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The geometry of resonance tongues: a singularity theory approach

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Abstract
Resonance tongues and their boundaries are studied for nondegenerate and
(certain) degenerate Hopf bifurcations of maps using singularity theory methods
of equivariant contact equivalence and universal unfoldings. We recover
the standard theory of tongues (the nondegenerate case) in a straightforward
way and we find certain surprises in the tongue boundary structure when
degeneracies are present. For example, the tongue boundaries at degenerate
singularities in weak resonance are much blunter than expected from the
nondegenerate theory. Also at a semi-global level we find 'pockets' or 'flames'
that can be understood in terms of the swallowtail catastrophe.

Mathematics Subject Classification: 37G15, 37G40, 34C25

1. Introduction

This paper focuses on resonance tongues obtained by Hopf bifurcation from a fixed point of a
map. More precisely, Hopf bifurcations of maps occur at parameter values where the Jacobian
of the map has a critical eigenvalue that is a root of unity $e^{2\pi pi/q}$, where p and q are coprime
integers with $q \geqslant 3$ and $|p| < q$. Resonance tongues themselves are regions in parameter space
near the point of Hopf bifurcation where periodic points of period q exist and tongue boundaries
consist of critical points in parameter space where the q-periodic points disappear, typically
in a saddle-node bifurcation. We assume, as is usually done, that the critical eigenvalues are
simple with no other eigenvalues on the unit circle. Moreover, usually just two parameters
are varied; the effect of changing these parameters is to move the eigenvalues about an open
region of the complex plane.

1.1. Several contexts

Resonance tongues arise in several different contexts, depending, e.g. on whether the dynamics is dissipative, conservative or reversible. Generally, resonance tongues are domains in parameter space, with periodic dynamics of a specified type (regarding period of rotation number, stability, etc). In each case, the tongue boundaries are part of the bifurcation set. We mention here two standard ways that resonance tongues appear.

1.1.1. Hopf bifurcation from a periodic solution. Let

$$\frac{\mathrm{d}X}{\mathrm{d}t} = F(X)$$

be an autonomous system of differential equations with a periodic solution $Y(t)$ having its Poincaré map P centred at $Y(0) = Y_0$. For simplicity we take $Y_0 = 0$, so $P(0) = 0$. A Hopf bifurcation occurs when eigenvalues of the Jacobian matrix $(dP)_0$ are on the unit circle and resonance occurs when these eigenvalues are roots of unity $\mathrm{e}^{2\pi \mathrm{p}i/q}$. Strong resonances occur when $q < 5$. Except at strong resonances, Hopf bifurcation leads to the existence of an invariant circle for the Poincaré map and an invariant torus for the autonomous system. This is usually called a Naimark–Sacker bifurcation. At weak resonance points the flow on the torus has very thin regions in parameter space (between the tongue boundaries) where this flow consists of a phase-locked periodic solution that winds around the torus q times in one direction (the direction approximated by the original periodic solution) and p times in the other.

1.1.2. Periodic forcing of an equilibrium. Let

$$\frac{\mathrm{d}X}{\mathrm{d}t} = F(X) + G(t)$$

be a periodically forced system of differential equations with 2π-periodic forcing $G(t)$. Suppose that the autonomous system has a hyperbolic equilibrium at $Y_0 = 0$; i.e. $F(0) = 0$. Then the forced system has a 2π-periodic solution $Y(t)$ with initial condition $Y(0) = Y_0$ near 0. The dynamics of the forced system near the point Y_0 is studied using the stroboscopic map P that maps the point X_0 to the point $X(2\pi)$, where $X(t)$ is the solution to the forced system with initial condition $X(0) = X_0$. Note that $P(0) = 0$ in coordinates centred at Y_0. Again resonance can occur as a parameter is varied when the stroboscopic map undergoes Hopf bifurcation with critical eigenvalues equal to roots of unity. Resonance tongues correspond to regions in parameter space near the resonance point where the stroboscopic map has q-periodic trajectories near 0. These q-periodic trajectories are often called subharmonics of order q.

1.2. Background and sketch of results

The types of resonances mentioned here have been much studied; we refer to Takens [37], Newhouse *et al* [33], Arnold [2] and references therein. For more recent work on strong resonance, see Krauskopf [30]. In general, these works study the complete dynamics near resonance, not just the shape of resonance tongues and their boundaries. Similar remarks can be made on studies in Hamiltonian or reversible contexts, such as Broer and Vegter [16] or Vanderbauwhede [39]. As in our paper, in many of these references some form of singularity theory is used as a tool.

The problem we address is how to find resonance tongues in the general setting, without being concerned by stability, further bifurcation and similar dynamical issues. It turns out that contact equivalence in the presence of $\mathbf{Z}_q$ symmetry is an appropriate tool for
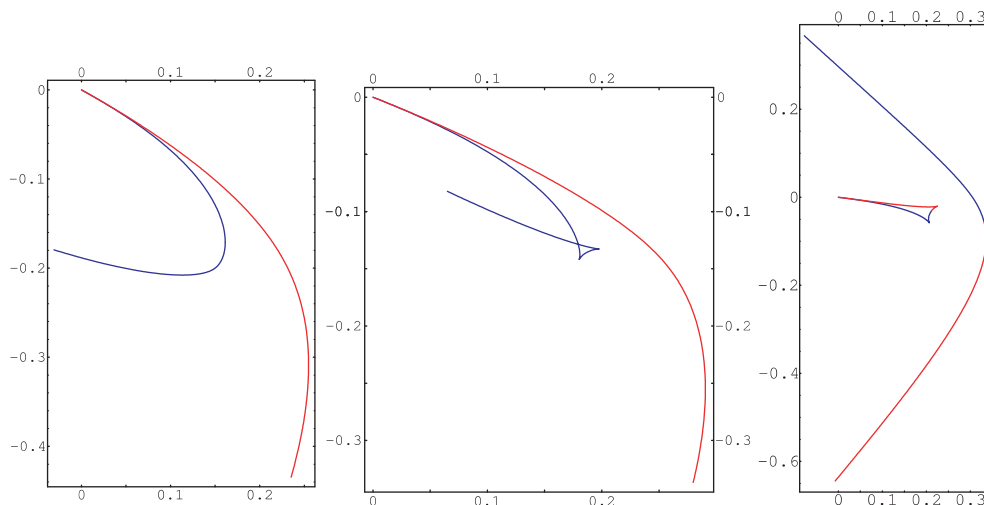
Figure 1. Resonance tongues with pocket- or flame-like phenonmena near a degenerate Hopf bifurcation through $e^{2\pi ip/q}$ in a family depending on two complex parameters. Fixing one of these parameters at various (three) values yields a family depending on one complex parameter, with resonance tongues contained in the plane of this second parameter. As the first parameter changes, these tongue boundaries exhibit cusps (middle picture), and even become disconnected (rightmost picture). The small triangle in the rightmost picture encloses the region of parameter values for which the system has four q-periodic orbits. These phenomena are explained in section 5.

this, when first a Liapunov–Schmidt reduction is utilized, see Golubitsky *et al* [24, 25]. The main question asks for the number of q-periodic solutions as a function of parameters, and each tongue boundary marks a change in this number. In the next subsection, we briefly describe how this reduction process works. The singularity theory and analysis that is needed to study the reduced equations is developed in sections 2 and 5; the longer singularity theory proofs are postponed to appendix A.

It turns out that the standard, nondegenerate cases of Hopf bifurcation [2, 37] can be easily recovered by this method. When $q \geqslant 7$ we are able to treat a degenerate case, where the third-order terms in the reduced equations, the 'Hopf coefficients', vanish. We find pocket- or flame-like regions of four q-periodic orbits in addition to the regions with only zero or two, compare figure 1. In addition, the tongue boundaries contain new cusp points and in certain cases the tongue region is blunter than in the nondegenerate case. These results are described in detail in section 5, also compare figure 2. Section 6 is devoted to concluding remarks and further questions.

1.3. Related work

The geometric complexity of resonance domains has been the subject of many studies of various scopes. Some of these, like this paper, deal with quite universal problems while others restrict themselves to interesting examples. As opposed to this paper, often normal form theory is used to obtain information about the nonlinear dynamics. In this context the normal forms automatically are $\mathbf{Z}_q$-equivariant.

1.3.1. Chenciner's degenerate Hopf bifurcation.

In [19–21] Chenciner considers a two-parameter unfolding of a degenerate Hopf bifurcation. Strong resonances to some finite order
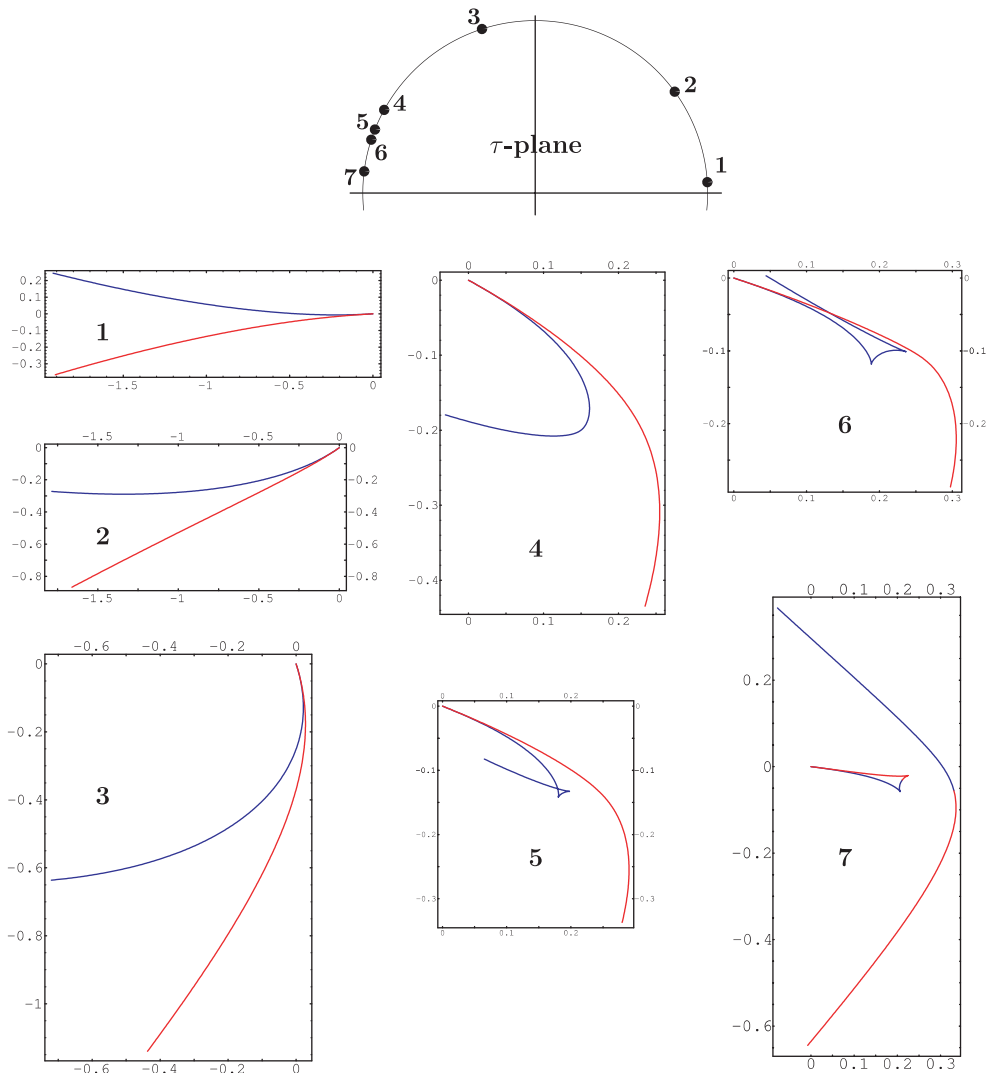
Figure 2. Top: the τ-plane. Frames 1–7: resonance tongues in the σ-plane for various values of $\tau = 0.1\,\mathrm{e}^{2\pi\mathrm{i}\theta}$. (1) $\theta = 0.01$, (2) $\theta = 0.10$, (3) $\theta = 0.30$, (4) $\theta = 0.42$, (5) $\theta = 0.44$, (6) $\theta = 0.45$, (7) $\theta = 0.48$.

are excluded in the 'rotation number' ω_0 at the central fixed point. In [21] for sequences of 'good' rationals p_n/q_n tending to ω_0, corresponding periodic points are studied with the help of $\mathbf{Z}_{q_n}$-equivariant normal form theory. For a further discussion of the codimension k Hopf bifurcation compare Broer and Roussarie [11].

1.3.2. The geometric program of Peckam et al.   The research program reflected in [31, 32, 35, 36] views resonance 'tongues' as projections on a 'traditional' parameter plane of (saddle-node) bifurcation sets in the product of parameter and phase space. This approach has the same spirit as ours and many interesting geometric properties of 'resonance tongues' are discovered and explained in this way. We note that the earlier result [34] on higher-order degeneracies in a period-doubling uses $\mathbf{Z}_2$ equivariant singularity theory.

Particularly, we would like to mention the results of [36] concerning a class of oscillators with doubly periodic forcing. It turns out that these systems can have coexistence of periodic attractors (of the same period), giving rise to 'secondary' saddle-node lines, sometimes enclosing a flame-like shape. In the present, more universal, approach we find similar complications of traditional resonance tongues, compare figure 2 and its explanation in section 5.

1.3.3. Related work by Broer et al. In [15], an even smaller universe of annulus maps is considered, with Arnold's family of circle maps as a limit. Here 'secondary' phenomena are found that are similar to the ones discussed presently. Indeed, apart from extra saddle-node curves inside tongues also many other bifurcation curves are detected.

We like to mention related results in the reversible and symplectic settings regarding parametric resonance with periodic and quasi-periodic forcing terms by Afsharnejad [1] and Broer *et al* [3,4,7–10,12–14,16]. Here the methods use Floquet theory, obtained by averaging, as a function of the parameters.

Singularity theory (with left–right equivalences) is used in various ways. First of all, it helps to understand the complexity of resonance tongues in the stability diagram. It turns out that crossing tongue boundaries, which may give rise to instability pockets, are related to Whitney folds as these occur in two-dimensional maps. These problems already occur in the linearized case of Hill's equation. The question is whether these phenomena can be recovered by methods as developed in this paper. Finally, in the nonlinear cases, application of Z_2- and D_2-equivariant singularity theory helps to get dynamical information on normal forms.

1.4. Finding resonance tongues

Our method for finding resonance tongues—and tongue boundaries—proceeds as follows. Find the region in parameter space corresponding to points where the map P has a q-periodic orbit; i.e. solve the equation $P^q(x) = x$. Using a method due to Vanderbauwhede (see [39,40]), we can solve for such orbits by Liapunov–Schmidt reduction. More precisely, a q-periodic orbit consists of q points $x_1, \ldots, x_q$ where

$$P(x_1) = x_2, \qquad \ldots, \qquad P(x_{q-1}) = x_q, \qquad P(x_q) = x_1.$$

Such periodic trajectories are just zeros of the map

$$\widehat{P}(x_1, \ldots, x_q) = (P(x_1) - x_2, \ldots, P(x_q) - x_1).$$

Note that $\widehat{P}(0) = 0$, and that we can find all zeros of $\widehat{P}$ near the resonance point by solving the equation $\widehat{P}(x) = 0$ by Liapunov–Schmidt reduction. Note also that the map $\widehat{P}$ has Z_q symmetry. More precisely, define

$$\sigma(x_1, \ldots, x_q) = (x_2, \ldots, x_q, x_1).$$

Then observe that

$$\widehat{P}\sigma = \sigma\widehat{P}.$$

At 0, the Jacobian matrix of $\widehat{P}$ has the block form

$$J = \begin{pmatrix} A & -I & 0 & 0 & \cdots & 0 & 0 \\ 0 & A & -I & 0 & \cdots & 0 & 0 \\ & & & \vdots & & & \\ 0 & 0 & 0 & 0 & \cdots & A & -I \\ -I & 0 & 0 & 0 & \cdots & 0 & A \end{pmatrix},$$

where $A = (dP)_0$. The matrix J automatically commutes with the symmetry σ and hence J can be block diagonalized using the isotypic components of irreducible representations of $\mathbf{Z}_q$. (An isotypic component is the sum of the $\mathbf{Z}_q$ isomorphic representations. See [25] for details. In this instance all calculations can be done explicitly and in a straightforward manner.) Over the complex numbers it is possible to write these irreducible representations explicitly. Let ω be a qth root of unity. Define V_ω to be the subspace consisting of vectors

$$[x]_\omega = \begin{pmatrix} x \\ \omega x \\ \vdots \\ \omega^{q-1} x \end{pmatrix}.$$

A short calculation shows that

$$J[x]_\omega = [(A - \omega I)x]_\omega.$$

Thus, J has zero eigenvalues precisely when A has qth roots of unity as eigenvalues. By assumption, A has just one such pair of complex conjugate qth roots of unity as eigenvalues.

Since the kernel of J is two-dimensional—the simple eigenvalue assumption in the Hopf bifurcation—it follows using Liapunov–Schmidt reduction that solving the equation $\widehat{P}(x) = 0$ near a resonance point is equivalent to finding the zeros of a reduced map from $\mathbf{R}^2 \to \mathbf{R}^2$. We can, however, naturally identify $\mathbf{R}^2$ with $\mathbf{C}$, which we do. Thus, we need to find the zeros of a smooth implicitly defined function

$$g : \mathbf{C} \to \mathbf{C},$$

where $g(0) = 0$ and $(dg)_0 = 0$. Moreover, assuming that the Liapunov–Schmidt reduction is done to respect symmetry, the reduced map g commutes with the action of σ on the critical eigenspace. More precisely, let ω be the critical resonant eigenvalue of $(dP)_0$; then

$$g(\omega z) = \omega g(z). \tag{1.1}$$

Since p and q are coprime, ω generates the group $\mathbf{Z}_q$ consisting of all qth roots of unity. So g is $\mathbf{Z}_q$-equivariant.

We propose to use $\mathbf{Z}_q$-equivariant singularity theory to classify resonance tongues and tongue boundaries.

2. $\mathbf{Z}_q$ singularity theory

In this section, we develop normal forms for the simplest singularities of $\mathbf{Z}_q$-equivariant maps g of the form (1.1). To do this, we need to describe the form of $\mathbf{Z}_q$-equivariant maps, contact equivalence, and finally the normal forms.

2.1. The structure of $\mathbf{Z}_q$-equivariant maps

We begin by determining a unique form for the general $\mathbf{Z}_q$-equivariant polynomial mapping. By Schwarz's theorem [25] this representation is also valid for C^∞ germs.

Lemma 2.1. *Every $\mathbf{Z}_q$-equivariant polynomial map $g : \mathbf{C} \to \mathbf{C}$ has the form*

$$g(z) = K(u, v)z + L(u, v)\bar{z}^{q-1}, \tag{2.1}$$

where $u = z\bar{z}$, $v = z^q + \bar{z}^q$, and K, L are uniquely defined complex-valued function germs.

Proof. It is known that every real-valued $\mathbf{Z}_q$-invariant polynomial $h : \mathbf{C} \to \mathbf{R}$ is a function of u, v, w where $w = i(z^q - \bar{z}^q)$. Since

$$w^2 = 4u^q - v^2,$$

it follows that invariant polynomials can be written uniquely in the form

$$h(z) = A(u, v) + B(u, v)w. \tag{2.2}$$

Similarly, every $\mathbf{Z}_q$-equivariant polynomial mapping $g : \mathbf{C} \to \mathbf{C}$ is well known to have the form

$$g(z) = K(z)z + L(z)\bar{z}^{q-1},$$

where K and L are complex-valued invariant functions. Since

$$w = iv - 2i\bar{z}^q \qquad \text{and} \qquad w = 2iz^q - iv$$

it follows that

$$
\begin{aligned}
wz &= ivz - 2iu\bar{z}^{q-1}, \\
w\bar{z}^{q-1} &= 2iu^{q-1}z - iv\bar{z}^{q-1}.
\end{aligned} \tag{2.3}
$$

Thus, we can assume that K and L are complex-valued invariant functions that are independent of w. ∎

Let $\mathcal{E}_{u,v}$ be the space of complex-valued map germs depending on u and v. Then (2.1) implies that we can identify $\mathbf{Z}_q$-equivariant germs g with pairs $(K, L) \in \mathcal{E}_{u,v}^2$.

When finding period q points, we are led by Liapunov–Schmidt reduction to a map g that has the form (2.1) and satisfies $K(0, 0) = 0$. By varying two parameters we can guarantee that $K(0, 0) \in \mathbf{C}$ can vary arbitrarily in a neighbourhood of 0.

2.2. $\mathbf{Z}_q$ contact equivalences

Singularity theory approaches the study of zeros of a mapping near a singularity by implementing coordinate changes that transform the mapping to a 'simple' normal form and then solving the normal form equation. The kinds of transformations that preserve the zeros of a mapping are called contact equivalences. More precisely, two $\mathbf{Z}_q$-equivariant germs g and h are $\mathbf{Z}_q$-contact equivalent if

$$h(z) = S(z)g(Z(z)), \tag{2.4}$$

where $Z(z)$ is a $\mathbf{Z}_q$-equivariant change of coordinates and $S(z) : \mathbf{C} \to \mathbf{C}$ is a real linear map for each z that satisfies

$$S(\gamma z)\gamma = \gamma S(z) \tag{2.5}$$

for all $\gamma \in \mathbf{Z}_q$. A characterization of the function S is given in lemma A.1

2.3. Normal form theorems

In this paper, we consider two classes of normal forms—the codimension two standard for resonant Hopf bifurcation and one more degenerate singularity that has a degeneracy at cubic order. These singularities all satisfy the nondegeneracy condition $L(0, 0) \neq 0$; we explore this case first.

Theorem 2.2. *Suppose that*

$$h(z) = K(u, v)z + L(u, v)\bar{z}^{q-1},$$

where $K(0, 0) = 0$.

$q \geqslant 5$. *If $K_u L(0, 0) \neq 0$, then h is $\mathbf{Z}_q$ contact equivalent to*

$$g(z) = |z|^2 z + \bar{z}^{q-1} \tag{2.6}$$

with universal unfolding

$$G(z, \sigma) = (\sigma + |z|^2)z + \bar{z}^{q-1}. \tag{2.7}$$

$q = 4$. *Let $a = |K_u(0, 0)/L(0, 0)|$. If $L(0, 0) \neq 0$ and $a \neq 0, 1$, then h is $\mathbf{Z}_4$ contact equivalent to*

$$g(z) = auz + \bar{z}^3,$$

where $0 < a < 1$ or $1 < a$ is a moduli parameter. The universal unfolding of g is

$$G(z) = (\sigma + au)z + \bar{z}^3, \tag{2.8}$$

where $\sigma \approx 0$ is complex.

$q = 3$. *If $L(0, 0) \neq 0$, then the normal form for $q = 3$ is*

$$g(z) = \bar{z}^2,$$

which has codimension two, and the universal unfolding of g is

$$G(z) = \sigma z + \bar{z}^2. \tag{2.9}$$

Theorem 2.3. *Suppose that*

$$h(z) = K(u, v)z + L(u, v)\bar{z}^{q-1},$$

where $K(0, 0) = 0 = K_u(0, 0)$ and $q \geqslant 7$. If $K_{uu}(0, 0)L(0, 0) \neq 0$, then h is $\mathbf{Z}_q$ contact equivalent to

$$g(z) = |z|^4 z + \bar{z}^{q-1} \tag{2.10}$$

with universal unfolding

$$G(z, \sigma, \tau) = (\sigma + \tau |z|^2 + |z|^4)z + \bar{z}^{q-1}, \tag{2.11}$$

where $\sigma, \tau \in \mathbf{C}$.

The proofs of theorems 2.2 and 2.3 are given in appendix A.

3. Resonance domains

In this section, we compute boundaries of resonance domains corresponding to universal unfoldings of the form

$$G(z) = b(u)z + \bar{z}^{q-1}. \tag{3.1}$$

By definition, the tongue boundary is the set of parameter values where local bifurcations in the number of period q points take place; and, typically, such bifurcations will be saddle-node bifurcations. For universal unfoldings of the simplest singularities the boundaries of these parameter domains have been called tongues, since the domains have the shape of a tongue, with its tip at the resonance point. Below we show that our method easily recovers resonance tongues in the standard least degenerate cases. Then, we study a more degenerate singularity and show that the usual description of tongues needs to be broadened.

Tongue boundaries of a $p : q$ resonance are determined by the following system:

$$\bar{z}G = 0,$$
$$\det(dG) = 0. \tag{3.2}$$

This follows from the fact that local bifurcations of the period q orbits occur at parameter values where the system $G = 0$ has a singularity, i.e. where the rank of dG is less than two. Recalling that

$$u = z\bar{z}, \qquad v = z^q + \bar{z}^q, \qquad w = i(z^q - \bar{z}^q), \tag{3.3}$$

we prove the following theorem, which is independent of the form of $b(u)$.

Theorem 3.1. *For universal unfoldings (3.1), equations (3.2) have the form*

$$|b|^2 = u^{q-2},$$
$$b\bar{b}' + \bar{b}b' = (q - 2)u^{q-3}. \tag{3.4}$$

Proof. Begin by noting that $G(z) = 0$ implies that

$$\bar{z}G(z) = bu + \bar{z}^q = 0.$$

Therefore,

$$\bar{z}^q = -bu \qquad \text{and} \qquad z^q = -\bar{b}u.$$

It follows that

$$v = -(b + \bar{b})u \qquad \text{and} \qquad w = -i(b - \bar{b})u.$$

Hence, the identity $v^2 + w^2 = 4u^q$ implies

$$|b|^2 = u^{q-2},$$

which is the first result to be shown.

Next we compute $\det(dG) = 0$ by recalling that in complex coordinates

$$\det(dG) = |G_z|^2 - |G_{\bar{z}}|^2.$$

Observe that

$$|G_z|^2 = (b + b'u)(\bar{b} + \bar{b}'u)$$
$$= |b|^2 + (b\bar{b}' + \bar{b}b')u + |b'|^2u^2$$

and

$$|G_{\bar{z}}|^2 = (b'z^2 + (q-1)\bar{z}^{q-2})(\bar{b}'\bar{z}^2 + (q-1)z^{q-2})$$
$$= |b'|^2u^2 - (q-1)(b\bar{b}' + \bar{b}b')u + (q-1)^2u^{q-2}.$$

Therefore,

$$0 = \det(dG) = |b|^2 + q(b\bar{b}' + \bar{b}b')u - (q-1)^2u^{q-2}. \tag{3.5}$$

The second equation in (3.4) now follows from (3.5) by applying the first equation in (3.4), which has already been proved. ■

We now use (3.4) to determine tongue boundaries for the universal unfoldings classified in theorems 2.2 and 2.3.

4. Nondegenerate cases: resonance tongues

Here we recover several classical results on the geometry of resonance tongues, in this context of Hopf bifurcation. To begin, we discuss weak resonances $q \geqslant 5$, where a $(q/2) - 1$ cusp forms the tongue-tip and where the concept of resonance tongue remains unchallenged. Note that similar tongues are found in the Arnold family of circle maps [2], also compare Broer *et al* [15]. Then, we consider the strong resonances $q = 3$ and 4, where we again recover known results on the shape of the resonance domain. For a more complete discussion of the dynamics near these resonance points see [2, 37].

4.1. The nondegenerate singularity when $q \geqslant 5$

We first investigate the nondegenerate case $q \geqslant 5$ given in (2.7). Here

$$b(u) = \sigma + u,$$

where $\sigma = \mu + i\nu$. We shall compute the tongue boundaries in the (μ, ν)-plane in the parametric form $\mu = \mu(u)$, $\nu = \nu(u)$, where $u \geqslant 0$ is a local real parameter.

Short computations show that

$$|b|^2 = (\mu + u)^2 + \nu^2,$$
$$b\bar{b}' + \bar{b}b' = 2(\mu + u).$$

Then theorem 3.1 gives us the following parametric representation of the tongue boundaries:

$$\mu = -u + \frac{q - 2}{2} u^{q-3},$$
$$\nu^2 = u^{q-2} - \frac{(q - 2)^2}{4} u^{2(q-3)}.$$

In this case the tongue boundaries at $(\mu, \nu) = (0, 0)$ meet in the familiar $(q - 2)/2$ cusp

$$\nu^2 \approx (-\mu)^{q-2}. \tag{4.1}$$

It is to this and similar situations that the usual notion of resonance tongue applies: inside the sharp tongue a pair of period q orbits exists and these orbits disappear in a saddle-node bifurcation at the boundary.

4.2. The nondegenerate singularity when $q = 4$

In this case by (2.8) we have that

$$b(u) = \sigma + au,$$

where with a a modulus parameter such that $a > 0$ and $a \neq 1$. Short computations show that

$$|b|^2 = (\mu + au)^2 + \nu^2,$$
$$b\bar{b}' + \bar{b}b' = 2a(\mu + au).$$

From (3.4) we obtain

$$\mu = \frac{1 - a^2}{a} u,$$
$$\nu^2 = \frac{a^2 - 1}{a^2} u^2.$$

These equations again give a parametric description of the resonance domain in the (μ, ν) plane. When $a > 1$ we find tongue boundaries formed by the lines

$$\mu = \pm\sqrt{a^2 - 1}\,\nu.$$

So the tongues bound a wedge approaching a half line as a approaches 1. When $0 < a < 1$ there are no tongue boundaries—period 4 points exist for all values of (μ, ν).

4.3. The nondegenerate singularity when $q = 3$

In this case we have

$$b = \sigma.$$

Again we recover the familiar result that no tongues exist, but that period 3 points exist for all values of (μ, ν).

More precisely, there is a period 3 trajectory, corresponding to a solution of (2.9), for every $\sigma \neq 0$. To verify this point, set $\sigma = re^{i\theta}$. Then, set $z = e^{-i\theta/3}y$ in (2.9) obtaining:

$$ry + \bar{y}^2 = 0,$$

where r is real and positive. It follows that $y = -r$ is a nonzero solution.

5. Degenerate singularities when $q \geqslant 7$

5.1. Tongue boundaries in the degenerate case

The next step is to analyse a more degenerate case, namely, the singularity

$$g(z) = u^2 z + \bar{z}^{q-1},$$

when $q \geqslant 7$. We recall from (2.11) that a universal unfolding of g is given by $G(z) = b(u)z + \bar{z}^{q-1}$, where

$$b(u) = \sigma + \tau u + u^2. \tag{5.1}$$

Here σ and τ are complex parameters, which leads to a real four-dimensional parameter space. As before, we set $\sigma = \mu + i\nu$ and consider how the tongue boundaries in the (μ, ν)-plane depend on the complex parameter τ. In this discussion, we calculate the tongue boundaries with computer assistance, using the parametric forms of μ and ν given by theorem 5.2. We will see that a new complication occurs in the tongue boundaries for certain τ, namely, cusp bifurcations occur at isolated points of the fold (saddle-node) lines. The interplay of these cusps is quite interesting and challenges some of the traditional descriptions of resonance tongues when $q = 7$ and presumably for $q \geqslant 7$.

We recall that the tongue boundaries are determined by equations (3.2). For the specific choice (5.1) for b it follows that the solution of these equations is the discriminant set

$$D_P = \left\{(\sigma, \tau) \in \mathbf{C} \times \mathbf{C} \mid P(u, \sigma, \tau) = \frac{\partial P}{\partial u}(u, \sigma, \tau) = 0, \text{ for some } u \in \mathbf{R}\right\}$$

of the polynomial P, defined by

$$P(u, \sigma, \tau) = (u^2 + \tau u + \sigma)(u^2 + \bar{\tau}u + \sigma) - u^{q-2}.$$

Putting $\tau = \alpha + i\beta$ and $\sigma = \mu + i\nu$, with $\alpha, \beta, \mu, \nu \in \mathbf{R}$, we obtain the following expression for the family P:

$$P(u, \alpha, \beta, \mu, \nu) = u^4 - u^{q-2} + 2\alpha u^3 + (\alpha^2 + \beta^2 + 2\mu)u^2 + 2(\alpha\mu + \beta\nu)u + (\mu^2 + \nu^2). \tag{5.2}$$

We try to obtain a parametrization of D_P by solving the system of equations

$$P(u, \alpha, \beta, \mu, \nu) = \frac{\partial P}{\partial u}(u, \alpha, \beta, \mu, \nu) = 0, \qquad (5.3)$$

for μ and ν. This gives us a solution of the form

$$(\mu, \nu) = (\mu(u, \alpha, \beta), \nu(u, \alpha, \beta)),$$

which defines a curve in the (μ, ν)-plane, for fixed (α, β).

To this end we consider the solution set of (5.3) as a perturbation of the solution set of the system

$$P_0(u, \alpha, \beta, \mu, \nu) = \frac{\partial P_0}{\partial u}(u, \alpha, \beta, \mu, \nu) = 0,$$

where

$$P_0(u, \sigma, \tau) = (u^2 + \tau u + \sigma)(u^2 + \bar{\tau} u + \sigma) = P(u) + u^{q-2}.$$

Lemma 5.1. *The discriminant set of P_0 is the hypersurface in $\mathbf{R}^4$ parametrized by $\Gamma_0 : \mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}^4$, with $\Gamma_0(u, \alpha, \beta) = (\alpha, \beta, \mu_0(u, \alpha, \beta), \nu_0(u, \alpha, \beta))$ defined by*

$$\begin{aligned}
\mu_0(u) &= -u(\alpha + u), \\
\nu_0(u) &= -\beta u.
\end{aligned} \qquad (5.4)$$

Proof. First, we observe that every root of $P_0(u, \sigma, \tau) = 0$ has multiplicity at least two, so $P_0(u, \sigma, \tau) = 0$ implies $(\partial P_0/\partial u)(u, \sigma, \tau) = 0$. Now $P_0(u, \tau, \sigma) = 0$ implies $\mu = \mu_0(u, \alpha, \beta)$ and $\nu = \nu_0(u, \alpha, \beta)$, with μ_0 and ν_0 as in (5.4). This completes the proof. ∎

Theorem 5.2. *The discriminant set of P is the union of the two hypersurfaces $D_P^{\pm}$ in $\mathbf{R}^4$ parametrized by $\Gamma_{\pm} : \mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}^4$, with $\Gamma_{\pm}$ of the form*

$$\Gamma_{\pm}(u, \alpha, \beta) = (\alpha, \beta, \mu_0(u, \alpha, \beta) + M_{\pm}(u, \alpha, \beta), \nu_0(u, \alpha, \beta) + N_{\pm}(u, \alpha, \beta))$$

with

$$\begin{aligned}
M_{\pm}(u, \alpha, \beta) &= \frac{(q-2)u^{q-3}(\alpha + 2u) \pm \beta u^{(q-2)/2}\sqrt{4D(u, \alpha, \beta) - (q-2)^2 u^{q-4}}}{2D(u, \alpha, \beta)}, \\
N_{\pm}(u, \alpha, \beta) &= \frac{(q-2)\beta u^{q-3} \mp (\alpha + 2u)u^{(q-2)/2}\sqrt{4D(u, \alpha, \beta) - (q-2)^2 u^{q-4}}}{2D(u, \alpha, \beta)},
\end{aligned} \qquad (5.5)$$

where

$$D(u, \alpha, \beta) = (\alpha + 2u)^2 + \beta^2.$$

Proof. We try to obtain a solution of (5.3) by perturbing the parametrization (5.4) by putting

$$\begin{aligned}
\mu &= \mu_0(u, \alpha, \beta) + M, \\
\nu &= \nu_0(u, \alpha, \beta) + N.
\end{aligned}$$

Plugging this into (5.3) yields the following system of equations for M and N:

$$\begin{aligned}
M^2 + N^2 - u^{q-2} &= 0, \\
2\alpha M + 2\beta N + 4Mu - (q-2)u^{q-3} &= 0.
\end{aligned}$$

This system has two pairs of real solutions $(M_{\pm}(u, \alpha, \beta), N_{\pm}(u, \alpha, \beta))$ defined by (5.5). ∎

In the next subsection this parametrization is used to obtain pictures of the resonance regions for various values of $\tau = \alpha + i\beta$.

5.2. Degenerate tongue boundaries when $q = 7$

Below we explore the tongue boundaries numerically for fixed τ. Note that the discriminant set D_P is invariant under the transformation $(\sigma, \tau) \mapsto (\bar{\sigma}, \bar{\tau})$. Thus, we may assume that $\mathrm{Im}(\tau) \geqslant 0$. To perform these calculations, we set $\tau = \rho e^{2\pi i \theta}$, where $0 \leqslant \theta \leqslant 0.5$.

We use `Mathematica` to plot the curves given by theorem 5.2 for various values of τ. Fixing $\rho = 0.1$ we choose several representative values of $\theta \in (0, 0.5)$. In figure 2, left column, we consider $\theta = 0.01, 0.10$ and 0.30, thereby traversing the first quadrant, and see a resonance tongue that begins to rotate and bend. Inside the cusped region there are two (locally defined) period 7 trajectories that annihilate each other in a saddle-node bifurcation as the tongue boundary is crossed. This description is consistent with the description of resonance tongues in the least degenerate case. However, this description changes as τ nears the negative real axis.

In figure 2, middle column, we consider $\theta = 0.42$ and 0.44. The left branch of the tongue folds over on itself forming a singularity as θ is varied. As that singularity unfolds a small triangular 'pocket' or 'flame' emerges before $\theta = 0.44$. This pocket is defined by two cusps and a crossing of the boundary curve. Inside the pocket region, there are four trajectories of period 7 points. The bending continues at $\theta = 0.45$ where the two cusps approach the tongue boundary, (see figure 2) right column. Moreover, the tongue boundary emanating from the right cusp sweeps past the origin and opens the 'pocket'. Figure 3 depicts the sequence of bifurcations in more detail, and gives approximate θ-values of the corresponding bifurcations. Moreover, it contains an additional frame, labelled 5a, corresponding to the situation just after the tongue boundary emanating from the right cusp has swept past the origin.

Finally, we see in figure 2 that by $\theta = 0.48$, the two cusps and the tongue tip have formed a triangle that has detached from the rest of the boundary. In the four-dimensional (σ, τ) parameter space, this geometry is reminiscent of catastrophes like the swallowtail or elliptic umbilic—but in a $\mathbf{Z}_q$-equivariant setting. In the two-dimensional parameter space, two intersecting tongue branches switch when the right cusp touches the boundary; namely, the upper tongue branch from the cusp at zero and the upper tongue branch from the right cusp.

For τ near the negative real axis this analysis refers to the boundary of the pocket region shown in figure 2 (right column) that surrounds the region of the four period 7 trajectories. For this part of the boundary we find the expected thin resonance 'tongue'—but that 'tongue' does not refer to the boundary between regions where period 7 points exist and regions where they do not. In the next section we show that the four period 7 points occur for α arbitrarily close to 0.

5.3. Four q-periodic orbits

Our pictures of the resonance regions for various values of τ suggest the existence of a small 'triangle' in the σ-plane with interior points corresponding to the occurrence of four real roots near 0 of the equation $P(u, \sigma, \tau) = 0$. This triangle emerges for τ-values near the negative real axis. In this section we show that this region exists for τ-values arbitrarily near the origin. We do so by constructing a smooth curve in parameter space, emerging from the origin, such that the equation $P(u, \sigma, \tau) = 0$ has four real roots near $u = 0$ for parameter values (σ, τ) on this curve.

To this end we observe that the 'spine' of the resonance region is the curve parametrized by (5.4). This spine corresponds to parameter values (σ, τ) for which the polynomial $P_0(u, \sigma, \tau)$ has two double zeros. Indeed, a simple computation shows that

$$P_0(u, -\alpha t - t^2, -\beta t, \alpha, \beta) = (u - t)^2((u + t + \alpha)^2 + \beta^2). \tag{5.6}$$
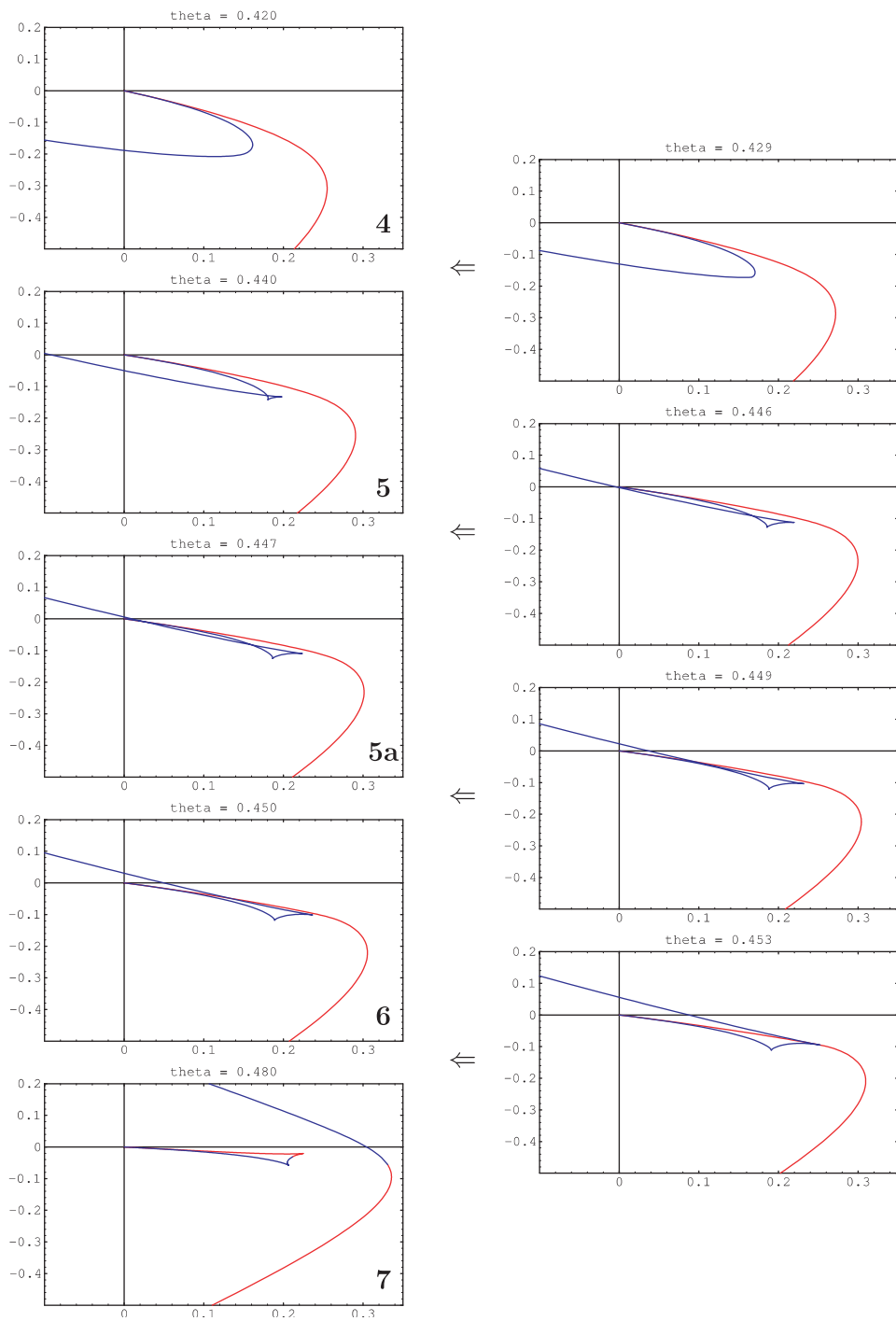
Figure 3. Birth and detachment of a triangular pocket corresponding to four period 7 points via a sequence of four bifurcations (right column). The pocket emerges due to a swallowtail bifurcation at $\theta \approx 0.429$. Its upper branch sweeps past the origin, at $\theta \approx 0.446$, after which the pocket is opened up at $\theta \approx 0.449$ via a tangency of two of its sides. Finally, the pocket is detached due to a branch switch at $\theta \approx 0.453$.

We now define a curve in parameter space by making t and β dependent on α, in such a way that four real zeros occur for parameters on this curve. Since our experiments indicate that four small real roots occur for τ near the negative real axis we put $\beta = 0$, and expect four real roots (where we have to restrict to $\alpha < 0$ if q is odd). Furthermore, the triangle enclosing σ-values for which four small real roots are observed shrinks to size 0 as τ tends to 0. Therefore, our second guess is $t = \alpha^2$. (In fact, we first tried $t = \alpha$, but the corresponding curve tends to 0 via the region of two small real roots.)

Summarizing, we consider the one-parameter family $p(u, \alpha)$ of polynomials in u, defined by

$$p(u, \alpha) = P(u, -\alpha^3 - \alpha^4, -\alpha^4, \alpha, 0). \tag{5.7}$$

In view of (5.6), this polynomial satisfies

$$p(u, \alpha) = (u - \alpha^2)^2 (u + \alpha + \alpha^2)^2 - u^{q-2}.$$

Theorem 5.3.

1. *For sufficiently small α the polynomial $p(\cdot, \alpha)$ has a pair $u_\pm(\alpha)$ of distinct real zeros near $0 \in \mathbf{R}$, satisfying*

$$u_\pm(\alpha) = \alpha^2 \pm \alpha^{q-3} + \mathrm{O}(\alpha^{q-2}), \tag{5.8}$$

2. *Moreover, for sufficiently small α there is an other pair $U_\pm(\alpha)$ of distinct real zeros near $0 \in \mathbf{R}$, satisfying*

$$U_\pm(\alpha) = -\alpha - \alpha^2 \pm |\alpha|^{(q-4)/2} + \mathrm{O}(|\alpha|^{(q-3)/2}) \tag{5.9}$$

for q odd, and $\alpha \leqslant 0$, and

$$U_\pm(\alpha) = -\alpha - \alpha^2 \pm \alpha^{(q-4)/2} + \mathrm{O}(\alpha^{(q-2)/2}) \tag{5.10}$$

for q even.

The proof is contained in appendix B.

Corollary 5.4. *There are parameter values (σ, τ) arbitrarily near $(0, 0) \in \mathbf{C} \times \mathbf{C}$ for which four period q trajectories exist.*

5.4. Four-dimensional geometric structure

We derived the parametrization (5.5) of the discriminant set D_P of the family P, given by (5.2). Using this parametrization, we obtained cross-sections of this singular surface D_P; see figure 2. In this section, we show that D_P is the pull-back (under a singular map) of the product of the swallowtail and the real line.

To this end, consider the discriminant set of the model family $Q : \mathbf{R}^4 \to \mathbf{R}$, defined by

$$Q(u, \kappa, \lambda, \varrho, \alpha) = u^4 - u^{q-2} + 2\alpha u^3 + \varrho u^2 + \lambda u + \kappa. \tag{5.11}$$

In fact, D_P is the pull-back of D_Q under the singular map $\psi : \mathbf{R}^4 \to \mathbf{R}^4$, defined by

$$\psi(\alpha, \beta, \mu, \nu) = (\mu^2 + \nu^2, \alpha\mu + \beta\nu, \alpha^2 + \beta^2 + 2\mu, \alpha). \tag{5.12}$$

We note in passing that the map $\psi : \mathbf{R}^4 \to \mathbf{R}^4$, given by (5.12), has rank 2 at $0 \in \mathbf{R}^4$. Its singular set is given by

$$S(\psi) = \{(\alpha, \beta, \mu, \nu) \mid \beta^2\mu - \alpha\beta\nu + \nu^2 = 0\}$$

and is diffeomorphic to the product of the real line and the Whitney umbrella $W = \{(\beta, \mu, \nu) \mid \beta^2\mu + \nu^2 = 0\}$ (see also figure 4).
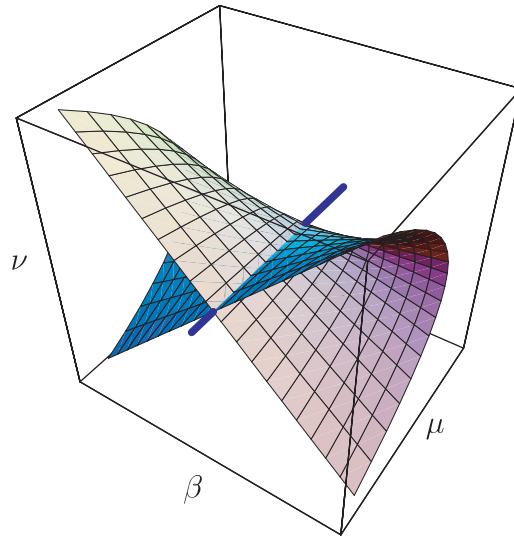
Figure 4. The Whitney umbrella $\beta^2\mu + \nu^2 = 0$.

The discriminant set D_Q is the product of the real line and a swallowtail, locally near $0 \in \mathbf{R}^4$. This follows from the fact that Q is a versal unfolding of the function $u \mapsto u^4 - u^{q-2}$, which is right-equivalent to the function u^4, locally near $0 \in \mathbf{R}$. A universal unfolding of the latter function is $\bar{Q} : \mathbf{R} \times \mathbf{R}^3 \to \mathbf{R}$, defined by

$$\bar{Q}(u, \kappa, \lambda, \varrho) = u^4 + \varrho u^2 + \lambda u + \kappa.$$

Since the discriminant set of $\bar{Q}$ is a (standard) swallowtail-surface, it follows from singularity theory that D_Q is diffeomorphic to the product of the real line and this swallowtail-surface.

To visualize $D_{\bar{Q}}$, we solve κ and λ from the equations

$$\bar{Q}(u, \kappa, \lambda, \varrho, \alpha) = \frac{\partial \bar{Q}}{\partial u}(u, \kappa, \lambda, \varrho, \alpha) = 0.$$

Thus we see that $D_{\bar{Q}}$ is the singular surface in $\mathbf{R}^4$ parametrized by

$$(u, \varrho, \alpha) \mapsto (\kappa, \lambda, \varrho, \alpha) = (-2\varrho u - 6au^2 - 4u^3 + 5u^4, \varrho u^2 + 4au^3 + 3u^4 - 4u^5, \varrho, \alpha),$$

$$(5.13)$$

with (u, ϱ, α) near $(0, 0, 0) \in \mathbf{R}^3$. In figure 5, we depict this surface, and recognize the swallowtail geometry.

5.5. Concluding remarks for the case $q \geqslant 7$

First, from the above description it is clear that the resonance domain generally is more complicated than the familiar tongue. Globally speaking the resonance domain is bounded by a saddle-node curve: 'inside' at least two q-periodic solutions exist that annihilate each other at the outer boundary. The inner boundary again consists of saddle-node curves, that meet at a cusp or at the 'tongue tip'. Inside the pocket- or flame-like triangular region four q-periodic orbits exist.
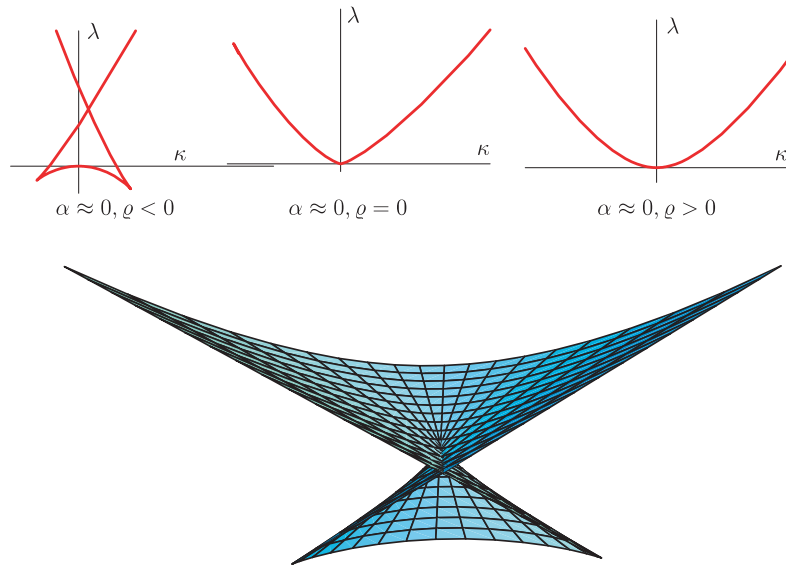
Figure 5. The discriminant set of the family $\bar{Q}(u, \kappa, \lambda, \varrho, \alpha) = u^4 - u^5 + 2\alpha u^3 + \varrho u^2 + \lambda u + \kappa$ parametrized by (5.13), locally near $0 \in \mathbf{R}^4$. This surface is the product of the real line and a swallowtail, near $0 \in \mathbf{R}^4$. Top row: cross-sections of $D_{\bar{Q}}$ with planes $(w, \alpha) = (w_0, \alpha_0)$. Bottom row: cross-sections of $D_{\bar{Q}}$ with hyperplanes $\alpha = \alpha_0$.

Second, the sharpness of the 'tongue' at the resonance $(\mu, \nu) = (0, 0)$ in the central singularity $\tau = 0$ can be computed directly from equations (5.5); it is

$$\nu \approx (-\mu)^{(q-2)/4}, \tag{5.14}$$

which is a $(q - 2)/4$ cusp. Note that the degenerate cusp is blunter than the $(q - 2)/2$ cusp we found in the standard nondegenerate case $q \geqslant 5$, see (4.1). In this unfolding, for $\tau \neq 0$ the 'tongue' tip will be a nondegenerate $(q - 2)/2$ cusp, where the leading coefficient depends on the value of τ.

Third, we comment on the scale of the present phenomena. From the numerical experiments, we see that for θ near 0.5, the description of the local geometry covers two scales—both of which are local. The secondary cusps and pockets or flames occur on a small scale. On a larger scale the resonance domain looks like a rather blunt cusp. These conclusions challenge the standard view of a sharp resonance tongue when $q \geqslant 5$. Compare this behaviour to the case where τ is in the first quadrant, i.e. where $0 < \theta < 0.25$. In this case, the tongues near the degeneracy $\tau = 0$ behave very much like tongues in the nondegenerate case.

6. Conclusions

We have developed a method, based on Liapunov–Schmidt reduction and $\mathbf{Z}_q$-equivariant singularity theory, to find period q resonance tongues in a Hopf bifurcation for dissipative maps. We recovered the standard nondegenerate results [2, 37], but also illustrated the method on the degenerate case $q \geqslant 7$. We now address various issues related to these methods and results that we aim to pursue in future research.

First, we would like to get a better understanding of the way in which the four-dimensional geometry, in particular the swallowtail catastrophe and the Whitney umbrella, determine the possible pockets or flames in the resonance tongues. Related to this, we would like to address

the question of strong resonances $q \leqslant 6$ in the degenerate case; this method can be adapted to these cases. To compute and visualize higher dimensional resonance domains we envision applying a combination of analytic and experimental techniques.

Second, our methods can be extended to other contexts, in particular, to cases where extra symmetries, including time reversibility, are present. This holds both for Liapunov–Schmidt reduction and $\mathbf{Z}_q$ equivariant singularity theory. In this respect Golubitsky *et al* [23], Knobloch and Vanderbauwhede [27, 29], and Vanderbauwhede [38] are helpful.

Third, there is the issue of how to apply our results to a concrete family of dynamical systems. Golubitsky and Schaeffer [24] describe methods for obtaining the Taylor expansion of the reduced function $g(z)$ in terms of the Poincaré map P and its derivatives. These methods may be easier to apply if the system is a periodically forced second-order differential equation, in which case the computations again may utilize parameter-dependent Floquet theory.

For numerical experiments our results may be helpful in the following way. If the third order 'Hopf coefficients' vanish at a specific parameter value, then, for nearby parameter values, we expect resonance tongues as described in section 5.

Finally, in this paper, we have studied only degeneracies in tongue boundaries. It would also be interesting to study low codimension degeneracies in the dynamics associated to the resonance tongues. Such a study will require tools that are more sophisticated than the singularity theory ones that we have considered here.

Acknowledgments

Appendix A. Proofs of the singularity theory theorems

This section divides into three parts. In the short first part we complete our description of $\mathbf{Z}_q$-equivariant contact equivalence by describing the structure of the mappings S satisfying the equivariance condition (2.5). The second part is devoted to deriving explicit generators for the tangent space of a $\mathbf{Z}_q$-equivariant mapping under contact equivalence. The last part uses the general 'tangent space constant' and 'universal unfolding theorems' of equivariant singularity theory [22, 24, 25] to derive the normal form theorems in section 2.

Lemma A.1. *Every map germ $S(z) : \mathbf{C} \to \mathbf{C}$ that satisfies (2.5) has the form*

$$S(z)y = \alpha(u, v, w)y + (\varphi(u, v, w)z^2 + \psi(u, v, w)\bar{z}^{q-2})\bar{y},$$

where α, φ, ψ are complex-valued and have the form (2.2).

Proof. A real linear map from $\mathbf{C}$ to $\mathbf{C}$ can be written as $y \mapsto \alpha y + \beta \bar{y}$ where $\alpha, \beta \in \mathbf{C}$. It follows that the linear map $S(z)$ has the form

$$S(z)(y) = \alpha(z)y + \beta(z)\bar{y}.$$

The equivariance condition (2.5) implies that

$$\alpha(\gamma z) = \alpha(z) \qquad \text{and} \qquad \beta(\gamma z) = \gamma^2 \beta(z).$$

The equivariance condition on $\beta(z)$ implies that

$$\beta(z) = \varphi(z)z^2 + \psi(z)\bar{z}^{q-2},$$

where φ and ψ are complex $\mathbf{Z}_q$-invariant functions. We may assume that α, φ, ψ are functions of u, v, w that are affine linear in w (see (2.2)). ∎

Of course, since $Z(z)$ is $\mathbf{Z}_q$-equivariant, it has the form

$$Z(z) = \zeta(z)z + \xi(z)\bar{z}^{q-1},$$

where ζ and ξ are complex $\mathbf{Z}_q$-invariant functions. Thus $\mathbf{Z}_q$ contact equivalences are determined by five complex-valued $\mathbf{Z}_q$-invariant functions α, φ, ψ, ζ, ξ.

The $\mathbf{Z}_q$ tangent space.   Equivariant singularity theory has two main theorems that are used to determine $\mathbf{Z}_q$-equivariant normal forms. Let $T(g)$ be the 'tangent space' of g, which we define formally in definition A.2. The tangent space constant theorem states that if $p : \mathbf{C} \to \mathbf{C}$ is $\mathbf{Z}_q$-equivariant and

$$T(g + tp) = T(g)$$

for all $t \in \mathbf{R}$, then $g + p$ is contact equivalent to g. The universal unfolding theorem states that if $V = \mathbf{R}\{p_1(z), \ldots, p_k(z)\}$ is a complementary subspace to $T(g)$ in the space of germs of $\mathbf{Z}_q$-equivariant mappings, then

$$G(z, \alpha_1, \ldots, \alpha_k) \equiv g(z) + \alpha_1 p_1(z) + \cdots + \alpha_k p_k(z)$$

is a universal unfolding of g.

The tangent space to a $\mathbf{Z}_q$-equivariant germ g is generated by infinitesimal versions of S and Z in (2.4), as follows.

Definition A.2. *The tangent space of $g(z)$ consists of all germs of the form*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}g_t(z)\right|_{t=0},$$

where $g_t(z)$ is a one-parameter family of germs that are contact equivalent to $g(z)$ and $g_0 = g$.

The assumption that g_t is a one-parameter family of germs that are contact equivalent to g means that

$$g_t(z) = S(z, t)g(Z(z, t)),$$

where S and Z satisfy the appropriate equivariance conditions. The assumption that $g_0 = g$ allows us to assume that $S(z, 0) = 1$ and $Z(z, 0) = z$. When computing the tangent space of g, linearity implies that we need consider only the deformations defined by the five invariant functions in the general equivalence—three coming from $S(z)$ and two from $Z(z)$. Thus

Lemma A.3. *The tangent space $T(g)$ is generated by 16 generators with coefficients that are real-valued $\mathbf{Z}_q$-invariant functions of u and v. The first 12 generators are obtained by multiplying each of the following three generators by 1, w, i, and iw:*

$$g(z) \quad z^2\overline{g(z)} \quad \bar{z}^{q-2}\overline{g(z)}. \tag{A.1}$$

The remaining four generators are:

$$(dg)_z(z) \quad (dg)_z(iz) \quad (dg)_z(\bar{z}^{q-1}) \quad (dg)_z(i\bar{z}^{q-1}). \tag{A.2}$$

Proof. Lemma A.1, coupled with (2.2), states that the function S is generated by six generators with coefficients that are functions of u and v. Lemma 2.1 states that Z is generated by two generators. Using the chain rule in the differentiation required by definition A.2 leads to the last four generators. ∎

Our next task is to compute the generators of $T(g)$ using the identification of a $\mathbf{Z}_q$ equivariant $g(z)$ with the pair of complex-valued functions $(K(u, v), L(u, v)) \in \mathcal{E}_{u,v}^2$.

Proposition A.4. *Let g be identified with the pair* $(K, L) \in \mathcal{E}_{u,v}^2$ *where* K, L *are complex-valued functions. Then, in terms of* (K, L), *there are 12 generators of* $T(g)$. *The first eight generators are obtained by multiplying the four generators*

$$(K, L) \quad (u\bar{K} + v\bar{L}, -u\bar{L}) \quad (u^{q-2}\bar{L}, \bar{K}) \quad (u^{q-1}L, -uK - vL)$$

by 1 *and* i. *The last four generators are*

$$(2uK_u + qvK_v, qL + 2uL_u + qvL_v)$$

$$(i(2u^{q-1}L_v + vK_v), -i(2uK_v + L + vL_v))$$

$$(2qu^{q-1}K_v + vK_u + (q-1)u^{q-2}L, K + 2qu^{q-1}L_v + vL_u)$$

$$(-i(vK_u + 2u^{q-1}L_u + (q-1)u^{q-2}L), i(K + 2uK_u + vL_u)).$$

Proof. The generators associated to multiplication by S are easiest to determine. Just compute

$$g(z) = Kz + L\bar{z}^{q-1}, \qquad z^2\overline{g(z)} = u\bar{K}z + \bar{L}z^q z, \qquad \bar{z}^{q-2}\overline{g(z)} = \bar{K}\bar{z}^{q-1} + u^{q-2}\bar{L}z.$$

Using $z^q z = vz - u\bar{z}^{q-1}$, we see that these generators correspond to:

$$(K, L) \quad (u\bar{K} + v\bar{L}, -u\bar{L}) \quad (u^{q-2}\bar{L}, \bar{K}). \tag{A.3}$$

To determine the generators with a w factor, use (2.3) to compute

$$w(K, L) = i(vK + 2u^{q-1}L, -2uK - vL). \tag{A.4}$$

Using (A.4), it follows that

$$w(u\bar{K} + v\bar{L}, -u\bar{L}) = \mathrm{i}(uv\bar{K} + (v^2 - 2u^q)\bar{L}, -2u^2\bar{K} - uv\bar{L}),$$
$$w(u^{q-2}\bar{L}, \bar{K}) = i(2u^{q-1}\bar{K} + u^{q-2}v\bar{L}, -v\bar{K} - 2u^{q-1}\bar{L}).$$

Note that

$$w(u\bar{K} + v\bar{L}, -u\bar{L}) = iv(u\bar{K} + v\bar{L}, -u\bar{L}) - 2iu^2(u^{q-2}\bar{L}, \bar{K}).$$

Thus, using (A.3), we see that this generator is redundant. Similarly,

$$w(u^{q-2}\bar{L}, \bar{K}) = 2iu^{q-2}(u\bar{K} + v\bar{L}, -u\bar{L}) - iv(u^{q-2}\bar{L}, \bar{K}).$$

So, this generator is also redundant.

To find the tangent space generators (A.2) corresponding to $Z(z)$, we first compute

$$K_z = K_u\bar{z} + qK_vz^{q-1},$$
$$K_{\bar{z}} = K_uz + qK_v\bar{z}^{q-1},$$

where $K = K(u, v)$ is $\mathbf{Z}_q$-invariant. Similar formulae hold for L. The tangent space generators from domain coordinate changes are

$$(dg)_z(\zeta(z)z) \quad \text{and} \quad (dg)_z(\xi(z)\bar{z}^{q-1}), \tag{A.5}$$

where

$$(dg)_z(y) = g_z(z)y + g_{\bar{z}}(z)\bar{y}.$$

So we compute

$$g_z = K + K_zz + L_z\bar{z}^{q-1}$$
$$= K + (K_u\bar{z} + qK_vz^{q-1})z + (L_u\bar{z} + qL_vz^{q-1})\bar{z}^{q-1}$$
$$= K + uK_u + qu^{q-1}L_v + qK_vz^q + L_u\bar{z}^q.$$

Also

$$g_{\bar{z}} = K_{\bar{z}}z + (q-1)L\bar{z}^{q-2} + L_{\bar{z}}\bar{z}^{q-1}$$
$$= (K_u z + qK_v\bar{z}^{q-1})z + (q-1)L\bar{z}^{q-2} + (L_u z + qL_v\bar{z}^{q-1})\bar{z}^{q-1}$$
$$= K_u z^2 + (quK_v + (q-1)L + uL_u + qL_v\bar{z}^q)\bar{z}^{q-2}.$$

The first generator from (A.5) is

$$(dg)_z(\zeta z) = g_z\zeta z + g_{\bar{z}}\bar{\zeta}\bar{z}$$
$$= \zeta(K + uK_u + qu^{q-1}L_v + qK_v z^q + L_u\bar{z}^q)z$$
$$\quad + \bar{\zeta}(K_u z^2 + (quK_v + (q-1)L + uL_u + qL_v\bar{z}^q)\bar{z}^{q-2})\bar{z}$$
$$= \zeta(K + uK_u + qu^{q-1}L_v + qK_v z^q + L_u\bar{z}^q)z + \zeta uL_u\bar{z}^{q-1}$$
$$\quad + \bar{\zeta}uK_u z + \bar{\zeta}(quK_v + (q-1)L + uL_u + qL_v\bar{z}^q)\bar{z}^{q-1}$$
$$= \zeta(K + uK_u + qu^{q-1}L_v + qvK_v)z + \zeta u(L_u - qK_v)\bar{z}^{q-1}$$
$$\quad + \bar{\zeta}u(K_u - qL_v u^{q-2})z + \bar{\zeta}(quK_v + (q-1)L + uL_u + qvL_v)\bar{z}^{q-1}, \qquad (A.6)$$

using $\bar{z}^q = v - z^q$ and $z^q = v - \bar{z}^q$. Evaluating (A.6) with $\zeta = 1$ and i yields the two generators

$$(K + 2uK_u + qvK_v, (q-1)L + 2uL_u + qvL_v)$$
$$(i(K + 2qu^{q-1}L_v + qvK_v), -i(2quK_v + (q-1)L + qvL_v)).$$

Note that the form of these generators can be simplified by subtracting the generator (K, L) from the first and (iK, iL) from the second (and then dividing by q).

The second generator from (A.5) is

$$(dg)_z(\xi\bar{z}^{q-1}) = g_z\xi\bar{z}^{q-1} + g_{\bar{z}}\bar{\xi}z^{q-1}$$
$$= \xi(K + uK_u + qu^{q-1}L_v + qK_v z^q + L_u\bar{z}^q)\bar{z}^{q-1}$$
$$\quad + \bar{\xi}(K_u z^2 + (quK_v + (q-1)L + uL_u + qL_v\bar{z}^q)\bar{z}^{q-2})z^{q-1}$$
$$= \xi qK_v u^{q-1}z + \xi(K + uK_u + qu^{q-1}L_v + L_u\bar{z}^q)\bar{z}^{q-1}$$
$$\quad + \bar{\xi}((quK_v + (q-1)L + uL_u)u^{q-2} + K_u z^q)z + \bar{\xi}qu^{q-1}L_v\bar{z}^{q-1}$$
$$= \xi(qK_v - L_u)u^{q-1}z + \xi(K + uK_u + qu^{q-1}L_v + vL_u)\bar{z}^{q-1}$$
$$\quad + \bar{\xi}((quK_v + (q-1)L + uL_u)u^{q-2} + vK_u)z + \bar{\xi}u(qu^{q-2}L_v - K_u)\bar{z}^{q-1}.$$
$$(A.7)$$

Evaluating (A.7) with $\xi = 1$ and $\xi = i$ yields the last two generators:

$$(2qu^{q-1}K_v + vK_u + (q-1)u^{q-2}L, K + 2qu^{q-1}L_v + vL_u)$$
$$(-i(vK_u + 2u^{q-1}L_u + (q-1)u^{q-2}L), i(K + 2uK_u + vL_u)). \qquad \blacksquare$$

A.1. Derivations of the normal forms

We now present the proofs of theorems 2.2 and 2.3.

Proof of theorem 2.2. Suppose that $L(0,0) \neq 0$. This case simplifies substantially because we can divide g by L obtaining a new g, namely

$$g(z) = K(u,v)z + \bar{z}^{q-1}. \qquad (A.8)$$

For the g in (A.8), the 12 generators for the tangent space $T(g)$ with real-valued $\mathbf{Z}_q$-invariant coefficients of the form $\varphi(u,v)$ are:

$$
\begin{array}{lll}
(K, 1) & (iK, i) & (K + 2uK_u + qvK_v, q-1) \\
(u\bar{K} + v, -u) & (iu\bar{K} + iv, -iu) & (i(K + qvK_v), -i(2quK_v + q - 1)) \\
(u^{q-2}, \bar{K}) & (iu^{q-2}, i\bar{K}) & (2qu^{q-1}K_v + (q-1)u^{q-2} + vK_u, K) \\
(u^{q-1}, -uK - v) & (iu^{q-1}, -i(uK + v)) & (-i((q-1)u^{q-2} + vK_u), i(K + 2uK_u)).
\end{array}
\qquad (A.9)
$$

Next we compute the generators for the tangent space modulo $(\mathcal{M}^2, \mathcal{M})$ where $\mathcal{M}$ is the maximal ideal in $\mathcal{E}_{u,v}$. Using $K(0, 0) = 0$ and $q \geqslant 3$, these generators are:

$$
\begin{array}{lll}
(K, 1) & (iK, i) & (K + 2uK_u + qvK_v, q - 1) \\
(v, 0) & (iv, 0) & (i(K + qvK_v), -i(q - 1)) \\
(u^{q-2}, 0) & (iu^{q-2}, 0) & ((q-1)u^{q-2} + vK_u, 0) \\
(0, 0) & (0, 0) & (-i((q-1)u^{q-2} + vK_u), 0).
\end{array}
\tag{A.10}
$$

The case $q \geqslant 5$. Let $K = au + bv + \cdots$ where $a, b \in \mathbf{C}$. On substituting K into (A.10), we obtain

$$
\begin{array}{lll}
(au + bv, 1) & (i(au + bv), i) & (3au + (q+1)bv, q - 1) \\
(v, 0) & (iv, 0) & (i(au + (q+1)bv), -i(q - 1)) \\
(0, 0) & (0, 0) & (av, 0) \\
(0, 0) & (0, 0) & (-iav, 0)
\end{array}
\tag{A.11}
$$

for the generators of $T(g)$ modulo $(\mathcal{M}^2, \mathcal{M})$.

We claim that $(\mathcal{M}, \mathcal{E})$ is contained in the tangent space. Using Nakayama's lemma, we need only show that $(\mathcal{M}, \mathcal{E})$ is contained in the tangent space modulo $(\mathcal{M}^2, \mathcal{M})$. Modulo $(\mathcal{M}^2, \mathcal{M})$, $(\mathcal{M}, \mathcal{E})$ is generated as a real vector space by the six vectors: $(u, 0)$, $(iu, 0)$, $(v, 0)$, $(iv, 0)$, $(0, 1)$, $(0, i)$. Note that the generators in (A.10) give us $(v, 0)$, $(iv, 0)$ directly. We now show that when $a \neq 0$, the four generators that do not have a u^{q-2} term in them already generate the remaining four vectors.

$$
\begin{array}{cc|cc}
(u, 0) & (iu, 0) & (0, 1) & (0, i) \\
\hline
a_R & a_I & 1 & \\
-a_I & a_R & & 1 \\
3a_R & 3a_I & q - 1 & \\
-a_I & a_R & & -q + 1
\end{array}
\tag{A.12}
$$

The determinant of the 4×4 matrix in (A.12) is $-q(q - 4)|a|^2$. Thus, when $a \neq 0$ and $q \neq 4$, $T(g) = (\mathcal{M}, \mathcal{E})$. It follows from the tangent space constant theorem that g is $\mathbf{Z}_q$-equivalent to (2.6) and from the universal unfolding theorem that (2.7) is a universal unfolding of (2.6).

The case $q = 4$. We compute the tangent space generators for this case. They are

$$
\begin{array}{lll}
(K, 1) & (iK, i) & (K + 2uK_u + 4vK_v, 3) \\
(u\bar{K} + v, -u) & (iu\bar{K} + iv, -iu) & (i(K + 4vK_v), -i(8uK_v + 3)) \\
(u^2, \bar{K}) & (iu^2, i\bar{K}) & (8u^3K_v + 3u^2 + vK_u, K) \\
(u^3, -uK - v) & (iu^3, -i(uK + v)) & (-i(3u^2 + vK_u), i(K + 2uK_u)).
\end{array}
\tag{A.13}
$$

We claim that $(\mathcal{M}^2, \mathcal{M}) \subset T(g)$ whenever $|a| \neq 0, 1$. To verify this claim, we show that

$$
(\mathcal{M}^2, \mathcal{M}) \subset T(g) + (\mathcal{M}^3, \mathcal{M}^2)
$$

and apply Nakayama's lemma. To verify the claim, set $K = au + bv + \cdots$ and observe that modulo $(\mathcal{M}^3, \mathcal{M}^2)$

$$
\begin{aligned}
(u^3, -uK - v) &= (0, -v), \\
(iu^3, -i(uK + v)) &= (0, -iv), \\
u(8u^3K_v + 3u^2 + vK_u, K) &\equiv (auv, 0), \\
v(8u^3K_v + 3u^2 + vK_u, K) &\equiv (av^2, 0), \\
u(-i(3u^2 + vK_u), i(K + 2uK_u)) &\equiv (-iauv, 0), \\
v(-i(3u^2 + vK_u), i(K + 2uK_u)) &\equiv (-iav^2, 0).
\end{aligned}
$$

Therefore, as long as $a \neq 0$, we have six of the ten vector space generators of $(\mathcal{M}^2, \mathcal{M})$ modulo $(\mathcal{M}^3, \mathcal{M}^2)$ contained in $T(g)$, namely,

$$(uv, 0) \quad (iuv, 0) \quad (v^2, 0) \quad (iv^2, 0) \quad (0, v) \quad (0, iv).$$

Next note that modulo these six generators and $(\mathcal{M}^3, \mathcal{M}^2)$, we have

	$(u^2, 0)$	$(iu^2, 0)$	$(0, u)$	$(0, iu)$
$u(K, 1)$	a_R	a_I	1	
$u(iK, i)$	$-a_I$	a_R		1
$(u^2, \bar{K})$	1		a_R	$-a_I$
$(iu^2, i\bar{K})$		1	a_I	a_R

The determinant of this 4×4 matrix is $(|a|^2 - 1)^2)$. Thus, as long as $|a| \neq 1$, we have verified the claim.

We can now compute the tangent space of g. Modulo $(\mathcal{M}^2, \mathcal{M})$ there are six generators of $(\mathcal{M}, \mathcal{E})$:

$$(u, 0) \quad (iu, 0) \quad (v, 0) \quad (iv, 0) \quad (0, 1) \quad (0, i).$$

Modulo $(\mathcal{M}, \mathcal{E})$ we compute (A.13) obtaining:

$$(au + bv, 1) \quad (i(au + bv), i) \quad (3au + 5bv, 3)$$
$$(v, 0) \qquad\qquad (iv, 0) \qquad\qquad (i(au + 5bv), -3i)$$
$$(0, 0) \qquad\qquad (0, 0) \qquad\qquad (av, 0)$$
$$(0, 0) \qquad\qquad (0, 0) \qquad\qquad (0, 0).$$

Note that $(v, 0)$ and $(iv, 0)$ are always in $T(g)$ and by taking appropriate multiples of these vectors we can eliminate any term involving a v in the first component. Thus $T(g) = (\mathcal{M}, \mathcal{E})$ plus linear combinations of the following vectors:

	$(u, 0)$	$(iu, 0)$	$(0, 1)$	$(0, i)$
$(au + bv, 1)$	a_R	a_I	1	
$(i(au + bv), i)$	$-a_I$	a_R		1
$(3au + 5bv, 3)/3$	a_R	a_I	1	
$(i(au + 5bv), -3i)$	$-a_I$	a_R		-3

It follows that when $|a| \neq 0, 1$, the tangent space

$$T(g) = (\mathcal{M}^2, \mathcal{M}) \oplus \mathbf{R}\{(v, 0), (iv, 0), (0, i), (au, 1), (iau, 0)\}.$$

Note that this tangent space does not change as long as a is fixed. It follows from the tangent space constant theorem that g is $\mathbf{Z}_4$ contact equivalent to

$$g(z) = auz + \bar{z}^3,$$

where $|a| \neq 0, 1$. Note that the codimension of $T(g)$ is three.

Next we make an explicit change of coordinates so that $a > 0$. Compute

$$\alpha g(\beta z) = \alpha a |\beta|^2 \beta uz + \alpha \bar{\beta}^3 \bar{z}^3.$$

By choosing $\alpha = \bar{\beta}^{-3}$ we can preserve $L = 1$ and obtain

$$\alpha g(\beta z) = \frac{|\beta|^2}{\bar{\beta}^2} auz + \bar{z}^3 = \hat{a}uz + \bar{z}^3,$$

where $\hat{a} > 0$ and $\hat{a} \neq 1$. Suppose $a = re^{i\theta}$; then choose $\beta = e^{-i\theta/2}$ so that $\hat{a} = r$. The universal unfolding of g is given by (2.8).

The case $q = 3$. The case $q = 3$ is the simplest case of all. We begin by computing the generators when $q = 3$. They are:

$$
\begin{array}{lll}
(K, 1) & (iK, i) & (K + 2uK_u + 3vK_v, 2) \\
(u\bar{K} + v, -u) & (iu\bar{K} + iv, -iu) & (i(K + 3vK_v), -i(6uK_v + 2)) \\
(u, \bar{K}) & (iu, i\bar{K}) & (6u^2 K_v + 3u + vK_u, K) \\
(u^2, -uK - v) & (iu^2, -i(uK + v)) & (-i(2u + vK_u), i(K + 2uK_u)).
\end{array}
\tag{A.14}
$$

We claim that $(\mathcal{M}, \mathcal{E})$ is always contained in the tangent space of g. To verify the claim, compute the generators in (A.14) modulo $(\mathcal{M}^2, \mathcal{M})$, obtaining:

$$
\begin{array}{lll}
(K, 1) & (iK, i) & (K + 2uK_u + 3vK_v, 2) \\
(v, 0) & (iv, 0) & (i(K + 3vK_v), -2i) \\
(u, 0) & (iu, 0) & (3u + vK_u, 0) \\
(0, 0) & (0, 0) & (-i(2u + vK_u), 0).
\end{array}
\tag{A.15}
$$

It is transparent that we obtain all of the generators of $(\mathcal{M}, \mathcal{E})$ modulo $(\mathcal{M}^2, \mathcal{M})$. It follows that $T(g) = (\mathcal{M}, \mathcal{E})$—independent of K.

Proof of theorem 2.3. We assume that $K(0, 0) = K_u(0, 0) = 0$ and that $K = cu^2 + bv + \cdots$. We claim that $(\mathcal{M}^3 + \mathcal{M}\langle v \rangle, \mathcal{M}) \subset T(g)$. We begin by computing the generators in (A.9) modulo $(\mathcal{M}^4 + \mathcal{M}^2\langle v \rangle, \mathcal{M}^2)$. Need $q \geqslant 6$ here.

$$
\begin{array}{lll}
(K, 1) & (iK, i) & (K + 2uK_u + qvK_v, q - 1) \\
(u\bar{K} + v, -u) & (iu\bar{K} + iv, -iu) & (i(K + qvK_v), -i(2quK_v + q - 1)) \\
(0, \bar{K}_v v) & (0, i\bar{K}_v v) & (vK_u, K_v v) \\
(0, -v) & (0, -iv) & (-ivK_u, iK_v v).
\end{array}
\tag{A.16}
$$

We can transform the generators in (A.16) to a new set of generators modulo $(\mathcal{M}^4 + \mathcal{M}^2\langle v \rangle, \mathcal{M}^2)$ by using the fact that $(0, v)$, $(0, iv)$, $(K, 1)$, (iK, i) are in $T(g)$. We obtain:

$$
\begin{array}{lll}
(K, 1) & (iK, i) & ((2 - q)K + 2uK_u + qvK_v, 0) \\
(u(K + \bar{K}) + v, 0) & (iu(K + \bar{K}) + iv, 0) & q(i(K + vK_v), -2iuK_v) \\
(0, 0) & (0, 0) & (vK_u, 0) \\
(0, v) & (0, iv) & (-ivK_u, 0).
\end{array}
\tag{A.17}
$$

Note that modulo $(\mathcal{M}^4 + \mathcal{M}^2\langle v \rangle, \mathcal{M}^2)$

$$
\begin{aligned}
u(u(K + \bar{K}) + v, 0) &= (uv, 0) \\
v(u(K + \bar{K}) + v, 0) &= (v^2, 0) \\
u(iu\bar{K} + iv, -iu) &= (iuv, 0) \\
v(iu\bar{K} + iv, -iu) &= (iv^2, 0).
\end{aligned}
$$

The remaining vector space generators of $(\mathcal{M}^3 + \mathcal{M}\langle v \rangle, \mathcal{M})$ modulo $(\mathcal{M}^4 + \mathcal{M}^2\langle v \rangle, \mathcal{M}^2)$ are $(u^3, 0)$, $(iu^3, 0)$, $(0, u)$, $(0, iu)$. We can rewrite multiples of our 12 tangent space generators in terms of these four basis vectors as:

	$(u^3, 0)$	$(iu^3, 0)$	$(0, u)$	$(0, iu)$
$u(K, 1)$	c_R	c_I	1	
$u(iK, i)$	$-c_I$	c_R		1
$((2 - q)K + 2uK_u + qvK_v, 0)$	$(6 - q)c_R$	$(6 - q)c_I$		
$u(i(K + vK_v), -2iuK_v)$	$-c_I$	c_R		

Assuming that $q \geqslant 7$ and $c \neq 0$, we have verified our claim.

Next, we compute the $T(g)$. To do this, we compute the generators of $T(g)$ modulo $(\mathcal{M}^3 + \mathcal{M}\langle v \rangle, \mathcal{M})$ obtaining

$$
\begin{array}{lll}
(cu^2 + bv, 1) & (i(cu^2 + bv), i) & ((2-q)K + 2uK_u + qvK_v, 0) \\
(v, 0) & (iv, 0) & (i(K + vK_v), 0) \\
(0, 0) & (0, 0) & (0, 0) \\
(0, 0) & (0, 0) & (0, 0).
\end{array}
\tag{A.18}
$$

Since $(v, 0)$ and $(iv, 0)$ are always in the tangent space we can simplify the generators to

$$
\begin{array}{lll}
(cu^2, 1) & (icu^2, i) & ((6-q)cu^2, 0) \\
(v, 0) & (iv, 0) & (icu^2, 0).
\end{array}
\tag{A.19}
$$

Since $q \geqslant 7$ and $c \neq 0$, it follows that $T(g) = (\mathcal{M}^2 + \langle v \rangle, \mathcal{E})$. Since $T(g)$ is independent of all terms in g, we can use the tangent space constant theorem to prove the existence of a $\mathbf{Z}_q$ contact equivalence of g to the normal form $cu^2 z + z^{q-1}$, as long as $c \neq 0$. We can now rescale this equation so that $c = 1$. The universal unfolding theorem gives us the desired universal unfolding. ∎

Appendix B. Proof of theorem 5.3

The proof proceeds in several steps, each giving an approximation improving the one from the preceding step.

Claim 1. There is a neighbourhood V of $(u, \alpha) = (0, 0)$ and positive constant c such that, for $(u, \alpha) \in V$ and $p(u, \alpha) = 0$ we have $|u| \leqslant c|\alpha|$.

Suppose this claim does not hold, then there is a sequence (u_n, α_n) tending to $(0, 0)$, such that (i) $|u_n/\alpha_n|$ tends to infinity, and (ii) $p(u_n, \alpha_n) = 0$. Observe that $p(u_n, \alpha_n) = u_n^4 Q(u_n, \alpha_n)$, with

$$
Q(u_n, \alpha_n) = (1 - t_n^2)(1 + t_n + \alpha_n t_n)^2 - u^{q-6},
$$

where we put $t_n = \alpha_n / u_n$. Since t_n tends to 0, we see that $Q(u_n, \alpha_n)$ tends to 1, and hence we conclude that $p(u_n, \alpha_n) \neq 0$. This contradiction proves claim 1.

Putting $p(\alpha u, \alpha) = \alpha^4 p_1(u, \alpha)$, with

$$
p_1(u, \alpha) = (u - \alpha)^2 (1 + u + \alpha)^2 - \alpha^{q-6} u^{q-2},
$$

the first claim allows us to conclude that we find all zeros of $p(\cdot, \alpha)$ near 0 by considering the bounded zeros of $p_1(\cdot, \alpha)$. Since $p_1(u, 0) = u^2(u + 1)^2$, we see that the bounded zeros of $p_1(\cdot, \alpha)$ are near 0 or 1, for α near 0. We look for zeros near 0 in claims 2 and 3, and for zeros near 1 in claim 4.

Claim 2. There is a neighbourhood V of $(u, \alpha) = (0, 0)$ and a constant $c > 0$ such that, for $(u, \alpha) \in V$ and $p_1(u, \alpha) = 0$, we have $|u| \leqslant c|\alpha|$.

The proof is similar to that of claim 1. So suppose this claim does not hold, then there is a sequence (u_n, α_n) tending to $(0, 0)$, such that (i) $|u_n/\alpha_n|$ tends to infinity, and (ii) $p_1(u_n, \alpha_n) = 0$. Observe that $p_1(u_n, \alpha_n) = u_n^2 Q_1(u_n, \alpha_n)$, with

$$
Q_1(u_n, \alpha_n) = (1 - t_n^2)(1 + u_n + \alpha_n)^2 - \alpha_n^{q-6} u_n^{q-4}.
$$

Again $t_n = \alpha_n / u_n$, so t_n tends to 0. Since $Q_1(u_n, \alpha_n)$ tends to 1, we see that $p_1(u_n, \alpha_n) \neq 0$. This contradiction proves claim 2.

In view of claim 2 we put $p_1(\alpha u, \alpha) = \alpha^2 p_2(u, \alpha)$, where

$$p_2(u, \alpha) = (u - 1)^2 (1 + \alpha + \alpha u)^2 - \alpha^{2q-10} u^{q-2}.$$

As before claim 2 allows us to conclude that the zeros of $p_1(\cdot, \alpha)$ near 0 can be found by looking for the bounded zeros of $p_2(\cdot, \alpha)$. Since $p_2(u, 0) = (u - 1)^2$, we see that these bounded zeros are near 1.

Claim 3. There is a neighbourhood V of $(u, \alpha) = (1, 0)$ and a constant $c > 0$ such that, for $(u, \alpha) \in V$ and $p_2(u, \alpha) = 0$, we have $|u - 1| \leqslant c\alpha^{q-5}$.

Again suppose the claim is false, then there is a sequence (u_n, α_n) tending to $(1, 0)$, such that (i) $|u_n - 1|/\alpha_n^{q-5}$ tends to infinity, and (ii) $p_2(u_n, \alpha_n) = 0$. Observe that $p_2(u_n, \alpha_n) = (u_n - 1)^2 Q_2(u_n, \alpha_n)$, with

$$Q_2(u_n, \alpha_n) = (1 + \alpha_n + \alpha_n u_n)^2 - t_n^{q-5} u_n^{q-2},$$

where $t_n = \alpha_n^{q-5}/(u_n - 1)$, so t_n tends to zero. Therefore $Q_2(u_n, \alpha_n)$ tends to 1, and hence $p_2(u_n, \alpha_n) \neq 0$. This contradiction proves the claim.

In view of claim 3, we observe that $p_2(1 + \alpha^{q-5}u, \alpha) = \alpha^4 p_3(u, \alpha)$, where

$$p_3(u, \alpha) = u^2 (1 + 2\alpha + \alpha^{q-4}u)^2 - (1 + \alpha^{q-5}u)^{q-2}.$$

Since $p_3(u, 0) = u^2 - 1$, we see that for α near 0, the bounded solutions of $p_3(u, \alpha)$ are near $u = \pm 1$.

Claim 4. There is a neighbourhood $V_\pm$ of $(u, \alpha) = (\pm 1, 0)$ such that, for $(u, \alpha) \in V_\pm$, the equation $p_3(u, \alpha) = 0$ has a unique solution $u = \pm 1 + O(\alpha)$. Outside $V_\pm$ there are no bounded real solutions.

Indeed, since

$$p_3(\pm 1, 0) = 0,$$
$$\frac{\partial p_3}{\partial u}(\pm 1, 0) = \pm 2 \neq 0,$$

the implicit function theorem allows us to conclude that, for (u, α) near $(\pm 1, 0)$, the equation $p_3(u, \alpha) = 0$ has a unique real solution $u = \pm 1 + O(\alpha)$.

The existence of a pair of zeros of the form (5.8) follows from claims 2, 3 and 4.

Now we derive the existence of the second pair of real zeros, and derive its asymptotic expansion for small negative α.

Claim 5. There is a neighbourhood V of $(u, \alpha) = (-1, 0)$ and positive constant c such that, for $(u, \alpha) \in V$ and $p_1(u, \alpha) = 0$ we have $|u + 1 + \alpha| \leqslant c|\alpha|^{q/2-3}$.

Suppose this claim does not hold, then there is a sequence (u_n, α_n) tending to $(-1, 0)$, such that

$$\frac{|u_n + 1 + \alpha_n|}{|\alpha_n|^{q/2-3}} \text{ tends to infinity}$$

and

$$p_1(u_n, \alpha_n) = 0.$$

Observe that $0 = p_1(u_n, \alpha_n) = |\alpha_n|^{q-6} Q(u_n, \alpha_n)$, with

$$Q(u_n, \alpha_n) = \frac{(u_n + 1 + \alpha_n)^2}{|\alpha_n|^{q-6}}(u_n - \alpha_n)^2 + \sigma u_n^{q-2},$$

with

$$\sigma = \begin{cases} +1, & \text{if } q \text{ is odd and } \alpha \leqslant 0, \\ -1, & \text{otherwise.} \end{cases}$$

Since $Q(u_n, \alpha_n)$ tends to infinity, we have derived a contradiction, which proves the claim.

To apply the result of claim 5, we distinguish the cases in which q is odd and q is even, respectively.

If q is odd, we see that $\alpha^{q-6}u^{q-6} = (u - \alpha)^2(u + 1 + \alpha)^2 \geqslant 0$, so $\alpha \leqslant 0$ if $u \approx -1$. Therefore we put $\alpha = -\gamma^2$, and observe that

$$p_1(-1 + \gamma^2 + \gamma^{q-6}u, -\gamma^2) = \gamma^{2q-12}Q(u, \gamma),$$

where

$$Q(u, \gamma) = u^2(-1 + 2\gamma^2 + \gamma^{q-6}u)^2 + (-1 + \gamma^2 + \gamma^{q-6}u)^{q-2}.$$

In particular, $Q(u, 0) = u^2 - 1$, so, according to the implicit function theorem, the equation $Q(u, \gamma) = 0$ has two real solutions $u = \bar{U}_\pm(\gamma)$ for small γ, satisfying $\bar{U}_\pm(\gamma) = \pm 1 + \mathrm{O}(\gamma)$.

In view of claims 1 and 5 these zeros correspond to the zeros $U_\pm(\alpha)$ satisfying (5.9). If q is even, we again apply claim 5 and observe that

$$p_1(-1 - \alpha + \alpha^{q/2-3}u, \alpha) = \alpha^{q-6}Q(u, \alpha),$$

where

$$Q(u, \alpha) = u^2(-1 - 2\alpha + \alpha^{q/2-3}u)^2 - (-1 - \alpha + \alpha^{q/2-3})^{q-2}.$$

Again, $Q(u, 0) = u^2 - 1$, so the implicit function theorem gives us two unique solutions $u = \bar{U}_\pm(\alpha) = -1 - \alpha \pm \alpha^{q/2-2} + \mathrm{O}(\alpha^{q/2-1})$.

In view of claims 1 and 5 these zeros correspond to the zeros $U_\pm(\alpha)$ satisfying (5.10). ∎

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