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Published in:
Proceedings of the 6th IFAC-Symposium on Nonlinear Control Systems

Publication date:
2004

A REACTIVE PORT-HAMILTONIAN CIRCUIT DESCRIPTION AND ITS CONTROL IMPLICATIONS

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Abstract: This paper first addresses the question when a given (possibly nonlinear) RGLC circuit can be rewritten as a port-Hamiltonian (PH) system—with state variables the inductor currents and capacitor voltages instead of the fluxes and charges, respectively. The question has an affirmative answer for a class of circuits that fulfills a certain regularity condition. This class includes circuits where all dynamic elements are linear, and the associated resistors and conductors are passive—though possibly nonlinear. Interestingly, the resulting Hamiltonian function is related with the circuits instantaneous reactive power associated with the inductors and capacitors. This novel circuit representation, called a *reactive* port-Hamiltonian description, naturally suggests a new set of non–standard passive outputs, which are shown to be useful for the design of reactive power compensation schemes. A Van der Pol oscillator circuit is used to illustrate the developments throughout the paper.

Keywords: Passivity, Stabilization, Nonlinear Systems, RGLC circuits, Hamiltonian Systems, Brayton- Moser Circuits, Van der Pol oscillator.

1. PRELIMINARIES

In the early sixties, J.K. Moser (Moser, 1960) developed a mathematical analysis to study the stability of circuits containing tunnel diodes.[1] His method was based on a certain 'potential function', which was four years later generalized and coined 'mixed-potential' by the same author, together with his colleague R.K. Brayton, in (Brayton and Moser, 1964). Basically, their theory is based on the observation that the differential equations describing the behavior of a large class of nonlinear RGLC circuits can be written in the form

$$Q(x)\dot{x} = \nabla_x P(x). \tag{1}$$

Here $x = \mathrm{col}(i_L, v_C)$, where $i_L = \mathrm{col}(i_{L_1}, \ldots, i_{L_\sigma})$ represents the currents through the σ inductors (L), and $v_C = \mathrm{col}(v_{C_1}, \ldots, v_{C_\rho})$ the voltages across the ρ capacitors (C), respectively. The notation $\nabla_x P(x)$ denotes the gradient of the scalar function $P : \mathbb{R}^{\sigma+\rho} \to \mathbb{R}$, i.e., $\nabla_x = \partial/\partial x$. This function—called the mixed-potential—captures all the necessary information about the topological structure (circuit graph), and the characteristics of the resistive elements contained in the circuit. The function $P(x) = P(i_L, v_C)$ has the units of power and is constructed as

$$P(i_L, v_C) = A(i_L) - B(v_C) + N(i_L, v_C), \tag{2}$$

where $A : \mathbb{R}^\sigma \to \mathbb{R}$ and $B : \mathbb{R}^\rho \to \mathbb{R}$ are the current potential (content) related with the current-controlled resistors (R) and voltage sources, and the voltage potential (co-content) related with the voltage-controlled resistors (i.e., conductors, G) and current sources, respectively. More specifically, the content and co-content are defined by the integrals

$$\int_0^{i_L} \hat{v}_R(i'_L)di'_L, \quad \int_0^{v_C} \hat{i}_G(v'_C)dv'_C,$$

[1] It should be mentioned that related ideas where already contained in a paper by Stöhr in the early fifties (see (Marten *et al.*, 1992) for some historical remarks).

where $\hat{v}_R(i_L)$ and $\hat{i}_G(v_C)$ are the characteristic functions of the resistors and conductors, respectively.

The function $N : \mathbb{R}^{\sigma+\rho} \to \mathbb{R}$ is determined by the interconnection of the inductors and capacitors: $N(i_L, v_C) = \sum_{j=1}^{\sigma} \sum_{k=1}^{\rho} \gamma_{jk} i_{L_j} v_{C_k}$, where γ_{jk} represents the interconnection between i_{L_j} and v_{C_k}. Furthermore, the matrix $Q(x) = Q(i_L, v_C)$ contains the incremental values of the inductors and capacitors, i.e.,

$$Q(i_L, v_C) = \begin{bmatrix} -L(i_L) & 0 \\ 0 & C(v_C) \end{bmatrix}. \qquad (3)$$

In the remaining of the paper we will restrict to *linear* passive inductors and capacitors, in this case L and C are constant positive definite matrices.

2. A REACTIVE PORT-HAMILTONIAN DESCRIPTION

The proposition below forms the basis for the main results of this paper. Basically, we show that, under some physically interpretable conditions, the Brayton-Moser model (1) can be rewritten as a port-Hamiltonian system with dissipation. [2] There are at least two motivations for rewriting (1) in a port-Hamiltonian form. First, in the resulting port-Hamiltonian description the state variables are the inductor currents and capacitor voltages instead of their fluxes and charges. For control applications, where the usual measured quantities are voltages and currents, this constitutes a clear practical advantage. Also, as will be shown later on, the new model naturally suggests a new set of passive port variables, fundamentally different from the ones identified with the classical model (Van der Schaft, 2000), where the associated storage function is not energy but a reactive power-like function.

2.1 The New Model

Let us define [3]

$$D(x) = \begin{bmatrix} \nabla^2_{i_L} A(i_L) & 0 \\ 0 & \nabla^2_{v_C} B(v_C) \end{bmatrix}, \qquad (4)$$

and

$$J = \begin{bmatrix} 0 & \gamma \\ -\gamma^T & 0 \end{bmatrix}, \qquad (5)$$

where γ is a matrix with elements γ_{jk} as defined in Section 1—the matrix J coincides with the usual structure matrix for conventional port-Hamiltonian systems in (Van der Schaft, 2000). On the other hand, if the resistors and conductors are passive with twice differentiable characteristic functions, $D(x) \geq 0$, and we will show below that this matrix can be interpreted as some nonlinear version of the usual dissipation matrix. The main assumption throughout the paper is that $J - D(x)$ is regular for all x (see Remark 1).

Proposition 1. Consider the Brayton-Moser equations (1) with linear passive inductors and capacitors. Assume that $J - D(x)$ is regular. Then, the Brayton-Moser equations (1) can be transformed into an autonomous port-Hamiltonian system

$$\dot{x} = \left[\tilde{J}(x) - \tilde{D}(x)\right] \nabla_x \tilde{P}(x), \qquad (6)$$

with

$$\tilde{J}(x) = \frac{1}{4}\left\{ \left[J - D(x)\right]^{-1} - \left[J - D(x)\right]^{-T} \right\},$$

which represents the 'structure' matrix, satisfying the skew–symmetry property $\tilde{J}(x) = -\tilde{J}^T(x)$, and

$$\begin{aligned}\tilde{D}(x) &= -\frac{1}{4}\left\{ \left[J - D(x)\right]^{-1} + \left[J - D(x)\right]^{-T} \right\} \\ &= \tilde{D}^T(x).\end{aligned}$$

The Hamiltonian for the circuit $\tilde{P} : \mathbb{R}^{\sigma+\rho} \to \mathbb{R}$ is given by

$$\tilde{P}(x) = \nabla_x^T P(x) M \nabla_x P(x), \qquad (7)$$

where M is a symmetric and positive-definite matrix of the form

$$M = \begin{bmatrix} L & 0 \\ 0 & C \end{bmatrix}^{-1}.$$

Furthermore, if the resistors and conductors are passive with twice differentiable characteristic functions $\tilde{D}(x) \geq 0$, hence (6) defines an autonomous port-Hamiltonian system with dissipation.

Proof. The key observation here is that the dynamics of (1) can alternatively be written as

$$\tilde{Q}(x)\dot{x} = \nabla_x \tilde{P}(x), \qquad (8)$$

where $\tilde{Q}(x)$ is defined by

$$\tilde{Q}(x) = 2\nabla_x^2 P(x) M Q, \qquad (9)$$

and $\tilde{P}(x)$ is of the form (7). From (1) and (8) it is obvious that to establish this claim it suffices to proof that $\tilde{Q}(x) Q^{-1} \nabla_x P(x) = \nabla_x \tilde{P}(x)$. The latter equality can be verified by noting that the gradient of $\tilde{P}(x)$ satisfies $\nabla_x \tilde{P}(x) = 2\nabla_x^2 P(x) M \nabla_x P(x)$.

Furthermore, since the Hessian of $P(x)$ equals

$$\nabla_x^2 P(x) = \begin{bmatrix} \nabla^2_{i_L} A(i_L) & \gamma \\ \gamma^T & -\nabla^2_{v_C} B(v_C) \end{bmatrix}$$

and

$$MQ = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix},$$

it is easily verified using (9) that $\tilde{Q}(x) = 2\left[J - D(x)\right]$.

[2] For an excellent treatment of port-Hamiltonian systems, the interested reader is referred to (Van der Schaft, 2000).

[3] We denote the Hessian of a scalar function by $\nabla_x^2 = \partial^2/\partial x^2$.

Under the assumption of regularity of $J-D(x)$, we can invert this matrix an define

$$\tilde{J}(x)-\tilde{D}(x)=\frac{1}{2}\left[J-D(x)\right]^{-1} \tag{10}$$

Then, we use the fact that every square matrix can be decomposed into a symmetric and a skew-symmetric part as done in the proposition.

For passive resistors and conductors $D(x)\geq 0$, and thus $\tilde{D}(x)\geq 0$. Hence, the system (6) is dissipative. This concludes the proof. ◀

The following remarks are in order:

Remark 1. We notice that the only condition needed for the derivation of (6) is that $\tilde{J}(x)-\tilde{D}(x)$, or equivalently $J-D(x)$, is full-rank. This full-rank condition does not seem restrictive in physical applications. On one hand, it will be verified if all inductors and capacitors are leaky, which is the case for real-life inductors that always possess some resistance. On the other hand, if $\tilde{J}(x)-\tilde{D}(x)$ is rank deficient then the circuit has equilibria at points which are not extrema of (7), and consequently of the original mixed potential (2). See e.g. (Ortega *et al.*, 2002; Jeltsema *et al.*, 2003*a*) for a similar discussion in the conventional port-Hamiltonian framework.

Remark 2. It is interesting to point out that, in contrast to conventional port-Hamiltonian systems, the (independent) external voltage and current sources are captured in both $\tilde{P}(x)$ and $\tilde{Q}(x)$, and therefore do not appear as external (port-)signals in the equation like in (Van der Schaft, 2000). For that reason, (6) may be considered as an autonomous Hamiltonian system. However, if a circuit is driven by one or more time-dependent sources, the resulting Hamiltonian description is extended as

$$\dot{x}=\tilde{Q}^{-1}(x,t)\nabla_x\tilde{P}(x,t). \tag{11}$$

This will be illustrated in Example 1.

In the following subsection we will provide a physical interpretation of the new Hamiltonian (7).

2.2 Total Instantaneous Reactive Power

To justify the title above, we make the following observation. The new Hamiltonian (7) can, by substitution of (1), be written in the form

$$\tilde{P}(x(t))=v_L^T(t)\frac{di_L}{dt}(t)+i_C^T(t)\frac{dv_C}{dt}(t), \tag{12}$$

where v_L and i_C represent the voltages across the inductors and the currents through the capacitors, respectively. If, for simplicity, we restrict the discussion to a linear and time-invariant (LTI) RGLC circuit driven by a single sinusoidal voltage source $v_S(t)=v_S'\cos(\omega t)$, we know from e.g., (Desoer and Kuh, 1969), that the total supplied (average) power is defined by

$$\overline{p}_S=\frac{1}{T}\int_0^T i_S(t)v_S(t)dt, \tag{13}$$

where the quantity $i_S(t)v_S(t)$ equals the total supplied instantaneous power $p_S(t)$, with $i_S(t)=i_S'\cos(\omega t\pm\phi)$, and ϕ the displacement angle between the port signals $i_S(t)$ and $v_S(t)$. Using basic trigonometric identities, (13) can be rewritten in term of the RMS-values[4] of the supplied voltage and current as $\overline{p}_S=I_SU_S\cos\phi$. If the displacement angle $\phi\neq 0$, one can decompose I_S into two components: a real component I_{S_r} and a imaginary component I_{S_i} as $I_S^2=I_{S_\mathrm{r}}^2+I_{S_\mathrm{i}}^2$, where $I_{S_\mathrm{r}}=I_S\cos\phi$ and $I_{S_\mathrm{i}}=I_S\sin\phi$, respectively. Hence, (13) can alternatively be written as $\overline{p}_S=I_{S_\mathrm{r}}U_S$.

A very important quantity in the study of electric circuits is the total supplied (average) reactive power. This quantity, denoted by $\overline{q}_S$, is classically defined as the product of the imaginary current component I_{S_i} and U_S, i.e., $\overline{q}_S:=I_{S_\mathrm{i}}U_S$, or equivalently, $\overline{q}_S:=I_SU_S\sin\phi$. Interestingly, it is easily proved that in a similar fashion as (13), we may relate to $\overline{q}_S$ an alternative definition involving time-derivatives of the port variables $i_S(t)$ or $v_S(t)$, i.e.,

$$\overline{q}_S=\frac{1}{T}\int_0^T q_S(t)dt, \tag{14}$$

where $q_S(t)$ is either represented by

$$\frac{1}{\omega}i_S(t)\frac{dv_S}{dt}(t),\ \text{or}\ -\frac{1}{\omega}v_S(t)\frac{di_S}{dt}(t).$$

This fact establishes the relationship between the average behavior of the port variables $i_S(t),v_S(t)$ and the classical reactive power.

The above discussion suggests that we can give to $q_S(t)$ an interpretation of supplied instantaneous reactive power. Hence, regarding (12), the quantities $v_L^Tdi_L/dt$ and $i_C^Tdv_C/dt$ can also be considered as some generalized powers related to the energy storing elements in the circuit. Moreover, as illustrated in the simple example below, we relate (12) with the total instantaneous reactive power.

Example 1. Consider the LTI RC circuit shown in Fig. 1. The circuit is driven by a time-varying voltage source $v_S(t)$. The dynamics can be described in the form (1) as follows. Since there are no inductors and no current-controlled resistors, we have that the

[4] For any periodic signal $u(t)$ the RMS (root-mean-square) value is defined as

$$U:=\sqrt{\frac{1}{T}\int_0^T u^2(t)dt}.$$

For a purely sinusoidal signal, i.e., $u(t)=\hat{u}\cos(\omega t)$, the RMS value is simply given by $U=\hat{u}/\sqrt{2}$.

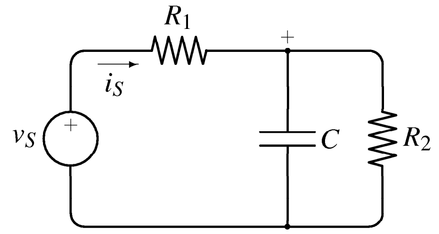

Fig. 1. A simple voltage-driven linear RC circuit.

content $A(i_L) = 0$ and $N(i_L, v_C) = 0$. However, the circuit's co-content is defined by

$$B(v_C,t) = \tfrac{1}{2} G v_C^2 - G_1 v_C v_S(t),$$

where $G_1 = 1/R_1$ and $G = 1/R_1 + 1/R_2$. Thus, for this circuit we have that $P(v_C,t) = -B(v_C,t)$, which yields for the dynamics that

$$C\dot{v}_C = \nabla_{v_C} P(v_C,t) = G_1 v_S(t) - G v_C.$$

According to Proposition 1, the circuit dynamics can equivalently be expressed as

$$\dot{v}_C = \tilde{Q}^{-1} \nabla_{v_C} \tilde{P}(v_C,t),$$

with $\tilde{Q} = -2G$, and $\tilde{P}(v_C,t) = C^{-1}\big(G_1 v_S(t) - G v_C\big)^2$. It is clear that $\tilde{P}(t) = i_C(t)\dot{v}_C(t)$. Now consider its time-derivative, i.e.,

$$\dot{\tilde{P}}(t) = -2G\dot{v}_C^2(t) + 2\frac{G_1}{C}\big(G_1 v_S(t) - G v_C(t)\big)\dot{v}_S(t).$$

For this simple example we can actually compute an explicit solution for $\tilde{P}(t)$. Indeed, the latter equation can be expressed in terms of $\tilde{P}(t)$ as

$$\dot{\tilde{P}}(t) = -2\frac{G}{C}\tilde{P}(t) + 2\sqrt{\frac{G_1^2}{C}\tilde{P}(t)}\dot{v}_S(t).$$

Consider then the case when $v_S(t) = E\cos(\omega t)$, then the solution of the differential equation above is easily obtained as

$$\tilde{P}(t) = \frac{\omega^2 G_1^2 C E^2}{(G^2 + \omega^2 C^2)^2} \times \big[\omega C \cos(\omega t) - G \sin(\omega t)\big]^2 + \varepsilon_t,$$

where ε_t are exponentially decaying terms due to the initial conditions.

As expected, the average value of $\tilde{P}$ coincides—up to a factor ω—with the classical average reactive power associated with the capacitor, in the sense that

$$\frac{1}{T}\int_0^T \tilde{P}(t)dt = \omega \bar{q}_C,$$

(compare with (14)) as computed for instance in Example 11.7 of (Carlo and Lin, 2001). Fig. 2 demonstrates the reactive power flow for some particular values of the circuit components. The classical average reactive power $\bar{q}_C$ associated with the capacitor is obtained by averaging the product $I_{C_i} U_C$ with a running window over one cycle of the fundamental frequency ω.

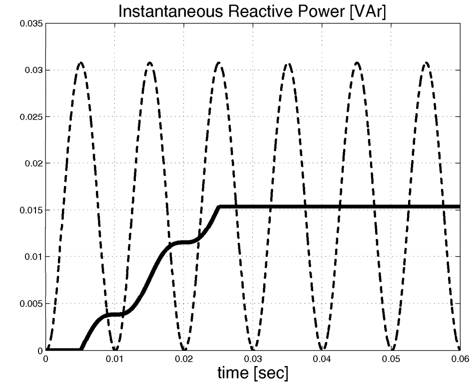

Fig. 2. Reactive power flow for the circuit of Example 1, with $E = 1$V, $\omega = 100\pi$ rad/sec., $R_1 = 1\Omega$, $R_2 = 100\Omega$, and $C = 100\mu$F: $\tilde{P}(t)/\omega$ (dashed), and the classical reactive power $\bar{q}_C$ (solid).

From the discussion above we see that the Hamiltonian function (12), and thus (7), is related to the the total instantaneous reactive power in the circuit, providing some justification to the following definition.

Definition 1. A system of the form (6), together with (7), is called a reactive port-Hamiltonian system. Furthermore, if the resistors and conductors are passive, it is called a reactive port-Hamiltonian system with dissipation.

In the remaining sections we illustrate the usefulness of (6) by highlighting some theoretical properties and potential implications for control.

3. APPLICATION 1: TOWARDS A REGULATION PROCEDURE OF INSTANTANEOUS REACTIVE POWER

So far, we have introduced a new port-Hamiltonian equation set that admits the interpretation of a reactive power-like description (recall that the original port-Hamiltonian equations, as defined in (Van der Schaft, 2000), have the interpretation of an energy description since the Hamiltonian function equals the total stored energy). In the proposition below we generalize the ideas illustrated in Example 1, and derive a simple expression for the time evolution of the total instantaneous power—that highlights the role of dissipation and suggests a procedure to regulate it with the inclusion of regulated voltage and/or current sources. For ease of presentation we will assume first that the external sources, which are contained in $A(i_L)$ and/or $B(v_C)$, are constant.

Proposition 2. Consider a RGLC circuit described by (6). Assume that the voltage sources and/or current sources are constant. Then, along the trajectories of the circuit we have that the rate of change of the reactive Hamiltonian satisfies

$$\begin{aligned}\dot{\tilde{P}}(x(t)) \;=\; &-2\left[\frac{di_L}{dt}\right]^T \nabla^2_{i_L}A(i_L)\left[\frac{di_L}{dt}\right] \\ &-2\left[\frac{dv_C}{dt}\right]^T \nabla^2_{v_C}B(v_C)\left[\frac{dv_C}{dt}\right].\end{aligned} \quad (15)$$

In particular, if the resistors and conductors contained in the circuit are passive, we have that $\dot{\tilde{P}}(x(t)) \leq 0$.

Proof. Since the external sources are constant by assumption, equation (15) follows directly by pre-multiplying (6) by $\dot{x}^T\tilde{Q}(x)$, i.e.,

$$\dot{x}^T\tilde{Q}(x)\dot{x} = \dot{x}^T\nabla_x\tilde{P}(x),$$

where we notice that $\dot{x}^T\nabla_x\tilde{P}(x) = \dot{\tilde{P}}(x(t))$. If the resistors and conductors are passive, $A(i_L) \geq 0$ and $B(v_C) \geq 0$, hence, the symmetric part of $\tilde{Q}(x)$ is negative semi-definite. ◀

Notice that the previous observations remain valid if we include current-dependent voltage sources, with characteristic function $v_{S_d} = \hat{v}_{S_d}(i_L)$, in series with the inductors and/or voltage-dependent current sources, with characteristic function $i_{S_d} = \hat{i}_{S_d}(v_C)$, in parallel with the capacitors. Indeed, the expressions (12) and (15) remain valid if we replace $A(i_L)$ and $B(v_C)$ with the new content and co-content functions

$$\tilde{A}(i_L) = A(i_L) - \int_0^{i_L}\hat{v}_{S_d}(i'_L)di'_L \quad (16)$$

and

$$\tilde{B}(v_C) = B(v_C) - \int_0^{v_C}\hat{i}_{S_d}(v'_C)dv'_C, \quad (17)$$

respectively. The characteristic functions $\hat{v}_{S_d}(i_L)$ and $\hat{i}_{S_d}(v_C)$ can be chosen by the designer. As indicated in (15), and illustrated in the example below, these control actions enter through the Hessians of the content and co-content functions. Henceforth, the reactive Hamiltonian $\tilde{P}(x)$ can be regulated via a suitable selection of the 'slopes' of the characteristic functions of the sources.

Example 2. Consider the circuit realization of a Van der Pol oscillator circuit shown in Fig. 3. It is easily shown that the reactive Hamiltonian reads

$$\tilde{P}(i_L, v_C) = \frac{1}{L}\dot{v}_C^2 + \frac{1}{C}\Big(i_L - \hat{i}_G(v_C)\Big)^2, \quad (18)$$

while its time evolution is determined by

$$\dot{\tilde{P}}(x(t)) = -2\nabla_{v_C}\hat{i}_G(v_C)\dot{v}_C^2.$$

As indicated above, the total stored instantaneous reactive power in the circuit can be 'controlled' adding regulated sources. For instance, let us add a voltage-dependent current source in parallel with the capacitor

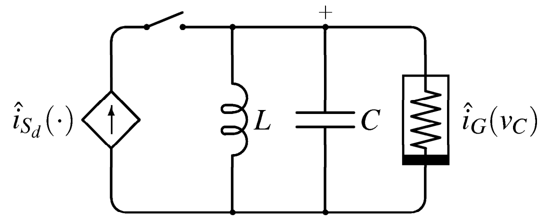

Fig. 3. Circuit realization of a Van der Pol oscillator. The nonlinear resistor is usually characterized by a function $\hat{i}_G(v_C) = \alpha v_C(v_C^2 - \beta)$, with $\alpha, \beta \in \mathbb{R}$.

by closing the switch in Fig. 3. Let the control action be given by $i_{S_d} = \hat{i}_{S_d}(v_C)$, with $\hat{i}_{S_d} : \mathbb{R} \to \mathbb{R}$ a function to be defined. One can easily verify that the quantities of Proposition 2 remain unaffected, and that only the circuit's co-content function $B(v_C)$ has to be changed to

$$\tilde{B}(v_C) = \underbrace{\int_0^{v_C}\hat{i}_G(v'_C)dv'_C}_{B(v_C)} - \int_0^{v_C}\hat{i}_{S_d}(v'_C)dv'_C.$$

The rate of change of the total instantaneous reactive power now becomes

$$\dot{\tilde{P}}(x(t)) = -2\nabla_{v_C}\Big(\hat{i}_G(v_C) - \hat{i}_{S_d}(v_C)\Big)\dot{v}_C^2.$$

The previous expression shows how we can modify the total instantaneous reactive power via a suitable selection of the 'slope' of the function $\hat{i}_{S_d}(v_C)$. A similar effect is obtained, but now modulated by the quantity $(di_L/dt)^2$, placing a current-dependent voltage source in series with the inductor (see e.g. Fig. 4).

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4. INPUT-OUTPUT REPRESENTATION AND PASSIVITY

For control applications it is convenient to write the equations with the manipulated inputs appearing explicitly. For that purpose, we need to extract the controllable sources from the reactive Hamiltonian. This is easily done as follows.

4.1 Input-Output Representation

Consider the Brayton-Moser equations (1). Let the content $A(i_L)$ (resp. co-content $B(v_C)$) be composed of two parts: a resistive content function $A_D(i_L)$ (resp. conductive co-content function $B_D(v_C)$) and an 'interaction' content function $A_S(i_L)$ (resp. co-content function $B_S(v_C)$). Consequently, the mixed-potential $P(x)$ can be decomposed into two parts: an 'internal' part $P_D(i_L, v_C) = A_D(i_L) - B_D(v_C) + N(i_L, v_C)$, and an 'interaction' part $P_S(i_L, v_C) = -A_S(i_L) + B_S(v_C)$. Hence, (1) can be written as

$$Q\dot{x} = \nabla_x P_D(x) + \nabla_x P_S(x).$$

A similar discussion holds for the reactive Hamiltonian description (6), as illustrated in the following proposition.

Proposition 3. The reactive port-Hamiltonian system (6) admits an input-output representation of the form

$$\begin{aligned} \dot{x} &= \tilde{Q}^{-1}(x)\nabla_x \tilde{P}_D(x) + \tilde{g}(x)u \\ y &= \tilde{g}^T(x)\nabla_x \tilde{P}_D(x), \end{aligned} \quad (19)$$

where $\tilde{P}_D(x) = \nabla_x P_D(x) M \nabla_x P_D(x)$ is the internal reactive Hamiltonian, $\tilde{Q}^{-1}(x)$ is defined in (9), and the forcing term is given by

$$\tilde{g}(x)u \triangleq Q^{-1}\nabla_x P_S(x), \quad (20)$$

with $\tilde{g}(x) \in \mathbb{R}^{n\times m}$ and $u \in \mathbb{R}^m$ representing the external voltage and/or current sources, and $y \in \mathbb{R}^m$ represents the output of the system.

Proof. The proof for the 'internal' part (i.e., $u = 0$) follows along the same lines of the proof of Proposition 1, while (20) follows by construction. ◄

Let us illustrate the latter using our example.

Example 3. Consider again the controlled Van der Pol circuit of Example 2, but now with a regulated voltage source $v_S = \hat{v}_S(i_L)$ in series with the inductor (Fig. 4). Since $A_D(i_L) = 0$, the modified content (16) reads

$$\tilde{A}(i_L) = -A_S(i_L) = -\int_0^{i_L} \hat{v}_{S_d}(i'_L)di'_L.$$

In order to obtain an input-output description we set $P_S(i_L) = -A_S(i_L)$, while $\tilde{P}_D(i_L, v_C)$ equals (18). Consequently, the controlled Van der Pol circuit in the form (19) reads

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} i \\ v \end{bmatrix} &= \frac{1}{2}\begin{bmatrix} -\nabla_{v_C}\hat{i}_G(v_C) & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \nabla_{i_L}\tilde{P}_D(i_L,v_C) \\ \nabla_{v_C}\tilde{P}_D(i_L,v_C) \end{bmatrix} \\ &\quad + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u, \quad u = \hat{v}_{S_d}(i_L) \\ y &= \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}^T \begin{bmatrix} \nabla_{i_L}\tilde{P}_D(i_L,v_C) \\ \nabla_{v_C}\tilde{P}_D(i_L,v_C) \end{bmatrix}, \end{aligned} \quad (21)$$

To this end, it is interesting to observe that the natural ("reactive power conjugated") output for the latter reactive port-Hamiltonian system equals

$$y = \frac{1}{LC}\left(i_L - \hat{i}_G(v_C)\right),$$

which, by using (1), can also be written as $y = \dot{v}_C/L$. Hence, there is a 'natural' differentiation in the output. This observation will be of key importance in the following section.

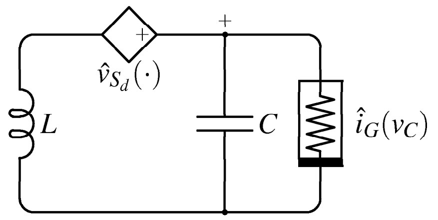

Fig. 4. Van der Pol oscillator with a regulated voltage source.

4.2 *Yet Another Passivity Property*

It is well-known that arbitrary interconnections of passive resistors, inductors and capacitors define passive systems, with power port variables the external sources as inputs and the capacitor voltages and/or inductor currents as outputs. The related storage function equals the total stored energy (Desoer and Kuh, 1969). Based on the mixed-potential, we have identified in (Jeltsema *et al.*, 2003*b*), (Ortega *et al.*, 2003) a class of RLC circuits for which it is possible to 'add a differentiation' to the port terminals preserving passivity—with a new storage function that is directly related to the circuit power. The class verifying these new passivity properties is identified in terms of an order relation between the magnetic and the electric energy and has led to the paradigm of a new control strategy, called power-shaping control (Ortega *et al.*, 2003).

The following proposition reveals a new passivity property, which is independent of the energy relations. It yields a set of passive outputs, different from the ones in (Jeltsema *et al.*, 2003*b*), that will be shown to be useful for control purposes in the next section.

Proposition 4. Consider the reactive port-Hamiltonian input-output system (19). If $A_D(i_L)$ and $B_D(v_C)$ are non-negative (i.e., the resistors and conductors are passive), then the circuit defines a passive system with port variables (u, y), where

$$y = \tilde{g}^T \nabla_x \tilde{P}_D(x), \quad (22)$$

and nonnegative storage function $\tilde{P}_D(x)$.

Proof. The proof consists in showing that

$$\tilde{P}_D[x(t)] - \tilde{P}_D[x(0)] \le \int_0^t u^T(t')y(t')dt'. \quad (23)$$

First, we notice that differentiation of $\tilde{P}_D(x)$ along the trajectories of (19) yields

$$\dot{\tilde{P}}_D(x) = \nabla_x^T \tilde{P}_D(x)\tilde{Q}^{-1}(x)\nabla_x \tilde{P}_D(x) + u^T y \quad (24)$$

Under the assumption that $A_D(i_L) \ge 0$ and $B_D(v_C) \ge 0$, we have that the symmetric part of $\tilde{Q}^{-1}(x) \le 0$. Hence, the proof is completed integrating (24) from 0 to t. ◄

Remark 3. Notice that we may interpret the inequality (23) as a reactive power-balance inequality.

Example 4. For the controlled Van der Pol oscillator of Fig. 4 we found in (21) that

$$y = \frac{1}{LC}\Big(i_L - \hat{i}_G(v_C)\Big) = \frac{1}{L}\dot{v}_C.$$

Hence, according to (24), we have

$$\dot{\tilde{P}}_D(i_L, v_C) = -2\nabla_{v_C}\hat{i}_G(v_C)\dot{v}_C^2 + \frac{1}{L}\hat{v}_{S_d}(i_L)\dot{v}_C.$$

If $i_L(0) = v_C(0) = 0$ and $\nabla_{v_C}\hat{i}_G(v_C) \geq 0$, then

$$\tilde{P}_D\big[i_L(t), v_C(t)\big] \leq \frac{1}{L}\int_0^t \dot{v}_C(t')v_{S_d}(t')dt'. \tag{25}$$

Hence, for all values of v_C for which $\nabla_{v_C}\hat{i}_G(v_C)$ is non-negative, the Van der Pol circuit defines a passive system with port variables $\big(v_{S_d}, \dot{v}_C/L\big)$ and storage function $\tilde{P}_D(i_L, v_C)$. On the other hand, it is clear that if $\nabla_{v_C}\hat{i}_G(v_C) \leq 0$, the circuit can be rendered passive by defining a suitable control $v_{S_d} = \hat{v}_{S_d}(i_L)$ that dominates the term $||2\nabla_{v_C}\hat{i}_G(v_C)\dot{v}_C^2||$ such that (25) is satisfied for all (i_L, v_C). (Compare to the method of Example 2.)

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5. APPLICATION 2: PI(D) CONTROL

Motivated by the foregoing discussion, we have the following proposition: [5]

Proposition 5. Consider the reactive port-Hamiltonian input-output system (19) with passive resistors and conductors. Assume $\tilde{P}_D(x)$ admits a local minimum, that we denote $x_\star$. [6] Let $\phi : \mathbb{R}^n \to \mathbb{R}^m$ be defined by

$$\begin{aligned} \phi(x,\xi) &= -K_P\tilde{g}^T(x)\nabla_x\tilde{P}_D(x) - K_I\xi(x) \\ \dot{\xi} &= \tilde{g}^T(x)\nabla_x\tilde{P}_D(x), \end{aligned} \tag{26}$$

with $K_P, K_I \in \mathbb{R}^{m\times m}$ some positive definite symmetric matrices. Then, the system (19) in closed-loop with the control $u = \phi(x,\xi)$, has $x_\star$ as an asymptotically stable equilibrium point.

Proof. The closed-loop dynamics reads

$$\begin{cases} \dot{x} = \tilde{Q}^{-1}(x)\nabla_x\tilde{P}_D(x) + \tilde{g}(x)\phi(x,\xi) \\ \dot{\xi} = \tilde{g}^T(x)\nabla_x\tilde{P}_D(x) \quad (= y). \end{cases} \tag{27}$$

Next, we define the Lyapunov function candidate

$$V(x,\xi) = \tilde{P}_D(x) + \frac{1}{2}\xi^T K_I \xi,$$

and differentiate $V(x,\xi)$ along the trajectories of (27), i.e.,

$$\begin{aligned} \dot{V}(x,\xi) = \nabla_x^T\tilde{P}_D(x)\Big[&\tilde{Q}^{-1}(x) \\ &-\tilde{g}^T(x)K_P\tilde{g}(x)\Big]\nabla_x\tilde{P}_D(x), \end{aligned} \tag{28}$$

Since the symmetric part of $\tilde{Q}^{-1}(x)$ is negative semi-definite by assumption and $\tilde{g}^T(x)K_P\tilde{g}(x) > 0$, we have that $\dot{V}(x,\xi) \leq 0$, for all x. Hence, we conclude that y and ξ are bounded, and thus $x_\star$ is still (Lyapunov) stable. The claim that the equilibrium point is asymptotically stable follows directly from LaSalle's invariance principle. ◀

At a first glance, the proposed control strategy is of course structurally equivalent to the well-know and widely used PI (proportional-integral) control. However, since the present construction of the control action is achieved through some kind of shaping process of the reactive power in the circuit, and more importantly, since it involves a completely different set of outputs, the application of PI control in the reactive port-Hamiltonian context seems novel. Moreover, depending on the structure of the circuit, the passive outputs may 'naturally' contain derivative actions on the signals (see e.g., Example 4, where y was found to be $y = \dot{v}_C/L$). This means that adding an integral term to the 'differentiated' output signals results in a proportional feedback, while a proportional feedback of a signal containing derivatives results in a differentiating action. Indeed, as will be illustrated, using our Van der Pol circuit example, the integral action reduces to a simple proportional controller, while the proportional part of the control reduces to a differentiating (D) action.

Example 5. To motivate the use of the previously developed theory, consider again the Van der Pol oscillator of Fig. 4. In applications, the characteristic function of the conductor is usually defined by $\hat{i}_G(v_C) = \alpha v_C(v_C^2 - \beta)$, where α and β are some (constant) design parameters. It is easily observed that

$$\nabla_{v_C}\hat{i}_G(v_C)\begin{cases} \text{passive} & \text{if } |v_C| > \sqrt{\beta} \\ \text{non-passive} & \text{if } |v_C| < \sqrt{\beta} \end{cases}, \tag{29}$$

According to (29), the circuit is only passive, with port variables $\big(v_{S_d}, \dot{v}_C/L\big)$, for all $|v_C| > \sqrt{\beta}$. However, straightforward application of Proposition 5 yields that the circuit dynamics in closed-loop with the PI controller (26) are given by

$$\begin{aligned} -L\frac{di_L}{dt} &= K_P'\dot{v}_C + \big(1 + K_I'\big)v_C \\ C\frac{dv_C}{dt} &= i_L - \hat{i}_G(v_C), \end{aligned} \tag{30}$$

[5] Proposition 5 has close relations with Proposition 9, pp. 49 of (Rodriguez, 2002), where a similar type of control is proposed for conventional pre-controlled port-Hamiltonian systems.

[6] Note that $x_\star$ is clearly an equilibrium point of the open loop system, that is furthermore stable.

where we have defined $K'_P = K_P/L$ and $K'_I = K_I/L$, respectively. Regarding the previous discussion, it is indeed directly recognized that the P-action, i.e., the term $-K'_P\dot{v}_C$, actually represents a D-action, while the I-action,

$$-K'_I \int_0^t \dot{v}_C(t')dt' = -K'_I v_C,$$

reduces to a simple P-action. (Thus, the PI controller acts as a PD controller.) Substitution of the second equation of (30) into the first and rearranging the terms yields

$$-L\frac{di_L}{dt} = K''_P i_L + v_C - \left(K''_P \nabla_{v_C}\hat{i}_G(v_C) - K'_I\right)v_C,$$

where we have defined $K''_P = K'_P/C$. It is directly noticed that $-K''_P i_L$ represents a damping term, which plays a similar role as if there where a real resistor connected in series with the inductor. Note that the third right-hand term, $\left(K''_P \nabla_{v_C}\hat{i}_G(v_C) - K'_I\right)v_C$, destroys the reciprocal structure, and thus the port-Hamiltonian/Brayton-Moser form of the closed-loop system. Now, suppose that we select a voltage depending gain $K'_I = \hat{K}'_I(v_C) = K''_P \nabla_{v_C}\hat{i}_G(v_C)$, the resulting closed-loop dynamics become

$$-L\frac{di_L}{dt} = K''_P i_L + v_C$$
$$C\frac{dv_C}{dt} = i_L - \hat{i}_G(v_C).$$

Evaluating (28) along the closed-loop dynamics suggests that if we choose K''_P large enough, i.e., choose K''_P such that it dominates $\nabla_{v_C}\hat{i}_G(v_C)$, the circuit will be stable for all (i_L, v_C). Indeed, we at least need to require that $K''_P > \sqrt{L/C}$, which is sufficient for stability (see (Brayton and Moser, 1964), Thm. 3, pp. 19).

6. CONCLUDING REMARKS

In this paper we have shown that, under some regularity assumptions on the mixed-potential function, the Brayton-Moser equations can be transformed into a port-Hamiltonian system with dissipation—with state variables inductor currents and capacitor voltages, and with Hamiltonian a function related with the reactive power of the circuit. For that reason, and with some obvious abuse of notation, the new description is coined a *reactive* port-Hamiltonian system. Furthermore, the reactive port-Hamiltonian framework naturally suggests a new passivity property that—unlike the passivity property of (Jeltsema *et al.*, 2003*b*)—does not impose any order relationships between the electric and magnetic energies. The new passivity property has been shown to be potentially useful for some control application, including the challenging and widely elusive reactive power compensation problem.

Although we have restricted here to the case of linear active elements, the main ideas apply as well to the nonlinear case. Indeed, it is easy to show that the effect of the nonlinearities appears as some additional terms on the diagonal of the matrix $\tilde{Q}(x)$, that is

$$\tilde{Q}(x) = 2\begin{bmatrix} -\nabla^2_{i_L}A(i_L) + \star & \gamma \\ -\gamma^T & -\nabla^2_{v_C}B(v_C) - \star \end{bmatrix},$$

denoted here with a '$\star$'. Given that these terms have a complicated expression we have preferred to present here the linear case which, as shown in the paper, has very nice and physically intuitive interpretations.

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