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A Note on Stabilization via Communication Channel in the presence of Input Constraints

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Abstract— The problem of asymptotic stabilization of a process via communication channel under control input constraints is studied. A solution is proposed which provides encoders, decoders and controllers accomplishing global asymptotic stabilization of the closed-loop system provided that a suitable number of bits is used to encode the information generated by the process. The proposed solution shows interesting features: it employs a number of bits for encoding equal to the relative degree of the system and works in the presence of an adaptive transmission rate.

I. INTRODUCTION

Controlling systems over networks is becoming a problem of increasing importance (cf. e.g. [5]). Systems in a network are typically interconnected through communication channels with finite bandwith. As a consequence several issues arise, of both practical and theoretical interest. For instance, information can be encoded only through a finite number of values (thus yielding quantization errors), delays and erasures are likely to happen, etc. In order to deal with these severe limitations, *ad-hoc* control strategies must be devised. Several authors have devoted studies to the problem, considering systems with linear or nonlinear dynamics in a deterministic or stochastic framework (see, to cite a few, [7], [6], [22], [23], [16], [3], [9], [21], [25], [26], [4], [2], [12] and references therein). Less attention has been paid to the problem of controlling a system through communication channels in the presence of additional constraints, such as input saturation, with some notable exceptions (cf. e.g. [8], [17]). In this contribution, the problem is examined for a chain of integrators cascaded to a zero dynamics, and a solution is provided (that is encoders, decoders and controllers are designed so as to stabilize the system through a communication channel with finite data rate while fulfilling the constraints on the control input) without relying on numerical algorithms and computational effort, but rather exploiting a robustness property of saturated feedback systems and adopting a suitable family of devices to encode and decode information generated by the process. The robustness property which is particularly convenient to tackle the problem under study is an $\mathcal{L}_2$ to $\mathcal{L}_\infty$ stability property of saturated feedback systems ([24]) in the presence of encoding errors (see proof of Proposition 1 below). Using a number of bits to encode information that is larger than that used in [8] to achieve practical convergence (via input quantization), our approach allows to

accomplish asymptotic convergence of the state to the origin. The encoding/decoding methods employed in this paper are mainly inspired by the results of [22], [11]. Namely, we borrow from [11] the idea to encode the state of continuous-time systems and transmit the encoded information at fixed (discrete) sampling times and, upon reception and decoding by the decoder, reconstruct the "inter-sampling" behavior of the process to control, before a new encoded piece of information is received. On the other hand, we adopt the idea of [22] — where control of discrete-time systems under communication constraints is considered — to explicitly take into account the Jordan form of the linear subsystem of the process (the chain of integrators) in order to conveniently assess the number of bits needed to encode information. The result can be proven for a variety of cases which include state and output feedback, ISS ([18], [19]) and integral ISS ([20]) zero dynamics, and taking into account a communication channel with finite bandwidth, delay and adaptive (i.e. *time-varying*) transmission rate.

Section II introduce the class of systems we consider and a technical result on robust stabilization of saturated systems. Encoders and decoders are introduced in III. The main result of the paper is stated and proved in Section IV. Conclusions are drawn in Section V.

II. PRELIMINARIES

Consider a system of the form

$$\begin{aligned}
\dot{z} &= f(z, x_1, \ldots, x_r, \mathrm{sat}(u)) \\
\dot{x}_1 &= x_2 \\
&\vdots \\
\dot{x}_{r-1} &= x_r \\
\dot{x}_r &= \mathrm{sat}(u) \,,
\end{aligned} \tag{1}$$

where $z \in \mathbb{R}^{n-r}$, $u \in \mathbb{R}$, $f(z, x_1, \ldots, x_r, \mathrm{sat}(u))$ is a smooth mapping, and $\mathrm{sat}(\cdot)$ is a saturation function, i.e. a globally Lipschitz function satisfying $|\mathrm{sat}(u+v) - \mathrm{sat}(u)| \leq \min\{a|v|, b\}$ and $|\mathrm{sat}(u) - u| \leq au^T \mathrm{sat}(u)$ for some $a > 0$ and $b > 0$ and for all $u, v \in \mathbb{R}^m$.

For the z-subsystem in (1) the following property is assumed

Assumption 1: The z-subsystem in (1) is input-to-state stable (ISS) with respect to the input vector $v = \mathrm{col}(x_1, \ldots, x_r, \mathrm{sat}(u))$, i.e. there exist a class-$\mathcal{KL}$ function $\beta(\cdot, \cdot)$ and a class-$\mathcal{K}$ function $\gamma(\cdot)$ such that, for any input $v(\cdot) \in \mathcal{L}_\infty$ and any initial condition $z_0 \in \mathbb{R}^{n-r}$, the response

$z(\cdot)$ of the system from $z(0) = z_0$ and under the action of $v(\cdot)$ satisfies

$$|z(t)| \leq \max\{\beta(|z_0|, t), \gamma(\|v(\cdot)\|_\infty\} . \lhd \qquad (2)$$

For the chain of r integrators in (1), several stabilization results with saturated inputs exist. We exploit in what follows the following statement, which points out the existence of a controller which renders the closed-loop system robust with respect to a class of disturbances. We state the result for systems of the form

$$
\begin{aligned}
\dot{x}_1 &= x_2 + p_1 d \\
&\vdots \\
\dot{x}_{r-1} &= x_{r-1} + p_{r-1} d \\
\dot{x}_r &= \text{sat}(u) + p_r d ,
\end{aligned}
\qquad (3)
$$

with $d \in \mathbb{R}$ and $p_1, \ldots, p_r$ real numbers. For this class of system, the following result is well known (cf. [24], Theorem 2.2).

Theorem 1: ([24]) For system (3) there exists a feedback law of the form

$$u = \chi(x) = \text{sat}(K_1 x + \text{sat}(K_2 x + \text{sat}(\ldots + \text{sat}(K_n x)))) , \qquad (4)$$

with K_i suitable $\mathbb{R}^{1 \times r}$ matrices, class-$\mathcal{K}_\infty$ functions $\gamma_2^0(\cdot)$, $\gamma_2^d(\cdot)$, $\gamma_\infty^0(\cdot)$, $\gamma_\infty^d(\cdot)$, such that, for each $x_0 \in \mathbb{R}^r$ and each $d(\cdot) \in \mathcal{L}_2$, the response of the closed-loop from the initial condition $x(0) = x_0$ exists for all $t \geq 0$ and satisfies [1]

$$
\begin{aligned}
\|x(\cdot)\|_2 &\leq \max\{\gamma_2^0(|x_0|), \gamma_2^d(\|d\|_2)\} \\
\|x(\cdot)\|_\infty &\leq \max\{\gamma_\infty^0(|x_0|), \gamma_\infty^d(\|d\|_2)\} .
\end{aligned}
\qquad (5)
$$

When sensors and controllers are separated by means of a communication channel with finite bandwidth, then the controller can not access to the exact value of the state to be used in the feedback law. This happens because at each given time only a limited number of bits are available to encode the value of the state at that time. This inevitably introduces quantization errors. The effect of this quantization effect can be contained by suitably encoding the information. The description of the encoding (and decoding) procedure adopted in our scheme can be found in the remaining part of this section and at the beginning of the next one. The following ideas have been inspired mainly by [22] and [9].

Assume that the process is allowed to send a package of B bits over the network every T units of time, where T is a given positive real number. For instance, if the transmission is using a digital network sampling data at 8 kHz, with 8 bits/sample, then we have $B = 8$ and $T = 1/8000$. At each time kT at which the process is allowed to send data through the channel, the vector $x(kT)$ can be represented using only 2^B possible values (*words* or *symbols*). The set of these symbols will be denoted by Σ and – to fix the

ideas – taken equal to the set of binary representations of the first 2^B natural numbers. The device which at each time kT maps $x(kT)$ into one of the symbols in Σ is called *encoder* and denoted by $\mathcal{E}$. The symbol generated by the encoder is then transmitted through the channel to the receiver. At the reception, a value $\hat{x}(kT)$ is generated starting from symbol $s(kT)$ by a device called the *decoder*, which utilizes $\hat{x}(kT)$ also to generate the estimate of $x(\cdot)$ over the interval $[kT, (k+1)T)$ (cf. (12) below). The estimate $\hat{x}(\cdot)$ produced by the decoder is then used by the controller to achieve the desired control objective.

III. ENCODERS AND DECODERS FOR A CHAIN OF INTEGRATORS

In this section, we introduce devices to encode (and decode) information generated by the continuous-time chain of integrators

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
&\vdots \\
\dot{x}_{r-1} &= x_r \\
\dot{x}_r &= \text{sat}(u) ,
\end{aligned}
\qquad (6)
$$

which, in order to be as concise as possible, will be referred to as

$$\dot{x} = Fx + G\text{sat}(u) , \qquad (7)$$

with obvious significance of the symbols.

Encoder. At each time step kT the encoder constructs the hyper-rectangle $\Omega(kT) \subset \mathbb{R}^r$ with centroid $C(kT) \in \mathbb{R}^r$ and edges whose lengths are given by the entries of the range vector $L(kT) \in \mathbb{R}_+^r$. Each edge of the hyper-rectangle is divided into 2 parts, so that the hyper-rectangle is uniformly partitioned into 2^r parts. Assume that at each time step, $x(kT) \in \Omega(kT)$ (we shall see in Lemma 1 that this results in no loss of generality). Then a sub-region in $\Omega(kT)$ will exists to which the state $x(kT)$ belongs and the centroid of this subregion will be denoted by $\hat{x}(kT)$. It is simple to give an expression to this centroid. As a matter of fact, there exist numbers $b_1, \ldots, b_r$, with $b_i \in \{-1, +1\}$ for all $i = 1, \ldots, r$, such that

$$
\hat{x}(kT) = C(kT) + \begin{bmatrix} b_1 L_1(kT)/4 \\ \vdots \\ b_r L_r(kT)/4 \end{bmatrix} . \qquad (8)
$$

The symbol $s(kT)$ to be sent through the channel is taken equal to the following sequence of 0's and 1's:

$$s(kT) = (\begin{array}{ccc} \tilde{b}_1 & \ldots & \tilde{b}_r \end{array})$$

where

$$
\tilde{b}_i = \begin{cases} b_i & \text{if } b_i = 1 \\ b_i + 1 & \text{if } b_i = -1 . \end{cases}
$$

We therefore conclude that, in this case, $B = r$. The values of $C(\cdot)$ and $L(\cdot)$ used by the encoder to construct $\Omega(kT)$ are

specified by the following equations:

$$\frac{d}{dt}\bar{x}(t) = F\bar{x}(t) + G\mathrm{sat}(u(t)),$$
$$t \in [kT, (k+1)T), k \geq 0, \qquad (9)$$
$$\bar{x}(kT) = \hat{x}(kT), k \geq 0,$$

where[2] $C(0) := \bar{x}(0^-) = 0$ and vector $\hat{x}(kT)$ is defined as in (8), having set

$$C(kT) = \bar{x}(kT^-).$$

The range vector satisfies the following equation:

$$\frac{d}{dt}L(t) = 0, t \in [kT, (k+1)T), k \geq 0,$$
$$L((k+1)T) = \Lambda L(kT), k \geq 0,$$
$$\qquad (10)$$

with

$$\Lambda = \begin{pmatrix} 1/2 & T/2 & \dots & T^{r-1}/(2(r-1)!) \\ 0 & 1/2 & \dots & T^{r-2}/(2(r-2)!) \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1/2 \end{pmatrix} \qquad (11)$$

and initial value given by $L_i(0) \geq 2|x_i(0)|$ for $i = 1, \dots, r$. Equations (9), (10) are well-posed. In fact, at each time kT, the encoder first receives $x(kT)$ from a sample-and-hold device, then acquires vectors $C(kT) = \bar{x}(kT^-)$ (obtained from the solution of (9) integrated over the interval $[(k-1)T, kT)$ and extended to time $t = kT$) and $L(kT)$. These are used to compute the quantization region $\Omega(kT)$ and to calculate $\hat{x}(kT)$. The latter is used to reinitialize (9) and compute $\bar{x}((k+1)T^-)$ by integrating (9) over the sampling interval $[kT, (k+1)T)$. The procedure is then repeated.

Decoder. In order to decode the symbol $s(kT)$ transmitted by the encoder, the decoder needs to know the hyper-rectangle $\Omega(kT)$, that is the values of the centroid $C(kT)$ and dynamic range $L(kT)$. To this purpose the following is assumed:

Assumption 2: Vector $L(0)$ is available to the decoder. ◁

Furthermore, the decoder implements the following equations:

$$\frac{d}{dt}\bar{x}_d(t) = F \cdot \bar{x}_d(t) + G\mathrm{sat}(u(t)),$$
$$t \in [kT, (k+1)T), k \geq 0, \qquad (12)$$
$$\bar{x}_d(kT) = \hat{x}(kT), k \geq 0,$$

with $\bar{x}_d(0^-) = 0$, and

$$\frac{d}{dt}L_d(t) = 0, t \in [kT, (k+1)T), k \geq 0,$$
$$L_d((k+1)T) = \Lambda L_d(kT), k \geq 0,$$
$$\qquad (13)$$

with $L_{d,i}(0) = L_i(0)$, and matrix Λ as in (11). We claim that equations (12), (13) are well-posed because $\hat{x}(kT)$ can

be determined by the decoder. In fact, by definition, $C(0) = \bar{x}_d(0^-)$ and $L(0) = L_d(0)$. Therefore, by (8), $\hat{x}(0)$ can be determined from the knowledge of $s(0)$. We proceed now by induction. Suppose now that for some $k \geq 0$, $\hat{x}(jT)$, for all $0 \leq j \leq k$, has been determined by the decoder. Then one can solve (12) over the interval $[kT, (k+1)T)$. Clearly, $\bar{x}(t) = \bar{x}_d(t)$ for all $t \in [kT, (k+1)T)$. In particular, $\bar{x}_d((k+1)T^-) = C((k+1)T)$. On the other hand, $L_d(kT) = L(kT)$ for all $k \geq 0$ trivially. Therefore, $\hat{x}((k+1)T)$ can be determined by the decoder from the knowledge of $s(k+1)T)$. We conclude that the claim is true. Note also that, $\bar{x}(t) = \bar{x}_d(t)$ for all $t \geq 0$ and $L(kT) = L_d(kT)$ for all $k \geq 0$.

Remark. Formulas (9) and (12) show that both the encoder and the decoder must have access to the input $u(\cdot)$ over the interval $[kT, (k+1)T)$ for all $k \geq 0$. This may not be a too demanding requirement when the input $u(\cdot)$ is the control law used to stabilize the process. As a matter of fact, we shall see in the next section that $u(\cdot)$ is a globally Lipschitz function of the variable $\bar{x}_d(\cdot)$, i.e. $u = \chi(\bar{x}_d)$. As a consequence, both encoder and decoder can *exactly* reconstruct $u(\cdot)$ provided that the control function $\chi(\cdot)$ is known to both the encoder and the decoder ($\bar{x}_d(\cdot)$ is known to the encoder because $\bar{x}(\cdot) = \bar{x}_d(\cdot)$). For a thorough discussion on how the type of knowledge the encoder possesses of the process and of the signals acting on it affects the performance of the encoder, we refer the reader to [22] and [23]. ◁

The next result proves that the decoder and encoder thus designed allow to estimate the state of the plant.

Lemma 1: Consider system (7). The estimate $\bar{x}(\cdot)$, generated by the decoder (12), (13) starting from state $x(\cdot)$ encoded by the encoder (9), (10), is such that:

(i) $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) \geq 0$ such that

$$|x(0)| \leq |L(0)|/2 \leq \delta(\varepsilon) \Rightarrow$$
$$|x(t) - \hat{x}(t)| \leq \varepsilon \ \forall t \geq 0;$$

(ii) $\forall \varepsilon > 0$, $\exists t(\varepsilon) \geq 0$ such that

$$|x(t) - \hat{x}(t)| \leq \varepsilon \ \forall t \geq t(\varepsilon).$$

Furthermore, signal $|x(\cdot) - \hat{x}(\cdot)|$ satisfies

$$\|x(\cdot) - \hat{x}(\cdot)\|_2 \leq \eta |L(0)|, \qquad (14)$$

for some positive constant η.

Proof: As a first step, we will prove that $x(kT)$ belongs to the region $\Omega_d(kT)$ for all $k \geq 0$ [3]. At time $t = 0$, as $x(0)$ lies in the quantization region by construction, we have

$|x_j(0) - \hat{x}_j(0)| \le L_{d,j}(0)/4$, for each $j = 1, \ldots, r$. We proceed now by induction and prove that

$$|x_j(kT) - \hat{x}_j(kT)| \le \frac{L_{d,j}(kT)}{4} , \qquad (15)$$

for all $k \ge 0$ and for each $j = 1, \ldots, r$. Suppose that (15) holds true for some k and for each $j = 1, \ldots, r$. For $t \in [kT, (k+1)T)$, we have[4]

$$|x_j(t) - \bar{x}_{d,j}(t)| = |(e^{F(t-kT)})_j(x(kT) - \bar{x}_d(kT))| =$$

$$|\sum_{\ell=0}^{r-j} (e^{F(t-kT)})_{j,j+\ell}(x_{j+\ell}(kT) - \hat{x}_{d,j+\ell}(kT))| \le$$

$$\sum_{\ell=0}^{r-j} 2\Lambda_{j,j+\ell}|x_{j+\ell}(kT) - \hat{x}_{d,j+\ell}(kT)| . \qquad (16)$$

Using the inductive hypothesis (15), and recalling that $\bar{x}_d(kT) = \hat{x}_d(kT) = \hat{x}(kT)$, we obtain

$$|x_j(t) - \bar{x}_{d,j}(t)| \le \sum_{\ell=0}^{r-j} 2\Lambda_{j,j+\ell} \frac{L_{d,j+\ell}(kT)}{4} = \qquad (17)$$

$$(\Lambda L_d(kT)/2)_j = L_{d,j}((k+1)T)/2 ,$$

where the inequality in particular holds for $t = (k+1)T^-$ by continuity, i.e.

$$|x_j((k+1)T^-) - \bar{x}_{d,j}((k+1)T^-)| =$$

$$|x_j((k+1)T) - \bar{x}_{d,j}((k+1)T^-)| \le L_{d,j}((k+1)T)/2 .$$

This implies that $x((k+1)T)$ lies in the quantization region $\Omega((k+1)T)$ and therefore

$$|x_j((k+1)T) - \hat{x}_j((k+1)T)| \le L_{d,j}((k+1)T)/4 .$$

The latter proves that (15) holds for all $k \ge 0$ and for each $j = 1, \ldots, r$. Moreover, (17) also shows that, for $t \in [kT, (k+1)T)$ and $k \ge 0$,

$$|x(t) - \bar{x}_d(t)| \le |\Lambda L_d(kT)|/2 .$$

Noting that $L_d(kT)$ converges exponentially to zero as k diverges to infinity, one can immediately realize that the thesis holds. In particular, denoted by $\tilde{\mu}$ and $\lambda < 1$ two positive constants for which $|L_d(kT)| \le \tilde{\mu}\lambda^k|L_d(0)|$, we have, for $t \in [kT, (k+1)T)$ and $k \ge 0$,

$$|x(t) - \bar{x}_d(t)| \le \tilde{\mu}\lambda^{k+1}|L_d(0)|/2 . \qquad (18)$$

Hence, for all $t \ge 0$, $|x(t) - \bar{x}_d(t)| \le \tilde{\mu}\lambda|L_d(0)|/2$.
From the last two inequalities, (i) and (ii) are promptly shown. As a matter of fact, the latter inequality implies (i) by setting $\delta(\varepsilon) = \varepsilon/(\tilde{\mu}\lambda)$. On the other hand, the former inequality implies that for all $t \ge 0$

$$|x(t) - \bar{x}_d(t)| \le \tilde{\mu}e^{-\tilde{\lambda}t}|L_d(0)|/2 , \qquad (19)$$

---

[4]In what follows, $(e^{F(t-kT)})_j$ denotes row j of matrix $e^{F(t-kT)}$, whereas $(e^{F(t-kT)})_{j,j+\ell}$ denotes its element $(j, j+\ell)$.

with $\tilde{\lambda} = |\ln \lambda|/T$, from which (ii) is immediately derived. As far as the last part of the thesis is concerned, it is enough to notice that

$$\int_0^\infty |x(t) - \bar{x}_d(t)|^2 dt = \sum_{k=0}^\infty \int_{kT}^{(k+1)T} |x(t) - \bar{x}_d(t)|^2 dt \le$$

$$\frac{(\tilde{\mu}|L_d(0)|)^2 T}{4} \sum_{k=0}^\infty \lambda^{2(k+1)} = \frac{(\tilde{\mu}|L_d(0)|)^2 T}{4} \frac{\lambda^2}{1-\lambda^2} =$$

$$\frac{\tilde{\mu}^2 T}{4} \frac{\lambda^2}{1-\lambda^2} |L_d(0)|^2 .$$

$$(20)$$

This concludes the proof. ∎

Remark. The proof of the lemma, namely formula (19), also shows that

$$\|x(\cdot) - \bar{x}_d(\cdot)\|_\infty \le \tilde{\mu}|L_d(0)|/2 . \quad \triangleleft$$

Remark. In the proof it is the component-wise analysis of the evolution of $x(\cdot) - \bar{x}_d(\cdot)$ (cf. e.g. formula (16)) which allows to achieve less conservative results on the number of bits used to encode the state information and also to show that such a number of bits is independent of the length of the sample period T. $\triangleleft$

Remark. Encoders (decoders) analogous to (9), (10) ((12), (13)) can be designed for any linear continuous-time systems so that results similar to the one above hold (see e.g. [13]). $\triangleleft$

IV. MAIN RESULT

Upon introducing encoders and decoders, we are now in the position to state and prove the main result of the paper:

Proposition 1: Consider system (1)

$$\begin{aligned}
\dot{z} &= f(z, x_1, \ldots, x_r, \text{sat}(u)) \\
\dot{x}_1 &= x_2 \\
&\vdots \\
\dot{x}_{r-1} &= x_r \\
\dot{x}_r &= \text{sat}(u) ,
\end{aligned}$$

and let Assumption 1 hold. Then, there exists suitable matrices $K_i \in \mathbb{R}^{1 \times n}$ for which the response of system (1) in closed-loop with the control law

$$u = \chi(\bar{x}_d) := K_1\bar{x}_d + \text{sat}(K_2\bar{x}_d + \text{sat}(\ldots + \text{sat}(K_r\bar{x}_d))) , \qquad (21)$$

where $\bar{x}_d(\cdot)$ is the estimate generated by decoder (12), (13) starting from state $x(\cdot)$ encoded by encoder (9), (10), satisfies:

(i) $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ such that:

$$|\text{col}(z(0), x(0))| \le |L_d(0)|/2 \le \delta(\varepsilon) \Rightarrow$$

$$|\text{col}(z(t), x(t))| \le \varepsilon \ \forall t \ge 0;$$

(ii) $\forall \varepsilon > 0$, $\exists t(\varepsilon) > 0$ such that:

$$|\mathrm{col}(z(t), x(t))| \leq \varepsilon, \ \forall t \geq t(\varepsilon) .$$

Proof: Closed-loop system (1), (21) has the form

$$
\begin{aligned}
\dot{z} &= f(z, x_1, \ldots, x_r, \mathrm{sat}(\chi(\bar{x}_d))) \\
\dot{x}_1 &= x_2 \\
&\vdots \\
\dot{x}_{r-1} &= x_r \\
\dot{x}_r &= \mathrm{sat}(\chi(x)) + d ,
\end{aligned}
\tag{22}
$$

where

$$d = \mathrm{sat}(\chi(\bar{x}_d)) - \mathrm{sat}(\chi(x)) .$$

Note that by definition of $\mathrm{sat}(\cdot)$, there exists a constant $\varrho > 0$ for which

$$|d| \leq \varrho |x - \bar{x}_d| ,$$

and hence, by (14),

$$\|d(\cdot)\|_2 \leq \bar{\varrho} |L_d(0)| ,\tag{23}$$

with $\bar{\varrho}$ a suitable positive constant. In view of (i) in Lemma 1, for any $\varepsilon_1 > 0$ there exists $\delta(\varepsilon_1) > 0$ such that $|x_0| \leq |L_d(0)|/2 \leq \delta(\varepsilon_1)$ implies

$$\|\bar{x}_d(\cdot)\|_\infty \leq \varepsilon_1 + \|x(\cdot)\|_\infty .$$

Hence, keeping in mind Assumption 1 and the easily proven inequality

$$\|v(\cdot)\|_\infty \leq \|x(\cdot)\|_\infty + \hat{\gamma}\|CNT_d(\cdot)\|_\infty ,$$

with $\hat{\gamma}$ some positive constant, if $|x_0| \leq |L_d(0)|/2 \leq \delta(\varepsilon_1)$ then

$$\|z(\cdot)\|_\infty \leq \max\{\beta(|z_0|, 0), \gamma((1 + \hat{\gamma})\|x(\cdot)\|_\infty + \hat{\gamma}\varepsilon_1)\} .\tag{24}$$

Let $\varepsilon_2 > 0$ and $\delta(\varepsilon_2) > 0$ be such that $|z_0| \leq \delta(\varepsilon_2)$ implies $\beta(|z_0|, 0) \leq \varepsilon_2$. From Theorem 1 it is known that

$$\|x(\cdot)\|_\infty \leq \max\{\gamma_\infty^0(|x_0|), \gamma_\infty^d(\|d(\cdot)\|_2)\} .$$

Inequality (23) then yields

$$\|x(\cdot)\|_\infty \leq \max\{\gamma_\infty^0(|x_0|), \gamma_\infty^d(\bar{\varrho}|L_d(0)|)\} .$$

Then for any $\varepsilon_3 > 0$ there exists $\delta(\varepsilon_3) > 0$ such that $|x_0| \leq |L_d(0)|/2 \leq \delta(\varepsilon_3)$ implies $\|x(\cdot)\|_\infty \leq \varepsilon_3$. Fix now $\varepsilon > 0$. Correspondingly fix

$$
\varepsilon_1 = \frac{\gamma^{-1}(\varepsilon/(2\sqrt{2}))}{2\hat{\gamma}}, \quad \varepsilon_2 = \frac{\varepsilon}{\sqrt{2}},
$$
$$
\varepsilon_3 = \min\{\frac{\gamma^{-1}(\varepsilon/(2\sqrt{2}))}{2(1 + \hat{\gamma})}, \frac{\varepsilon}{\sqrt{2}}\} ,
$$

where we are assuming without loss of generality $\gamma(\cdot)$ to be a class-$\mathcal{K}_\infty$ function. Take now

$$\delta(\varepsilon) = \min\{\delta(\varepsilon_1), \delta(\varepsilon_2), \delta(\varepsilon_3)\} .$$

Then, $|\mathrm{col}(z_0, x_0)| \leq |L_d(0)|/2 \leq \delta(\varepsilon)$ yields $|\mathrm{col}(z(t), x(t))| \leq \varepsilon$ for all $t \geq 0$.

As far as attractivity is concerned, note that, by Lemma 1, $d(\cdot) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. By Theorem (1), $x(\cdot) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. Hence, $x(\cdot) \in \mathcal{L}_\infty$ as well and by Barbalat's lemma $x(\cdot)$ asymptotically converges to the origin. This in turn implies the same convergence property for $v = \mathrm{col}(x_1, \ldots, x_r, \mathrm{sat}(\chi(\bar{x}_d)))$, since $\bar{x}_d(\cdot)$ is asymptotically converging to $x(\cdot)$ which is converging to zero. As a consequence, by the well-known convergent-input convergent-state property of ISS systems, Assumption 1 guarantees the state $z(\cdot)$ to converge to zero as well. ∎

V. CONCLUSIONS

It has been shown how to design encoders, decoders and controllers which allow to stabilize a class of systems when sensors and actuators are separated by a communication channel with finite bandwidth and control input constraints must be fulfilled. The class of systems which has been considered have a well-defined uniform relative degree and an ISS (or iISS) zero dynamics. Note that requiring the zero dynamics to be ISS can be a restrictive condition. Recent advances in stability of cascaded systems ([1]) have pointed out the possibility of relaxing such assumption by requiring iISS for the zero dynamics (iISS is a weaker assumption than ISS — cf. [20]). The same program of [1] can be pursued for the problem at hand. Namely, it is possible to consider the case in which system (1) has an integral input-to-state stable dynamics with a suitable gain function (see [15] for details) and study whether or not the control law developed in this paper is still able to achieve asymptotic stability for the closed-loop system. The study rests on proving a local exponential decay for the solutions of the $\bar{x}_d$-subsystem (12) in closed-loop with controller (21). As a consequence, it can be proven ([15]) that the closed-loop cascade (1), (21) does retain asymptotic stability as it was the case for the system with an ISS zero dynamics. The basic result can be extended to deal with various additional scenarios, such as those in which only output measurement is available and the channel exhibits transmission delay and adaptive transmission rate ([15]). Additional realistic features of the communication channel can be taken into account. Asymptotic stabilization using encoded saturated feedback can also be shown for the class of nonlinear feed-forward systems of the form:

$$
\dot{x} = \begin{pmatrix}
A_1 \underline{x}_1 + f_1(\underline{x}_2, \underline{x}_3, \ldots, \underline{x}_\nu, u) \\
A_2 \underline{x}_2 + f_2(\underline{x}_3, \ldots, \underline{x}_\nu, u) \\
\vdots \\
A_{\nu-1} \underline{x}_{\nu-1} + f_{\nu-1}(\underline{x}_\nu, u) \\
A_\nu \underline{x}_\nu + f_\nu(u)
\end{pmatrix} ,
$$

with $\underline{x}_i \in \mathbb{R}^{n_i}$, $x := (\underline{x}_1, \ldots, \underline{x}_\nu) \in \mathbb{R}^n$ and A_i critically stable. Interestingly, even in this case, the number of bits needed to decode the information equals the dimension of the system and hence it is independent of the length of the sample

period. See [14]. The results can be extended to stabilize — using bounded control — systems which are exponentially unstable, although local or semi-global results are likely to emerge.

The procedure adopted here to estimate the state of a chain of integrators from encoded information can be adapted to deal with any linear system and used to synthesize observers and controllers for problems such as fault detection and output regulation (cf. e.g. [13]). Further study must be pursued to understand to what extent the approach in [8] can lead to a smaller number of bits for information encoding and can be used to address the case in which also input undergoes encoding (papers [9], [10] are especially relevant to the latter regard).

VI. REFERENCES

[1] M. Arçak, D. Angeli, and E.D. Sontag. A unifying integral iss framework for stability of nonlinear cascades. *SIAM Journal on Control and Optimization*, 40(6):1888–1904, 2002.

[2] A. Bicchi, A. Marigo, and B. Piccoli. On the reachability of quantized control systems. *IEEE Trans. Aut. Contr.*, 2002.

[3] R.W. Brockett and D. Liberzon. Quantized feedback stabilization of linear systems. *IEEE Trans. Aut. Contr.*, 45:1279–1289, 2000.

[4] D.F. Delchamps. Extracting state information from a quantized output record. *Systems & Control Letters*, 13:365–372, 1989.

[5] L. Bushnell (Ed.). Networks and control. *IEEE Control Systems Magazine*, 21(1):22–23, 2001.

[6] N. Elia and S.K. Mitter. Stabilization of linear systems with limited information. *IEEE Trans. Aut. Contr.*, 46(9):1384–1400, September 2001.

[7] X. Feng and K.A. Loparo. Active probing for information in control systems with quantized state measurements: A minimum entropy approach. *IEEE Trans. Aut. Contr.*, 42(2):216–238, february 1997.

[8] G. Kaliora and A. Astolfi. Stabilization with positive and quantized control. In *Proceedings of the 41st IEEE Conf. on Dec. and Contr.*, pages 1892–1897, 2002.

[9] D. Liberzon. Stabilization by quantized state or output feedback: A hybrid control approach. In *Proceedings of the 15th IFAC Triennal World Congress*, 2002.

[10] D. Liberzon. Hybrid feedback stabilization of systems with quantized signals. To appear in *Automatica*, 2003.

[11] D. Liberzon. On stabilization of linear systems with limited information. *IEEE Trans. Aut. Contr.*, 48(2):304–307, February 2003.

[12] G. N. Nair and R. J. Evans. Stabilization with data-rate-limited feedback: Tightest attainable bounds. *Systems & Control Letters*, 41(1):49–56, September 2000.

[13] C. De Persis. Detecting faults from encoded information. To appear in *Proceedings of IEEE Conference on Decision and Control*, 2003.

[14] C. De Persis. n-bit stabilization of n-dimensional nonlinear feed-forward systems. Preprint, 2003.

[15] C. De Persis. Stabilization under input constraints via communication channel. Submitted to *SIAM J. Contr. Optim.*, 2003.

[16] I.R. Petersen and A.V. Savkin. Multi-rate stabilization of multivariable discrete-time linear systems via a limited capacity communication channel. In *Proceedings of the 40th Conf. on Dec. and Contr.*, pages 304–309, 2001.

[17] D.E. Quevedo, J.A. De Dona, and G.C. Goodwin. Receding horizon linear quadratic control with finite constraint sets. In *Proceedings of the 15th IFAC World Congress*, 2002.

[18] E.D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Trans. Aut. Contr.*, 34, 1989.

[19] E.D. Sontag. Further facts about input-to-state stabilization. *IEEE Trans. Aut. Contr.*, 35:473–476, 1990.

[20] E.D. Sontag. Comments on integral variants of iss. *Systems & Control Letters*, 34, 1998.

[21] J. Sur and B.E. Paden. State observer for linear time-invariant systems with quantized output. *ASME J. Dyn. Sys. Measur. and Contr.*, 120:423–426, 1998.

[22] S. Tatikonda and S. Mitter. Control under communication constraints: Part i. Submitted to *IEEE Trans. Aut. Contr.*, February 2002.

[23] S. Tatikonda and S. Mitter. Control under communication constraints: Part ii. Submitted to *IEEE Trans. Aut. Contr.*, February 2002.

[24] A.R. Teel. On l2 performance induced by feedbacks with multiple saturation. *ESIAM: Control, Optimization and Calculus of Variations*, 1, 1996.

[25] W.S. Wong and R.W. Brockett. Systems with finite communication bandwidth constraints i: State estimation problems. *IEEE Trans. Aut. Contr.*, 42(9):1294–1299, 1997.

[26] W.S. Wong and R.W. Brockett. Systems with finite communication bandwidth constraints ii: Stabilization with limited information feedback. *IEEE Trans. Aut. Contr.*, 44:1049–1053, 1999.