k\right]^{+}-\left[X-S_M\right]^{+}} \quad (8)$$

where state $1\leq k\leq N$. ||

Note the similarity between equations (6) and (8). Note also that the conditions in part 1 of Lemma 2 are close to risk-neutrality, which occurs if and only if $\frac{q_j}{\pi_j} = \frac{1}{1+r_f}$ for all states *j*. If the put price is unbiased for all exercise prices *X*, then there is risk-neutrality, so that the forward price is also unbiased.[4]

Both forward and put unbiasedness

Up to now we have only considered the possibility of forward unbiasedness or put unbiasedness. Forward and put unbiasedness occurs if both $Cov\left(\frac{\tilde{q}}{\tilde{\pi}}, \tilde{S}\right) = 0$ and $Cov\left(\frac{\tilde{q}}{\tilde{\pi}}, \left[X - \tilde{S}\right]^{+}\right) = 0$. If we abstract from risk-neutrality, this only occurs if there are two restrictions on the probability distribution, and we simultaneously have to solve $F = \frac{\sum_{j=1}^{N} q_j S_j}{\sum_{j=1}^{N} q_j} = \sum_{j=1}^{N} \pi_j S_j$ and

$P(X) \cdot \left(1 + r_f\right) = \frac{\sum_{j=1}^{N} q_j \left[X - S_j\right]^{+}}{\sum_{j=1}^{N} q_j} = \sum_{j=1}^{N} \pi_j \left[X - S_j\right]^{+}$. This occurs if there are two states *k* and *l* (with *k* < *l*) for which the following restrictions hold:

---

4 Of course, in risk-neutrality all asset prices are unbiased.

$$\pi_k = \frac{\frac{1}{S_k}\left(\sum_{j=1:\, j\neq k,l}^{N} \pi_j S_j + F - S_l\left(\left(\left(-\sum_{j=1:\, j\neq k,l}^{N} \pi_j S_j + F\right)[X - S_k]^+\right) + \sum_{j=1:\, j\neq k,l}^{N} \pi_j [X - S]^+ - \frac{P(X)}{\sum_{j=1}^{N} q_j}\right)\right)}{S_l [X - S_k]^+ - S_k [X - S_l]^+}$$

$$\pi_l = \frac{S_k\left(\frac{\left(-\sum_{j=1:\, j\neq k,l}^{N} \pi_j S_j + F\right)[X - S_k]^+}{S_k} + \sum_{j=1:\, j\neq k,l}^{N} \pi_j [X - S_j]^+ - \frac{P(X)}{\sum_{j=1:\, j\neq k,l}^{N} q_j}\right)}{S_l [X - S_k]^+ - S_k [X - S_l]^+} \tag{9}$$

where state $1 \le k < l \le N$.

Given these restrictions on the probability distribution both forward contracts as well as put options are unbiased predictors of the expected payoff. Contrary to the traditional belief that unbiasedness of derivative instruments only occurs under risk-neutrality, it can also occur for certain probability distributions.

3. Optimal production and risk management by a risk-averse producer

Up to this point we have proved some general statements about state price mathematics. We now introduce a producer who maximizes a utility function which includes production and hedging.[5] We assume that the producer's state prices (denoted $\{q_j^P\}$) are not necessarily the same as the market state prices which determine the prices of the forward contract and the put contract (from now on we denote these prices by $\{q_j^M\}$). In order to derive optimal production and risk

[5] We assume that both the market and the producer have the same subjective state probabilities π.

management decisions we must first say something about the market conditions. We say that markets are complete if the producer's state prices and the market state prices are the same.[6] In Theorem 1 we prove that in complete markets there is no advantage to the producer in hedging, either by using puts or by using forward contracts.

The market is incomplete if the producer's and the market state prices are not the same. In this case, the producer disagrees with the market about the pricing of forward contracts and put options, and the producer's valuation of forward and put contracts (by using his private state prices) differs from that of the market.

The case of complete markets

The producer faces uncertainty because the future price of goods sold is random. At time 0 the producer, with initial wealth W_0, chooses the output y to his production function; these inputs cost $C(y)$, where C is a strictly convex cost function. At time 1, uncertainty regarding the commodity price is resolved; in each state of the world, the producer produces output y and realizes the proceeds from his sales given the stochastic commodity price.

If the producer has access to forward and put markets, he has to choose how many forward contracts (n_F) and how many puts (n_P) to buy or sell in order to solve the following problem:

$$\begin{aligned}
&Max\, E\left[U(\tilde{c})\right] = U(c_0) + \delta\sum_{j=1}^{N}\pi_j U(c_j) \\
&s.t. \\
&c_0 = W_0 - C(y) - n_P P(X) \\
&c_j = S_j \cdot y + n_P\left[X - S_j\right]^+ + n_F\left(F - S_j\right)
\end{aligned} \tag{10}$$

Before turning to the optimal production and risk management decisions we first examine the covariance factors $Cov\left(\frac{\tilde{q}^P}{\tilde{\pi}}, \tilde{S}\right)$ and $Cov\left(\frac{\tilde{q}^P}{\tilde{\pi}}, \left[X - \tilde{S}\right]^+\right)$.

---

[6] This will happen, for example, if the producer is also the representative consumer.

Lemma 3: Suppose the producer does not have access to either put or forward markets. If the producer's state prices are derived from his marginal rates of substitution, then the covariance term $Cov\left(\frac{\tilde{q}^P}{\tilde{\pi}}, \tilde{S}\right) < 0$ and $Cov\left(\frac{\tilde{q}^P}{\tilde{\pi}}, \left[X - \tilde{S}\right]^+\right) > 0$. Thus for this case both the put and the forward prices are biased.

Proof: When the cost function is strictly convex there is a unique solution for optimal production y^* and to state prices $q_j^P = \delta \frac{\pi_j U'\left(S_j \cdot y^*\right)}{U'\left(W_0 - C\left(y^*\right)\right)}$. This implies that the covariance factor can be written as:

$$Cov\left(\frac{q_j^P}{\pi_j}, S_j\right) = Cov\left(\frac{\delta \frac{\pi_j U'\left(c_j^*\right)}{U'\left(c_0^*\right)}}{\pi_j}, S_j\right) = Cov\left(\delta \frac{U'\left(c_j^*\right)}{U'\left(c_0^*\right)}, S_j\right),$$

where c^* denotes consumption given optimal production. Given a strictly concave utility function, $\frac{U'\left(c_j^*\right)}{U'\left(c_0^*\right)}$ is a decreasing function in *S*, which results in a negative covariance factor $Cov\left(\frac{\tilde{q}^P}{\tilde{\pi}}, \tilde{S}\right)$ and a positive covariance term $Cov\left(\frac{\tilde{q}^P}{\tilde{\pi}}, \left[X - \tilde{S}\right]^+\right)$.||

Hedging with puts and forwards

The first-order conditions of (10) with respect to output, forwards, and puts are given by:

$$\frac{dE\left[U\left(\tilde{c}\right)\right]}{dy} = -C'\left(y\right)U'\left(c_0\right) + \delta\sum_{j=1}^{N}\pi_j S_j U'\left(c_j\right) = 0$$
$$\frac{dE\left[U\left(\tilde{c}\right)\right]}{dn_F} = \delta\sum_{j=1}^{N}\pi_j U'\left(c_j\right)\left(S_j - F\right) = 0 \tag{11}$$
$$\frac{dE\left[U\left(\tilde{c}\right)\right]}{dn_P} = -U'\left(c_0\right)P\left(X\right) + \delta\sum_{j=1}^{N}\pi_j U'\left(c_j\right)\left[X - S_j\right]^+ = 0$$

Theorem 1: If the producer is representative in the sense that his marginal rates of substitution are equal to the market state prices, then an incremental purchase of the forward or the put contract does not increase the producer's welfare. In this case neither hedging with puts or forwards is preferable one over the other.

Proof: As shown in Lemma 3, there is a unique solution to the optimal production decision which gives state prices $q_j = \delta\frac{\pi_j U'\left(S_j \cdot y^*\right)}{U'\left(W_0 - C\left(y^*\right)\right)}$. Now suppose the producer is trying to decide whether to add a small quantity n_F forwards or n_P puts to his position. His expected utility will now be:

$$E\left[U\left(\tilde{c}\left(n_F, n_P\right)\right)\right] = U\left(W_0 - C\left(y^*\right) - n_P P\left(X\right)\right) + \delta\sum_{j=1}^{N}\pi_j U\left(S_j \cdot y^* + n_F \cdot \left(S_j - F\right) + n_P\left[X - S_j\right]^+\right)$$

Consider the choice of forward contracts first. Using a standard first-order Taylor series expansion, it is clear that:

$$\begin{aligned} E\left[U\left(\tilde{c}\left(n_F, 0\right)\right)\right] &= U\left(W_0 - C\left(y^*\right)\right) + \delta\sum_{j=1}^{N}\pi_j U\left(S_j \cdot y^* + n_F \cdot \left(S_j - F\right)\right) \\ &= E\left[U\left(\tilde{c}\left(0,0\right)\right)\right] + \delta n_F\sum_{j=1}^{N}\pi_j U'\left(S_j \cdot y^*\right)\cdot\left(S_j - F\right) \end{aligned}$$

where $E\left[U\left(\tilde{c}\left(0,0\right)\right)\right]$ is the expected utility and y^* is the optimal production in case the firm does not hedge. It is obvious that hedging adds value if and only if

$\delta n_F \sum_{j=1}^{N} \pi_j U'\left(S_j \cdot y^*\right) \cdot \left(S_j - F\right) > 0$. However, dividing the previous equation by $U'\left(c\left(0,0\right)\right)$ yields $\delta n_F \sum_{j=1}^{N} \pi_j \frac{U'\left(S_j \cdot y^*\right)}{U'\left(c\left(0,0\right)\right)} \cdot \left(S_j - F\right) = n_F \sum_{j=1}^{N} q_j \cdot \left(S_j - F\right)$, which equals zero by definition of the forward price. This proves the theorem for forwards. The proof for puts is similar. ||

Thus, if the producer is the representative agent, hedging with forward contracts or put options does not improve his personal utility of wealth, since buying or selling financial instruments is always a zero-NPV investment. There will be welfare improvements only if the producer's implicit state prices differ from the pricing for the forward/put contracts. Thus, hedging can only add value if there is some kind of market incompleteness.

The case of incomplete markets

When markets are incomplete, the producer's marginal rates of substitution are different from the market prices. In this case the use of forwards and put options can lead to welfare improvements. Suppose that the producer's state prices are derived from his marginal rates of substitution from maximization problem (10), leading to state prices $q_j^P = \delta \frac{\pi_j U'\left(S_j \cdot y^*\right)}{U'\left(W_0 - C\left(y^*\right)\right)}$. Furthermore, suppose that the market forward price is unbiased. Then:

$$F = \frac{\sum_{j=1}^{N} q_j^M S_j}{\sum_{j=1}^{N} q_j^M} = \sum_{j=1}^{N} \pi_j S_j \qquad (12)$$

If the producer agrees with the market valuation, then:

$$F = \frac{\sum_{j=1}^{N} q_j^P S_j}{\sum_{j=1}^{N} q_j^P} = \sum_{j=1}^{N} \pi_j S_j \tag{13}$$

However, by Lemma 3, $F < \sum_{j=1}^{N} \pi_j S_j$. From this we conclude that if the forward price is unbiased (using market state prices), then the producer views the market forward price as being overpriced. A similar statement is true for puts.

Thus, if the market forward price is unbiased, then the producer thinks that the market forward price is too high and the put price is too low. Therefore the optimal forward position for the producer is to short the forward contract and to go long the put contract.

Theorem 2: If the market forward price is unbiased and markets are incomplete, the producer will engage in a full hedge. There will be separation between the production and hedging decision even though the producer perceives the forward price to be overpriced.[7]

Proof: We consider a producer who maximizes his expected utility using forwards only:

$$\begin{aligned} &Max\, E\left[U\left(\tilde{c}\right)\right] = U\left(c_0\right) + \delta \sum_{j=1}^{N} \pi_j U\left(c_j\right) \\ &s.t. \\ &c_0 = W_0 - C\left(y\right) \\ &c_j = S_j \cdot y + n_F \cdot \left(S_j - F^M\right) \end{aligned} \tag{14}$$

[7] Note that we explicitly distinguish between the market forward price, which is given to the producer, and his private valuation which he would be willing to pay for the forward contract given his private state prices.

where $F^M = \frac{\sum_{j=1}^{N} q_j^M S_j}{\sum_{j=1}^{N} q_j^M} = \sum_{j=1}^{N} \pi_j S_j > F^P = \frac{\sum_{j=1}^{N} q_j^P S_j}{\sum_{j=1}^{N} q_j^P}$ implying that the producer perceives the market forward price to be overpriced (by Lemma 3). The superscripts *M* and *P* are added to distinguish between the market forward price and the producer's private valuation of the forward price. The first order conditions for the producer are given by:

$$\begin{aligned} \frac{dE\left[U(\tilde{c})\right]}{dy} &= -C'(y)U'(c_0) + \delta\sum_{j=1}^{N} \pi_j S_j U'(c_j) = 0 \\ \frac{dE\left[U(\tilde{c})\right]}{dn_F} &= \delta\sum_{j=1}^{N} \pi_j U'(c_j)\left(S_j - F^M\right) = 0 \end{aligned} \tag{15}$$

However, since both the producer and the market face the same probability distribution, the producer perceives the market forward price as being unbiased and the optimization problem becomes a standard decision. Look at the choice of the number of forward contracts first. The first-order condition can be rewritten as:

$$\begin{aligned} \frac{dE\left[U(\tilde{c})\right]}{dn_F} &= \delta\sum_{j=1}^{N} \pi_j U'(c_j)\left(S_j - F^M\right) \\ &= \delta E\left[U'(c_j)\left(S_j - F^M\right)\right] \\ &= \delta E\left[U'(c_j)\right] E\left[\left(S_j - F^M\right)\right] + Cov\left(U'(c_j), \left(S_j - F^M\right)\right) \end{aligned}$$

Since $E\left[\left(S_j - F^M\right)\right]$ is zero by definition of an unbiased market forward price, the first order condition is zero if and only if $Cov\left(U'(c_j), \left(S_j - F^M\right)\right) = 0$. If the producer engages in a full hedge, i.e., $n_F = -y$, marginal utility is constant and the covariance term will be zero. Thus, even though the producer perceives the forward to be overpriced, he still engages in a full hedge. Now consider the optimal production decision. Dividing the first-order condition through $U'(c_0)$ yields:

$$\frac{dE\left[U(\tilde{c})\right]}{dy} = -C'(y) + \delta\sum_{j=1}^{N}\pi_j S_j \frac{U'(c_j)}{U'(c_0)} = -C'(y) + E\left[\tilde{S}\cdot\delta\frac{U'(\tilde{c})}{U'(c_0)}\right]$$

$$= -C'(y) + E\left[\tilde{S}\right]E\left[\delta\frac{U'(\tilde{c})}{U'(c_0)}\right] + Cov\left(\tilde{S}, \delta\frac{U'(\tilde{c})}{U'(c_0)}\right)$$

However since the producer engages in a full hedge, $U'(\tilde{c})$ is constant which allows us to rewrite the first order condition as:[8]

$$\frac{dE\left[U(\tilde{c})\right]}{dy} = -C'(y) + E\left[\tilde{S}\right]E\left[\delta\frac{U'(\bar{c})}{U'(c_0)}\right] + Cov\left(\tilde{S}, \delta\frac{U'(\bar{c})}{U'(c_0)}\right)$$

$$= -C'(y) + F^M \cdot \frac{1}{1+r_f}$$

$$= -C'(y) + S_0$$

This completes the proof. ||

By Theorem 2, production takes place up to the point where the marginal costs equal the current spot price (or alternatively: production takes place up to the point where the forward marginal costs equal the market forward price). The optimal hedge is a full hedge and the optimal production decision is to produce until the marginal costs of production equal the current spot price. Thus, the producer's risk preferences do not influence his optimal production decision, nor his optimal forward hedge. This proof differs from the traditional papers on optimal hedging and production, in which the producer agrees with the market valuation of the forward.[9] Even though the producer does not agree with the market valuation of the forward contract (according to his private valuation the forward is overpriced) he still engages in a full hedge and there is separation between the production and hedging decision.

---

[8] A bar ($\bar{\ }$) is used to denote a nonrandom variable.

[9] See e.g., Benninga, Eldor, and Zilcha (1983, 1985).

Lemma 4: If the market valuation of the put is unbiased and if producer and market agree on the interest rate, then the producer will view the put as underpriced. Thus the optimal hedging position for the producer will be long in the put.[10]

Proof: If the market valuation of the put is unbiased we must have the general restriction on the probability distribution, as given in equation (11).[11] We will now show, by contradiction, that the producer cannot agree with the market valuation of the put. If the market put price is unbiased then we have $P(X)\cdot(1+r_f)=\frac{\sum_{j=1}^{N} q_j^M [X-S_j]^+}{\sum_{j=1}^{N} q_j^M}=\sum_{j=1}^{N}\pi_j [X-S_j]^+$. Now we will show that the producer does not agree with the market valuation of the put. To see this, we proceed by contradiction by saying that if the producer would agree with the market put price, then:

$$P(X)\cdot(1+r_f)=\frac{\sum_{j=1}^{N} q_j^P [X-S_j]^+}{\sum_{j=1}^{N} q_j^P}=\sum_{j=1}^{N}\pi_j [X-S_j]^+$$

However, from Lemma 3, $P(X)\cdot(1+r_f)>\sum_{j=1}^{N}\pi_j [X-S_j]^+$, from which we can conclude that if the market put price is unbiased, then the producer will view the put option as being underpriced. This implies that the optimal put position will be long. ||

Theorem 3: If the market put price is unbiased, then the optimal put position depends on producer access to the bond market. This can result in a full hedge, in an overhedge, and in an underhedge.

---

[10] Remember that an unbiased put price means that the forward put price equals the expected payoff.
[11] We assume that at least one state of the world generates a positive payoff.

Proof: If the producer can use put options only, his maximization problem becomes:

$$
\begin{aligned}
& Max\, E\left[U\left(\tilde{c}\right)\right] = U\left(c_0\right) + \delta \sum_{j=1}^{N} \pi_j U\left(c_j\right) \\
& s.t. \\
& c_0 = W_0 - C\left(y\right) - n_P P^M \\
& c_j = S_j \cdot y + n_P \cdot \left[X - S_j\right]^+
\end{aligned}
\tag{16}
$$

which results in the following first order conditions:

$$
\begin{aligned}
& \frac{dE\left[U\left(\tilde{c}\right)\right]}{dy} = -C'\left(y\right)U'\left(c_0\right) + \delta \sum_{j=1}^{N} \pi_j S_j U'\left(c_j\right) = 0 \\
& \frac{dE\left[U\left(\tilde{c}\right)\right]}{dn_P} = -U'\left(c_0\right)P^M + \delta \sum_{j=1}^{N} \pi_j U'\left(c_j\right)\left[X - S_j\right]^+ = 0
\end{aligned}
\tag{17}
$$

Consider the choice of the optimal number of put options first. The question is, whether the producer—as in the case of hedging with forwards discussed above—will engage in a full hedge. Dividing the first order condition by $U'\left(c_0\right)$ and rewriting shows us:

$$
\begin{aligned}
\frac{dE\left[U\left(\tilde{c}\right)\right]}{dn_P} &= -P^M + \delta \sum_{j=1}^{N} \pi_j \frac{U'\left(c_j\right)}{U'\left(c_0\right)}\left[X - S_j\right]^+ = -P^M + E\left[\delta \frac{U'\left(c_j\right)}{U'\left(c_0\right)}\left[X - S_j\right]^+\right] \\
&= -P^M + E\left[\delta \frac{U'\left(c_j\right)}{U'\left(c_0\right)}\right] E\left[\left[X - S_j\right]^+\right] + Cov\left(\delta \frac{U'\left(c_j\right)}{U'\left(c_0\right)}, \left[X - S_j\right]^+\right) \\
&= -\frac{E\left[\left[X - S_j\right]^+\right]}{1 + r_f} + E\left[\delta \frac{U'\left(c_j\right)}{U'\left(c_0\right)}\right] E\left[\left[X - S_j\right]^+\right] + Cov\left(\delta \frac{U'\left(c_j\right)}{U'\left(c_0\right)}, \left[X - S_j\right]^+\right) \\
&= E\left[\left[X - S_j\right]^+\right] \cdot \left(E\left[\delta \frac{U'\left(c_j\right)}{U'\left(c_0\right)}\right] - \frac{1}{1 + r_f}\right) + Cov\left(\delta \frac{U'\left(c_j\right)}{U'\left(c_0\right)}, \left[X - S_j\right]^+\right)
\end{aligned}
$$

Case 1: If the producer has full access to the bond market then $E\left[\delta \frac{U'(c_j)}{U'(c_0)}\right] = \frac{1}{1+r_f}$. This means that the covariance term $Cov\left(\delta \frac{U'(c_j)}{U'(c_0)}, \left[X - S_j\right]^+\right)$ must be zero in order to have an optimum. As shown by Battermann, Braulke, Broll, and Schimmelpfennig (2000) this results in overhedging.

Case 2: If the producer would like to borrow but cannot (e.g., because he has restricted borrowing) the term $E\left[\delta \frac{U'(c_j)}{U'(c_0)}\right] < \frac{1}{1+r_f}$. This means that at a producer optimum:

$$\frac{dE\left[U(\tilde{c})\right]}{dn_P} = \underbrace{E\left[\left[X - S_j\right]^+\right]}_{>0} \cdot \underbrace{\left(E\left[\delta \frac{U'(c_j)}{U'(c_0)}\right] - \frac{1}{1+r_f}\right)}_{<0} + \underbrace{Cov\left(\delta \frac{U'(c_j)}{U'(c_0)}, \left[X - S_j\right]^+\right)}_{>0} = 0$$

In order to set the marginal utility = 0, the Cov > 0; this means that for this case the producer buys fewer puts than in case 1, but still has a positive put position by Lemma 4. Since setting the covariance equal to zero (as in Case 1) leads to overhedging we know that choosing the optimal number of put options can lead to the possibility of full hedging, underhedging, and overhedging.

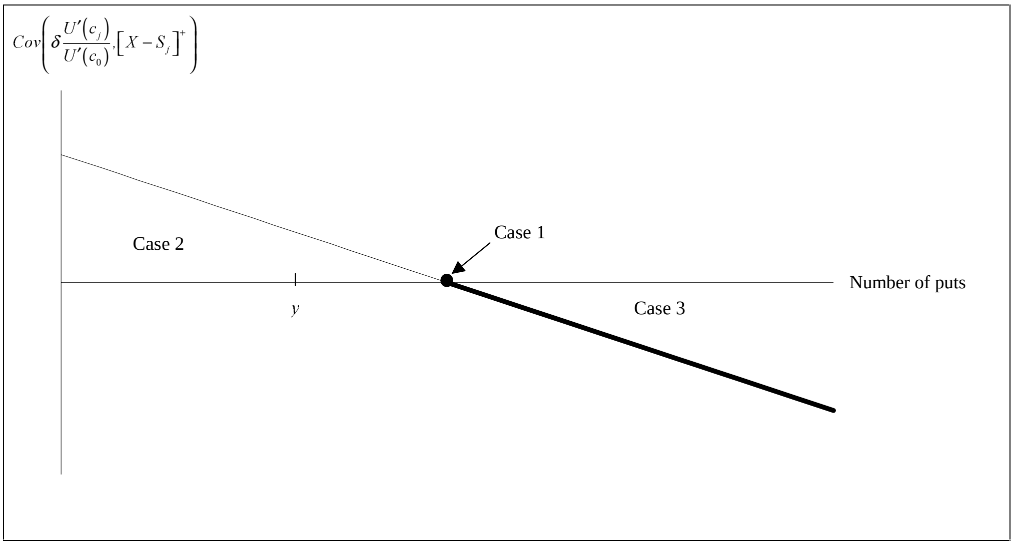
Case 3: If the producer would faces a binding lending constraint, then the term $E\left[\delta \frac{U'(c_j)}{U'(c_0)}\right] > \frac{1}{1+r_f}$. In this case:

$$\frac{dE\left[U(\tilde{c})\right]}{dn_P} = \underbrace{E\left[\left[X - S_j\right]^+\right]}_{>0} \cdot \underbrace{\left(E\left[\delta \frac{U'(c_j)}{U'(c_0)}\right] - \frac{1}{1+r_f}\right)}_{>0} + Cov\left(\delta \frac{U'(c_j)}{U'(c_0)}, \left[X - S_j\right]^+\right) = 0$$

Thus at an optimum, the covariance term must be negative, which means that the producer overhedges. Note that in this case, the producer overhedges even more then in case 1. ||

The results are summarized in Figure 1.

Figure 1: Optimal hedging in case of put options



The benchmark case where the producer has full access to the bond market is in given by the thick dot and results in overhedging. The thin part of the line is the case where the producer has restricted borrowing. This will result in the possibility of underhedging, full hedging and overhedging. If the producer faces a restricted lending constraint, his optimal put position is given by the thick part of the line. In this case, the producer overhedges, and even more than in the benchmark case 1.

Theorem 4: If the producer uses put options in his optimization problem he will decrease total production.

Proof: A long position in puts increases future consumption in the states of the world where the put ends up in the money. This means that buying puts (weakly) decreases the producer's implicit state prices because of two reasons. Recall that the producer's implicit state prices are given by:

$$q_j^P = \delta\pi \frac{U'(c_j)}{U'(c_0)}$$

Since buying puts increases future consumption in the "bad" states of the world, the numerator for these states of the world decreases because of declining marginal utility. Since buying puts also decreases current consumption the denominator increases which will make the state prices go down for every state of the world. Thus, buying more puts decreases all implicit state prices. Now suppose we have an equilibrium in which the producer does not use puts, so that $n_P = 0$:

$$\frac{dE\left[U(\tilde{c})\right]}{dy} = -C'(y)U'(c_0) + \delta\sum_{j=1}^{N}\pi_j S_j U'(c_j) = 0$$

$$\Rightarrow C'(y) = \sum_{j=1}^{N}\delta\pi_j \frac{U'(c_j)}{U'(c_0)} S_j$$

If we now add put options by increasing n_P, this decreases the implicit state prices and therefore decreases y. Puts therefore lead to a reduction in output. ||

Previous theorems have discussed the (separate) use of puts and forwards in the producer's decision. The next theorem proves properties of the combined use of these instruments.

Theorem 5: If the market forward price and the market put price are unbiased, the producer will decrease his total production, buy put options and fully hedge his total output with forward contracts.

Proof: If the producer can use unbiased forwards and puts he has to solve the following maximization problem:

$$\begin{aligned} &Max\, E\left[U(\tilde{c})\right] = U(c_0) + \delta\sum_{j=1}^{N}\pi_j U(c_j) \\ &s.t. \\ &c_0 = W_0 - C(y) - n_P P^M(X) \\ &c_j = S_j \cdot y + n_P\left[X - S_j\right]^+ + n_F\left(F^M - S_j\right) \end{aligned} \tag{18}$$

resulting in the following first-order conditions:

$$\begin{aligned} \frac{dE\left[U\left(\tilde{c}\right)\right]}{dy} &= -C'\left(y\right)U'\left(c_0\right) + \delta\sum_{j=1}^{N}\pi_j S_j U'\left(c_j\right) = 0 \\ \frac{dE\left[U\left(\tilde{c}\right)\right]}{dn_F} &= \delta\sum_{j=1}^{N}\pi_j U'\left(c_j\right)\left(S_j - F^M\right) = 0 \\ \frac{dE\left[U\left(\tilde{c}\right)\right]}{dn_P} &= -U'\left(c_0\right)P^M\left(X\right) + \delta\sum_{j=1}^{N}\pi_j U'\left(c_j\right)\left[X - S_j\right]^+ = 0 \end{aligned} \tag{19}$$

Consider the choice of the number of puts first. As shown in Theorem 3, the optimal number of put options depends on the producer's access to the bond market, which can result in underhedging, overhedging, and full hedging. Furthermore, similar to Theorem 4, this results in the producer lowering his output related to the no-hedging case. Finally, the second part of equation (19) is solved if the producer, as can be expected, fully hedges his total output. ||

Contrary to the traditional papers on optimal hedging and producing, there is a hedging role for both forwards and options, even if there is only one stochastic factor.

4. Conclusions

In this paper we examine two issues which have caused some confusion in the optimal hedging literature. We derive general conditions under which market prices for forward and put contracts can both be unbiased; this is to hold under risk neutrality or if a technical condition related on the state probabilities holds.

Our second line of research relates to optimal hedging by a producer who can use both puts and forwards to hedge production. This problem has interest mainly in the case where the producer's state prices are different from the market state prices; if both market and private state prices are identical, then hedging by producers is not an issue.

When the market state prices differ from the producer's private state prices, we show that unbiasedness of the forward price will lead to a full hedge, even though the producer will consider the forward contract to be overpriced. Unbiasedness of the put price will lead to the producer taking a long position in the put. The optimal put position depends on the producer's access to the bond market. This can lead to underhedging, full hedging and overhedging. Furthermore, if the producer uses puts to hedge his price exposure, optimal production will decrease. Finally, if both the prices of forward contracts as well as put options are unbiased there is a hedging role for put options.

References

Baillie, R.T., Bollerslev, T., 2000. The forward premium anomaly is not as bad as you think. Journal of International Money and Finance 19, 471-488.

Baron, D.P., 1970. Price uncertainty, utility and industry equilibrium in pure competition. International Economic Review 11, 463-480.

Battermann, H.L., Braulke, M., Broll, U., Schimmelpfennig, J., 2000. The preferred hedge instrument. Economics Letters 66, 85-91.

Beja, A., 1972. On systematic and unsystematic components of financial risk. Journal of Finance 27, 37-45.

Benninga, S.Z., Eldor, R., Zilcha, I., 1983. Optimal hedging in the futures market under price uncertainty. Economics Letters 13, 141-145.

Benninga, S.Z., Eldor, R., Zilcha, I., 1985. Optimal international hedging and output policies in commodity and currency forward markets. Journal of International Money and Finance 4, 537-552.

Chiang, T.C., Trinidad, J.A., 1997. Risk and international parity conditions: a synthesis from consumption-based models. International Economic Journal 11, 73-87.

Danthine, J.P., 1978. Information, futures prices and stabilizing speculation. Journal of Economic Theory 17, 79-98.

Feder, G., Just, R.E., Schmitz, A., 1980. Futures markets and the theory of the producer under price uncertainty. Quarterly Journal of Economics 97, 317-328.

Franke, G., Stapleton, R.C. Subrahmanyam, M.G., 1998. Who buys and who sells options: the role and pricing of options in an economy with background risk. Journal of Economic Theory 82, 89-109.

Holthausen, D.M., 1979. Hedging and the competitive firm under price uncertainty. American Economic Review 69, 989-995.

Lapan, H., Moschini, G., 1994. Futures hedging under price, basis, and production risk. American Journal of Agricultural Economics 76, 465-77.

Lapan, H., Moschini, G., Hanson, S.D., 1991. Production, hedging, and speculative decisions with options and futures markets. American Journal of Agricultural Economics 73, 66-74.

McKinnon, R.I., 1967. Futures markets, buffer stocks, and income stability for primary producers. Journal of Political Economy 75, 844-861.

Moschini, G, Lapan, H., 1995. The hedging role of options and futures under joint price, basis, and production risk. International Economic Review 36, 1025-1049.

Sandmo, A., 1971. On the theory of the competitive firm under price uncertainty. American Economic Review 61, 65-73.

Wu, Y., Zhang, H., 1997. Forward premiums as unbiased predictors of future currency depreciation: a non-parametric analysis. Journal of International Money and Finance 16, 609-623.