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Vegter, Gert

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32 COMPUTATIONAL TOPOLOGY

Gert Vegter

INTRODUCTION

Topology studies point sets and their invariants under continuous deformations, invariants such as the number of connected components, holes, tunnels, or cavities. Metric properties such as the position of a point, the distance between points, or the curvature of a surface, are irrelevant to topology. A high level description of the main concepts and problems in topology is given in Section 32.1. Computational topology deals with the complexity of such problems, and with the design of efficient algorithms for their solution, in case these problems are tractable. These algorithms can deal only with spaces and maps that have a finite representation. To this end we consider simplicial complexes and maps (Section 32.2) and CW-complexes (Section 32.3). Section 32.4 deals with algebraic invariants of topological spaces, which are in general easier to compute than topological invariants. Mapping (embedding) a topological space 1–1 into another space may reveal some of its topological properties. Several types of embeddings are considered in Section 32.5. Section 32.6 deals with the classification of immersions of a space into another space. These maps are only *locally* 1–1, and hence more general than embeddings. Section 32.7 constitutes a brief introduction to Morse theory.

Many computational problems in topology are undecidable (in the sense of complexity theory). The mathematical literature of this century contains many (beautiful) topological algorithms, usually reducing to decision procedures, in many cases with exponential-time complexity. The quest for efficient algorithms for topological problems has started rather recently. Most of the problems in computational topology still await an efficient solution.

32.1 TOPOLOGICAL SPACES AND MAPS

Topology deals with the classification of spaces that are the same up to some equivalence relation. We introduce these notions, and describe some classes of topological problems.

GLOSSARY

Space: In this chapter a *topological space* (or *space*, for short) is a subset of some Euclidean space \mathbb{R}^d , endowed with the topology of \mathbb{R}^d .

Map: A function $f : X \rightarrow Y$ from a space X to a space Y is a map if f is continuous.

Homeomorphism: A 1–1 map $h : X \rightarrow Y$, with a continuous inverse, is called

a homeomorphism from X to Y (or: between X and Y).

Topological equivalence: Two spaces are topologically equivalent (or *homeomorphic*) if there is a homeomorphism between them.

Embedding: A map $f : X \rightarrow Y$ is an embedding if f is a homeomorphism onto its image. We say that X can be (topologically) embedded in Y .

Homotopy of maps: Two maps $f_0, f_1 : X \rightarrow Y$ are homotopic if there is a map $F : X \times [0, 1] \rightarrow Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$, for all $x \in X$.

Homotopy equivalence: Two spaces X and Y are homotopy-equivalent if there are maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that gf and fg are homotopic to the identity mappings on X and Y , respectively. Obviously topological equivalence implies homotopy equivalence.

Topological/homotopy invariant: A map ζ associating a number, or a group, $\zeta(X)$ to a space X , is a topological invariant (resp. homotopy invariant) if $\zeta(X_1)$ and $\zeta(X_2)$ are equal, or isomorphic, for topologically equivalent (resp. homotopy-equivalent) spaces X_1 and X_2 .

Contractibility: A space is contractible if it is homotopy-equivalent to a point.

Unit interval \mathbb{I} : The interval $[0, 1]$ in \mathbb{R} .

Ball: Open d -ball: $\mathbb{B}^d = \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_1^2 + \dots + x_d^2 < 1\}$. Closed d -ball: $\overline{\mathbb{B}}^d$ is the closure of \mathbb{B}^d .

Half ball: $\mathbb{B}_+^d = \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_1^2 + \dots + x_d^2 < 1 \text{ and } x_d \geq 0\}$.

Sphere: $\mathbb{S}^d = \{(x_1, \dots, x_{d+1}) \in \mathbb{R}^{d+1} \mid x_1^2 + \dots + x_{d+1}^2 = 1\}$ is the d -sphere. It is the boundary of the $(d+1)$ -ball.

Manifold: A space X is a d -dimensional (topological) manifold (also: d -manifold) if every point of X has a neighborhood homeomorphic to \mathbb{B}^d . X is a d -manifold *with boundary* if every point has a neighborhood homeomorphic to \mathbb{B}^d or \mathbb{B}_+^d .

Surface: A 2-dimensional manifold, with or without boundary. A *closed surface* is a surface without boundary.

Curve: A curve in X is a continuous map $\mathbb{I} \rightarrow X$. For $x_0 \in X$, a x_0 -based *closed curve* c is a curve for which $c(0) = c(1) = x_0$.

BASIC TOPOLOGICAL PROBLEMS AND APPLICATIONS

Topological equivalence and classification: Decide whether a space belongs to (is topologically equivalent to an element of) a class of known objects.

Application: Object recognition in computer vision.

Homotopy equivalence: Decide whether two spaces are homotopy-equivalent, or whether a curve in X is contractible (the *contractibility problem*).

Applications: α -hull, skeletons; see [Ede94]. Concurrent computing; see [HS94a].

Embedding: Decide whether X can be embedded in Y . If so, construct an embedding.

Application: Graph drawing (Chapter 52), VLSI-layout, and wire routing.

Extension of maps: Let A be a subspace of X . Decide whether a map $f : A \rightarrow Y$ can be extended to X (i.e., whether there is a map $F : X \rightarrow Y$ whose restriction to A is f).

Lifting of maps: Let $f : A \rightarrow X$ and $p : Y \rightarrow X$ be maps. Decide whether there is a map $F : A \rightarrow Y$ such that $pF = f$.

Application: Inverse kinematics problems and tracking algorithms in robotics; see [Bak90] and Section 48.1.

32.2 SIMPLICIAL COMPLEXES

Computation requires finite representation of topological spaces. Representing a space by a simplicial complex corresponds to the idea of building the space from simplices. Simplicial complexes may be considered as combinatorial objects, with a straightforward data structure for their representation. See also Section 18.1.

GLOSSARY

Geometric simplex: A geometric k -simplex σ_k is the convex hull of a set A of $k + 1$ independent points a_0, \dots, a_k in some Euclidean space \mathbb{R}^d (so $d \geq k$). A is said to **span** the simplex σ_k . A simplex spanned by a subset A' of A is called a **face** of σ_k . The face is proper if $\emptyset \neq A' \neq A$. The **dimension** of the face is $|A'| - 1$. A 0-dimensional face is called a **vertex**, a 1-dimensional face is called an **edge**. The union σ_k^i , $0 \leq i \leq k$, of all faces of dimension at most i is called the **i -skeleton** of σ_k . In particular σ_k^0 is the set of vertices, and $\sigma_k^k = \sigma_k$. An **orientation** of σ_k is induced by an ordering of its vertices, denoted by $\langle a_0, \dots, a_k \rangle$, as follows: For any permutation π of $0, \dots, k$, the orientation $\langle a_{\pi(0)}, \dots, a_{\pi(k)} \rangle$ is equal to $(-1)^{\text{sign}(\pi)} \langle a_0, \dots, a_k \rangle$, where $\text{sign}(\pi)$ is the number of transpositions of π (so any simplex has two distinct orientations). If τ is a $(k-1)$ -dimensional face of σ , obtained by omitting the vertex a_i , then the **induced orientation** on τ is $(-1)^i \langle a_0, \dots, \hat{a}_i, \dots, a_k \rangle$, where the hat indicates omission of a_i .

Geometric simplicial complex K : A finite set of simplices in some Euclidean space \mathbb{R}^m , such that (i) if σ is a simplex of K and τ is a face of σ , then τ is a simplex of K , and (ii) if σ and τ are simplices of K , then $\sigma \cap \tau$ is either empty or a common face of σ and τ . The **dimension** of K is the maximum of the dimensions of its simplices. The **underlying space** of K , denoted by $|K|$, is the union of all simplices of K , endowed with the subspace topology of \mathbb{R}^m . The **i -skeleton** of K , denoted by K^i , is the union of all simplices of K of dimension at most i . A **subcomplex** L of K is a subset of K that is a simplicial complex.

Combinatorial simplicial complex: A pair $\mathcal{K} = (V, \Sigma)$, where V contains finitely many elements, called vertices, and Σ is a collection of subsets of V , called (combinatorial) simplices, with the property that any subset of a simplex is a simplex. The dimension of a simplex is one less than the number of vertices it contains. The dimension of \mathcal{K} is the maximum of the dimensions of its simplices.

Geometric realization: A geometric simplicial complex K in \mathbb{R}^m is called a geometric realization (in \mathbb{R}^m) of the combinatorial simplicial complex $\mathcal{K} = (V, \Sigma)$ if there is a 1–1 correspondence $f : V \rightarrow K^0$, such that $A \subset V$ is a simplex of \mathcal{K} iff $f(A)$ spans a simplex of K . Furthermore \mathcal{K} is called the **abstraction** of K .

Triangulation: A triangulation of a topological space X is a pair (K, h) , where K is a geometric simplicial complex and h is a homeomorphism from the underlying space $|K|$ to X .

Barycentric subdivision: The **barycenter** (center of mass) of a geometric k -simplex with vertices a_0, \dots, a_k in \mathbb{R}^m is the point $1/(k+1) \sum_{i=0}^k a_i$. The barycentric subdivision of a geometric simplicial complex K is defined inductively: (i) the barycentric subdivision of the 0-skeleton σ^0 is σ^0 itself; (ii) if σ is an i dimensional face of K , $i > 0$, then σ is subdivided into the collection of simplices $C(b, \tau)$, for all simplices τ in the barycentric subdivision of the $(i-1)$ -skeleton of σ . Here $C(b, \tau)$ is the convex hull of $b \cup \tau$ and b the barycenter of σ .

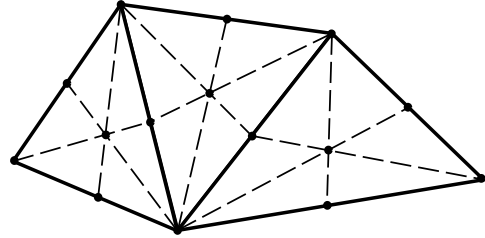


FIGURE 32.2.1
Barycentric subdivision.

The (first) barycentric subdivision of a simplicial complex K is the simplicial complex $s(K)$ obtained by barycentric subdivision of all simplexes of K ; see Figure 32.2.1. The i th barycentric subdivision of K , $i > 1$, is defined inductively as $s(s^{i-1}(K))$. A simplicial complex L is called a **refinement** of K if $L = s^i(K)$, for some $i \geq 0$.

Simplicial map: A simplicial map between simplicial complexes K and L is a function $f : |K| \rightarrow |L|$ such that (i) if a is a vertex of K then $f(a)$ is a vertex of L ; (ii) if a_0, \dots, a_k are vertices of a simplex of K , then the convex hull of $f(a_0), \dots, f(a_k)$ is a simplex of L (whose dimension may be less than k); and (iii) f is linear on each simplex: if $x = \sum_{i=0}^k \lambda_i a_i$ is a point in a simplex with vertices a_0, \dots, a_k , then $f(x) = \sum_{i=0}^k \lambda_i f(a_i)$.

Simplicial equivalence: Two simplicial complexes K and L are simplicially equivalent iff there are simplicial maps $f : |K| \rightarrow |L|$ and $g : |L| \rightarrow |K|$ such that gf is the identity on $|K|$ and fg is the identity on $|L|$.

Piecewise linear (PL)-equivalence: Two simplicial complexes K and L are called piecewise linearly equivalent (PL-equivalent, for short) if there is a refinement K' of K and L' of L such that K' and L' are simplicially equivalent.

Orientation of a simplicial manifold: An orientation of a simplicial complex K , whose underlying space is a d -manifold, is a choice of orientation for each simplex of K , such that, if τ is a $(d-1)$ -face of two distinct d -simplices σ_1 and σ_2 , then the orientation on τ induced by σ_1 is the opposite of the orientation induced by σ_2 . The manifold is called **orientable** if it has a triangulation that has an orientation, otherwise it is **nonorientable**.

Euler characteristic: (Combinatorial definition; cf. Section 32.4) The Euler characteristic of a simplicial d -complex K , denoted by $\chi(K)$, is the number $\sum_{i=0}^d (-1)^i \alpha_i$, where α_i is the number of i -simplices of K .

Polygonal schema for a surface: Let $\mathcal{M}_g(a_1, b_1, \dots, a_g, b_g)$ be a regular $4g$ -gon, whose successive edges are labeled $a_1, b_1, \bar{a}_1, \bar{b}_1, \dots, a_g, b_g, \bar{a}_g, \bar{b}_g$. Edge x is directed counterclockwise, edge \bar{x} clockwise. The space obtained by identifying edges x and \bar{x} , as indicated by their direction, is a closed oriented surface, denoted by \mathbb{M}_g ; see e.g., [Sti93, Chapter 1.4]. This surface, called the orientable surface of genus g , is homeomorphic to a 2-sphere with g handles.

Let $\mathcal{N}_g(a_1, \dots, a_g)$ be the regular $2g$ -gon whose successive edges are labeled $a_1, a_1, \dots, a_g, a_g$. Identifying edges in pairs, as indicated by their oriented labels, yields a closed nonorientable surface, denoted by \mathbb{N}_g . This surface, called the nonorientable surface of genus g , is homeomorphic to a 2-sphere with g cross-caps.

The labeled polygon $\mathcal{M}_g(\mathcal{N}_g)$ is called the polygonal schema of $\mathbb{M}_g(\mathbb{N}_g)$. \mathbb{M}_1 is the *torus*, \mathbb{N}_1 is the *projective plane*, \mathbb{N}_2 is the *Klein bottle*.

Minimal triangulation: A triangulation of a surface is called minimal if it has no contractible edges (i.e., contracting an edge yields a subdivision that is not a triangulation).

EXAMPLES

1. A **graph** is a 1-dimensional simplicial complex. The complete graph with n vertices is the 1-skeleton of an $(n-1)$ -simplex: $K_n = \sigma_{n-1}^1$.
2. Every connected, compact 1-manifold is topologically equivalent to \mathbb{S}^1 or \mathbb{I} .
3. The Delaunay triangulation of a set of points in general position in \mathbb{R}^d is a simplicial complex.

BASIC PROPERTIES

1. Every triangulation of an orientable manifold has an orientation (i.e., the definition of orientability does not depend on the particular triangulation).
2. The Euler characteristic is a homotopy (and hence a topological) invariant (cf. Section 32.4).
3. A simplicial 2-complex is (topologically equivalent to) a closed surface iff every edge is incident with two faces, and the faces around a vertex can be ordered as f_0, \dots, f_{k-1} so that there is exactly one edge incident with both f_i and f_{i+1} (indices modulo k).
4. An oriented closed surface X is topologically equivalent to \mathbb{S}^2 if $\chi(X) = 2$, or to \mathbb{M}_g if $\chi(X) \neq 2$, where g is uniquely determined by $\chi(X) = 2 - 2g$. A nonorientable closed surface X is topologically equivalent to \mathbb{N}_g , with $\chi(X) = 2 - g$. The number g is called the **genus** of the surface.
5. Every surface has finitely many minimal triangulations. (This number is 1 for \mathbb{S}^2 , 2 for the projective plane, and 22 for the torus; cf. Section 21.2.)
6. A simplicial complex is a 3-manifold without boundary iff every 2-simplex is incident with exactly two 3-simplices and $\chi(M) = 0$. See [Fom91, p. 184].

7. Every combinatorial simplicial d -complex has a geometric realization in \mathbb{R}^{2d+1} .
8. Two geometric realizations K_1 and K_2 of a combinatorial simplicial complex \mathcal{K} are simplicially equivalent (therefore the topology of K does not depend on the Euclidean space in which \mathcal{K} is geometrically realized).
9. A simplicial map $f : |K| \rightarrow |L|$ is continuous. Hence both simplicial equivalence and PL-equivalence imply topological equivalence.
10. **Hauptvermutung:** Two simplicial complexes are PL-equivalent iff their underlying spaces are topologically equivalent. The Hauptvermutung is true if the underlying spaces are manifolds of dimension ≤ 3 , and open for manifolds of dimension exceeding 3. It is false for general simplicial complexes, see Milnor [Mil61]. (*Reidemeister torsion* is a PL-invariant, but not a topological invariant [DFN90, pp. 156, 372].)

ALGORITHMS, DATA STRUCTURES, AND COMPLEXITY

Representation of spaces: The *Delaunay complex* D_X is a geometric simplicial complex which is, under some conditions, homotopically (or even topologically) equivalent to a given subspace X of some Euclidean space \mathbb{R}^d . See [ES94]. For applications of simplicial complexes to geometric modeling, see [Ede94].

Classification of surfaces: The Euler characteristic and orientability of a triangulated surface with n simplices can both be computed in $O(n)$ time.

Polygonal schema for a surface of genus $g > 0$: Given a triangulation of a closed orientable (nonorientable) surface of genus $g > 0$ with n triangles, there is a sequence of $O(n)$ elementary transformations (called **cross-cap** or **handle normalizations**) that turns the triangulation into a polygonal schema of the form $\mathcal{M}_g(\mathcal{N}_g)$. This sequence of transformations can be computed in $O(n \log n)$ time [VY90].

Minimal triangulations of a surface: For a triangulation of a surface of genus g with n triangles, a sequence of $O(n)$ edge contractions leading to a minimal triangulation, can be computed in $O(n \log n)$ time [Sch91]. Therefore the classification problem for triangulated surfaces with n -triangles can be solved in $O(n)$ time; see property (4) above.

Isomorphism (simplicial equivalence): The homeomorphism problem for 2-complexes is equivalent to the graph-isomorphism problem [ÓWW00]. It is unknown whether the graph-isomorphism problem is solvable in polynomial time (in the size of the graphs). See [vL90].

PL-equivalence: Deciding whether two arbitrary simplicial d -manifolds are PL-equivalent is unsolvable for $d \geq 4$ [Sti93, Chapter 9].

OPEN PROBLEMS

1. Design an algorithm that determines whether a simplicial 3-manifold is topologically equivalent to \mathbb{S}^3 . This is a hard problem; see [VKF74] for partial results.

2. Design an algorithm that computes all minimal triangulations for a surface of genus g .
3. Determine the minimal size of a triangulation for a triangulable d -manifold [BK87, Sar87].

32.3 CELL COMPLEXES

Although simplicial complexes are convenient representations of topological spaces from an algorithmic point of view, they usually have many simplices. If a representation with a smaller number of cells is desirable, CW-complexes seem appropriate. See also Section 18.4.

GLOSSARY

Attaching cells to a space: Let X and Y be topological spaces, such that $X \subset Y$. We say that Y is obtained by attaching a (finite) collection of k -cells to X if $Y \setminus X$ is the disjoint union of a finite number of open k -balls $\{e_i^k \mid i \in I\}$, with the property that, for each i in the index set I , there is a map $f_i : \mathbb{B}^k \rightarrow \bar{e}_i^k$, called the **characteristic map** of the cell e_i^k , such that $f_i(\mathbb{S}^{k-1}) \subset X$ and the restriction $f_i \mid \mathbb{B}^k$ is a homeomorphism $\mathbb{B}^k \rightarrow e_i^k$. (Note: $\bar{\mathbb{B}}^k$ need not be homeomorphic to \bar{e}_i^k .)

Cell complex (CW complex): A (finite) CW-decomposition of a topological space X is a finite sequence

$$\emptyset = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X^d = X \quad (32.3.1)$$

such that (i) X^0 is a finite set of points, called the 0-cells of X ; (ii) for $k > 0$, X^k is obtained from X^{k-1} by attaching a finite number of k -cells to X^{k-1} . The connected components of $X^k \setminus X^{k-1}$ are called the k -cells of X . The space X is called a (finite) CW-complex. The dimension of X is the maximal dimension of the cells of X . A finite CW-complex is called **regular** if the characteristic map of each cell is a homeomorphism. (“CW” stands for “Closure-finite with the Weak topology.”)

EXAMPLES AND ELEMENTARY PROPERTIES

1. The d -sphere ($d > 0$) is a CW-complex, obtained by attaching a d -cell to a point p (so $X^k = \{p\}$, for $0 \leq k < d$, and $X^d = \mathbb{S}^d$). This CW-complex is not regular: the characteristic map of the d -cell maps the boundary of \mathbb{B}^d to a single point.
2. The **orientable surface** \mathbb{M}_g of genus $g > 1$ is a CW-complex with one 0-cell, $2g$ 1-cells, and one 2-cell. Let the 1-cells be $a_1, b_1, \dots, a_g, b_g$, endowed with an orientation (direction). The characteristic map of the 2-cell is uniquely determined by attaching the labeled $4g$ -gon $\mathcal{M}_g(a_1, b_1, \dots, a_g, b_g)$ (cf. Section 32.2) to the 1-skeleton by mapping an edge to the 1-cell with the same

label, so that the directions of the edge and the 1-cell correspond. See [VY90]. The $2g$ 1-cells are curves on the surface, disjoint except at their common endpoint (which is the 0-cell). These curves are called *canonical generators* of the surface (see Section 32.4 for a justification of this nomenclature). The total number of cells is $2g + 2$, whereas the total number of simplices in a triangulation is at least $10g - 10 + \Theta(\sqrt{g})$ [JR80].

3. The *nonorientable surface* \mathbb{N}_g of genus $g > 1$ is a CW-complex, with one 0-cell, g 1-cells, and one 2-cell. The characteristic map of the 2-cell is obtained from the polygonal schema represented by the $2g$ -gon $\mathcal{N}_g(a_1, \dots, a_g)$.
4. A geometric simplicial complex is a regular CW-complex.
5. The dual map of a triangulation of a surface is a regular CW-complex, but not a simplicial complex.
6. Examples of CW-complexes arising in computational geometry are: arrangements of hyperplanes in \mathbb{R}^d (after addition of a point at infinity), the visibility complex [PV93], the free space of a polygonal robot moving amid polygonal obstacles (see [SS83] and Chapter 47 of this Handbook), and the zero-set of a generic polynomial defined on $\mathbb{S}^d \subset \mathbb{R}^{d+1}$.

ALGORITHMS AND DATA STRUCTURES

Representation: A data structure for the representation and manipulation of a finite, d -manifold CW-complex is described in [Bri93].

CW-decomposition of surfaces from triangulations: For a triangulated surface of genus g , with a total of n simplices, a set of canonical generators (cf. property (2)) can be computed in $O(gn)$ time, which is optimal in the worst case [VY90]. Two algorithms achieving this time complexity have been implemented; See [LPVV01].

Each of the g or $2g$ canonical generators is represented by a polygonal curve whose vertices are on the 1-skeleton, while its other points are in the interior of a 2-simplex. In some cases the total number of edges of a single generator is $O(n)$. This method can be used to construct covering surfaces of m sheets in time $O(gnm)$ time and space; see also Section 32.4.

CW-decomposition in motion planning: A general method to solve motion planning problems is the construction of a cell decomposition (Equation 32.3.1) of the free space X of the robot, together with a *retraction* $r : X \rightarrow X^k$ of X onto a low-dimensional skeleton, such that there is a motion from initial position $x_0 \in X$ to final position $x_1 \in X$ iff there is a motion from $r(x_0)$ to $r(x_1)$. This may be regarded as a reduction of the degrees of freedom of the robot. Because in general the complexity of the motion planning problem is exponential in the number of degrees of freedom, this approach simplifies the problem. For more details on the cell decomposition method in motion planning, see Section 47.1.

32.4 ALGEBRAIC TOPOLOGY

In algebraic topology one associates homotopy-invariant groups (homology and homotopy groups) to a space, and homotopy-invariant homomorphisms to maps

between spaces. In passing from topology to algebra one may lose information since topologically distinct spaces may give rise to identical algebraic invariants. However, one gains on the algorithmic side, since the algebraic counterpart of an intractable topological problem may be tractable.

32.4.1 SIMPLICIAL HOMOLOGY GROUPS

Historically speaking, simplicial homology groups were among the first invariants associated with topological spaces. They are conceptually and algorithmically appealing. Modern algebraic topology usually deals with singular and cellular homology groups, which are more convenient from a mathematical point of view.

GLOSSARY

Ordered simplex: Let the vertices of a simplicial complex K be ordered v_0, \dots, v_m . A k -simplex of K with vertices v_{i_0}, \dots, v_{i_k} , $i_0 < \dots < i_k$ is represented by the symbol $[v_{i_0}, \dots, v_{i_k}]$, and called an ordered simplex.

Simplicial chain: If G is an abelian group, then an (ordered) simplicial k -chain is a formal sum of the form $\sum_j a_j \sigma_j$, with $a_j \in G$ and σ_j the symbol of a k -simplex in K . With the obvious definition for addition, the set of all (ordered) simplicial k -chains forms a (free) abelian group $C_k(K, G)$, called the group of (ordered) simplicial k -chains of K . If $G = \mathbb{Z}$, the group of integers, an element of $C_k(K, G)$ is called an *integral k -chain*.

Boundary operator: The boundary operator $\partial_k : C_k(K, G) \rightarrow C_{k-1}(K, G)$ is defined as follows. For a single (ordered) k -simplex $\sigma = [v_{i_0}, \dots, v_{i_k}]$, let $\partial_k \sigma = \sum_{h=0}^{k-1} (-1)^h [v_{i_0}, \dots, \hat{v}_{i_h}, \dots, v_{i_k}]$, and then let ∂_k be extended linearly, viz., $\partial_k(\sum_j a_j \sigma_j) = \sum_j a_j \partial_k \sigma_j$. The boundary operator is a homomorphism of groups. It satisfies $\partial_k \partial_{k+1} = 0$.

Simplicial k -cycles: $Z_k(K, G) = \ker \partial_k$ is called the group of (ordered) simplicial k -cycles.

Simplicial k -boundaries: $B_k(K, G) = \text{im } \partial_{k+1}$ is called the group of (ordered) simplicial k -boundaries. Since the boundary of a boundary is 0, B_k is a subgroup of $Z_k(K, G)$.

Simplicial homology groups: The group $H_k(K, G) = Z_k(K, G)/B_k(K, G)$ is the k th (simplicial) homology group of K . This is a purely combinatorial object, since in fact it is defined for abstract simplicial complexes. If $G = \mathbb{Z}$, these groups are called *integral homology groups*, usually denoted by $H_k(K)$. If G is a field (such as \mathbb{R}), then $H_k(K, G)$ is a vector space.

Homology groups of a triangulable topological space: $H_k(X, G) = H_k(K, G)$, if K is a simplicial complex triangulating X . This definition is independent of the triangulation K : if $h_i : K_i \rightarrow X$, $i = 1, 2$, are two triangulations of X , then $H_k(K_1, G) = H_k(K_2, G)$.

Betti numbers: The k th Betti number $\beta_k(K)$ of a simplicial complex K is the dimension of the real vector space $H_k(K, \mathbb{R})$. (For an alternative definition, see [Bre93, Chapter IV.1].)

Euler characteristic: The Euler characteristic $\chi(K)$ of a simplicial d -complex K is defined by $\chi(K) = \sum_{i=0}^d (-1)^i \beta_i(K)$. This definition is equivalent to the one of Section 32.2.

EXAMPLES

1. The n -sphere ($n > 0$): $H_k(\mathbb{S}^n, \mathbb{Z}) = \mathbb{Z}$, if $k = 0$ or n , and 0 otherwise.
2. *Orientable surface*: For $g \geq 0$, $H_0(\mathbb{M}_g, G) = H_2(\mathbb{M}_g, G) = G$, $H_1(\mathbb{M}_g, G) = \oplus_{i=1}^{2g} G$, $H_k(\mathbb{M}_g, G) = 0$ for $k > 2$. Taking $G = \mathbb{R}$ we see that $\chi(\mathbb{M}_g) = 2 - 2g$.
3. *Nonorientable surface*: For $g \geq 0$, $H_0(\mathbb{N}_g, \mathbb{Z}) = \mathbb{Z}$, $H_1(\mathbb{N}_g, \mathbb{Z}) = \oplus_{i=1}^{g-1} \mathbb{Z} \oplus \mathbb{Z}_2$, $H_k(\mathbb{N}_g, \mathbb{Z}) = 0$ for $k \geq 2$. $H_0(\mathbb{N}_g, \mathbb{R}) = \mathbb{R}$, $H_1(\mathbb{N}_g, \mathbb{R}) = \oplus_{i=1}^{g-1} \mathbb{R}$, $H_2(\mathbb{N}_g, \mathbb{R}) = 0$. Hence, $\chi(\mathbb{N}_g) = 2 - g$.

BASIC PROPERTIES

1. Homology is a homotopy invariant: if X_1 and X_2 are homotopy-equivalent, then $H_k(X_1) = H_k(X_2)$ for all k . In particular, Betti numbers and the Euler characteristic are homotopy invariants.
2. For a simplicial d -complex K : $H_k(K, G) = 0$ for $k > d$.
3. Let $\alpha_i(K)$ be the number of i -simplices of a simplicial d -complex K . Then $\chi(K) = \sum_{i=0}^d (-1)^i \alpha_i(K)$. This justifies the definition of χ in Section 32.2.

COMPUTING BETTI NUMBERS AND HOMOLOGY GROUPS

See Table 32.4.1 for the algorithmic complexity of computing the Betti numbers of several important types of spaces. The paper [DG98] also presents a method of computing a basis for the first and second homology groups of a complex in \mathbb{R}^3 of size n , in time $O(\bar{g}n^2)$, where the integer \bar{g} is an invariant of the complex, with $\bar{g} < n$.

Bounds on the sum of the Betti numbers of closed semialgebraic sets are given in [Bas99], as well as a single-exponential-time algorithm for computing the Euler characteristic of arbitrary closed semialgebraic sets.

TABLE 32.4.1 Complexity of computing Betti numbers.

TYPE OF SPACE	COMPLEXITY	SOURCE
Simplicial subcomplex of \mathbb{S}^3 of size n	$O(n\alpha(n))$	[DE95]
Simplicial complex in \mathbb{R}^3 of size n	$O(n)$	[DG98]
Sparse simplicial complex of size n	$O(n^2)$ (probabilistic)	[DC91]
Semialgebraic set, defined by m poly's (deg $\leq d$) on \mathbb{R}^n , n fixed	polynomial in m, d	[SS83]

32.4.2 HOMOTOPY GROUPS

Homotopy groups usually provide more information than homology groups, but are generally harder to compute. The main object is the fundamental group, whose computation requires some combinatorial group theory.

GLOSSARY

Fundamental group: The space of x_0 -based curves on X is endowed with a group structure by (group multiplication) $(u_1 \cdot u_2)(t) = u_1(2t)$, if $0 \leq t \leq \frac{1}{2}$, and $u_2(2t - 1)$ if $\frac{1}{2} \leq t \leq 1$, and (inverse) $u^{-1}(t) = u(1 - t)$.

This group structure can be extended to homotopy classes of x_0 -based curves: If u, v are homotopic, then u^{-1} and v^{-1} are homotopic, and if u_i and v_i , $i = 1, 2$, are homotopic, then $u_1 \cdot u_2$ and $v_1 \cdot v_2$ are homotopic (homotopies respect the basepoint x_0). The group of homotopy classes of closed x_0 -based curves is called the fundamental group (or, the **first homotopy group**) of (X, x_0) , and is denoted by $\pi_1(X, x_0)$. If X is connected, the definition is independent of the basepoint. Then the fundamental group is denoted by $\pi_1(X)$.

Combinatorial definition of the fundamental group: If X is a connected space with triangulation K and vertices a_0, \dots, a_m , then the fundamental group has generators g_{ij} , one per ordered 1-simplex $[a_i, a_j]$, and relations $g_{ij}g_{jk}g_{ik}^{-1} = 1$, one for each ordered 2-simplex $[a_i, a_j, a_k]$ [Mau70, Chapter 3]. See [Sti93] for an introduction to combinatorial group theory.

k th homotopy group: Let $s_0 \in \mathbb{S}^k$, for $k \geq 1$. The space of homotopy classes of basepoint-preserving maps $(\mathbb{S}^k, s_0) \rightarrow (X, x_0)$ can be endowed with a group structure. The group is called the k th homotopy group of (X, x_0) , and is denoted by $\pi_k(X, x_0)$.

Word problem for a group G : Given a (finitely generated) group generated by g_1, \dots, g_k (the alphabet), and a finite set of relations of the form $g_1^{m_1} \cdots g_k^{m_k} = 1$ (rewrite rules) with $m_i \in \mathbb{Z}$, decide whether a given word of the form $g_1^{n_1} \cdots g_k^{n_k}$ represents the unit element 1.

Covering space: A continuous map $p: Y \rightarrow X$ is a covering map if every point $x \in X$ has a connected neighborhood U such that for each connected component V of $p^{-1}(x)$ the restriction of p to V is a homeomorphism $V \rightarrow U$. Y is called a covering space of X . If the cardinality n of $p^{-1}(U)$ is finite, Y is called an n -sheeted cover of X . This number is the same for all $x \in X$.

Universal covering space: A connected covering space Y of X is called universal if $\pi_1(Y) = 0$.

EXAMPLES

1. *The n -sphere* ($n > 0$): $\pi_1(\mathbb{S}^n, s_0) = \mathbb{Z}$ if $n = 1$, and 0 otherwise.
2. *Orientable surface of genus $g \geq 1$:* $\pi_1(\mathbb{M}_g)$ is generated by $2g$ generators $a_1, b_1, \dots, a_g, b_g$, with the single relation $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1$.
3. *Nonorientable surface of genus $g \geq 1$:* $\pi_1(\mathbb{N}_g)$ is generated by g generators a_1, \dots, a_g , with the single relation $a_1 a_1 \cdots a_g a_g = 1$.
4. *Universal covering space:* The universal covering space of \mathbb{S}^1 is \mathbb{R} , with covering map $p: \mathbb{R} \rightarrow \mathbb{S}^1$ defined by $p(t) = (\cos t, \sin t)$. The universal covering space of the projective plane \mathbb{P} is \mathbb{S}^2 , the covering map being antipodal identification. The plane is the universal covering space of \mathbb{M}_g and \mathbb{N}_g , $g > 0$.

BASIC PROPERTIES

1. The homotopy groups are homotopy invariants.
2. The first integral homology group is the abelianized fundamental group.
3. The fundamental group of a simplicial complex is the fundamental group of its 2-skeleton.
4. For every finitely generated group G there is a finite simplicial 2-complex K and a 4-manifold M such that $\pi_1(K) = G$ and $\pi_1(M) = G$.
5. Homotopy invariants are topological invariants, but not vice versa. For example, the *lens spaces* $L(5, 1)$ and $L(5, 2)$ are not homotopy-equivalent, but do have isomorphic homology and homotopy groups [Bre93, Chapter VI].
6. Let Y be a universal covering space of X with covering map $p : Y \rightarrow X$, and let $y_0 \in Y$ and $x_0 = p(y_0) \in X$. Every curve $c : \mathbb{I} \rightarrow X$ with $c(0) = x_0$ has a unique lift $\bar{c} : \mathbb{I} \rightarrow Y$ with $\bar{c}(0) = y_0$. Furthermore, a closed curve c is contractible in X iff \bar{c} is a closed curve in Y , i.e., $\bar{c}(1) = y_0$ [Sti93, Chapter 6]. This is the basis of Dehn's algorithm for the contractibility problem on surfaces (see below).

ALGORITHMS AND COMPLEXITY

Undecidability of homeomorphism problem: The word problem for general groups is undecidable. Hence the contractibility problem for general simplicial 2-complexes, and for manifolds of dimension ≥ 4 , is undecidable [Sti93]. A slight variation even proves that the homeomorphism problem for 4-manifolds is undecidable.

Contractibility problem for surfaces: Determine whether a curve with k edges on a triangulated surface \mathbb{M}_g of size n is contractible, and, if so, construct a contraction.

Dey and Schipper [DS95] implement Dehn's algorithm in $O(n + k \log g)$ time and $O(n + k)$ space by constructing a finite portion of the covering surface of \mathbb{M}_g , for $g > 1$, and determining whether the lift of the curve to the covering space is closed. These algorithms can also be applied to solving the homotopy problem for curves on a surface.

The paper [DG99] presents an algorithmic solution of the word problem for fundamental groups of the orientable surfaces \mathbb{M}_g , if $g \neq 2$, and of the nonorientable surfaces \mathbb{N}_g , if $g \neq 3, 4$. This algorithm yields a method to decide whether a curve on such a surface is contractible in $O(n + k)$ time and space, which is optimal.

Representation problem: There is an algorithm that decides whether a homotopy class of curves contains a simple closed curve. The algorithm of [Chi72] can be turned into a polynomial-time algorithm using methods similar to those of [Sch92] and [VY90]. (Poincaré had already given a condition for a *homology* class of a curve on a surface to contain a simple closed curve. This can also be turned into a polynomial algorithm along similar lines.)

Homotopy of polygonal paths among points in the plane: Several algorithms determine whether two polygonal paths in the plane with n points removed

are homotopic. Hershberger and Snoeyink [HS94b] construct part of the covering space to compute minimum length curves that are homotopy-equivalent to a given curve, in $\Theta(n^2)$ time, where n is the number of point-shaped holes and the input curve consists of at most n edges. Cabello, Liu, Mantler, and Snoeyink [CLMS02] present an $O(n \log n)$ algorithm to test whether two *simple* paths, with the same endpoints, are homotopic. See Section 27.2.

32.5 EMBEDDING SIMPLICIAL COMPLEXES

Embeddability problems are important for their own sake, but also for computations. Especially important algorithmically is the problem of embedding a simplicial complex in a Euclidean space of lowest dimension. See also Section 21.1.

GLOSSARY

Simplicial embedding of a simplicial complex K in simplicial complex L : A simplicial map $f : |K| \rightarrow |L|$ that is a topological embedding.

Geometric embedding of a simplicial complex K in \mathbb{R}^d : A simplicial equivalence $f : K \rightarrow L$, where L is a geometric simplicial complex in \mathbb{R}^d .

Piecewise-linear (PL) embedding of a simplicial complex K in a simplicial complex L : A simplicial embedding of a refinement K' of K in a refinement L' of L . If L is a geometric simplicial complex in \mathbb{R}^d , we say that K can be PL-embedded in \mathbb{R}^d .

PL-minimality: A simplicial complex is PL-minimal in \mathbb{R}^d if it is not PL-embeddable in \mathbb{R}^d , but every proper subcomplex can be PL-embedded in \mathbb{R}^d .

Genus of a graph: The orientable (nonorientable) genus of a graph G is the minimal genus of an orientable (nonorientable) surface in which G is PL-embeddable.

Book: A book with p pages is a simplicial complex consisting of p triangles sharing a common edge (and nothing else).

Page number of a graph: Minimal number of pages of a book in which the graph is PL-embeddable.

32.5.1 PL-EMBEDDINGS

BASIC RESULTS

1. A simplicial d -complex that is topologically embeddable in \mathbb{R}^{2d} is also PL-embeddable in \mathbb{R}^{2d} [Web67].
2. For $d \geq 3$, a simplicial d -complex K is PL-embeddable in \mathbb{R}^{2d} iff its *van Kampen obstruction class* $o(K) = 0$. ($o(K)$ is an element of the $2d$ th cohomology group of the symmetric product of K minus the diagonal; see [vK33, Sha57].) If K is a triangulation of a d -manifold, then $o(K) = 0$, so K can be embedded in \mathbb{R}^{2d} [Whi44].

3. **Kuratowski's theorem:** a graph G is PL-embeddable in the plane iff K_5 and $K_{3,3}$ are not PL-embeddable in G . The graphs K_5 and $K_{3,3}$ are called **forbidden minors** for planarity.
4. Every orientable triangulated surface can be PL-embedded in \mathbb{R}^3 . Every nonorientable triangulated surface can be PL-embedded in \mathbb{R}^4 , but not in \mathbb{R}^3 (for a simple proof of the latter, see [Mae93]).
5. Kuratowski's theorem can be rephrased by saying that K_5 and $K_{3,3}$ are the only PL-minimal 1-complexes in \mathbb{R}^2 . For each $n \geq 2$ and each d , with $n + 1 \leq d \leq 2n$, there are countably many nonhomeomorphic n -complexes that are all PL-minimal in \mathbb{R}^d [Zak69].
6. There is a finite set of forbidden minors for PL-embeddability in a surface of fixed genus g [RS90].
7. The page-number of a graph is $O(g)$ [HI92].

ALGORITHMS AND COMPLEXITY

PL-embeddability of graphs: It can be decided in $O(n \log n)$ time whether a graph with n vertices is planar (PL-embeddable in the plane). In $O(n \log n)$ time a geometric embedding in the plane can be constructed [HT74].

Graph genus: The graph genus problem is NP-complete [Tho89].

OPEN PROBLEMS

1. Give an efficient algorithm that computes the van Kampen obstruction $o(K)$ for a simplicial d -complex K with a total of n simplices. Find an algorithm that constructs a PL-embedding (of reasonable complexity) for K in case $o(K) = 0$.
2. Design an efficient algorithm that determines whether a simplicial d -complex can be PL-embedded in \mathbb{R}^k , for $d \leq k < 2d$.

32.5.2 GEOMETRIC EMBEDDINGS

MAIN RESULTS

1. Every simplicial d -complex can be geometrically embedded in \mathbb{R}^{2d+1} .
2. Every simplicial 1-complex (graph) that is PL-embeddable in \mathbb{R}^2 can be geometrically embedded in \mathbb{R}^2 (**Fáry's theorem**).
3. For each $d \geq 2$ there is a simplicial d -complex that is PL-embeddable in \mathbb{R}^{d+1} , but not geometrically embeddable in \mathbb{R}^{d+1} [Duk70].
4. All minimal triangulations of the 2-sphere and the torus can be geometrically embedded in \mathbb{R}^3 [BW93]. All minimal triangulations of the projective plane can be geometrically embedded in \mathbb{R}^4 [BW93].

ALGORITHMS

Geometric embeddability of a graph: It can be decided in $O(n \log n)$ time whether a simplicial 1-complex (graph) with n cells (edges and vertices) can be geometrically embedded in the plane. If such an embedding exists, it can be constructed in $O(n \log n)$ time [HT74].

OPEN PROBLEMS

1. Can every minimal triangulation (see Section 32.2) of the surface of genus g be geometrically embedded in \mathbb{R}^3 (cf. [BW93])?
2. Design an efficient (polynomial-time) algorithm that determines whether a simplicial d -complex can be geometrically embedded in \mathbb{R}^k , for $d \leq k \leq 2d$.
3. Prove or disprove: If a simplicial d -complex is PL-embeddable in \mathbb{R}^{2d} , then it is geometrically embeddable in \mathbb{R}^{2d} .
4. Is there a constant c such that the c th barycentric subdivision of any simplicial complex K whose underlying space can be PL-embedded in \mathbb{R}^d , can be geometrically embedded in \mathbb{R}^d ? Recall that there are examples of simplicial complexes that are PL-embeddable in \mathbb{R}^d , but not geometrically embeddable.

32.5.3 KNOTS

GLOSSARY

Knot: A PL-embedding of a polygon in \mathbb{R}^3 .

Spanning surface of a knot: A PL-embedded orientable surface in \mathbb{R}^3 , whose boundary is the knot (also called a *Seifert surface*).

Trivial knot: A knot with a spanning surface that is PL-equivalent to a disk.

Genus of a knot: Minimum possible genus of a spanning surface. (The genus of a spanning surface is the genus of the closed orientable surface obtained by attaching a disk—cf. Section 32.3—along the boundary of the spanning surface. In particular, a trivial knot has genus 0.)

ALGORITHMS AND COMPLEXITY

1. A spanning surface for a polygonal knot with n vertices can be constructed in $O(n^2)$ time (Seifert's construction [Liv93]).
2. There is an algorithm that solves the *knot triviality* problem (or, unknotting problem), i.e., that decides whether a polygonal knot with n vertices is trivial, in $O(\exp(cn^2))$ time and $O(n^2 \log n)$ space, for some positive constant c (the Haken-Hemion unknottedness algorithm; see [Hem92]). The Jaco-Tollefson

unknottedness algorithm [JT95] decides this question in at most $O(\exp(c'n))$ time and $O(n^2 \log n)$ space, for some positive constant c' . The knot triviality problem is in **NP**, [HLP99].

3. The genus problem for a polygonal knot is in **PSPACE** [HLP99].

OPEN PROBLEM

Knot triviality: Is the knot triviality problem NP-complete?

32.6 IMMERSIONS

GLOSSARY

Immersion: Let K and L be simplicial complexes. A PL-map $f : |K| \rightarrow |L|$ is called an immersion if it is locally injective (i.e., every point $p \in |K|$ has a neighborhood in $|K|$ on which f is 1-1). We say that $|K|$ is immersed in $|L|$. An immersion of $|K|$ in \mathbb{R}^d is defined similarly.

Regular equivalence of immersions: Two immersions f_0 and f_1 of $|K|$ in $|L|$ (or \mathbb{R}^d) are regularly equivalent if there is a homotopy F , between f_0 and f_1 , defined on $|K| \times \mathbb{I}$, such that f_t , defined by $f_t(x) = F(x, t)$, is an immersion of $|K|$ in $|L|$ (or \mathbb{R}^d).

Winding number: Consider a polygon P with n vertices, immersed in the plane. Let its exterior angles $\theta_1, \dots, \theta_n$, be measured with sign. The winding number of P is $w(P) = \frac{1}{2\pi} \sum_{i=1}^n \theta_i \in \mathbb{Z}$ (the total number of turns of its tangent vector). P may be considered as the image of a PL-immersion $c : \mathbb{S}^1 \rightarrow \mathbb{R}^2$, for which we define $w(c) = w(P)$.

BASIC RESULTS

1. Every PL-embedding is an immersion.
2. Every simplicial d -manifold can be immersed in \mathbb{R}^{2d-1} [Whi44].
3. Two immersions $c_1, c_2 : \mathbb{S}^1 \rightarrow \mathbb{R}^2$ are regularly equivalent iff $w(c_1) = w(c_2)$ (a theorem of Whitney-Graustein).
4. There are two regular equivalence classes of immersions $\mathbb{S}^1 \rightarrow \mathbb{S}^2$, viz the curves that go once and twice along the equator of \mathbb{S}^2 .
5. Smale [Sma58a] associates with each immersion $c : \mathbb{S}^1 \rightarrow \mathbb{M}_g$ an element $W(c)$ of the fundamental group of the unit tangent bundle $S^1(\mathbb{M}_g)$ of \mathbb{M}_g that is a complete invariant for the regular equivalence class of c . This element $W(c)$ may be considered the generalization of the winding number of an immersion of \mathbb{S}^1 in \mathbb{R}^2 . For related definitions, see [Chi72, MC93].
6. All immersions of \mathbb{S}^2 in \mathbb{R}^3 are regularly equivalent. See [Sma58b] and [Fra87, Phi66] for pictures and constructions.

7. More generally, there are 4^g regular equivalence classes of immersions of an oriented closed surface in \mathbb{R}^3 [JT66]. See [Phi66] for pictures of the 4 classes of immersions of the torus \mathbb{M}_1 in \mathbb{R}^3 .

ALGORITHMS

Kinkfree deformations of immersed curves in \mathbb{R}^2 : If $w(P_1) = w(P_2)$ for planar polygons P_1 and P_2 with a total of n vertices, there is a sequence of $O(n)$ “elementary” moves that realizes a regular equivalence between P_1 and P_2 . This sequence can be computed in $O(n \log n)$ time [Veg89]. This algorithm can be adapted to construct a regular equivalence between two polygonal curves on \mathbb{S}^2 .

Regular closed curves on \mathbb{M}_g , $g > 0$: There is an algorithm that determines in polynomial time whether two PL-immersions $\mathbb{S}^1 \rightarrow \mathbb{M}_g$ are regularly equivalent [Chi72, MC93].

OPEN PROBLEMS

1. *Regular deformations of curves on a surface:* Design an optimal algorithm that determines whether two PL-immersions $\mathbb{S}^1 \rightarrow \mathbb{M}_g$ are regularly equivalent, and, if so, construct such an equivalence.
2. *Immersions of \mathbb{S}^2 in \mathbb{R}^3 :* Design an efficient algorithm that constructs a regular equivalence between two arbitrary PL-immersions of \mathbb{S}^2 in \mathbb{R}^3 .
3. *Immersions of \mathbb{M}_g in \mathbb{R}^3 :* Design an algorithm that determines whether two immersions of \mathbb{M}_g in \mathbb{R}^3 are regularly equivalent. Extend the method to the construction of such an equivalence.

32.7 MORSE THEORY

Finite dimensional Morse theory deals with the relation between the topology of a smooth manifold and the critical points of smooth real-valued functions on the manifold. It is the basic tool for the solution of fundamental problems in differential topology. Recently, basic notions from Morse theory have been used in the study of the geometry and topology of large molecules.

GLOSSARY

Differential of a smooth map between Euclidean spaces: A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is called *smooth* if it has derivatives of all orders. A map $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called smooth if its component functions are smooth. The **differential** of φ at $q \in \mathbb{R}^n$ is the *linear map* $d\varphi_q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined as follows. For $v \in \mathbb{R}^n$, let $\alpha : I \rightarrow \mathbb{R}^n$, with $I = (-\varepsilon, \varepsilon)$ for some positive ε , be defined by $\alpha(t) = \varphi(q + tv)$; then $d\varphi_q(v) = \alpha'(0)$. If $\varphi(x_1, \dots, x_n) = (\varphi_1(x_1, \dots, x_n), \dots, \varphi_m(x_1, \dots, x_n))$, then the differential $d\varphi_q$ is represented by the **Jacobian matrix**

$$\begin{pmatrix} \frac{\partial \varphi_1}{\partial x_1}(q) & \cdots & \frac{\partial \varphi_1}{\partial x_n}(q) \\ \vdots & & \vdots \\ \frac{\partial \varphi_m}{\partial x_1}(q) & \cdots & \frac{\partial \varphi_m}{\partial x_n}(q) \end{pmatrix}.$$

Submanifold of \mathbb{R}^n : If $m \leq n$, a subset M of \mathbb{R}^n is an m -dimensional smooth submanifold of \mathbb{R}^n if, for each $p \in M$, there is an open set V in \mathbb{R}^n containing p , and a map $\varphi : U \rightarrow M \cap V$ from an open subset U in \mathbb{R}^m onto $V \cap M$, such that (i) φ is a smooth homeomorphism and (ii) the differential $d\varphi_q : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is injective for each $q \in U$. The map φ is called a **parametrization** of M at p .

Tangent space of a manifold: A smooth curve through a point p on a smooth submanifold M of \mathbb{R}^n is a smooth map $\alpha : I \rightarrow \mathbb{R}^n$, with $I = (-\varepsilon, \varepsilon)$ for some positive ε , satisfying $\alpha(t) \in M$ for $t \in I$ and $\alpha(0) = p$. A **tangent vector** of M at p is the tangent vector $\alpha'(0)$ of some smooth curve $\alpha : I \rightarrow M$ through p . The set $T_p M$ of all tangent vectors of M at p is the tangent space of M at p .

If $\varphi : U \rightarrow M$ is a smooth parametrization of M at p , with $0 \in U$ and $\varphi(0) = p$, then $T_p M$ is the m -dimensional subspace $d\varphi_0(\mathbb{R}^m)$ of \mathbb{R}^n , which passes through $\varphi(0) = p$. Let $\{e_1, \dots, e_m\}$ be the standard basis of \mathbb{R}^m , and define the tangent vector $\bar{e}_i \in T_p M$ by $\bar{e}_i = d\varphi_0(e_i)$. Then $\{\bar{e}_1, \dots, \bar{e}_m\}$ is a basis of $T_p M$.

Smooth function on a submanifold: A function $f : M \rightarrow \mathbb{R}$ on an m -dimensional smooth submanifold M of \mathbb{R}^n is smooth at $p \in M$ if there is a smooth parametrization $\varphi : U \rightarrow M \cap V$, with U an open set in \mathbb{R}^m and V an open set in \mathbb{R}^n containing p , such that the function $f \circ \varphi : U \rightarrow \mathbb{R}$ is smooth. A function on a manifold is called smooth if it is smooth at every point of the manifold.

Critical point: A point $p \in M$ is a critical point of a smooth function $f : M \rightarrow \mathbb{R}$ if there is a local parametrization $\varphi : U \rightarrow \mathbb{R}^n$ of M at p , with $\varphi(0) = p$, such that 0 is a critical point of $f \circ \varphi : U \rightarrow \mathbb{R}$ (i.e., the differential of $f \circ \varphi$ at q is the zero function on \mathbb{R}^n). This condition does not depend on the particular parametrization. A real number $c \in \mathbb{R}$ is a **regular value** of f if $f(p) \neq c$ for all critical points p of f , and a **critical value** if $f^{-1}(c)$ contains a critical point of f .

Hessian at a critical point: Let M be a smooth submanifold of \mathbb{R}^n , and let $f : M \rightarrow \mathbb{R}$ be a smooth function. The Hessian of f at a critical point p is the quadratic form $H_p f$ on $T_p M$ defined as follows. For $v \in T_p M$, let $\alpha : (-\varepsilon, \varepsilon) \rightarrow M$ be a curve with $\alpha(0) = p$ and $\alpha'(0) = v$. Then

$$H_p f(v) = \left. \frac{d^2}{dt^2} \right|_{t=0} f(\alpha(t)).$$

Let $\varphi : U \rightarrow M$ be a smooth parametrization of M at p , with $0 \in U$ and $\varphi(0) = p$, and let $v = v_1 \bar{e}_1 + \cdots + v_m \bar{e}_m \in T_p M$, where $\bar{e}_i = d\varphi_0(e_i)$. Then

$$H_p f(v) = \sum_{i,j=1}^m \frac{\partial^2 (f \circ \varphi)}{\partial x_i \partial x_j} (0) v_i v_j.$$

In particular, the matrix of $H_f(p)$ with respect to this basis is

$$\begin{pmatrix} \frac{\partial^2(f \circ \varphi)}{\partial x_1^2}(0) & \cdots & \frac{\partial^2(f \circ \varphi)}{\partial x_1 \partial x_m}(0) \\ \vdots & & \vdots \\ \frac{\partial^2(f \circ \varphi)}{\partial x_1 \partial x_m}(0) & \cdots & \frac{\partial^2(f \circ \varphi)}{\partial x_m^2}(0) \end{pmatrix}. \quad (32.7.1)$$

Nondegenerate critical point: The critical point p of $f : M \rightarrow \mathbb{R}$ is nondegenerate if the Hessian $H_p f$ is nondegenerate. The **index** of the critical point p is the number of negative eigenvalues of the Hessian at p . If M is 2-dimensional, then a critical point of index 0, 1, or 2, is called a **minimum**, **saddle point**, or **maximum**, respectively.

Morse function: A smooth function on a manifold is a Morse function if all critical points are nondegenerate. The **k th Morse number** of a Morse function f , denoted by $\mu_k(f)$, is the number of critical points of f of index k .

EXAMPLES

1. \mathbb{R}^m is a smooth submanifold of \mathbb{R}^n , for $m \leq n$. For $m < n$, we identify \mathbb{R}^m with the subset $\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_{m+1} = \dots = x_n = 0\}$ of \mathbb{R}^n .

2. The quadratic function $f : \mathbb{R}^m \rightarrow \mathbb{R}$ defined by

$$f(x_1, \dots, x_m) = -x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_m^2$$

is a Morse function, with a single critical point $(0, \dots, 0)$. This point is a nondegenerate critical point, since the Hessian matrix at this point is $\text{diag}(-2, \dots, -2, 2, \dots, 2)$, with k entries on the diagonal equal to -2 . In particular, the index of the critical point is k .

3. \mathbb{S}^{m-1} is a smooth submanifold of \mathbb{R}^m . A smooth parametrization of \mathbb{S}^{m-1} at $(0, \dots, 0, 1) \in \mathbb{S}^{m-1}$ is given by $\varphi : U \rightarrow \mathbb{R}^m$, with

$$U = \{(x_1, \dots, x_{m-1}) \in \mathbb{R}^{m-1} \mid x_1^2 + \dots + x_{m-1}^2 < 1\}$$

and

$$\varphi(x_1, \dots, x_{m-1}) = (x_1, \dots, x_{m-1}, \sqrt{1 - x_1^2 - \dots - x_{m-1}^2}).$$

In fact, φ is a parametrization at every point of the upper hemisphere, i.e., the intersection of \mathbb{S}^{m-1} and the upper half space $\{(y_1, \dots, y_m) \mid y_m > 0\}$.

4. The **height function** on \mathbb{S}^{m-1} , defined by $f(y_1, \dots, y_m) = y_m$ for $(y_1, \dots, y_m) \in \mathbb{S}^{m-1}$, is a Morse function. With respect to the parametrization φ the expression of the height function is $f \circ \varphi(x_1, \dots, x_{m-1}) = \sqrt{1 - x_1^2 - \dots - x_{m-1}^2}$, so that the only critical point of f on the upper hemisphere is $(0, \dots, 0, 1)$. The Hessian matrix (32.7.1) is the diagonal matrix $\text{diag}(-1, -1, \dots, -1)$, so that $(0, \dots, 0, 1)$ is a critical point of index $n - 1$. Similarly, the other critical point is $(0, \dots, 0, -1)$, which is of index 0.

5. The torus M in \mathbb{R}^3 , obtained by rotating a circle in the x, y -plane with center $(0, R, 0)$ and radius r around the x -axis, is a smooth 2-manifold. Let $U = \{(u, v) \mid -\pi/2 < u, v < 3\pi/2\} \subset \mathbb{R}^2$, and let the map $\varphi : U \rightarrow \mathbb{R}^3$ be defined by

$$\varphi(u, v) = (r \sin u, (R - r \cos u) \sin v, (R - r \cos u) \cos v).$$

Then φ is a parametrization at all points of M , except for points on one latitudinal and one longitudinal circle. The height function on M is the function $h : M \rightarrow \mathbb{R}$ defined by

$$\tilde{h}(u, v) = h(\varphi(u, v)) = (R - r \cos u) \cos v,$$

so that the singular points of h are as shown in Table 32.7.1.

TABLE 32.7.1 Singularities of the height function for a torus.

(u, v)	$\varphi(u, v)$	TYPE OF SINGULARITY
$(0, 0)$	$(0, 0, R - r)$	saddle point
$(0, \pi)$	$(0, 0, -R + r)$	saddle point
$(\pi, 0)$	$(0, 0, R + r)$	maximum
(π, π)	$(0, 0, -R - r)$	minimum

BASIC RESULTS

- Regular level sets.** Let M be an m -dimensional submanifold of \mathbb{R}^n , and let $f : M \rightarrow \mathbb{R}$ be a smooth function. If $c \in \mathbb{R}$ is a regular value of f , then $f^{-1}(c)$ is a regular $(m - 1)$ -dimensional submanifold of \mathbb{R}^n .

For $a \in \mathbb{R}$, let $M_a = \{q \in M \mid f(q) \leq a\}$. If f has no critical values in $[a, b]$, for $a < b$, then the subsets M_a and M_b of M are homotopy-equivalent.

- The Morse Lemma.** Let M be a smooth m -dimensional submanifold of \mathbb{R}^n , and let $f : M \rightarrow \mathbb{R}$ be a smooth function on M with a nondegenerate critical point p of index k . Then there is a smooth parametrization $\varphi : U \rightarrow M$ of M at p , with U an open neighborhood of $0 \in \mathbb{R}^m$ and $\varphi(0) = p$, such that

$$f \circ \varphi(x_1, \dots, x_m) = f(p) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_m^2.$$

In particular, a critical point of index 0 is a local minimum of f , whereas a critical point of index m is a local maximum of f .

- Abundance of Morse functions.** (i) *Morse functions are generic.* Every smooth compact submanifold of \mathbb{R}^n has a Morse function. (In fact, if we endow the set $C^\infty(M)$ of smooth functions on M with the so-called Whitney topology, then the set of Morse functions on M is an open and dense subset of $C^\infty(M)$. In particular, there are Morse functions arbitrarily close to any smooth function on M .)

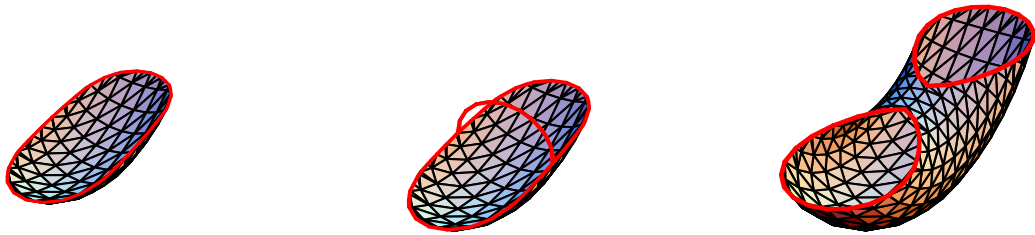
(ii) *Generic height functions are Morse functions.* Let M be an m -dimensional submanifold of \mathbb{R}^{m+1} (e.g., a smooth surface in \mathbb{R}^3). For $v \in \mathbb{S}^m$, the **height function** $h_v : M \rightarrow \mathbb{R}$ with respect to direction v is defined by $h_v(p) = \langle v, p \rangle$. The set of v for which h_v is not a Morse function has measure zero in \mathbb{S}^m .

- Passing critical levels.** Let $f : M \rightarrow \mathbb{R}$ be a smooth Morse function with exactly one critical level in (a, b) , and let a and b be regular values of f . Then

M_b is homotopy-equivalent to M_a with a cell of dimension k attached (cf. Section 32.3), where k is the index of the critical point in $f^{-1}([a, b])$. See Figure 32.7.1.

FIGURE 32.7.1

Passing a critical level of index 1 corresponds to attaching a 1-cell. Here M is the 2-torus embedded in \mathbb{R}^3 , in standard vertical position, and f is the height function with respect to the vertical direction. Left: M_a , for a below the critical level of the lower saddle point of f . Middle: M_a with a 1-cell attached to it. Right: M_b , for b above the critical level of the lower saddle point of f . This set is homotopy-equivalent to the set in the middle part of the figure.



5. **Morse inequalities.** Let f be a Morse function on a compact m -dimensional smooth submanifold of \mathbb{R}^n . For each k , $0 \leq k \leq m$, the k th Morse number of f dominates the k th Betti number of M :

$$\mu_k(f) \geq \beta_k(M).$$

The Morse numbers of f are related to the Betti numbers and the Euler characteristic of M by the following identity:

$$\sum_{k=1}^m (-1)^k \mu_k(f) = \sum_{k=1}^m (-1)^k \beta_k(M) = \chi(M).$$

32.8 SOURCES AND RELATED MATERIAL

FURTHER READING

[Sti93]: Low dimensional topology, including some knot theory and relationships with combinatorial group theory. Good starting point for exploration of topology; nice historical setting.

[Fom91]: User-friendly introduction to algebraic topology and the classification problem for manifolds.

[ST34]: A classic, dealing with combinatorial algebraic topology.

[Mau70]: Extensive treatment of simplicial complexes and simplicial algebraic topology.

[Bre93]: Modern textbook on algebraic topology, especially as related to topological aspects of manifold theory.

[FK97]: Introduction to concepts from differential geometry and topology. Well illustrated.

- [GP74]: Introduction to differential topology.
- [Mil73]: The classic monograph on Morse theory.
- [DEG99]: A survey on computational topology, containing links to applications.
- [Ede01]: Introduces the language of combinatorial topology, and applies it to mesh generation and simplification problems.
- [Epp01]: A web page, containing web pointers and course notes on knot theory, primarily related to geometry.

RELATED CHAPTERS

- Chapter 14: Topological methods
- Chapter 15: Face numbers of polytopes and complexes
- Chapter 18: Polyhedral maps
- Chapter 29: Computational real algebraic geometry
- Chapter 63: Biological applications of computational topology

REFERENCES

- [Bak90] D.R. Baker. Some topological problems in robotics. *Math. Intelligencer*, 12:66–76, 1990.
- [Bas99] S. Basu. On Bounding the Betti numbers and computing the Euler characteristic of semi-algebraic Sets. *Discrete Comput. Geom.*, 22:1–18, 1999.
- [BK87] U. Brehm and W. Kühnel. Combinatorial manifolds with few vertices. *Topology*, 26:465–473, 1987.
- [Bre93] G.E. Bredon. *Topology and Geometry*, volume 139 of *Grad. Texts in Math.* Springer-Verlag, New York, 1993.
- [Bri93] E. Brisson. Representing geometric structures in d dimensions: topology and order. *Discrete Comput. Geom.*, 9:387–426, 1993.
- [BW93] U. Brehm and J.M. Wills. Polyhedral manifolds. In P.M. Gruber and J.M. Wills, editors, *Handbook of Convex Geometry*, pages 535–554. Elsevier, Amsterdam, 1993.
- [CLMS02] S. Cabello, Y. Liu, A. Mantler, and J. Snoeyink. Testing homotopy for paths in the plane. In *Proc. 18th Annu. ACM Sympos. Comput. Geom.*, pages 160–169, 2002.
- [Chi72] D.R.J. Chillingworth. Winding numbers on surfaces, I. *Math. Ann.*, 196:218–249, 1972.
- [DC91] B.R. Donald and D.R. Chang. On the complexity of computing the homology type of a triangulation. In *Proc. 32nd Annu. IEEE Sympos. Found. Comput. Sci.*, pages 650–662, 1991.
- [DE95] C.J.A. Delfinado and H. Edelsbrunner. An incremental algorithm for Betti numbers of simplicial complexes on the 3-sphere. *Comput. Aided Geom. Design*, 12:771–784, 1995.
- [DEG99] T.K. Dey, H. Edelsbrunner, and S. Guha. Computational topology. In B. Chazelle, J.E. Goodman, and R. Pollack, editors, *Advances in Discrete and Computational Geometry*, volume 223 of *Contemp. Math.*, pages 109–143. Amer. Math. Soc., Providence, 1999.
- [DFN90] B.A. Dubrovin, A.T. Fomenko, and S.P. Novikov. *Modern Geometry—Methods and Applications, Part III*, volume 124 of *Grad. Texts in Math.* Springer-Verlag, New York, 1990.
- [DG98] T.K. Dey and S. Guha. Computing homology groups of simplicial complexes. *J. Assoc. Comput. Mach.*, 45:266–287, 1998.

- [DG99] T.K. Dey and S. Guha. Transforming curves on surfaces. *J. Comput. System. Sci.*, 58:297–325, 1999.
- [DS95] T.K. Dey and H. Schipper. A new technique to compute polygonal schema for 2-manifolds with applications to null-homotopy detection. *Discrete Comput. Geom.*, 14:93–110, 1995.
- [Duk70] R.A. Duke. Geometric embedding of complexes. *Amer. Math. Monthly*, 77:597–603, 1970.
- [Ede94] H. Edelsbrunner. Modeling with simplicial complexes: Topology, geometry, and algorithms. In *Proc. 6th Canad. Conf. Comput. Geom.*, pages 36–44, 1994.
- [Ede01] H. Edelsbrunner. *Geometry and Topology for Mesh Generation*, volume 6 of *Cambridge Monographs Appl. Comput. Math.* Cambridge University Press, 2001.
- [ES94] H. Edelsbrunner and N.R. Shah. Triangulating topological spaces. In *Proc. 10th Annu. ACM Sympos. Comput. Geom.*, pages 285–292, 1994.
- [Epp01] D. Eppstein. The Geometry Junkyard: Knot Theory. <http://www.ics.uci.edu/eppstein/junkyard/knot.html>
- [FK97] A.T. Fomenko and T.L. Kunii. *Topological Modeling for Visualization*. Springer-Verlag, New York, 1997.
- [Fom91] A. Fomenko. *Visual Geometry and Topology*. Springer-Verlag, New York, 1991.
- [Fra87] G.K. Francis. *A Topological Picturebook*. Springer-Verlag, New York, 1987.
- [GP74] V. Guillemin and A. Pollack. *Differential Topology*. Prentice Hall, Englewood Cliffs, 1974.
- [Hem92] G. Hemion. *The Classification of Knots and 3-dimensional Spaces*. Oxford Univ. Press, New York, 1992.
- [HI92] L.S. Heath and S. Istrail. The pagenumber of genus g graphs is $O(g)$. *J. Assoc. Comput. Mach.*, 39:479–501, 1992.
- [HLP99] J. Hass, J.C. Lagarias, and N. Pippenger. The computational complexity of knot and link problems. *J. Assoc. Comput. Mach.*, 46:185–211, 1999.
- [HS94a] M. Herlihy and N. Shavit. Applications of algebraic topology to concurrent computation. *SIAM News*, pages 10–12, December 1994.
- [HS94b] J. Hershberger and J. Snoeyink. Computing minimum length paths of a given homotopy class. *Comput. Geom. Theory Appl.*, 4:63–98, 1994.
- [HT74] J. Hopcroft and R.E. Tarjan. Efficient planarity testing. *J. Assoc. Comput. Mach.*, 21:549–568, 1974.
- [JR80] M. Jungerman and G. Ringel. Minimal triangulations of orientable surfaces. *Acta Math.*, 145:121–154, 1980.
- [JT66] I. James and E. Thomas. On the enumeration of cross-sections. *Topology*, 5:95–114, 1966.
- [JT95] W. Jaco and J.L. Tollefson. Algorithms for the complete decomposition of a closed 3-manifold. *Illinois J. Math.*, 39:358–406, 1995.
- [Liv93] C. Livingston. *Knot Theory*, volume 24 of *Carus Math. Monographs*. Math. Assoc. Amer., 1993.
- [LPVV01] F. Lazarus, M. Pocchiola, G. Vegter, and A. Verroust. Computing a canonical polygonal schema of an orientable triangulated surface. In *Proc. 17th Annu. ACM Sympos. Comput. Geom.*, pages 80–89, 2001.
- [Mae93] H. Maehara. Why is P^2 not embeddable in R^3 ? *Amer. Math. Monthly*, 100:862–864, 1993.
- [Mau70] S.R.F. Maunder. *Algebraic Topology*. Van Nostrand Reinhold, London, 1970.
- [MC93] M. McIntyre and G. Cairns. A new formula for winding number. *Geom. Dedicata*, 46:149–160, 1993.

- [Mil61] J. Milnor. Two complexes which are homeomorphic but combinatorially distinct. *Ann. of Math.*, 74:575–590, 1961.
- [Mil73] J. Milnor. *Morse Theory*, volume 51 of *Ann. of Math. Stud.*. Princeton University Press, 1973.
- [ÓWW00] C. Ó'Dúnlaing, C. Watt, and D. Wilkins. Homeomorphism of 2-complexes is equivalent to graph isomorphism. *Internat. J. Comput. Geom. Appl.*, 10:453–476, 2000.
- [Phi66] A. Phillips. Turning a surface inside out. *Sci. Amer.*, 214:112–120, May 1966.
- [PV93] M. Pocchiola and G. Vegter. The visibility complex. In *Proc. 9th Annu. ACM Sympos. Comput. Geom.*, pages 328–337, 1993.
- [RS90] N. Robertson and P.D. Seymour. Graph minors VIII. A Kuratowski theorem for general surfaces. *J. Combin. Theory Ser. B*, 48:255–288, 1990.
- [Sar87] K.S. Sarkaria. Heawood inequalities. *J. Combin. Theory Ser. A*, 46:50–78, 1987.
- [Sch91] H. Schipper. Generating triangulations of 2-manifolds. In *Computational Geometry—Methods, Algorithms and Applications: Proc. 7th Internat. Workshop Comput. Geom. (CG '91)*, volume 553 of *Lecture Notes in Comput. Sci.*, pages 237–248. Springer-Verlag, Berlin, 1991.
- [Sch92] H. Schipper. Determining contractibility of curves. In *Proc. 8th Annu. ACM Sympos. Comput. Geom.*, pages 358–367, 1992.
- [Sha57] A. Shapiro. Obstructions to the imbedding of a complex in a euclidean space. I. The first obstruction. *Ann. of Math.*, 66:256–269, 1957.
- [Sma58a] S. Smale. Regular curves on Riemannian manifolds. *Trans. Amer. Math. Soc.*, 87:492–512, 1958.
- [Sma58b] S. Smale. A classification of immersions of the two-sphere. *Trans. Amer. Math. Soc.*, 90:281–290, 1958.
- [SS83] J.T. Schwartz and M. Sharir. On the piano mover's problem: II. General techniques for computing topological properties of real algebraic manifolds. *Adv. in Appl. Math.*, 4:298–351, 1983.
- [ST34] H. Seifert and W. Threlfall. *Lehrbuch der Topologie*. Teubner, Leipzig, 1934. English translation, *A Textbook of Topology*. Academic Press, New York, 1980.
- [Sti93] J. Stillwell. *Classical topology and combinatorial group theory*, volume 72 of *Grad. Texts in Math.* Springer-Verlag, New York, 1993.
- [Tho89] C. Thomassen. The graph genus problem is NP-complete. *J. Algorithms*, 10:568–576, 1989.
- [Veg89] G. Vegter. Kink-free deformations of polygons. In *Proc. 5th Annu. ACM Sympos. Comput. Geom.*, pages 61–68, 1989.
- [vK33] E.R. van Kampen. Komplexe in euklidischen Räumen. *Abh. Math. Sem. Hamb.*, 9:72–78 and 152–153, 1933.
- [VKF74] I.A. Volodin, V.E. Kuznetsov, and A.T. Fomenko. The problem of discriminating algorithmically the standard three-dimensional sphere. *Russian Math. Surveys*, 29:71–172, 1974.
- [vL90] J. van Leeuwen. Graph algorithms. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*, volume A, pages 525–631. Elsevier, Amsterdam, 1990.
- [VY90] G. Vegter and C.K. Yap. Computational complexity of combinatorial surfaces. In *Proc. 6th Annu. ACM Sympos. Comput. Geom.*, pages 102–111, 1990.
- [Web67] C. Weber. Plongement des polyèdres dans le domaine métastable. *Comment. Math. Helv.*, 42:1–27, 1967.
- [Whi44] H. Whitney. The self-intersections of a smooth n -manifold in $2n$ -space. *Ann. of Math.*, 45:221–246, 1944.
- [Zak69] J. Zaks. On minimal complexes. *Pacific J. Math.*, 28:721–727, 1969.