# Method I

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# 2 Method I: Planar reduction

*We apply the planar reduction method to a general two degree of freedom system with optional symmetry, near equilibrium and close to resonance. As a leading example the spring-pendulum close to 1:2 resonance is used. The resulting planar model is computed explicitly, and the bifurcation curves obtained are compared to numerical simulations.*

## 2.1 Introduction

Our goal is to obtain a description of the dynamics and bifurcations of a Hamiltonian system near equilibrium. To reach this goal, we use in this chapter the so-called planar reduction method, which we describe below. The result is a polynomial Hamiltonian model system living on the plane. This system is easy to analyze, yielding a qualitative description of the original system. In particular, bifurcation curves are easy to find. What is new in the current approach is that in each step towards the final polynomial model, the simplifying transformations are computed explicitly. This allows us to pull back the final bifurcation curves to the original parameter- and phase-space, so that quantitative results for the bifurcations of the original system are obtained. These results can subsequently be checked against numerical simulations of the iso-energetic Poincaré map, of which the planar model is an integrable approximation (see e.g. [BCKV93]). The agreement of numerical data and the pulled-back bifurcation curve is good, especially for small excitations.

The reduction methods, used here and in the next chapter, consist of three parts: Birkhoff normalization, symmetry reduction, and singularity theory. In each of these stages the coordinate transformations are explicitly computed, and especially in the first and final stage this is rather involved. For these calculations we summon the computer's help, using algorithms described and developed in later chapters. Algorithms for Birkhoff normalization are described in Chap. 4, whereas Chaps. 6 and 7 deal with the computation of the transformations for the singularity theory stage.

The reduction methods can be applied to systems with an equilibrium at the origin, that have a nondegenerate quadratic part exhibiting a single resonance between two of the $n$ degrees of freedom. Optionally, the system may have (e.g. discrete) symmetries. The system is first subjected to the Birkhoff normalization

procedure. The result is a formal coordinate transformation, and a normalized system with a $\mathbb{T}^{n-1}$ torus symmetry. By Noether's theorem (which holds for Lagrangian systems; see e.g. [AM78] for the Hamiltonian version), the 1-parameter continuous symmetries are related to conserved quantities, in this context also called *momenta*, see [CS85]. After applying the Birkhoff procedure, $n-1$ of such independent conserved quantities can be found explicitly.

The large symmetry group and related conserved quantities imply that the system is integrable, but generically an $n$ degree of freedom system ($n \geq 2$) is not [BT89]. In fact, the formal transformation lifts to smooth coordinate transformations by a theorem of Borel and Schwarz [Bro81, Dui84, GSS88], but these do not form conjugations. However, they do transform the system to a *perturbed* version of the normalized system. This perturbation is flat in the phase variables, so that formally conserved quantities are actually adiabatic invariants, and solution curves of the integrable system stay close to those of the original system for a long time.

For two degrees of freedom the situation is even better. By KAM theory, there exist a fat Cantor set of tori with parallel dynamics. On this part of phase space, a smooth conjugation with the integrable system does exist. The KAM tori prevent chaotic solution curves from wandering through phase space, so that even these solutions stay within a bounded distance from the integrable system's tori for ever. This provides a justification for using the integrable approximation to study the full system; see also [LL92]. From here on, therefore, we shall ignore the flat perturbation.

Our aim in this chapter is to obtain formulas for bifurcation curves up to a certain degree. Therefore, we also need the Taylor series of the normalized system up to certain degree only. By performing the iteration in Birkhoff's procedure a finite number of times, we obtain a smooth (in fact, polynomial) transformation and *approximate* symmetries, which we make exact by truncating the system.

The second part, symmetric reduction, is well-known and goes back to the Kepler problem, see e.g. [AM78, Bro79, CS85, CB97, Mee85, Tak74b]. The idea is to divide out the symmetry, and regard the associated conserved quantities as parameters (also called *integrals*), a procedure known as *orbit space reduction*. Sometimes this reduction is done on the entire phase space, and sometimes on each leaf of the foliation defined by the levels of the integrals. Care has to be taken when the topological type of the leaf depends on the value of these integrals, and when the symmetry group does not act freely (i.e., when some points have nontrivial isotropy group); see [AM78, CS85] for details. The result is a reduced Hamiltonian system with one degree of freedom, and whose dynamics coincides with the projection of the dynamics of the original system onto the orbit space.

At this point the planar reduction method, and the energy-momentum method of Chap. 3 start to diverge: Both the reduced system, as well as its subsequent treatment with singularity theory, are different. The reduction method of this chapter now applies a symplectic transformation that reduces the system to the plane, which gives the method its name.

The last stage consists of normalizing the planar system by arbitrary right-transformations. Such transformations are not symplectic, and therefore do not yield conjugations. However for 1 degree of freedom systems they do yield *equivalences*, i.e., conjugations modulo smooth time-reparametrizations. In particular, bifurcations are preserved, as they involve equilibria in the reduced system. The result is a polynomial model with parameters, together with transformations and reparametrizations that connect it with the original system.

In the first part of this chapter we describe the method in general terms, after which it is applied to the leading example of the spring-pendulum, a two degree-of-freedom system with a $\mathbb{Z}_2 \times \mathbb{Z}_2$ spatial and reversing symmetry.

Large parts of this chapter have been published in [BHLV98, BLV98]. The planar reduction method as used here was introduced in [BCKV95, BCKV93]. It stands in a long tradition, see e.g. [Mee85] for a historical overview. We here also mention the method described in [Dui84] which will receive full treatment in Chap. 3; see also [VvdM95].

### 2.1.1 BCKV-restricted morphisms

An important aspect of the equivalence transformations used to simplify the planar system, is their treatment of the *distinguished parameter* $\lambda$, the conserved quantity resulting from Birkhoff normalization. It is a parameter to the planar system, but a special one since it depends on the original phase variables, hence the adjective *distinguished*. This special nature of $\lambda$ is reflected in the class of reparametrizations allowed on the planar system: It is required that the zero-level of $\lambda$ is preserved, and that reparametrizations of ordinary parameters do not depend on $\lambda$; see remark 2.6 and [BCKV95, BCKV93] for more details. Theorem 5.16 implements these restrictions, yielding for the case of the 1 : 2 resonance a normal form

$$x(x^2 + y^2) + (\lambda_1 + u_1)x + (\lambda_2 + u_2)y^2.$$

Here $\lambda_i$ and $u_i$ are distinguished and ordinary parameters, respectively. One consequence of the theorem is that a versal deformation requires at least two distinguished parameters. For our application we have only one at our disposal, a problem which is resolved by the *path formulation*, see [BCKV93, FSS98, Mon94]. The resulting normal form is given in Proposition 2.16, where one of the distinguished parameters depends explicitly on the other parameters, so that it traces out a path through the parameter space of the versal deformation.

## 2.2 Details of the planar reduction method

In this section we give a detailed outline of the planar reduction method. The method is applied to the spring-pendulum around the 1 : 2 resonance in Sect. 2.3.

**Table 2.1**    Overview of the planar reduction of the spring-pendulum in $1:2$ resonance

| Section: | | 2.2.3 | 2.2.4 | 2.2.5 | 2.2.6 | 2.2.7 |
|---|---|---|---|---|---|---|
| Context: | Original | Birkhoff normal form | Planar reduc- tion | Central normal form | Versal defor- mation | BCKV normal form |
| System: | $H^0$ | $H^n$ | $H^r$ | $H^c$ | $H^u$ | $H^B$ |
| Phase space: | $\mathbb{R}^4$ | $\mathbb{R}^4$ | $\mathbb{D}^2$ | $\mathbb{R}^2$ | $\mathbb{R}^2$ | $\mathbb{R}^2$ |
| Coefficients: | $a_i, i \geq 1$ | $b_i, i \geq 2$ | $b_i, i \geq 2$ | $d_i$ | — | $a_i$ |
| Parameters: | — | $b_1$ | $b_1, \lambda$ | $c_i, \lambda$ | $u_1, u_2$ | $\lambda, b_1$ |
| Symmetry: | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $\mathbb{Z}_2 \times \mathbb{S}_1$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |

During the reduction, the system, the parameters it depends on, the phase space on which it lives, and its symmetry change several times. A summary is given in table 2.1. Using this as a guide, we start this section by outlining the reduction procedure leading to the BCKV normal form. In 2.2.2 we give some notation that will be used in the sequel, after which we begin discussing the reduction method proper.

### 2.2.1 Overview

The starting point is a Hamiltonian $H^0$ with an equilibrium at the origin. It is supposed to be close to some resonance of the form $p : q$, and to depend on several coefficients $a_i$. Optionally, the system may be invariant (or reversing) under some symmetry group $\Gamma$, which is supposed to respect the symplectic structure. (Table 2.1 shows the symmetry group relevant for the spring-pendulum example in the various stages during reduction.)

The first step is to apply the Birkhoff procedure around the resonance, resulting in a system $H^n$ which has acquired a (formal) $\mathbb{S}^1$-symmetry. This step singles out a *detuning parameter* denoted by $b_1$, which measures the deviation from the resonance around which the Birkhoff procedure is performed. For convenience, the other coefficients are now denoted by $b_2, b_3, \ldots$, and depend on the $a_i$.

The system $H^n$ has two independent conserved quantities: $H^n$ itself, and the formal integral $\lambda$ conjugate to the cyclic variable associated to the $\mathbb{S}^1$ symmetry. In the cases we consider, this integral is equal to the quadratic part of $H^n$. Since $\lambda$ is conserved, trajectories of the system lie in level sets of $\lambda$. In the case that $H^0$ has an elliptic equilibrium at the origin, these level sets foliate the phase space, close to the origin, by compact sets that are homeomorphic to 3-spheres in $\mathbb{R}^4$. After dividing out the $\mathbb{S}^1$ symmetry, on a leaf with $\lambda$ nonzero and fixed, we get an $\mathbb{S}^2$ (see 2.3.3). We go to a planar system by flattening out this $\mathbb{S}^2$ to a disk with boundary $\mathbb{D}^2$. The boundary is the image of an $\mathbb{S}^1$-orbit which is singular with respect to the reduction, and is called the *singular circle*. It coincides with

a singularity in the coordinate transformation to the plane. The reduced system so obtained is denoted by $H^r$.

The diameter of the singular circle depends on $\lambda$, and vanishes when $\lambda$ does. If the planar coordinates are denoted $x, y$, then in $x, y, \lambda$-space the singular circles form a cone. From here on we ignore these singular circles, and consider the system in a full neighborhood of the origin in $\mathbb{R}^2 \times \mathbb{R}$. The system is most singular when $\lambda = 0$; this is called the *central singularity*, or *organizing center* [Mon94]. It is subjected to a transformation in order to bring it into a simple form. For the case of the spring-pendulum considered in Sect. 2.3, this is the $\mathbb{Z}_2$-symmetric hyperbolic umbilic. We are left with a deformation $H^c$ of this singularity, in terms of the parameters $b_1$ and $\lambda$.

There exists a versal deformation of the hyperbolic umbilic with only two parameters. (In the non-equivariant case one finds three.) This deformation is denoted by $H^u$. In Sect. 2.2.6 we find the reparametrizations that induce $H^c$ from $H^u$. This step is computationally involved, and is dealt with in the last two chapters. In this step we employ a standard basis and the corresponding division algorithm to compute the required morphisms efficiently.

Finally, we use the reparametrizations of Sect. 2.2.6 to compute the BCKV-restricted normal form $H^B$ of our system.

### 2.2.2 Some notation

**Symmetries and coordinate systems** In the sequel, we use Cartesian canonical coordinates $x_i, y_i$ as well as complex variables $z_i, \bar{z}_i$ and Hamiltonian polar coordinates $L_i, \phi_i$, because certain transformations take a simple form in one of these coordinates. Two $\mathbb{Z}_2$-symmetries will also play a role: a mirror symmetry $S$ which acts on the coordinates $x_2$, $y_2$ only, and a time-reversal symmetry $T$ that acts on the momentum coordinates $y_i$. The relations between the coordinates and symmetries are as follows:

$$
\begin{aligned}
z_i &= x_i + \mathrm{i}y_i = & \sqrt{2L_i}\mathrm{e}^{\mathrm{i}\phi_i}, & \qquad \bar{z}_i = x_i - \mathrm{i}y_i = & \sqrt{2L_i}\mathrm{e}^{-\mathrm{i}\phi_i} \\
\phi_i &= \tfrac{1}{2\mathrm{i}} \log \tfrac{z_i}{\bar{z}_i} = & \arctan \tfrac{y_i}{x_i}, & \qquad L_i = \tfrac{1}{2}z_i\bar{z}_i = & \tfrac{1}{2}\left(x_i^2 + y_i^2\right) \\
x_i &= \sqrt{2L_i}\cos\phi_i = \tfrac{1}{2}(z_i + \bar{z}_i), & & \qquad y_i = \sqrt{2L_i}\sin\phi_i = \tfrac{1}{2\mathrm{i}}(z_i - \bar{z}_i)
\end{aligned}
\tag{2.1}
$$

$T :$

$(x_1, x_2, y_1, y_2) \mapsto (x_1, x_2, -y_1, -y_2)$

$(z_1, \bar{z}_1, z_2, \bar{z}_2) \mapsto (\bar{z}_1, z_1, \bar{z}_2, z_2)$

$(L_1, \phi_1, L_2, \phi_2) \mapsto (L_1, -\phi_1, L_2, -\phi_2)$

$S :$

$(x_1, x_2, y_1, y_2) \mapsto (x_1, -x_2, y_1, -y_2)$

$(z_1, \bar{z}_1, z_2, \bar{z}_2) \mapsto (z_1, \bar{z}_1, -z_2, -\bar{z}_2)$

$(L_1, \phi_1, L_2, \phi_2) \mapsto (L_1, \phi_1, L_2, \phi_2 + \pi)$

**Parameters and coefficients** The dynamical systems we investigate depend on a number of variables. Certain variables are supposed to be constant during the evolution of the system, for example the mass of a pendulum. Throughout, we reserve the name *coefficient* for 'constant variables' that can take on arbitrary values, except possibly a few isolated ones excluded by non-degeneracy

conditions. The name *parameter* is reserved for 'constant variables' which are small. Asymptotic expansions are done in terms of phase space variables and parameters.

**Hamiltonian contexts** The Hamiltonian system $H$ we consider appears in several versions, in the corresponding stages of the normalization process. The current 'stage' or context is denoted by a superscript, e.g. $H^0$ for the original Hamiltonian, $H^n$ for the Birkhoff normal form.

**Big-oh notation** For brevity, we use the notation $O(|x, y|^k)$ to denote terms of total order $k$ and higher in $x$ and $y$. In standard notation, this would be $O((|x| + |y|)^k)$. Also, e.g. $O(|c_i, \lambda|^k)$ stands for $O((|c_1| + |c_2| + \cdots + |\lambda|)^k)$, when $c = (c_1, c_2, \ldots)$ is a vector of coefficients. This will be clear from the context.

**Formal power series and functions** In this work we often use formal power series. In order not to make the text unreadable, we shall all the same refer to them simply as *functions* or *maps* or *vector fields*. This is no problem since all operations on functions (maps, vector fields) are also allowed on formal power series, with the exception of conjugations with coordinate transformations that do not leave the origin fixed.

### 2.2.3 Birkhoff normalization

The first step in the reduction procedure is the application of the Birkhoff normal form. The quadratic part of the system's Hamiltonian $H^0$ determines the normal form. In particular, when the quadratic part is nondegenerate and nonresonant, the normal form system is integrable, and exhibits only trivial dynamics. On the other hand, if more than two harmonic oscillators are in resonance, reduction to 1 degree of freedom is not possible, and the planar reduction method cannot be used. We restrict our attention to the case of one resonance; for more remarks on this see the introduction to this chapter.

　　In Chap. 4 a description of the Birkhoff normal form procedure is given, together with algorithms that implement it. Here we give the results of the computation. The Birkhoff normal form of a Hamiltonian $H^0$ with an elliptic equilibrium at the origin, is determined chiefly by the kernel of $\mathrm{ad}_{H_2}$, where $H_2$ is the quadratic part of $H^0$. Since the Birkhoff normal form procedure generally yields a divergent power series, we work in the ring $R = \mathbb{R}[[z_i, \bar{z}_i]]$ of formal power series (see Sect. 2.1 for remarks). Here $z_i, \bar{z}_i$ are complex coordinates that make $\mathrm{ad}_{H_2}$ act diagonally with respect to a monomial basis on $R$.

　　When $H^0$ is invariant under some symmetry group $\Gamma$ (that respects the symplectic structure), the normal form procedure may be carried out within the ring of $\Gamma$-invariant power series $R^\Gamma$; see Chap. 4 remark 4.4. For the case of the symmetry groups considered above, we have the following result. (In general the problem of finding basic invariant polynomials for a given group action is difficult; see e.g. [Stu93].)

**Proposition 2.1.** *Let $H_2 = iz_1\bar{z}_1 + i\omega z_2\bar{z}_2$, where $\omega \neq 0$, and assume $\Gamma$ is either $\{Id\}$, $\{Id, T\}$ or $\{Id, S, T, ST\}$. If $\omega \notin \mathbb{Q}$, a Hilbert basis for the algebra $\ker \mathrm{ad}_{H_2}$ in $R^\Gamma$ is $\{z_1\bar{z}_1, z_2\bar{z}_2\}$. In the case that $\omega = p/q = P/Q$, where $p, P \neq 0$, $q, Q > 0$, $Q$ even and $\gcd(p, q) = \gcd(P, Q/2) = 1$, the Hilbert basis for $\ker \mathrm{ad}_{H_2}$ additionally contains*

a) $\Gamma = \{Id\}$: $\qquad z_1^p \bar{z}_2^q,\ \bar{z}_1^p z_2^q\ (p \geq 0) \qquad$ *or* $\qquad \bar{z}_1^{-p}\bar{z}_2^q,\ z_1^{-p}z_2^q\ (p \leq 0)$
b) $\Gamma = \{Id, T\}$: $\qquad z_1^p\bar{z}_2^q + \bar{z}_1^p z_2^q\ (p \geq 0) \qquad$ *or* $\qquad \bar{z}_1^{-p}\bar{z}_2^q + z_1^{-p}z_2^q\ (p \leq 0)$
c) $\Gamma = \{Id, S, T, ST\}$: $\quad z_1^P\bar{z}_2^Q + \bar{z}_1^P z_2^Q\ (p \geq 0)\ $ *or* $\ \bar{z}_1^{-P}\bar{z}_2^q + z_1^{-P}z_2^q\ (p \leq 0)$

A finite set of invariants generating the invariant ring of a Lie group is called a Hilbert basis; the terminology above is justified since $\ker \mathrm{ad}_{H_2}$ is the invariant ring of the Lie group consisting of exponentials of the vector field associated to $H_2$; see [Gat00, Hil93].

Note that since $S^2 = T^2 = Id$, the group $\Gamma$ is isomorphic to $\mathbb{Z}_2$ in case (b), and isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ in case (c).

Applying Proposition 2.1 to the spring-pendulum system, which has 2 degrees of freedom, we get the following:

**Proposition 2.2.** *(Birkhoff normal form) Let $H^0$ be a Hamiltonian on $\mathbb{R}^4$ with vanishing linear part, invariant under $T$ as defined in (2.1). Let $H_2^0 = iz_1\bar{z}_1 + i\omega z_2\bar{z}_2$ be its quadratic part, and assume that $\omega = \frac{P}{Q} = \frac{p}{q}$ with $Q$ even, $Q, q > 0$ and $\gcd(P, Q/2) = \gcd(p, q) = 1$. Let*

$$\psi = z_1^p \bar{z}_2^q + \bar{z}_1^p z_2^q \qquad \text{if } p > 0,$$
$$\psi = z_1^{-p} z_2^q + \bar{z}_1^{-p}\bar{z}_2^q \qquad \text{if } p < 0,$$
$$\psi = z_1^P \bar{z}_2^Q + \bar{z}_1^P z_2^Q \qquad \text{if } H^0 \text{ is } S\text{-invariant and } p > 0,$$
$$\psi = z_1^{-P} z_2^Q + \bar{z}_1^{-P}\bar{z}_2^Q \qquad \text{if } H^0 \text{ is } S\text{-invariant and } p < 0,$$

*then there exists a formal symplectic $T$-equivariant coordinate transformation $\phi$ such that*
$$H^n := H^0 \circ \phi = H_2^0 + f_0(z_1\bar{z}_1, z_2\bar{z}_2, \psi),$$
*where $f_0(\zeta_1, \zeta_2, \zeta_3) = i\alpha\zeta_3 +$ quadratic and higher order terms, for some $\alpha \in \mathbb{R}$. If $H^0$ is $S$-invariant, $\phi$ can be chosen to commute with $S$ too. The quadratic part $H_2^0$ is conserved under the flow of $H^n$, i.e., $H^n$ is invariant under the $\mathbb{S}^1$-action $A_\xi : (z_1, z_2) \mapsto (e^{qi\xi}z_1, e^{pi\xi}z_2)$, where $\xi \in \mathbb{S}^1 \equiv \mathbb{R}/2\pi\mathbb{Z}$. This action is nondegenerate except on the axes $z_1 = 0$ and $z_2 = 0$ on which points have isotropy subgroup (stabilizer) $\mathbb{Z}_p \subseteq \mathbb{S}^1$ and $\mathbb{Z}_q \subseteq \mathbb{S}^1$ respectively.*

**Remark 2.3.** *(Complex coordinates)* Both in Proposition 2.1 as in Proposition 2.2 we use complex coordinates $z_i, \bar{z}_i$, connected to Cartesian coordinates through (2.1). Whenever we use complex coordinates, we shall assume that the symplectic form is $dz \wedge d\bar{z}$. The transformation from $x_i, y_i$ to complex coordinates (with this symplectic form) is symplectic with *multiplier* 2i. Hence, real Hamiltonians in Cartesian (or Hamiltonian polar) coordinates correspond to purely imaginary Hamiltonians in complex coordinates. This explains why $H_2^0$ is purely imaginary in the propositions above.

### 2.2.4 Reduction to planar 1 degree-of-freedom system

We now discuss the reduction to the plane, for a system in $p : q$ resonance. The normalized system $H^n$ has an additional (formal) $\mathbb{S}^1$ symmetry, with action $(z_1, z_2) \mapsto (e^{iq\xi}z_1, e^{ip\xi}z_2)$ for $\xi \in \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$, and corresponding conserved quantity $H_2^0 = z_1\bar{z}_1 + \omega z_2\bar{z}_2$. This symmetry enables us to formally reduce to a one degree-of-freedom system. We first express the normalized system in Hamiltonian polar coordinates $L_i, \phi_i$ (see (2.1)):

$$(2.2) \qquad H^n(L, \phi) = L_1 + \omega L_2 + f_1 \left( L_1, L_2, L_1^{P/2} L_2^{Q/2} \cos(P\phi_1 - Q\phi_2) \right).$$

Here, and elsewhere in this section, the functions $f_i$ are of the same form as $f_0$ in Proposition 2.2, differing only by innocent linear changes of variables. Let $p = P/\gcd(P, Q), q = Q/\gcd(P, Q)$, and let $r, s$ be integers such that $pr - qs = 1$. Consider the following symplectic coordinate change:

$$(2.3) \qquad \begin{pmatrix} \tilde{L}_1 \\ \tilde{L}_2 \end{pmatrix} = \begin{pmatrix} r & s \\ q & p \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}, \qquad \begin{pmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{pmatrix} = \begin{pmatrix} p & -q \\ -s & r \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

The action of a symmetry group $\Gamma$ will also have to be transformed to the new variables. In these variables, the system, the acquired $\mathbb{S}^1$-action and the action of the symmetry group generators $S$ and $T$ that will be used in the sequel, take the following form:

$$H^n(\tilde{L}, \tilde{\phi}) = \frac{1}{q}\tilde{L}_2 + f_2(\tilde{L}_1, \tilde{L}_2, (p\tilde{L}_1 - s\tilde{L}_2)^{P/2}(-q\tilde{L}_1 + r\tilde{L}_2)^{Q/2} \cos(\gcd(P, Q)\tilde{\phi}_1)),$$

$T : (\tilde{\phi}_1, \tilde{\phi}_2) \mapsto (-\tilde{\phi}_1, -\tilde{\phi}_2),$

$S : (\tilde{\phi}_1, \tilde{\phi}_2) \mapsto (\tilde{\phi}_1 + q\pi, \tilde{\phi}_2 + r\pi),$

$\mathbb{S}^1\text{-action} : (\tilde{\phi}_1, \tilde{\phi}_2) \mapsto (\tilde{\phi}_1, \tilde{\phi}_2 + \xi),$

from which it is manifest that $\tilde{L}_2$ is conserved (since the conjugate variable $\tilde{\phi}_2$ is cyclic), indeed, $\tilde{L}_2 = qH_2^0$. Note that the transformation (2.3) is invertible; in particular, the $\mathbb{S}^1$-action is nondegenerate (except at certain points). This may be contrasted to the $q$-sheeted cover used in [BV92]. We now reduce to a planar system by dividing out the $\mathbb{S}^1$-symmetry generated by $\tilde{L}_2$, viewing $\tilde{L}_2$ as a *distinguished parameter* We denote the planar reduction of $H^n$ we get in this way by $H^r$. Calling $\tilde{L}_2$ a parameter is justified if we consider only small deviations from the system's lower equilibrium [1] for then the system has little energy, so $H^n$, and therefore $\tilde{L}_2$, is small. (See also remark 2.6.) To emphasize the role of $\tilde{L}_2$ as parameter, we write

$$\lambda := \tilde{L}_2 = qL_1 + pL_2$$

---

[1] i.e., the pendulum not moving and hanging straight down, with gravity balancing the spring force.

from now on. Next, we apply the translation $\bar{L}_1 = \tilde{L}_1 - \frac{s}{p}\lambda$, $\bar{\phi}_1 = \tilde{\phi}_1$, which in the context of planar Hamiltonian systems is a symplectic transformation. The Hamiltonian then becomes $H^r = \frac{\lambda}{q} + f_3(\bar{L}_1, \lambda, \bar{L}_1^{P/2}(\bar{L}_1 - \frac{\lambda}{pq})^{Q/2}\cos(\gcd(P,Q)\bar{\phi}_1))$. Finally, we return to Cartesian coordinates. Dropping the constant and hence dynamically irrelevant term $\lambda/q$, we get the following:

**Proposition 2.4.** *Under the assumptions of Proposition 2.2, let $H^n$ be a Hamiltonian in Birkhoff normal form. There exist coordinates $x, y, \lambda, \phi$ on $\mathbb{R}^4$ such that $\lambda$ is constant on orbits of $H^n$, and the projections of those orbits onto the $(x,y)$-plane coincide with those of a planar Hamiltonian system $H^r(x,y)$, with parameter $\lambda$ and independent of $\phi$, of the form*

$$P \text{ even:} \quad H^r = f_4\left(x^2 + y^2 \; , \; \lambda \; , \; (x^2+y^2)^{\frac{P}{2}-1}(x^2-y^2)\left(x^2+y^2-\frac{2\lambda}{pq}\right)^{Q/2}\right);$$

$$P \text{ odd:} \quad H^r = f_4\left(x^2 + y^2 \; , \; \lambda \; , \; (x^2+y^2)^{\frac{P-1}{2}}x\left(x^2+y^2-\frac{2\lambda}{pq}\right)^{Q/2}\right),$$

*where $f_4(\zeta_1, \zeta_2, \zeta_3) = b_1\zeta_3 + h.o.t.$*

**Remark 2.5.** *(Singular circle)* The coordinate transformation to Hamiltonian polar coordinates used in (2.2) is singular at the coordinate axes $L_1 = 0$ and $L_2 = 0$. These axes become $p\tilde{L}_1 - s\tilde{L}_2 = 0$ and $-q\tilde{L}_1 + r\tilde{L}_2 = 0$ in the transformed coordinates, and after translation $\bar{L}_1 = 0$ and $\bar{L}_1 = \lambda/pq$. The first singularity is removed by returning to Cartesian coordinates in the plane. The second singularity is called the *singular circle* (see Sect. 2.3.3). At this circle $L_2 = 0$ so that the coordinate $\phi_2$ is ill-defined, and therefore so is $\bar{\phi}_1 = p\phi_1 - q\phi_2$. In particular this implies that $H^r$ is constant on the circle; see also Sect. 2.3.3.

**Remark 2.6.** *(The parameter $\lambda$)* The adjective *distinguished* refers to the fact that $\lambda$ stems from the phase space of $H^n$, and is a parameter only for the reduced system, not for the original one. If we are interested in the geometry of the local level sets of $H^n$ on the full 4-dimensional phase space, we may not let reparametrizations of ordinary parameters depend on the distinguished parameter, see Sect. 2.2.7. This should be contrasted to the point of view taken in Sect. 2.2.6, where we classify the geometry of level sets in $\mathbb{R}^2$, and where it is permissible to treat $\lambda$ as an ordinary parameter. Note that in either setting we do not allow reparametrizations of $\lambda$ to depend on phase variables.

**Remark 2.7.** *(Symmetries)* When $q$ is even, the acquired $\mathbb{S}^1$ normal form symmetry group contains the reflection $\mathbb{Z}_2$ symmetry $S$ as a subgroup. Before reduction the symmetry group is therefore $\mathbb{S}^1 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{S}^1 \times \mathbb{Z}_2$, depending on the parity of $q$, leading to a symmetry group $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ for the reduced system. See also Proposition 2.1.

### 2.2.5 Reduction to the central singularity

At this point the system is reduced to a planar Hamiltonian $H^r$ depending on a distinguished parameter $\lambda$ and several ordinary parameters. Recall that parameters are supposed to be *small*. We now look at the 'degenerate' Hamiltonian that results when $\lambda$ and the other parameters vanish. This is called the *central singularity*, also known as the *organizing center* [FSS98, Mon94] and is denoted by $H_0^r$.

    In the cases we consider $H_0^r$ is *finitely determined*, i.e., a finite piece of the jet of $H_0^r$ is equivalent to $H_0^r$ itself, via a smooth planar coordinate transformation $\phi$. This transformation is independent of the parameters of $H^r$. Existence of this transformation is guaranteed by singularity theory. Since $H^r$ is invariant under the action of a symmetry group, this has to be taken into account by singularity theory; see section 5.3 for details. The result is a polynomial normal form $H_0^r \circ \phi$ of $H_0^r$.

    The *central singularity reduced normal form* is defined as $H^c := H^r \circ \phi$. At the origin of parameter space this is equal to the polynomial normal form, i.e., $H^c$ is in fact a deformation of this polynomial normal form. The final step in the reduction process is to find a versal deformation of this normal form which can serve as a model for $H^c$. This versal deformation is denoted by $H^u$ and forms the final model of the planar system.

### 2.2.6 Inducing the system from a versal deformation

With the versal deformation $H^u$ in hand, we can find bifurcation curves in terms of the model's parameters. In order to pull-back these curves to the original parameter space, we need the explicit transformations that induce $H^c$ from the versal deformation $H^u$. These transformations are used in the final step where we find the BCKV normal form, which incorporates the distinguished nature of the parameter $\lambda$.

    Let us denote the small *parameters* in $H^c$ by $\lambda$ and $c_1, c_2, \ldots$, and the *coefficients* by $d_1, d_2, \ldots$. For the moment we disregard the distinguished nature of $\lambda$, treating it, like the $c_i$, as an ordinary parameter (see remark 2.6), and for notational convenience we write $\lambda = c_0$.

    We want to find transformations that induce $H^c$ from the versal deformation $H^u$. Assume that $H^u(x, y, u_1, \ldots, u_k)$ is a versal deformation of the central singularity; see [BL75, Mon91]. It follows that there exists a pair of transformations $(\phi, \rho)$, where $\phi : \mathbb{R}^2 \times \mathbb{R}^c \times \mathbb{R}^d \to \mathbb{R}^2$ is a parameter-dependent coordinate transformation, and $\rho : \mathbb{R}^c \times \mathbb{R}^d \to \mathbb{R}^k$ is a reparametrization from $(c_i, d_j)$ to $(u_1, \ldots, u_k)$, such that

$$H^u(\phi(x, y, c_i, d_i), \rho(c_i, d_i)) = H^c(x, y, c_i, d_i).$$

These transformations obey the following additional constraints: $\phi$ is equivariant under the symmetry group, and both $\phi$ and $\rho$ are trivial at the central singularity,

i.e., $\phi(x, y, 0, d_i) = (x, y)$ and $\rho(0, d_i) = 0$. In Sect. 5.3.1 we give a necessary and sufficient condition for a deformation to be versal. It amounts to solvability of the well-known *infinitesimal stability* equation[2] adapted to our equivariant context.

In the case of the $1 : 2$ resonance, the central singularity is isomorphic to $x(x^2 + y^2)$, with a symmetry group $\mathbb{Z}_2$ acting on $\mathbb{R}^2$ via $(x, y) \mapsto (x, -y)$. A versal unfolding is $H^u = x(x^2 + y^2) + u_1 x + u_2 y^2$, and the condition boils down to: For every $\mathbb{Z}_2$-invariant germ $g$ vanishing at the origin there should exist $\mathbb{Z}_2$-invariant germs $\alpha_i(x, y)$, $i = 1, 2, 3$ and real numbers $u_1, u_2$ such that

$$(2.4) \quad g(x, y) \;=\; \alpha_1(x, y)x\frac{\partial f}{\partial x} + \alpha_2(x, y)y^2\frac{\partial f}{\partial x} + \alpha_3(x, y)y\frac{\partial f}{\partial y} + u_1 x + u_2 y^2.$$

Here $f = x(x^2 + y^2)$ is the central singularity. For this $f$ the condition is indeed satisfied; see Sect. 5.3.2.

Starting from the infinitesimal stability condition, versality is proved by invoking the Mather-Malgrange preparation theorem [Mar82, Poè76]. Our interest is not so much in proving existence, as in explicitly computing the transformations $\phi$ and $\rho$, up to a certain degree. An algorithm due to Kas and Schlessinger [KS72] accomplishes this; see Sect. 7.2.2. By using the fact that equations of the form (2.4) can be solved, it uses the solutions $\alpha_i$ and $u_i$ to iteratively build the transformations $\phi$ and $\rho$. This algorithm can be regarded as a constructive proof of the existence of a *formal* solution for $\phi$ and $\rho$.

Our ability to compute $\phi$ and $\rho$ now rests on our ability to compute solutions to (2.4). This can be done efficiently using standard bases; see Chap. 6 and Sect. 7.2.3.

### 2.2.7 BCKV normal form

BCKV theory classifies the family of systems $H^r$ as 2 degree-of-freedom systems. For a given member of the family (i.e., for certain values of the coefficients) it provides a normal form system, which is itself a 2 degree-of-freedom system. This should be contrasted to the deformation $H^u$, which classifies $H^r$ as a family of *planar* systems; see remark 2.6. The parameter $\lambda$ is a phase-space variable that is constant under flows of the system. In the BCKV normal form, reparametrizations are not allowed to depend on $\lambda$.

It turns out to be possible to use $H^u$ for constructing a suitable unfolding $H^B$ (see theorem 5.16), corresponding to a generic path (surface) in a more general parameter space. (See also [Mon94].) In this setting many more parameters are needed for versality. The path arises when the coefficients of such a versal normal form are expressed as functions of the available parameters. Moreover, these parameters will be expressed in the original (physical) constants of the system. This gives the natural set-up for the aforementioned perturbation problem.

---

[2] See [GG73]. Necessity of this condition is immediate by considering deformations of the form $H^u(x, y, 0) + c_1 g(x, y)$ for arbitrary (symmetric) $g$; see [Mar82, Prop. IV.3.2].
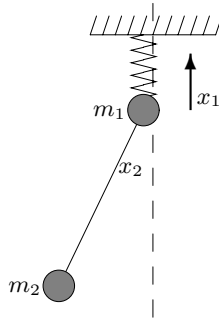
**Fig. 2.1**   The spring-pendulum with its axis of symmetry

See [BCKV93] for a general discussion. For the definition of BCKV-restricted unfoldings, see section 5.3.3.

## 2.3 Spring-pendulum in 1:2-resonance

This chapter ends with the application of the planar reduction method to the spring-pendulum system. We start by introducing the system, followed by the reduction to the polynomial normal form. At the end of this section we discuss the $1:2$-resonant dynamics of the spring-pendulum, and give bifurcation diagrams, numeric Poincaré sections and a comparisons of the predicted pulled-back bifurcation curves with numerical data.

### 2.3.1 The system

The spring-pendulum is a planar pendulum suspended by a spring constrained to move along the vertical axis. It is a typical two degree-of-freedom Hamiltonian system with a $\mathbb{Z}_2 \times \mathbb{Z}_2$ (time-reversal and reflection) symmetry.

We now describe the system. Masses are attached to both ends of the rod, while both rod and spring are massless. The configuration is given by the displacement of the suspension point and the angle of the pendulum with the vertical axis, denoted by $x_1$ and $x_2$. The potential energy is $U(x_1, x_2) = -m_2 gl \cos x_2 + \frac{1}{2}a^2 x_1^2$ when the origin is suitably chosen. The $m_i$ denote masses, $M^2 = m_1 + m_2$, $g$ the gravitational acceleration, $l$ the length of the pendulum and $a^2$ the spring coefficient. The Hamiltonian of the system, expressed in configuration coordinates $x_i$ and their conjugates $y_i$ reads

$$(2.5) \qquad H(x,y) = \tfrac{1}{2}a^2 x_1^2 - m_2 gl \cos x_2 + \tfrac{l^2 m_2 y_1^2 + M^2 y_2^2 - 2lm_2 y_1 y_2 \sin x_2}{m_2 l^2 (2M^2 + m_2[\cos(2x_2)-1])}.$$

This Hamiltonian exhibits two $\mathbb{Z}_2$-symmetries, i.e., we have a symmetry group $\Gamma := \mathbb{Z}_2 \times \mathbb{Z}_2$. Generators are a time-reversible symmetry denoted by $T$, and reflection symmetry in the vertical axis, denoted by $S$; here

$$T : (x_1, x_2, y_1, y_2) \mapsto (x_1, x_2, -y_1, -y_2),$$
$$S : (x_1, x_2, y_1, y_2) \mapsto (x_1, -x_2, y_1, -y_2).$$

(2.6)

Writing $H$ as a Taylor series in the $x_i$ and $y_i$ variables, and applying a rescaling of variables and time to tidy up the quadratic terms, we get:

**Proposition 2.8.** *Provided that $m_2 \neq 0$ and $a \neq 0$, by a rescaling of variables and time we can bring the Hamiltonian (2.5) into the form*

(2.7)
$$H^0(x, y) := \frac{x_1^2 + y_1^2}{2} + a_1 \frac{x_2^2 + y_2^2}{2} - 8a_2 x_2 y_1 y_2 - 16a_3 x_2^4 + 16a_4 x_2^2 y_1^2 + 16a_5 x_2^2 y_2^2 +$$
$$+ 32a_6 x_2^3 y_1 y_2 + 64a_7 x_2^6 + 64a_8 x_2^4 y_1^2 + 64a_9 x_2^4 y_2^2 + h.o.t.,$$

*where the h.o.t. are $O(|x, y|^7)$ terms, and the symplectic form is $\mathrm{d}x \wedge \mathrm{d}y$. Here $a_1 = \frac{\sqrt{g}M}{a\sqrt{l}}$ and $a_2 = \frac{1}{8al}$, and $H^0$ is invariant under $S$ and $T$.*

We use (2.7) as starting-point, with no conditions on the coefficients $a_i$. That is, we forget about the algebraic relations between the $a_i$ that exist for this particular system. The system (2.7) has the same qualitative form as the spring-pendulum system. In fact, for a proper choice of the coefficients $a_i$, and modulo a rescaling, the latter is a high order perturbation of (2.7).

The physical origin of the system imposes some constraints on the coefficients, for example $a_1 > 0$ and $a_2 > 0$. We will not use these. Instead, we keep an eye on the non-degeneracy conditions encountered during the calculations, allowing the $a_i$ to otherwise take arbitrary values. It turns out that some of these conditions are implied by the physical constraints.

### 2.3.2 Reduction

**Birkhoff normalization, reduction to a planar system** Let us denote by $H_2$ the quadratic part of $H^0$. At $a_1 = \frac{1}{2}$, the kernel of $\mathrm{ad}_{H_2}$ is generated as an algebra by $z_1 \bar{z}_1$, $z_2 \bar{z}_2$ and $z_1 \bar{z}_2^2 + \bar{z}_1 z_2^2$. These generators are invariant under the action of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetry generated by $S$ and $T$. After Birkhoff normalizing the spring-pendulum Hamiltonian can be written as

$$H^n = H_2 + f(z_1 \bar{z}_1, z_2 \bar{z}_2, z_1 \bar{z}_2^2 + \bar{z}_1 z_2^2).$$

In the new Birkhoff coordinates, the quadratic part of the Hamiltonian is an integral of motion. We denote this integral by $\lambda := 2H_2$, see also (2.9) below. Later on, we refer to this integral as the *distinguished parameter*. On the original phase space it has the expression

(2.8)
$$\lambda = \left(x_1^2 + y_1^2\right) + \frac{1}{2}\left(x_2^2 + y_2^2\right) + \frac{8a_2}{1 + 2a_1}\left(x_1 x_2^2 - 2x_2 y_1 y_2 - x_1 y_2^2\right)$$
$$+ \frac{4a_4}{a_1(1 - a_1^2)}\left((1 - 4a_1 + 2a_1^3)x_1^2 x_2^2 + (1 + 4a_1 - 2a_1^2 - 2a_1^3)x_2^2 y_1^2 + \right.$$
$$\left. (4a_1 - 8a_1^2)x_1 x_2 y_1 y_2 + (-1 + 2a_1^3)x_1^2 y_2^2 + (-1 + 2a_1^2 - 2a_1^3)y_1^2 y_2^2\right)$$
$$+ \frac{64a_2^2}{(1 + 2a_1)^2}\left((1 + a_1)x_1^2 - a_1 y_1^2\right)\left(x_2^2 + y_2^2\right)$$
$$+ \frac{4a_2^2}{a_1(1 + 2a_1)^2}\left(-2(3 + 4a_1)x_2^2 y_2^2 + (1 + 4a_1)(x_2^4 + y_2^4)\right)$$
$$+ \frac{(-10a_3 - 2a_5)x_2^4 + 12(a_3 + a_5)x_2^2 y_2^2 + (6a_3 - 2a_5)y_2^4}{a_1} + O(|x_i, y_i|^5).$$

We now transform to Hamiltonian polar coordinates. The generators $z_i \bar{z}_i$ and $z_1 \bar{z}_2^2 + \bar{z}_1 z_2^2$ take the form $L_1$, $L_2$ and $\sqrt{L_1}L_2 \cos(\phi_1 - 2\phi_2)$. The coordinates $L_1, L_2$ are globally defined, and the angles $\phi_1, \phi_2$ are well-defined for $L_1 \neq 0$ and $L_2 \neq 0$, respectively. To reduce to a planar system we apply the symplectic transformation in these coordinates (see (2.3)),

(2.9)          $$\bar{L}_1 = L_1, \quad \lambda = 2L_1 + L_2, \quad \bar{\phi}_1 = \phi_1 - 2\phi_2, \quad \bar{\phi}_2 = \phi_2.$$

This transformation is singular for $L_2 = 0$ and $L_1 \neq 0$, since then $\bar{L}_1 \neq 0$ whereas $\bar{\phi}_1$ is ill-defined. This happens at $\bar{L}_1 = \lambda/2$, and the corresponding circle (in the $\bar{L}_1, \bar{\phi}_1$ polar coordinates) is called the *singular circle*.

The normalized system is independent of $\bar{\phi}_2$, whereas the new coordinate $\bar{\phi}_1$ becomes constant in the unperturbed ($\omega = \frac{1}{2}$) linear flow. After this we return to complex coordinates by the usual transformation (2.1). In complex coordinates, transformation (2.9) reads

(2.10)                    $$z_1 = z_1' \frac{z_2'}{\bar{z}_2'}, \quad z_2 = z_2'\sqrt{1 - 2\frac{z_1'\bar{z}_1'}{z_2'\bar{z}_2'}},$$

where $z_i$ are the old complex coordinates. The singular circle in new coordinates is $z_1'\bar{z}_1' = \frac{1}{2}z_2'\bar{z}_2'$ (a codimension-1 subspace; it is a circle only after reduction).

**Proposition 2.9.** *(Planar reduction) After Birkhoff normalization and reduction to one degree of freedom, for the $1 : 2$ resonance ($a_1 \approx \frac{1}{2}$), the Hamiltonian (2.7) takes the form*

$$H^r = b_1\zeta_1 + b_2\zeta_2 + \frac{1}{b_3^3}\zeta_3 + b_4\zeta_1^2 + b_5\zeta_1\zeta_2 + b_6\zeta_2^2 +$$
$$b_7\zeta_1\zeta_3 + b_8\zeta_2\zeta_3 + b_9\zeta_1^3 + b_{10}\zeta_1^2\zeta_2 + b_{11}\zeta_1\zeta_2^2 + b_{12}\zeta_2^3 + b_{13}\zeta_3^2 + O(|z_i, \bar{z}_i|^y),$$

*where $\zeta_1 = x^2 + y^2$, $\zeta_2 = \lambda$, $\zeta_3 = x(x^2 + y^2 - \lambda)$, and the coefficients for the terms up to order four in the original phase coordinates are given by*

$$b_1 = \frac{1}{2} - a_1; \quad b_2 = a_1; \quad b_3 = \frac{1}{\sqrt[3]{a_2}};$$

$$b_4 = 8\left(\frac{a_2^2}{1 + 2a_1} - 3a_3 - a_4 + a_5\right); \quad b_5 = 8(6a_3 + a_4 - 2a_5);$$

$$b_6 = -8\left(\frac{a_2^2}{1 + 2a_1} + 3a_3 - a_5\right).$$

The special form for the coefficient of $\zeta_3$ was chosen for notational convenience in the formulas below. The coefficient $b_1$ vanishes at resonance ($a_1 = \frac{1}{2}$). It is considered to be small throughout, and is referred to as *detuning parameter*, since it measures the deviation from the resonant frequency.

**Remark 2.10.** *(Nondegeneracy conditions)*     From the expression of the $b_i$, the first nondegeneracy condition can be read off: $1 + 2a_1 \neq 0$. If we continue to normalize to higher orders, more conditions of the form $a_1 \neq p/q$ are found, where $p/q \in \mathbb{Q}$.

**Reduction to the central singularity** Because the system is planar now, we may use general ($\mathbb{Z}_2$-equivariant) planar transformation $\phi$ for further normalization, as opposed to just the symplectic ones. The resulting system is not dynamically conjugate but *equivalent* to the original, i.e., it is conjugate modulo state-dependent time-reparametrizations; for more remarks see [BCKV95, BHLV98, BLV98, BCKV93].

    The central singularity is defined by $b_1 = 0$ (resonance) and $\lambda = 0$. At this point the singularity still depends on the coefficients $b_2, b_3, \ldots$. In this section we bring the system at the central singularity in polynomial normal form $x(x^2 + y^2)$, which is independent of the $b_i$. This singularity is the $\mathbb{Z}_2$-invariant hyperbolic umbilic (see [PS78]), in Arnol'd's classification denoted by $D_4^+$.

    First, by a simple scaling transformation $\phi_0$ we can achieve that the Hamiltonian takes the form $H^{r'} := H^r_{b_1 = \lambda = 0} \circ \phi_0 = x(x^2 + y^2) + h.o.t.$

**Remark 2.11.** *(Nondegeneracy conditions)*     This is possible provided that the coefficient of the third-order terms (in $x, y$) are nonzero. This translates into the condition $a_2 \neq 0$ (see Proposition 2.9).

Next, we look for a near-identity planar morphism $\phi$ removing the *h.o.t.* from $H^{r'}$. This morphism should respect the $\mathbb{Z}_2$ symmetry $(x, y) \mapsto (x, -y)$. By a generalization of [Mar82, theorem III.5.2] incorporating the symmetry group, $H^{r'}$ is isomorphic to $x(x^2 + y^2)$; for details see Proposition 5.11 in Sect. 5.3.2.

    Armed with the knowledge that $\phi$ exists we set out to compute it, using the following iterative approach. Set $\phi_1(x) = x$, and assume that

(2.11) $$H^{r'} \circ \phi_k = x(x^2 + y^2) + O(|x, y|^{k+3})$$

for some $k$. To find $\phi_{k'}$ with $k' = k + 1$ we set $\phi_{k'} = \phi_k + \sum \alpha_i t_i$, where $\{t_i\}$ span the space of $\mathbb{Z}_2$-equivariant terms in $x, y$ of degree $k'$. This results in a

set of linear equations for the real numbers $\alpha_i$. By existence of the normalizing transformation, this set of equations is not over-determined, in fact it is usually under-determined.

**Proposition 2.12.** *There exists a coordinate transformation $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $H^c := H^r \circ \phi$ is of the form*

$$
\begin{aligned}
H^c = {} & (1 + c_1)x(x^2 + y^2) + c_2(x^2 + y^2) + c_3 x^4 + c_4 x^2 y^2 + c_5 y^4 + {} \\
& \lambda(d_1 x + d_2 x^2 + d_3 y^2 + d_4 x^3 + d_5 xy^2 + d_6 x^4 + d_7 x^2 y^2 + d_8 y^4) + {} \\
& \lambda^2(d_{12} x + d_{13} x^2 + d_{14} y^2) + h.o.t.,
\end{aligned}
$$

*where the h.o.t. are terms of order $O(|x, y|^5)$, $O(|c_i, \lambda|^3)$ and $O(|x, y|^3 |c, \lambda|^2)$. Here $d_i = d_i(b_j)$ are coefficients, and $c_i = c_i(b_j)$ are parameters, all of them rational expressions in the $b_i$, with the $c_i$ vanishing at $b_1 = 0$. Up to third order terms, the following is a suitable transformation $\phi$:*

$$
\phi : \begin{cases}
x \mapsto b_3 x - \frac{1}{3} b_3^5 b_4 x^2 + \left( \frac{1}{3} b_3^9 b_4^2 - \frac{1}{3} b_3^6 b_7 \right) x^3 - b_3^5 b_4 y^2 - \left( \frac{1}{3} b_3^9 b_4^2 + b_3^6 b_7 \right) xy^2, \\
y \mapsto b_3 y + \frac{2}{3} b_3^5 b_4 xy + \left( \frac{2}{3} b_3^9 b_4^2 - \frac{4}{3} b_3^6 b_7 x^2 y \right) x^2 y.
\end{cases}
$$

**Proof:**    This is a corollary to Proposition 5.11. The transformation $\phi$ was computed using the algorithm outlined above.                                                ■

We say that $H^c$ is in *central singularity reduced form*, i.e., at the central singularity $b_1 = \lambda = 0$ it reduces to the normal form $x(x^2 + y^2)$.

   We consider $\phi$ in the above proposition to be *fixed*, i.e., independent of parameters. It does depend on the coefficients $b_2, b_3, \dots$ however, since $H^r|_{\lambda = b_1 = 0}$ also depends on those. Some leading order parameters and coefficients of the deformation $H^c$ are:

$$
c_1 = -\frac{2}{3} b_1 b_3^6 b_4, \quad c_2 = b_1 b_3^2,
$$

$$
d_1 = -\frac{1}{b_3^2}, \quad d_2 = \frac{1}{3} b_3^2 (b_4 + 3 b_5), \quad d_3 = b_3^2 (b_4 + b_5).
$$

**Inducing the system from a versal deformation** From proposition 5.12 in Chap. 5 it follows that $H^u = x(x^2 + y^2) + u_1 x + u_2 y^2$ is a versal deformation of the hyperbolic umbilic $x(x^2 + y^2)$ (in the context of $\mathbb{Z}_2$-symmetric potential functions). Our system $H^c$ is now normalized to the extent that the algorithms of Sect. 7.2 may be applied. The result is as follows:

**Proposition 2.13.** *Let $H^c$ be a planar Hamiltonian depending on parameters $c_i$ and coefficients $d_i$, with central singularity $x(x^2 + y^2)$ at $c_0 = c_1 = \dots = 0$, symmetric under the $\mathbb{Z}_2$-action $(x, y) \mapsto (x, -y)$. A versal deformation of this central singularity is given by*

$$
H^u := x(x^2 + y^2) + u_1 x + u_2 y^2,
$$

*so that there exist $\phi$ and $\rho$ such that*

(2.12) $$H^c = H^u(\phi(x, y, c_i, d_i), \rho_1(c_i, d_i), \rho_2(c_i, d_i)),$$

*with $\phi(x, y, 0, d_i) = (x, y)$, $\rho(0, d_i) = (0, 0)$. To compute*

a) *$\phi$ modulo $O(|x, y|^A) + O(|c_i|^B)$, it is sufficient to know $H^c$ modulo $O(|x, y|^{A+2}) + O(|c_i|^B)$;*

b) *$\rho$ modulo $O(|c_i|^B)$, it is sufficient to know $H^c$ modulo $O(|c_i|^B) + O(|x, y|^3)$.*

*For system $H^c$ of Proposition 2.12, modulo $O(|c_i, \lambda|^3)$ terms, and writing $\lambda$ instead of $c_0$ again, the reparametrization $\rho$ reads*

$$u_1 = \left(-\frac{1}{3}c_2^2 + O(c_i^3)\right) + \lambda\left(d_1 - \frac{1}{3}c_1 d_1 - \frac{2}{3}c_2 d_2 + O(c_i^2)\right) +$$
$$\lambda^2\left(d_{12} - \frac{1}{3}d_2^2 - \frac{1}{3}d_1 d_4 + O(c_i)\right) + O(\lambda^3),$$

$$u_2 = \left(\frac{2}{3}c_2 - \frac{4}{9}c_2 c_1 + O(c_i^3)\right) +$$
$$\lambda\left(-\frac{1}{3}d_2 + d_3 + \frac{1}{9}c_3 d_1 - c_5 d_1 + \frac{2}{9}c_1 d_2 - \frac{2}{3}c_1 d_3 + \frac{5}{9}c_2 d_4 - c_2 d_5 + O(c_i^2)\right) +$$
$$\lambda^2\left(-\frac{1}{3}d_{13} + d_{14} + \frac{2}{9}d_2 d_4 + \frac{1}{3}d_3 d_4 - d_3 d_5 + \frac{1}{9}d_1 d_6 - d_1 d_8 + O(c_i)\right) + O(\lambda^3).$$

*The coordinate transformation $\phi$, modulo $O(|x, y|^3) + O(|c_i, \lambda|^2)$ terms, reads*

$$x \mapsto \frac{1}{3}c_2 + \frac{1}{3}d_2\lambda + \left(1 + \frac{1}{3}c_1 + \frac{1}{3}d_4\lambda\right)x + \left(\frac{1}{3}c_3 + \frac{1}{3}d_6\lambda\right)x^2 + (c_5 + d_8\lambda)y^2$$

$$y \mapsto \left(1 - \frac{1}{6}d_4\lambda + \frac{1}{2}d_5\lambda + \frac{1}{3}c_1\right)y + \left(\frac{1}{2}c_4 - \frac{3}{2}c_5 - \frac{1}{6}c_3 + \frac{1}{2}d_7\lambda - \frac{3}{2}d_8\lambda - \frac{1}{6}d_6\lambda\right)xy.$$

**Remark 2.14.** *(Relevant degree for $H^n$)*    To compute $\rho$ up to second order, it suffices to know $H^c$ modulo $O(|c_i, \lambda|^3) + O(|x, y|^3)$ terms. In turn, for this, $H^n$ modulo $O(|x, y|^7)$ terms suffices, as $\lambda$ is a quadratic polynomial on the phase space of $H^n$. To compute $\phi$ up to terms given in Proposition 2.13, it suffices to know $H^c$ modulo $O(|x, y|^5) + O(|c_i, \lambda|^2)$ terms, and again $H^n$ modulo $O(|x, y|^7)$ terms suffices.

**Remark 2.15.** *(Singular circle)*    In Sect. 2.3.3 the singular circle of $H^u$ is defined as the circular level set that touches the two saddle points arising for $u_1 < 0$ (see Fig. 2.4). By a topological argument, its pull-back by $\phi$ must coincide with the singular circle of $H^c$, defined as the set of singular points of (2.10). Up to the order in $x, y, c_i$ and $\lambda$ that $\phi$ and $\rho$ were computed, we verified that they indeed do.

**Proof** (of proposition 2.13): The first part is proved by inspecting the Kas and Schlessinger's algorithm described in section 7.2.2 and 7.2.3, and the division algorithm 6.14 of Sect. 6.3.7. The fact that $H^c$ is required up to order $A + 2$ in order to compute $\phi$ only up to degree $A$ is due to the first derivatives of the central singularity being of second degree. Similarly, in order to fix $\rho$, it is sufficient to compute $H^c$ up to degree 2 in $(x, y)$ as the deformation directions associated to $\rho_1$ and $\rho_2$ are of degree 2 or less (namely $x$ and $y^2$ respectively). A little computer algebra yields the second part. ∎

**BCKV normal form of $H^c$** The constructive proofs of Proposition 5.19 and Lemma 5.18 provide an algorithm for computing the BCKV normal form. Using the reparametrizations of Proposition 2.13, we choose for $\Upsilon$ the following:

$$\Upsilon(\lambda, c_i) = (u_2(\lambda, c_i) - u_2(\lambda, 0), c_1, u_2(0, c_i), c_3, c_4, \ldots),$$

which is invertible, and then $\tilde{\sigma}_2 := u_1 \circ \Upsilon^{-1}$. The result is the following:

**Theorem 2.16.** *(BCKV normal form:) The system $H^c$ of Proposition 2.12 is equivalent, modulo BCKV-restricted morphisms and reparametrizations, and modulo terms of order $O(|c_i, \lambda|^3)$, to*

$$H^B(x, y, \lambda, c_i) = x(x^2 + y^2) + y^2(\lambda + c_2) + x \times \left( -\frac{c_2^2}{3} + O(c_i^3) + \right.$$

$$\left( d_1 - \frac{c_1 d_1}{3} - \frac{2 c_2 d_2}{3} + O(c_i^2) \right) \beta \lambda + \left( d_{12} - \frac{d_2^2}{3} - \frac{d_1 d_4}{3} \right) \beta^2 \lambda^2 +$$

$$\left( \frac{d_1 d_{13}}{3} - d_1 d_{14} - \frac{2 d_1 d_2 d_4}{9} - \frac{d_1 d_3 d_4}{3} + d_1 d_3 d_5 - \frac{d_1^2 d_6}{9} + d_1^2 d_8 + O(c_i) \right) \beta^3 \lambda^2$$

$$\left. + O(\lambda^3) \quad \right),$$

*where $\beta = 9(-3d_2 + 9d_3 + c_1(2d_2 - 6d_3) + c_2(5d_4 - 9d_5) + c_3 d_1 - 9c_5 d_1)^{-1} + O(c_i^2)$ The coefficient of $x$ expressed in the $a_i$ reads:*

$$\tilde{\sigma}_2 = -\frac{1}{48\alpha}(1 - 2a_1)^2 + O((1 - 2a_1)^3) +$$

$$\lambda \left( \frac{-9a_2^2}{2\alpha\delta} + \frac{1}{4\alpha\delta^2} \left( 2a_2^4 - 144a_3^2 + 5a_4^2 - 6a_3(a_4 - 16a_5) + 2a_4 a_5 + \right. \right.$$

$$\left. -16a_5^2 + a_2^2(-30a_3 + a_4 + 26a_5) + 3a_2 a_6 \right)(1 - 2a_1) + O((1 - 2a_1)^2) \left. \right)$$

$$+ O(\lambda^2),$$

*where $\alpha = \sqrt[3]{2a_2^2}$ and $\delta = 2a_2^2 + 6a_3 - a_4 - 2a_5$.*

**Remark 2.17.** *(Nondegeneracy conditions)* The BCKV normal form is only well-defined if $\beta$ is, i.e., if $d_2 - 3d_3 \neq 0$. This translates into $a_2 \neq 0$ and $a_2^2 \neq (1 + 2a_1)(3a_3 + a_4 - a_5)$. For the spring-pendulum the first condition is trivial, the second one is not.

### 2.3.3 Dynamics and bifurcations

**The planar system** In this section we regard the system as a *planar* system depending on the detuning parameter $1 - 2a_1$ and distinguished parameter $\lambda$. This gives an integrable approximation to the dynamics of the iso-energetic, or equivalently[3] iso-$\lambda$, Poincaré map. In Sect. 2.2.6 we arrived at the planar versal normal form

$$H^u(x, y, u_1, u_2) = x(x^2 + y^2) + u_1 x + u_2 y^2.$$

The level sets of this deformation are organized by a special level set that factorizes into first- and second-degree algebraic curves crossing in the point $(x, y) = (-u_2, \pm\sqrt{-u_1 - 3u_2^2})$. These curves are given by $x = -u_2$ and $(x - \frac{1}{2}u_2)^2 + y^2 = -u_1 - \frac{3}{4}u_2^2$. For parameter values for which these curves cross, the second equation defines a circle separating compact level curves from unbounded ones. This circle is referred to as the *singular circle*, the reason being that it is the image of singular points of the transformation (2.10).

The deformation has the critical points $(x, y) = (\pm\sqrt{-(1/3)u_1}, 0)$ and $(-u_2, \pm\sqrt{-u_1 - 3u_2^2}$. Saddle-center and and Hamiltonian pitchfork bifurcations occur along the curves $u_1 = 0$ and $u_1 + 3u_2^2 = 0$, respectively (see figure 2.4). Plugging in the reparametrizations found in Proposition 2.13, Sect. 2.3.2, yields implicit equations for these bifurcation curves in the $(\lambda, 1 - 2a_1)$-plane. For practical reasons we choose to solve for $\lambda$ in terms of $1 - 2a_1$. The result is:

**Proposition 2.18.** *In the reduced system $H^c$ of Proposition 2.12, saddle-center and Hamiltonian pitchfork bifurcations respectively occur along the following curves in parameter space:*

(2.13)    $(u_1 = 0 :)$
$$\lambda = -\frac{(1 - 2a_1)^2 \left(4(5 + 8a_1)a_2^2 + (4a_1^2 - 1)(24a_3 + 5a_4 - 8a_5)\right)}{3456(1 + 2a_1)a_2^4} +$$
$$+ O((1 - 2a_1)^3),$$

(2.14)    $(u_1 + 3u_2^2 = 0 :)$
$$\lambda = \frac{(1 - 2a_1)^2}{64a_2^2} + \frac{(a_2^2 - a_4)(1 - 2a_1)^3}{128a_2^4} + O((1 - 2a_1)^4).$$

**Remark 2.19.** *(Phantom bifurcation)* The parameter $\lambda$ is nonnegative, and close to resonance ($a_1 \approx \frac{1}{2}$) the solution (2.13) is negative. In the system $H^0$, therefore, the corresponding bifurcation does not occur. This conclusion also follows from the observation that at the bifurcation (2.13) the singular circle disappears, whereas $H^r$ exhibits this singularity for all parameter values (as long as $\lambda > 0$).

The second solution does define a bifurcation, however. We continue with its description.

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[3] see [BCKV93, Sect. 4.2]
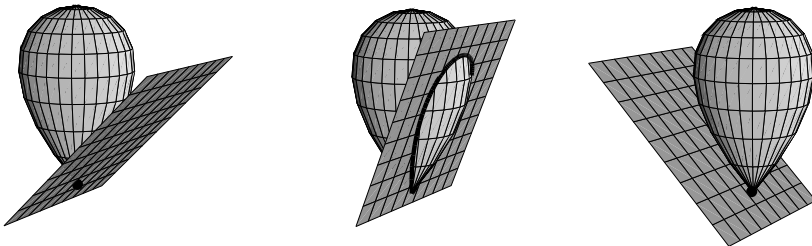
**Fig. 2.2** Qualitative picture of intersections of the reduced phase space with level sets of $H^n$ (here depicted as planes, which is the limiting situation for small energies, i.e., small balloons) through the singular point in $\rho_2, \psi, \chi$-space, for three values of the detuning parameter.

**Bifurcations and dynamical implications** First we discuss the bifurcation of the reduced system $H^c$ in the plane. If we let $a_1$ deviate sufficiently far from the resonant value $\frac{1}{2}$, the corresponding points in the $(u_1, u_2)$-plane in Fig. 2.4 will trace out a line that crosses the parabola twice, as $u_1$ is always negative.

Assume the parabola is crossed from below. Then at first the system has one maximum inside the singular circle, and a saddle point outside it. After the first Hamiltonian pitchfork bifurcation, two saddle points have formed on the singular circle, together with a minimum inside, with no critical points outside. The two saddle points have a heteroclinic connection because of the $\mathbb{Z}_2$ symmetry.

The second bifurcation destroys the maximum, leaving only a minimum inside the circle, and again a saddle outside of it.

**Topological remarks** *A priori* the spring-pendulum lives on the fixed-energy submanifold in $\mathbb{R}^4$, in our case $\mathbb{S}^3$. This sphere is homeomorphic to $\mathbb{D}^2 \times \mathbb{S}^1$ modulo an identification on $\partial\mathbb{D}^2 \times \mathbb{S}^1$.

The normalized Hamiltonian $H^n$ on $\mathbb{S}^3$ has an $\mathbb{S}^1$-symmetry. There is one $\mathbb{S}^1$-orbit on which points have stabilizer (or isotropy subgroup) $\mathbb{Z}_2$, see Proposition 2.2; all other points have trivial stabilizer. As an illustration of this topology, consider the following model of $\mathbb{S}^3$ in the form of a map $\mathbb{D}^2 \times \mathbb{S}^1 \to \mathbb{S}^3$ given by

$$(x, y, \phi) \mapsto (\sqrt{1 - r^2} \cos\phi, \sqrt{1 - r^2} \sin\phi, x\cos 2\phi - y\sin 2\phi, x\sin 2\phi + y\cos 2\phi).$$

(Here $r^2 = x^2 + y^2$ and $\mathbb{D}^2 = \{r^2 \leq 1\}$.) This map is surjective, and injective on the interior of its domain. It provides a correspondence between $\mathbb{S}^1$-invariant functions on $\mathbb{S}^3$ and functions on $\mathbb{D}^2$ that are constant on $\partial\mathbb{D}^2$, that is, functions on $\mathbb{S}^2$. This conclusion holds for any[4] nondegenerate $\mathbb{S}^1$-action on $\mathbb{S}^3$, and justifies viewing the bifurcations described above on $\mathbb{S}^2$. This is the content of the remark

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[4] There exist mutually non-homotopic, even non-homeomorphic, nondegenerate $\mathbb{S}^1$-actions on $\mathbb{S}^3$. This is in contrast to the case of $\mathbb{S}^2$: up to homotopy between actions there exists only 1 nondegenerate $\mathbb{S}^1$-action on $\mathbb{S}^2$. Here, a *nondegenerate* action is an action that maps nonidentity elements to nonidentity elements.

in Sect. 2.2, where we said that $\mathbb{S}^3$ divided out by an $\mathbb{S}^1$-action gives $\mathbb{S}^2$. After this division, the singular circle collapses to a single point on $\mathbb{S}^2$, and is referred to as the *pole*.

More precise information can be obtained by looking at the algebra. In general, the normalized Hamiltonian can be written as $H^n = f(\rho_1, \rho_2, \psi, \chi)$ where $\rho_1 = z_1 \bar{z}_1$, $\rho_2 = z_2 \bar{z}_2$, $\psi = z_1 \bar{z}_2^2 + \bar{z}_1 z_2^2$ and $\chi = (z_1 \bar{z}_2^2 - \bar{z}_1 z_2^2)/\mathrm{i}$ are invariants generating the $\mathbb{S}^1$-invariant functions (i.e., they form a Hilbert basis). Note that for time-reversible Hamiltonians the Birkhoff normal form is independent of $\chi$, which explains why $\chi$ does not appear in Proposition 2.2. The invariants satisfy the relation $\rho_1 \rho_2^2 = (\psi^2 + \chi^2)/4$. Moreover, reality conditions imply $\rho_1 \geq 0$ and $\rho_2 \geq 0$. The quadratic part $H_2$ is an integral of $H^n$, and without loss of generality we may reduce to $H_2 = 2\rho_1 + \rho_2 = \epsilon$, where $\epsilon$ is some small positive number. Then, the relation between the invariants defines a 2-dimensional manifold in the real space $\mathbb{R}^3 \ni (\rho_1, \psi, \chi)$, the *reduced phase space*, namely $(\epsilon - 2\rho_1)^2 \rho_1 = \frac{1}{4}(\psi^2 + \chi^2)$. Topologically it is a sphere in this real space, but it has a cone-like singularity at $\rho_2 = 0$ (see figure 2.2). This singularity has dynamical significance: it is always a fixed point.

We now interpret the bifurcations on this (topological) sphere. Levels of $H^n$ are surfaces in $\mathbb{R}^3 \ni (\rho_1, \psi, \chi)$ and intersect the reduced phase space in a curve; again, see Fig. 2.2. For small energies the level sets of $H^n$ will be approximately planar on the scale of the reduced phase space ('balloon'), and the intersection curves are smooth circles, except for the level of the pole.

As in the previous section, suppose we traverse the $(u_1, u_2)$-plane on the left of the $u_2$-axis crossing the parabola of Hamiltonian pitchfork bifurcations twice. First, the Hamiltonian has one maximum somewhere on the sphere, and a minimum at the pole. The homoclinic[5] connection appearing in the planar normal form after the first bifurcation (see Fig. 2.5) corresponds to a level curve passing through the pole. In this situation, the pole is no longer a minimum. At the second bifurcation the homoclinic connection disappears, implying that the pole is an extremum again, now a maximum.

**Dynamics of the spring-pendulum** At the pole, $\rho_2 = 0$ and $\rho_1$ is a maximum, corresponding to the pendulum oscillating vertically without swinging ($x_2 = 0$, see figure 2.1). Points on this periodic orbit have nontrivial stabilizer, $\mathbb{Z}_2$; in other words, the period of this orbit is (to first order) half that of the other periodic orbits. In the literature it is referred to as the *short periodic orbit*. Since the pole always exists and is always a fixed point, the short periodic orbit always exists. Far from resonance (outside the parabola of Fig. 2.4) the pole is an extremum of the Hamiltonian, so that this orbit is stable. It is unstable close to resonance.

In that situation, the spring-pendulum exhibits two stable periodic trajectories (the *long periodic orbits*), corresponding to the two extrema existing on the sphere (see the center picture in Fig. 2.2). Physically, the lower mass traces

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[5] The connection is not heteroclinic; see e.g. Fig. 2.2, center picture. In Fig. 2.5 the pole is blown up to a circle, so that there the connection *seems* to be heteroclinic.

out ∪-shaped and ∩-shaped paths, respectively. As the system moves away from resonance, one of these paths gets wider while the other gets narrower, until at bifurcation, the narrow one coincides with the short periodic orbit. After the bifurcation only one long periodic orbit remains (which is stable); the short periodic orbit has also become stable.



$a_1 = 0.40$          $a_1 = 0.50$

$a_1 = 0.68$          $a_1 = 0.70$
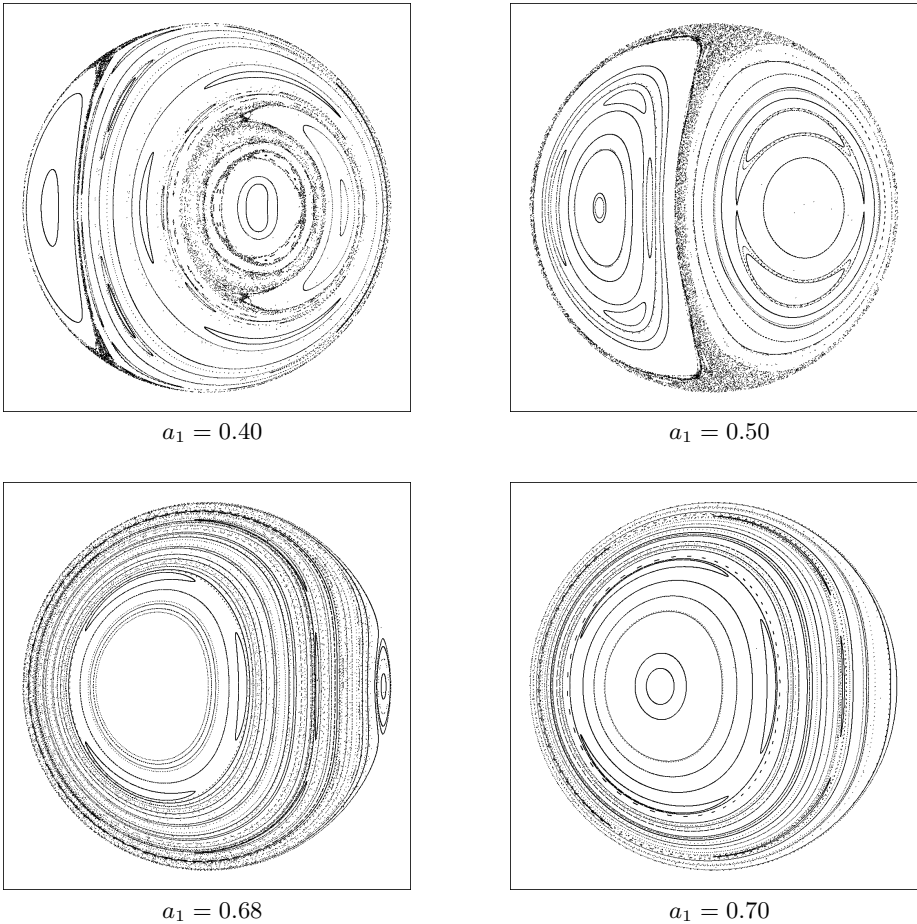
**Fig. 2.3**   Orbits of iso-energetic Poincaré map of $H^0$ near $1:2$ resonance, for various values of detuning parameter $1 - 2a_1$ (see Proposition 2.9). For these pictures we used $a_2 = 0.07$, $a_3 = 0.001$, other coefficients zero and $H^0 = 0.2$.

**Comparison with numerical simulations**   To check the results above, we integrated $H^0$ numerically, and plotted the iso-energetic Poincaré section $\phi_2 = 0$

for varying values of the energy and detuning parameter $a_1$. The resulting pictures, shown in figure 2.3, are similar to those found by computation and shown in Fig. 2.5. The differences (chaotic regions, subharmonics) are caused by the flat perturbation between the normalized $H^0$ and $H^n$, destroying integrability in $H^0$; see also remarks in Sect. 2.1.

To check (2.14), we located some bifurcation points, by varying the detuning parameter $a_1$ for fixed $H$, $a_2$ and $a_3$. Other $a_i$ were set to zero. The results are given in table 2.2. To compute $\lambda$ we used equation (2.8); we see that for these values of the energy, $\lambda = 2H$ to good approximation. The final column gives the bifurcation value of $\lambda$ given by (2.14) in each situation. The agreement with the measured value of $\lambda$ is very good, especially for small $H$, as expected.

**Bifurcation diagrams and comments** For the spring-pendulum in 1 : 2 resonance, the iso-energetic Poincaré map is shown in Fig. 2.3 for various values of the detuning parameter. Its main bifurcation occurs at $a_1 \approx 0.69$, where an elliptic equilibrium disappears. To find analytic expressions for this bifurcation value, first the planar normal form is computed, which turns out to be
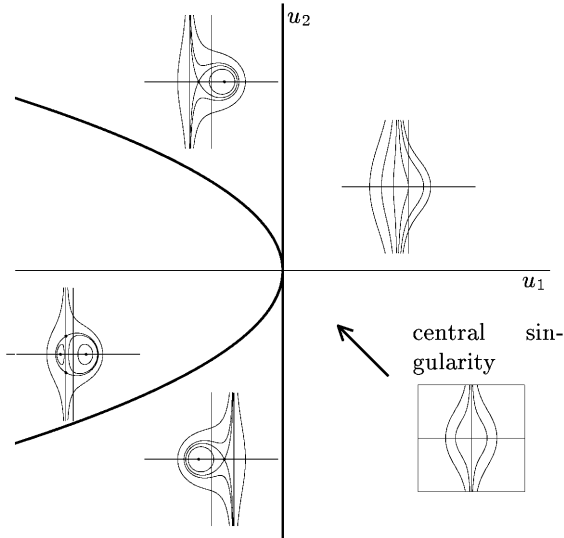
$$x(x^2 + y^2) + u_1 x + u_2 y^2.$$

Here $u_i$ are the unfolding parameters. Its bifurcation diagram is depicted in Fig. 2.4. The curve $u_1 = 0$ corresponds to a saddle-center bifurcation, while $u_1 + 3u_2^2 = 0$ corresponds to a Hamiltonian pitchfork bifurcation. Using the reparametrizations, it turns out that the bifurcation curve $u_1 = 0$ does not correspond to bifurcations in the original system. The other curve does, however. Fig. 2.5 gives the bifurcation diagram in original parameters. Grey areas are portions of the parameter- or phase-plane that do not correspond to a configuration of the original system.

**Table 2.2** Comparison of bifurcation values obtained numerically, and by means of the pull-back of the bifurcation curve, (2.14)

| $H$ | $a_1$ | $a_2$ | $a_3$ | $\lambda_{\text{measured}}$ | $\lambda_{\text{predicted}}$ |
|---|---|---|---|---|---|
| 0.01 | 0.5385 | 0.07 | 0.001 | 0.020 | 0.018 |
| 0.01 | 0.463 | 0.07 | 0.001 | 0.020 | 0.018 |
| 0.001 | 0.51255 | 0.07 | 0.001 | 0.00200 | 0.00198 |
| 0.001 | 0.4876 | 0.07 | 0.001 | 0.00200 | 0.00199 |
| 0.001 | 0.5365 | 0.2 | 0.001 | 0.00200 | 0.00201 |
| 0.001 | 0.465 | 0.2 | 0.001 | 0.00200 | 0.00198 |

**Fig. 2.4** Bifurcation diagram of the $\mathbb{Z}_2$-invariant hyperbolic umbilic $x(x^2 + y^2) + u_1 x + u_2 y^2$. Across the bifurcation lines saddle-center bifurcations occur. Across the parabola $u_1 + 3u_2^2 = 0$ a Hamiltonian pitchfork bifurcation occurs due to $\mathbb{Z}_2$ symmetry.
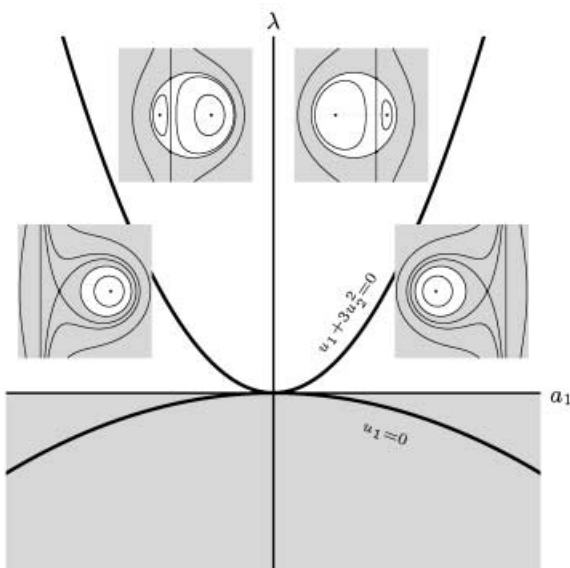


**Fig. 2.5** Bifurcation diagram of the planar reduced system $H^r$, obtained from pulling back the bifurcation diagram of Fig. 2.4 to original coordinates. Grey areas denote portions of phase- or parameter-space that do not correspond to phase points of the original system.