

University of Groningen

## Input-to-State Stabilizing MPC for Neutrally Stable Linear Systems subject to Input Constraints

Kim, Jung-Su; Yoon, Tae-Woong; Jadbabaie, Ali; Persis, Claudio De

*Published in:*  
Proceedings of the 43rd IEEE Conference on Decision and Control

**IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.**

*Document Version*  
Publisher's PDF, also known as Version of record

*Publication date:*  
2004

[Link to publication in University of Groningen/UMCG research database](#)

*Citation for published version (APA):*

Kim, J-S., Yoon, T-W., Jadbabaie, A., & Persis, C. D. (2004). Input-to-State Stabilizing MPC for Neutrally Stable Linear Systems subject to Input Constraints. In *Proceedings of the 43rd IEEE Conference on Decision and Control* (Vol. 5, pp. 5041-5046). University of Groningen, Research Institute of Technology and Management.

### Copyright

Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license. More information can be found on the University of Groningen website: <https://www.rug.nl/library/open-access/self-archiving-pure/taverne-amendment>.

### Take-down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Downloaded from the University of Groningen/UMCG research database (Pure): <http://www.rug.nl/research/portal>. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.

# Input-to-State Stabilizing MPC for Neutrally Stable Linear Systems subject to Input Constraints

Jung-Su Kim, Tae-Woong Yoon, Ali Jadbabaie and Claudio De Persis

**Abstract**—MPC(Model Predictive Control) is representative of control methods which are able to handle physical constraints. Closed-loop stability can therefore be ensured only locally in the presence of constraints of this type. However, if the system is neutrally stable, and if the constraints are imposed only on the input, global asymptotic stability can be obtained. A globally stabilizing finite-horizon MPC has lately been suggested for the neutrally stable systems using a non-quadratic terminal cost which consists of cubic as well as quadratic functions of the state. In this paper, an input-to-state-stabilizing MPC is proposed for the discrete-time input-constrained neutrally stable system using a non-quadratic terminal cost which is similar to that used in the global stabilizing MPC, provided that the external disturbance is sufficiently small. The proposed MPC algorithm is also coded using an SQP (Sequential Quadratic Programming) algorithm, and simulation results are given to show the effectiveness of the method.

## I. INTRODUCTION

MPC or model predictive control is a receding horizon strategy, where the control is computed via an optimization procedure at every sampling instant. It is therefore possible to handle physical constraints on the input and/or state variables through the optimization [18]. Over the last decade, there have been many stability results on constrained MPC. Moreover, explicit solutions to constrained MPC are proposed recently [19], [4]. These results reduce on-line computational burden regarded as a main drawback of MPC and extend the applicability of MPC to faster plants as in electrical applications.

Particular attention is paid in this paper to input-constrained systems as all real processes are subject to actuator saturation. Generally, it is not possible to stabilize input-constrained plants globally. However, if the unconstrained part of the system is neutrally stable<sup>1</sup>, then global stabilization can be achieved. A typical example is the so-called small gain control [20], [3], [5]; it is noted that the Lyapunov functions used for stability analysis are non-quadratic functions containing cubic as well as quadratic terms.

This work was not supported by any organization

Jung-Su Kim and Tae-Woong Yoon are with the Department of Electrical Engineering, Korea University, KOREA jsk, twy@cello.korea.ac.kr

Ali Jadbabaie is with the Department of Electrical and Systems Engineering University of Pennsylvania, USA jadbabai@seas.upenn.edu

Claudio De Persis is with the Dipartimento di Informatica e Sistemistica "A. Ruberti" Università di Roma "La Sapienza", depersis@dis.uniroma1.it

<sup>1</sup>All eigenvalues lie within the unit circle and those on the unit circle are simple.

Global stabilization of input-constrained neutrally stable systems is also possible via MPC; see e.g. [6]. As in [6], use of infinite horizons is generally thought to be inevitable. However, infinite horizon MPC can cause trouble in practice. For implementation, the optimization problem should be reformulated as a finite horizon MPC with a variable horizon, and it is not possible to predetermine a finite upper bound on the horizon. It is only fairly recently that globally stabilizing finite horizon MPC has been proposed for both continuous-time and discrete-time neutrally stable systems [12], [21], respectively. This late achievement is based on two observations; firstly, the stability of an MPC system is mostly proved by showing that the terminal cost is a control Lyapunov function [18], [11]. Secondly, the global stabilization of an input-constrained neutrally stable system can be achieved by using a non-quadratic Lyapunov function as mentioned above. By making use of these two facts, a new finite horizon MPC has been suggested in [12], [21], where a non-quadratic Lyapunov function as in [20], [3], [5] is employed as the terminal cost, thereby guaranteeing the closed-loop stability.

Recently, ISS (Input to State Stability) and its integral variant, iISS(integral Input to State Stability) have become important concepts in nonlinear systems analysis and design [13], [1], [2]. ISS and iISS imply that the nominal system is globally stable, and that the closed-loop system is robust against a bounded disturbance and a disturbance with finite energy, respectively. There have been some reports on ISS properties of MPC [17], [15], [14]. However, these results are limited in that plants are assumed to be open-loop stable in [14], and only local properties are obtained in [17], [15].

On the other hand, we derive in this paper a global ISS characterization for an input-constrained neutrally stable discrete-time plant with a restriction on the external disturbance, by showing that the optimal cost with a non-quadratic terminal cost is an ISS Lyapunov function. The proposed MPC algorithm is coded using an SQP (Sequential Quadratic Programming) algorithm, and simulation results are given to show the effectiveness of the method.

## II. AN OVERVIEW OF MPC

Following [18], a brief summary on MPC is given in this section. Consider a discrete-time system described by

$$x(k+1) = Ax(k) + Bu(k)$$

where  $x(k) \in \mathbb{R}^n$  and  $u(k) \in \mathbb{R}^m$  are the state and input, and  $(A, B)$  is assumed to be controllable. Defining

$$\mathbf{u}(k) = \{u(k|k), u(k+1|k), \dots, u(k+N-1|k)\},$$

the MPC law is obtained by minimizing with respect to  $\mathbf{u}(k)$

$$J_N(x(k), \mathbf{u}(k)) = \sum_{i=0}^{N-1} l(x(k+i|k), u(k+i|k)) + V(x(k+N|k))$$

subject to

$$\begin{aligned} x(k+i+1|k) &= Ax(k+i|k) + Bu(k+i|k), \quad x(k|k) = x(k) \\ x(k+i+1|k) &\in \mathcal{X}, \quad u(k+i|k) \in \mathcal{U}, \quad i \in [0, N-1] \\ x(k+N|k) &\in \mathcal{X}_f \subset \mathcal{X} \end{aligned}$$

where

$$l(x, u) = x^T Q x + u^T R u \quad (1)$$

with  $Q$  and  $R$  being positive definite,  $V(x(k+N|k))$  is the terminal cost, the sets  $\mathcal{U}$ ,  $\mathcal{X}$  represent the input and state constraints, and  $x(k+N|k) \in \mathcal{X}_f$  is the artificial terminal constraint employed for stability guarantees. Note that  $V(x)$  and  $\mathcal{X}_f$  are chosen such that  $V(x)$  is a control Lyapunov function in  $\mathcal{X}_f$ . The entire procedure is repeated at each sampling instant, i.e.

$$u(k) = u^*(k|k)$$

where  $u^*(k+i|k)$  is the optimal value of  $u(k+i|k)$ . The stability properties of the resulting closed-loop are summarized below.

**Theorem 1:** [18] For some local controller  $k_f : \mathcal{X}_f \rightarrow \mathbb{R}^m$ , suppose the following:

- A1.**  $\mathcal{X}_f$  is closed and  $0 \in \mathcal{X}_f \subset \mathcal{X}$ ;
- A2.**  $k_f(x) \in \mathcal{U}$ ,  $\forall x \in \mathcal{X}_f$  (feasibility);
- A3.**  $Ax + Bk_f(x) \in \mathcal{X}_f$ ,  $\forall x \in \mathcal{X}_f$  (invariance);
- A4.**  $V(Ax + Bk_f(x)) - V(x) + l(x, k_f(x)) \leq 0$ ,  $\forall x \in \mathcal{X}_f$ .

Then the optimization problem is guaranteed to be feasible for all  $k \geq 0$  as long as the initial state  $x(0)$  can be steerable to  $\mathcal{X}_f$  in  $N$  steps or less (i.e. the problem is feasible initially). In addition, the optimal cost  $J_N^*(k)$  at time  $k$ , i.e. the minimal value of  $J_N(x(k), \mathbf{u}(k))$  satisfies

$$J_N^*(k+1) - J_N^*(k) + l(x(k), u^*(k|k)) \leq 0,$$

thereby ensuring asymptotic stability of the closed-loop.

*Outline of proof:* Suppose that

$$\mathbf{u}^*(k) = \{u^*(k|k), \dots, u^*(k+N-1|k)\}$$

is the optimal (and thus feasible)  $\mathbf{u}(k)$  obtained at time  $k$ , and consider

$$\tilde{\mathbf{u}}(k+1) = \{u^*(k+1|k), \dots, u^*(k+N-1|k), k_f(x^*(k+N|k))\}$$

where

$$x^*(k+i+1|k) = Ax^*(k+i|k) + Bu^*(k+i|k), \quad i \in [0, N-1].$$

Note that  $x^*(k+1|k) = x(k+1)$  as  $u^*(k|k) = u(k)$  and that  $x^*(k+N|k) \in \mathcal{X}_f$ . It then follows from **A1** and **A2** that  $\tilde{\mathbf{u}}(k+1)$  is also feasible at time  $k+1$ , i.e. the feasibility

of the problem at time  $k+1$  is guaranteed by the feasibility at time  $k$ . Also from assumption **A4** and

$$J_N^*(k+1) = J_N(x(k+1), \mathbf{u}^*(k+1)) \leq J_N(x(k+1), \tilde{\mathbf{u}}(k+1)),$$

we have

$$\begin{aligned} J_N^*(k+1) &\leq J_N(x(k+1), \tilde{\mathbf{u}}(k+1)) \\ &= J_N^*(k) - l(x(k), u^*(k|k)) + l(x^*(k+N|k), k_f(x^*(k+N|k))) \\ &\quad + V(Ax^*(k+N|k) + Bk_f(x^*(k+N|k))) - V(x^*(k+N|k)) \\ &\leq J_N^*(k) - l(x(k), u^*(k|k)). \end{aligned} \quad (2)$$

This completes the proof.  $\blacksquare$

Theorem 1 shows that if  $\mathcal{X}_f$  is a feasible and invariant set for  $x(k+1) = Ax(k) + Bk_f(x(k))$ , MPC is stabilizing and its domain of attraction is the set of the initial state vectors which can be steerable to  $\mathcal{X}_f$  in  $N$  steps or less. An interesting consequence is that the MPC can be globally stabilizing if  $k_f(x)$  is found such that  $\mathcal{X}_f = \mathbb{R}^n$ . This is in fact possible if the unconstrained plant is neutrally stable and if constraints are imposed only on the input, i.e.  $\mathcal{X} = \mathbb{R}^n$ , as discussed on small gain control in the introduction.

### III. PRELIMINARIES

Some previous results on stabilization of neutrally stable systems subject to input saturations is introduced.

Consider the following neutrally stable plant

$$x(k+1) = Ax(k) + B \text{sat}(u(k)) \quad (3)$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ , and it is assumed as in the previous section that  $(A, B)$  is controllable and all the eigenvalues of  $A$  lie within and on the unit circle with those on the unit circle being simple. The saturation function  $\text{sat}(\cdot)$  is defined as follows:

$$\text{sat}(u) = [\text{sat}(u_1) \text{sat}(u_2) \dots \text{sat}(u_m)]^T$$

where

$$\text{sat}(u_i) = \begin{cases} u_{\max}, & u_i > u_{\max} \\ u_i, & |u_i| \leq u_{\max} \\ -u_{\max}, & u_i < -u_{\max} \end{cases}$$

and  $u_{\max}$  is a positive constant. Then for any  $L_u$  satisfying  $L_u u_{\max} > 1$ , we have

$$\|\text{sat}(u) - u\| \leq L_u u^T \text{sat}(u). \quad (4)$$

It also follows from the neutral stability that there exists a positive definite matrix  $M_c$  satisfying

$$A^T M_c A - M_c \leq 0. \quad (5)$$

Now globally stabilizing small gain control is given by

$$u(k) = -\kappa B^T M_c A x(k) \quad (6)$$

where  $\kappa (> 0)$  satisfies

$$\kappa B^T M_c B < I. \quad (7)$$

This control law is similar to those in [20], [3], [7]. It can then be shown that there exists a positive definite matrix  $M_q$  such that

$$(A - \kappa BB^T M_c A)^T M_q (A - \kappa BB^T M_c A) - M_q = -I. \quad (8)$$

The following two theorems state global asymptotic stability of the closed-loop resulting from the small gain control and MPC, respectively.

**Theorem 2:** [21] For the closed-loop system (3) and (6), there exists a Lyapunov function  $W(x)$  such that

$$\begin{aligned} W(x(k)) &= W_q(x(k)) + \lambda W_c(x(k)) \\ &= x^T(k) M_q x(k) + \lambda (x^T(k) M_c x(k))^{\frac{3}{2}} \\ W(x(k+1)) - W(x(k)) &\leq -\|x(k)\|^2 \end{aligned} \quad (9)$$

where

$$\lambda = \frac{2\kappa L_u \sigma_{\max}(A_c^T M_q B)}{\sqrt{\lambda_{\min}(M_c)}} \quad (10)$$

with  $\sigma_{\max}$  and  $\lambda_{\min}$  denoting the maximum singular value and the minimum eigenvalue, respectively.

**Theorem 3:** [21] Consider the neutrally stable plant (3) and the following MPC law:

$$\text{minimize } J_N(x(k), \mathbf{u}(k))$$

subject to

$$\begin{aligned} x(k+i+1|k) &= Ax(k+i|k) + Bu(k+i|k), \quad x(k|k) = x(k) \\ u(k+i|k) &= \text{sat}(u(k+i|k)), \quad i \in [0, N-1], \end{aligned}$$

where

$$V(x(k+N|k)) = \Theta W(x(k+N|k)),$$

and  $l(x, u)$  and  $W(x)$  are defined as in equations (1) and (9), respectively. Then, given any positive integer  $N$ , the closed-loop is globally asymptotically stable for any positive positive  $\Theta$  satisfying

$$\Theta \geq \Theta_0 := \lambda_{\max}(Q + \kappa^2 A^T M_c B R B^T M_c A).$$

#### IV. INPUT-TO-STATE STABILIZING MPC

In this section, an input-to-state stabilizing MPC is proposed such that the closed-loop system resulting from the proposed MPC is ISS under the assumption that the  $\infty$ -norm of the disturbance is sufficiently small. To this end, firstly ISS of the small gain control is obtained, which is then used to derive an ISS characterization of the proposed MPC. To begin with, the definition of ISS and a theorem regarding the Lyapunov characterization of ISS are stated below, before presenting our main results.

Consider the following discrete-time nonlinear system with an external disturbance

$$x(k+1) = f(x(k), w(k)). \quad (11)$$

**Definition 1:** [13] The system (11) is ISS if there exist  $\alpha, \gamma \in \mathcal{K}_\infty$  and  $\beta \in \mathcal{K}\mathcal{L}$  such that

$$\alpha(\|x(k)\|) \leq \beta(\|x(0)\|, k) + \gamma(\|w(k)\|). \quad (12)$$

**Theorem 4:** [13] The system (11) is ISS if and only if there exists an ISS Lyapunov function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  such that for  $\alpha_1, \alpha_2, \alpha_3, \sigma \in \mathcal{K}_\infty$ ,  $V$  satisfies

$$\alpha_1(\|x(k)\|) \leq V(x(k)) \leq \alpha_2(\|x(k)\|)$$

and

$$V(x(k+1)) - V(x(k)) \leq -\alpha_3(\|x(k)\|) + \sigma(\|w(k)\|). \quad (13)$$

**Remark 1:** In [21], a global stabilizing MPC is proposed by using the terminal cost function as the non-quadratic function (9) which is a global Lyapunov function. In the main result, a very similar non-quadratic function is employed as the terminal cost function to devise an input-to-state stabilizing MPC.

#### A. ISS of the small gain control

In this subsection, an ISS characterization of the small gain control is derived. Consider the following discrete-time neutrally stable plant with input saturation

$$x(k+1) = Ax(k) + B \text{sat}(u(k)) + w(k) \quad (14)$$

where  $w$  is an external disturbance. The next theorem shows that the small gain control (6) stabilizes (14) in the sense of ISS with a restriction on  $w(k)$ .

**Theorem 5:** For the closed-loop system (14) and (6), there exists an ISS Lyapunov function  $W(x)$  and a positive  $\delta$  such that for  $\|w\| \leq \delta$ ,

$$\begin{aligned} W(x(k)) &= W_q(x(k)) + \lambda W_c(x(k)) \\ &= x^T(k) M_q x(k) + \lambda (x^T(k) M_c x(k))^{\frac{3}{2}} \end{aligned} \quad (15)$$

$$\begin{aligned} W(x(k+1)) - W(x(k)) &\leq -\epsilon_W \|x(k)\|^2 + \beta_{w1} \|w(k)\| + \beta_{w2} \|w(k)\|^2 \end{aligned} \quad (16)$$

where  $\lambda$  is defined as in (10), and  $\epsilon_W$ ,  $\beta_{w1}$  and  $\beta_{w2}$  are some positive numbers.

*Proof:* Proceeding as in [20], [10], we obtain the differences of the quadratic and cubic terms as follows:

$$\begin{aligned} W_q(x(k+1)) - W_q(x(k)) &\leq -\|x\|^2 + a_1 2\|x\| L_u u^T \text{sat}(u) \\ &\quad + \epsilon_{xq} \|x\|^2 + \beta_{wq2} \|w\|^2 + \beta_{wq1} \|w\| \end{aligned} \quad (17)$$

$$\begin{aligned} W_c(x(k+1)) - W_c(x(k)) &\leq 2c_1 c_2 \|w\| \|x\|^2 + \epsilon_{xc} \|x\|^2 \\ &\quad + \beta_{wc1} \|w\| + \beta_{wc2} \|w\|^2 - \frac{a_2}{\kappa} \|x\| u^T \text{sat}(u) \end{aligned} \quad (18)$$

where  $\epsilon_{xq}$  and  $\epsilon_{xc}$  are arbitrarily small positive constants,  $\beta_{wq1}, \beta_{wq2}, \beta_{wc1}$  and  $\beta_{wc2}$  are positive,  $a_1 = \sigma_{\max}(A^T M_q B)$ ,  $a_2 = \sqrt{\lambda_{\min}(M_c)}$ ,  $c_1 = 2\sigma_{\max}(A^T M_c)$ , and  $c_2 = \sqrt{\lambda_{\max}(M_c)}$ . With  $\lambda$  in (10), we obtain

$$\begin{aligned} W(x(k+1)) - W(x(k)) &\leq \left( -1 + \lambda 2c_1 c_2 \|w\| \right) \|x\|^2 + \epsilon_x \|x\|^2 + \beta_{w2} \|w\|^2 + \beta_{w1} \|w\| \\ &\leq \left( -1 + \lambda 2c_1 c_2 \delta \right) \|x\|^2 + \epsilon_x \|x\|^2 + \beta_{w2} \|w\|^2 + \beta_{w1} \|w\| \end{aligned}$$

where  $\epsilon_x = \epsilon_{xq} + \lambda\epsilon_{xc}$ ,  $\beta_{w1} = \beta_{wq1} + \lambda\beta_{wc1}$ , and  $\beta_{w2} = \beta_{wq2} + \lambda\beta_{wc2}$ . Note that  $\epsilon_x$  can be made arbitrarily small as  $\epsilon_{xq}$  and  $\epsilon_{xc}$  are arbitrarily small. Now for  $\delta$  such that

$$-1 + \lambda 2c_1 c_2 \delta \leq -\epsilon_W - \epsilon_x \quad (19)$$

with  $0 < \epsilon_W < 1$ , we have

$$W(x(k+1)) - W(x(k)) \leq -\epsilon_W \|x\|^2 + \beta_{w2} \|w\|^2 + \beta_{w1} \|w\| \quad (20)$$

This completes the proof.  $\blacksquare$

**Remark 2:** As  $\epsilon_W$  and  $\epsilon_x$  can be made arbitrarily small in (19), the inequality in (20) is valid for any  $\delta$  strictly less than  $1/(2c_1 c_2 \lambda)$ .

**Remark 3:** The ISS characterization in Theorem 5 is similar to, but is slightly different from those in [20], [3]. In [3], the ISS characterization is obtained for the case where the external disturbance enters into the system in the following manner

$$x(k+1) = Ax(k) + Bsat(u+w).$$

In [20], the ISS characterization is derived for the matched case, i.e.

$$x(k+1) = Ax(k) + Bsat(u) + Bw.$$

Note also that Theorem 5 can be extended to a discrete time version of Proposition 14.1.5 of [10].

### B. ISS of the proposed MPC

On the basis of the ISS property of the small-gain control given in the previous subsection, we propose an input-to-state stabilizing MPC. The following theorem gives a global ISS characterization<sup>2</sup> with a restriction on the disturbance.

**Theorem 6:** Consider the neutrally stable plant (14) and the MPC law described in Theorem 3. Then, given any positive integer  $N$ , the closed-loop system is ISS for some positive  $\Theta$ , with a restriction on the external disturbance  $w(k)$ .

*Proof:* For notational simplicity, define

$$u_i^* = u^*(k+i|k) \quad \text{and} \quad x_i^* = x^*(k+i|k).$$

Note that

$$\mathbf{u}^*(k) = \{u_0^*, u_1^*, \dots, u_{N-1}^*\}.$$

As in the proof of Theorem 1, consider the following feasible vector at time  $k+1$

$$\tilde{\mathbf{u}}(k+1) = \{u_1^*, \dots, u_{N-1}^*, k_f(x_N^*)\}$$

where  $k_f(\cdot)$  is defined as

$$k_f(x) = -\text{sat}(\kappa B^T M_c A x), \quad (21)$$

Then, in view of

$$x(k+1|k+1) = x(k+1) = x^*(k+1|k) + w(k) = x_1^*,$$

<sup>2</sup>The previous results reported in [17], [15] provide local ISS characterizations; for the notion of local ISS, see [16] (page 192).

the resulting state predictions are given by

$$\begin{aligned} x(k+i|k+1) &= x_i^* + A^{i-1}w(k), \quad 1 \leq i \leq N \\ x(k+N+1|k+1) &= A(x_N^* + A^{N-1}w(k)) + Bk_f(x_N^*) \\ &= Ax_N^* + Bk_f(x_N^*) + A^N w(k) \end{aligned}$$

We write  $J_N(x(k+1), \tilde{\mathbf{u}}(k+1))$  as

$$J_N(x(k+1), \tilde{\mathbf{u}}(k+1)) \quad (22)$$

$$\begin{aligned} &= \sum_{i=1}^{N-1} l(x(k+i|k+1), u_i^*) + l(x(k+N|k+1), k_f(x_N^*)) \\ &\quad + \Theta W(Ax(k+N|k+1) + Bk_f(x_N^*)) \\ &= \sum_{i=1}^{N-1} l(x_i^* + A^{i-1}w, u_i^*) + l(x_N^* + A^{N-1}w, k_f(x_N^*)) \\ &\quad + \Theta W(Ax_N^* + Bk_f(x_N^*) + A^N w) \\ &= J_N^*(x, \mathbf{u}) - l(x_0^*, u_0^*) + M_1 + M_2 + M_3 \end{aligned} \quad (23)$$

where

$$\begin{aligned} M_1 &= l(x_N^* + A^{N-1}w, k_f(x_N^*)) \\ M_2 &= \Theta W(Ax_N^* + Bk_f(x_N^*) + A^N w) - \Theta W(x_N^*) \\ M_3 &= \sum_{i=1}^{N-1} \left\{ l(x_i^* + A^{i-1}w, u_i^*) - l(x_i^*, u_i^*) \right\}. \end{aligned}$$

We first consider  $M_1$  defined above as follows:

$$\begin{aligned} M_1 &= (x_N^* + A^{N-1}w)^T Q (x_N^* + A^{N-1}w) \\ &\quad + \text{sat}(\kappa B^T M_c A x_N^*)^T R \text{sat}(\kappa B^T M_c A x_N^*) \\ &\leq (x_N^* + A^{N-1}w)^T Q (x_N^* + A^{N-1}w) \\ &\quad + (\kappa B^T M_c A x_N^*)^T R (\kappa B^T M_c A x_N^*) \\ &= x_N^{*T} Q x_N^* + 2x_N^{*T} Q A^{N-1}w + w^T A^{N-1T} Q A^{N-1}w \\ &\quad + x_N^{*T} \kappa^2 A^T M_c B R B^T M_c A x_N^* \\ &\leq \epsilon_1 \|x_N^*\|^2 + \beta_1 \|w\|^2 \end{aligned} \quad (24)$$

where

$$\begin{aligned} \epsilon_1 &= \lambda_{\max}(Q + \kappa^2 A^T M_c B R B^T M_c A) + \epsilon_y \quad (25) \\ \beta_1 &= \frac{1}{\epsilon_y} \|Q A^{N-1}\| + \lambda_{\max}(A^{N-1T} Q A^{N-1}) \end{aligned}$$

and  $\epsilon_y$  is an arbitrary positive constant. Therefore,  $\epsilon_1$  can be made an arbitrary constant larger than  $\lambda_{\max}(Q + \kappa^2 A^T M_c B R B^T M_c A)$ . In view of Theorem 5, an upper bound on  $M_2$  can be obtained as

$$\begin{aligned} M_2 &= \Theta W(Ax_N^* + Bk_f(x_N^*) + A^N w) - \Theta W(x_N^*) \\ &\leq -\Theta \epsilon_2 \|x_N^*\|^2 + \Theta \beta_2 \|w\| + \Theta \beta_3 \|w\|^2 \end{aligned} \quad (26)$$

where  $\epsilon_2$  is an arbitrary positive constant satisfying  $0 < \epsilon_2 < 1$ ,  $\beta_2$  and  $\beta_3$  are positive, and  $\|w(k)\|$  is assumed to be less than or equal to a certain number. In order to give an upper bound on  $M_3$ , we consider

$$x_i^* = A^i x_0^* + W_{ci} U_i$$

where

$$U_i = [u_0^* \ u_1^* \ \cdots \ u_{i-1}^*]^T, \quad W_{ci} = [A^{i-1}B \ A^{i-2}B \ \cdots \ B].$$

Using this expression, we obtain an upper bound on  $M_3$  as follows:

$$\begin{aligned} M_3 &= \sum_{i=1}^{N-1} \left\{ (x_i^* + A^{i-1}w)^T Q (x_i^* + A^{i-1}w) - x_i^{*T} Q x_i^* \right\} \\ &= \sum_{i=1}^{N-1} \left\{ 2 \left( A^i x_0^* + W_{ci} U_i \right)^T Q A^{i-1} w + w^T A^{i-1T} Q A^{i-1} w \right\} \\ &= \sum_{i=1}^{N-1} \left\{ 2 \left( x_0^{*T} A^{iT} + U_i^T W_{ci}^T \right) Q A^{i-1} w \right. \\ &\quad \left. + w^T A^{i-1T} Q A^{i-1} w \right\} \\ &= \sum_{i=1}^{N-1} \left\{ 2 x_0^{*T} A^{iT} Q A^{i-1} w + 2 U_i^T W_{ci}^T Q A^{i-1} w \right. \\ &\quad \left. + w^T A^{i-1T} Q A^{i-1} w \right\} \\ &\leq \epsilon_3 \|x_0^*\|^2 + \beta_4 \|w\|^2 + \beta_5 \|w\| \end{aligned} \quad (27)$$

where  $\epsilon_3$  is an arbitrary positive constant, and  $\beta_4$  and  $\beta_5$  are positive. Note that we use  $2U_i^T W_{ci}^T Q A^{i-1} w \leq \beta_5 \|w\|$  above, which ensues from the fact that the inputs are bounded due to saturation. To sum up, (24), (26) and (27) result in an upper bound on the difference of the optimal cost as follows:

$$\begin{aligned} J_N^*(k+1) - J_N^*(k) &\leq J_N(x(k+1), \tilde{\mathbf{u}}(k+1)) - J_N(x(k), \mathbf{u}^*(k)) \\ &\leq -l(x_0^*, u_0^*) + \epsilon_3 \|x_0^*\|^2 + (\epsilon_1 - \Theta \epsilon_2) \|x_N^*\|^2 \\ &\quad + (\beta_1 + \Theta \beta_3 + \beta_4) \|w\|^2 + (\Theta \beta_2 + \beta_5) \|w\|. \end{aligned}$$

Therefore, if  $\Theta$  is set to a value larger than or equal to  $\frac{\epsilon_1}{\epsilon_2}$ , we have

$$J_N^*(k+1) - J_N^*(k) \leq -(\lambda_{\min}(Q) - \epsilon_3) \|x\|^2 + \beta_6 \|w\|^2 + \beta_7 \|w\|. \quad (28)$$

where  $\beta_6 = \beta_1 + \Theta \beta_3 + \beta_4$  and  $\beta_7 = \Theta \beta_2 + \beta_5$ . The proof is then completed by observing that  $\epsilon_3$  in (28) can be made arbitrarily small; note that this ISS characterization holds with a restriction on  $w$  as  $\|w\|$  is assumed to be sufficiently small when deriving (26) using Theorem 5.

## V. SIMULATION

To demonstrate the effectiveness of the proposed MPC scheme, we consider the following plant

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 0.8 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(k), \quad -1 \leq u \leq 1$$

where  $w(k)$  is an external disturbance. Note that the unconstrained part of the system is neutrally stable with

one integrator. For implementation, we employ an SQP algorithm in the optimization toolbox for Matlab. The MPC parameters used in the simulation are summarized below.

$$\begin{aligned} N &= 3, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 0.8, \\ M_c &= \begin{bmatrix} 0.06 & 0.3 \\ 0.3 & 2 \end{bmatrix}, \quad \kappa = \frac{0.95}{\lambda_{\max}(B^T M_c B)}, \\ \Theta &= 1.05 \cdot \Theta_0. \end{aligned}$$

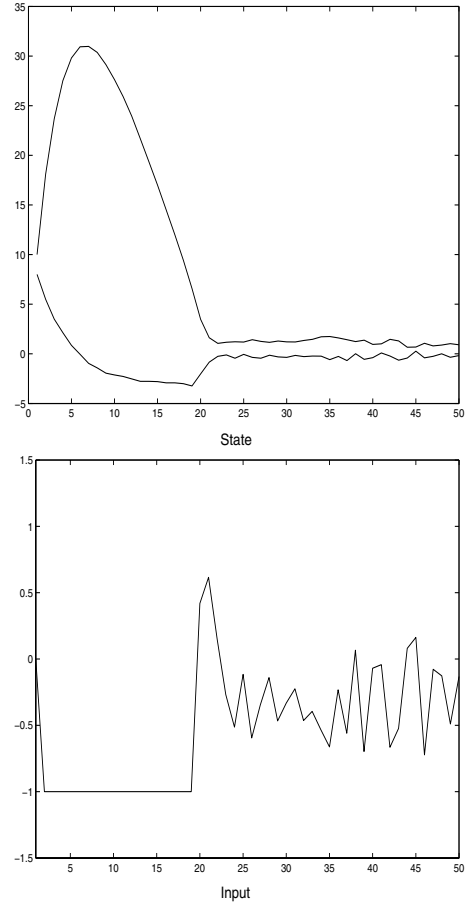


Fig. 1. State and input with  $x_0 = [10, 8]$  and  $w(k)$  random

Figure 1 shows the results for the case where the disturbance is a random bounded sequence ranging from 0 to 0.5. Despite the presence of the persistent disturbance, the output remains bounded as suggested by Theorem 6.

Figure 2 concerns the case where the disturbance has finite energy ( $w(k) = 5 \cdot 0.9^k$ ). As global stability is guaranteed by ISS and global stability implies the iISS property [1], the output is bounded and tends to zero as time goes by.

To sum up, the simulations here illustrate what the theorems of this paper state. The stability properties obtained are global, and as a consequence, there is no problem with feasibility. The closed-loop is also robust against the

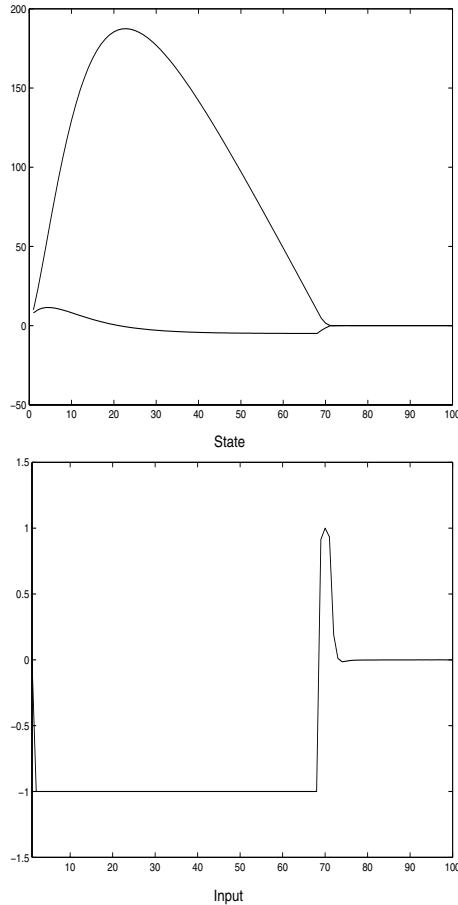


Fig. 2. State and input with  $x_0 = [10, 8]$  and  $w = 5 \cdot 0.9^k$

disturbance. This is in sharp contrast with finite-horizon quadratic MPC, where the domain of attraction does not cover the whole space and thus infeasibility can always occur due to the disturbance even when the initial state belongs to the domain of attraction.

## VI. CONCLUSION

In this paper, we have proposed an input-to-state-stabilizing MPC for input-constrained neutrally stable systems using a non-quadratic terminal cost under the assumption that the external disturbance is sufficiently small. Simulations using an SQP algorithm show the effectiveness of the proposed MPC. Possible directions for future research include switching-based adaptive MPC for uncertain input-constrained neutrally stable systems; it is expected that the ISS property obtained here may be used to develop a switching algorithm, leading to a stable adaptive system.

## REFERENCES

- [1] David Angeli, "Intrinsic robustness of global asymptotic stability", *Systems and Control Letters*, 38, pp. 297-307, 1999
- [2] David Angeli and Eduardo D. Sontag and Yuan Wang, "A characterization of Integral Input-to-State Stability", *IEEE Trans. on Automatic Control*, 45, pp. 1082-1097, 2000
- [3] Xiangyu Bao and Zongli Lin and E. Sontag "Finite gain stabilization of discrete-time linear systems subject to actuator saturation", *Automatica* 36, 269-277, 2000
- [4] A. Bemporad and M. Morari and V. Dua and E.N. Pistikopoulos, "The explicit linear quadratic regulator for constrained systems," *Automatica*, vol. 38, no. 1, pp. 3-20, 2002.
- [5] Y. Chitour and W. Liu and E. Sontag, "On finite gain stabilizability of linear systems subject to input saturation", *SIAM J. Control and Optimization* 34, 1190-1219, 1996
- [6] D. Chmielewski and V. Manousiouthakis, "Constrained Infinite-time Linear Quadratic Optimal Control," *Syst. & Contr. Let.*, vol. 29, p. 121, 1996
- [7] Jinhoon Choi, "On the stabilization of linear discrete time system subject to input saturation", *Systems and Control Letters*, Vol. 36, pp. 241-244, 1999
- [8] Claudio De Persis and Raffaella De Santis and A. Stephen Morse, "Supervisory control with state-dependent dwell-time logic and constraints", To appear in *Automatica*, 2004
- [9] Joao Hespanha and Daniel Liberzon and A. S. Morse, "Supervision of integral-input-to-state stabilizing controllers", *Automatica*, 38(8):1327-1335, Aug. 2002
- [10] A. Isidori, "Nonlinear Control Systems, Vol. II", Springer Verlag, London, 1999
- [11] A. Jadbabaie and J. Yu and J. Hauser, "Unconstrained receding horizon control of nonlinear systems" *IEEE Transactions on Automatic Control*, Vol 46, No 5, pp.776-783, 2001
- [12] Ali Jadbabaie and Claudio De Persis and Tae-Woong Yoon, "A globally stabilizing Receding Horizon Controller for Neutrally Stable Linear Systems with Input Constraints", In the *Proceedings of CDC* 2002
- [13] Jiang, Z.P. and Wang, Y. "Input-to-state stability for discrete-time nonlinear systems", *Automatica*, 37, 857-869, 2001
- [14] A. Jadbabaie and A. S. Morse, "On the ISS Property for Receding Horizon Control of Constrained Linear Systems", *Proceedings of the 2002 IFAC Congress*, July, 2002
- [15] E.C. Kerrigan and J.M. Maciejowski. "On robust optimization and the optimal control of constrained linear systems with bounded state disturbances", In *Proc. 2003 European Control Conference*, Cambridge, UK, September 2003
- [16] Hassan K. Khalil, "Nonlinear Systems", Prentice Hall, 3rd edition, 2002
- [17] D. Limon and Alamo, T. and Camacho, E.F. "Input to state stable MPC for constrained discrete time nonlinear systems with bounded additive uncertainties", In the *Proceedings of CDC* 2002
- [18] D.Q. Mayne and J. B. Rawlings and C. V. Rao and P. O. M. Scokaert, "Constrained Model Predictive Control: Stability and Optimality", *Automatica*, Vol. 36, pp 789-814, 2000.
- [19] M. Seron and G.C. Goodwin and J. De Dona, "Closed-form solution to a class of quadratic programmes with application to model predictive control" submitted to *IEEE Transactions on Automatic Control*, 2001
- [20] Guoyong Shi and Ali Saberi and Anton A. Stoorvogel, "On the  $L_p(l_p)$  stabilization of open-loop neutrally stable linear plants with input subject to amplitude saturation", *International Journal of Robust and Nonlinear Control*, Vol. 13, 2003
- [21] Jung-Su Kim, Tae-Woong Yoon, Ali Jadbabaie and Claudio De Persis "Finite Horizon Global Stabilizing MPC for Neutrally Stable Systems subject to Input Saturation", In the *Proceedings of ICCAS*, 2003