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# Renormalization Group, Non-Gibbsian states, their relationship and further developments.

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## Abstract:

We review what we have learned about the “Renormalization Group peculiarities” which were discovered more than twentyfive years ago by Griffiths and Pearce. We discuss which of the questions they asked have been answered and which ones are still widely open.

The problems they raised have led to the study of non-Gibbsian states (probability measures). We also mention some further related developments, which find applications in nonequilibrium questions and disordered models.

**Keywords:** Renormalization-Group peculiarities, non-Gibbsian measures.

## 1 Introduction

More than twentyfive years ago, Griffiths and Pearce [30, 31] discovered some unexpected mathematical difficulties in rigorously implementing many of the generally used real-space Renormalization Group transformations as maps on a space of Hamiltonians.

In this short review I plan to assess what we have learned about these problems since then. The relevant objects which have provided the most information are the so-called non-Gibbsian measures. Indeed, renormalizing Gibbs measures associated to some Hamiltonian can result in a non-Gibbsian measure for which no “reasonable” renormalized Hamiltonian can be found.

We will see that Renormalization-Group maps cannot be discontinuous, although they can be ill-defined.

Moreover, in any region of the phase diagram the question whether a particular transformation is well-defined or ill-defined turns out to be highly non-trivial. The ill-definedness of various Renormalization-Group maps can be expressed in the violation of the property of “quasilocality” in the renormalized states. The study and classification of such non-quasilocal states (—non-Gibbsian measures—) has led also to various results of mathematical interest, some of which we will mention further on. Some papers covering the area of non-Gibbsianness and Renormalization-Group peculiarities, also treating further related material are [104, 103, 98, 62, 112, 100, 21, 22, 23, 83, 51, 99, 48, 49, 38, 46] and references therein. The first systematic and extensive (an almost 300-page paper...) follow-up study appeared in 1993 [103]. Although many further results have since been proven, the conceptual point of view I present here is still essentially based on that paper.

Beyond Renormalization Group applications, non-Gibbsian measures have been found in various other circumstances. I will shortly mention some of those. At the end of the paper I mention some further open problems.

## 2 Gibbs measures and quasilocality

In this section we will describe some definitions and facts we will need about the theory of Gibbs measures. For a more extensive treatment we refer to [28] or [103].

We will consider spin systems on a lattice  $Z^d$ , where in most cases we will take a single-spin space  $\Omega_0$  which is finite. The configuration space of the whole system is  $\Omega = \Omega_0^{Z^d}$ . Configurations will be denoted by small Greek letters such as  $\sigma$  or  $\omega$ , and their coordinates at lattice site  $i$  are denoted by  $\omega(i)$  or  $\sigma(i)$ . A (regular) interaction  $\Phi$  is a collection of functions  $\Phi_X$  on  $\Omega_0^X$ ,  $X \in Z^d$  which is translation invariant and satisfy:

$$\sum_{0 \in X} |\Phi_X|_\infty < \infty \quad (1)$$

Formally Hamiltonians are given by

$$H^\Phi = \sum_{X \in Z^d} \Phi_X \quad (2)$$

Under the above regularity condition these type of expressions make mathematical sense if the sum is taken over all subsets having non-empty intersections with a finite volume  $\Lambda$ . For regular interactions one can define Gibbs measures as probability measures on  $\Omega$  having conditional probabilities which are described in terms of appropriate Boltzmann-Gibbs factors:

$$\frac{\mu(\sigma_\Lambda^1 | \omega_{\Lambda^c})}{\mu(\sigma_\Lambda^2 | \omega_{\Lambda^c})} = \exp -(\sum_X [\Phi_X(\sigma_\Lambda^1 \omega_{\Lambda^c}) - \Phi_X(\sigma_\Lambda^2 \omega_{\Lambda^c})]) \quad (3)$$

for each volume  $\Lambda$ ,  $\mu$ -almost every boundary condition  $\omega_{\Lambda^c}$  outside  $\Lambda$  and each pair of configurations  $\sigma_\Lambda^1$  and  $\sigma_\Lambda^2$  in  $\Lambda$ . As long as  $\Omega_0$  is compact, there always exists at least one Gibbs measure for every regular interaction; the existence of more than one Gibbs measure is one definition of the occurrence of a first-order phase transition of some sort. Thus the map from interactions to measures is one to at-least-one. Every Gibbs measure has the property that (for one of its versions) its conditional probabilities are continuous functions of the boundary condition  $\omega_{\Lambda^c}$ , in the product topology. It is a non-trivial fact that this continuity, which goes by the name “quasilocality” or “almost Markovianess”, in fact characterizes the Gibbs measures [95, 47], once one knows that all the conditional probabilities are bounded away from zero (that is, the measure is *nonnull* or has the *finite energy* property). In some examples it turns out to be possible to check this continuity (quasilocality) property quite explicitly. If a measure is a Gibbs measure for a regular interaction, this interaction is essentially uniquely determined. Thus the map from measures to interactions is one to at-most-one.

A second characterization of Gibbs measures uses the variational principle expressing that in equilibrium a system minimizes its free energy. A probabilistic formulation of this fact naturally occurs in terms of the theory of large deviations. A (third level) large deviation rate function is up to a constant and a sign equal to a free energy density. To be precise, let  $\mu$  be a translation invariant Gibbs measure, and let  $\nu$  be an arbitrary translation invariant measure. Then the relative entropy density  $i(\nu|\mu)$  can be defined as the limit:

$$i(\nu|\mu) = \lim_{\Lambda \rightarrow Z^d} \frac{1}{|\Lambda|} I_\Lambda(\nu|\mu) \quad (4)$$

where

$$I_\Lambda(\nu|\mu) = \int \log\left(\frac{d\nu_\Lambda}{d\mu_\Lambda}\right) d\nu_\Lambda \quad (5)$$

and  $\mu_\Lambda$  and  $\nu_\Lambda$  are the restrictions of  $\mu$  and  $\nu$  to  $\Omega_0^\Lambda$ . It has the property that  $i(\nu|\mu) = 0$  if and only if the measure  $\nu$  is a Gibbs measure for the same interaction as the base measure  $\mu$ . We can use this result in applications if we know for example that a known measure  $\nu$  cannot be a Gibbs measure for the same interaction as some measure  $\mu$  we want to investigate. For example, if  $\nu$  is a point measure, or if it is the case that  $\nu$  is a product measure and  $\mu$  is not, we can conclude from the statement:  $i(\nu|\mu) = 0$ , that  $\mu$  lacks the Gibbs property.

For another method of proving that a measure is non-Gibbsian because of having the “wrong” type of (in this case too small) large deviation probabilities, see [90].

### 3 Renormalization-Group maps: some examples

We will mostly consider the standard nearest neighbor Ising model with (formal) Hamiltonian

$$H = \sum_{\langle i,j \rangle} -\sigma(i)\sigma(j) - h\sum_i \sigma(i) \quad (6)$$

at inverse temperature  $\beta$ . The magnetic field strength is  $h$ . The dimension  $d$  in what follows will be at least 2.

We will consider various real-space Renormalization-Group or block-spin transformations which act on the Ising Gibbs measures. Our first question is to find the domain of definition of such transformations (that is: Is the first step of the renormalization program well-defined?). The question then is to find the renormalized interaction, that is the interaction associated to the transformed measure. This then should define the transformation at the level of interactions (Hamiltonians).

Although in applications the transformation needs to be iterated, and one would like to know about fixed points, domains of attraction etc, we will mostly restrict ourselves to considering a single transformation. Existence of the first step is of course necessary but far from sufficient for justification of an iterative procedure. For some recent work considers what happens after many iterations, see [5, 6, 7]. We mention e.g. that sometimes, even if the first step is ill-defined, after repeated transformations a transformed interaction can be found [78].

In contrast to what one might at first believe, the critical point can be either outside or inside the domain of definition of the Renormalization Group operator, and does in general not play any special role.

We divide the lattice into a collection of non-overlapping blocks. A Renormalization-Group transformation defined at the level of measures will be a probability kernel

$$T(\omega'|\omega) = \prod_{\text{blocks}} T(\omega'(j)|\omega(i); i \in \text{block}_j) \quad (7)$$

This means that the distribution of a block-spin depends only on the spins in the corresponding block, in other words the transformation is local in real space. The case of a deterministic transformation is included, by having a  $T$  which is either zero or one.

Renormalization-Group methods are widely in use to study phase transitions and in particular critical phenomena of various sorts (see for example [113, 68, 17, 26]), as was also testified by the workshop. Some good recent references in which the theory is explained, mostly at a physical level of rigour, but including some more careful statements about what actually has and has not been proven are [29, 2, 1].

1) One class of examples we will consider are (linear) block-average transformations. This means that the block-spins are the average spins in each block. Applied to Ising systems they suffer from the fact that the renormalized system has a different single-spin space from the original one. Despite this objection, the linearity makes these maps mathematically rather attractive, and they have often been considered. As long as we are not iterating the transformation we need not worry too much about the single-spin space changing, but see [5, 6, 7].

2) Majority rule and Kadanoff transformations.

In the case of majority rule transformations [84] applied to blocks containing an odd number of sites, the block spin is just given by the sign of the majority of the spins in the block. These transformations have been chosen often because of their numerical tractability.

The Kadanoff transformation is a soft version (a proper example of a stochastic transformation) of the majority rule:

$$T(\sigma'(j)|\sigma(i); i \in \text{block}_j) = C \exp [p\sigma'(j)\sum_{i \in \text{block}_j} \sigma(i)] \quad (8)$$

In the limit in which  $p$  goes to  $\infty$  the Kadanoff map reduces to a majority rule transformation. Once the block size is even, and there is the possibility of a tie (equally many pluses and minuses in the

block), one gives a prescription what happens in such a case. For example, a popular prescription is then to flip a coin to decide.

Majority rule transformations map Ising systems onto Ising systems, and have at least in principle the possibility to be iterated, and to have nontrivial fixed points.

3) Decimations and projections.

We will call a “decimation” taking the marginal of a measure restricted to the spins on a sublattice of the same dimension as the original system, (thus the block-spins are the spins in some periodic sub-lattice).

A “projection” will mean taking the marginal to a lower-dimensional sublattice. Projections are not Renormalization-Group maps proper, but share some mathematical properties of Renormalization-Group maps. See [69, 70, 90].

Although decimation transformations have the advantages both of being linear and of preserving the single-spin space, infinite iteration has the disadvantage that critical fixed points won’t occur. However, this problem does not show up after a finite number of applications of these maps, so we will here not too much worry about it.

## 4 Peculiarities: The investigations of Griffiths and Pearce

Griffiths and Pearce [30, 31] seem to be the first investigators who looked seriously at the question whether renormalized Hamiltonians exist in a precise mathematical sense. They found that for some real-space transformations like decimation or Kadanoff transformations in the low-density regime (that means strong magnetic fields in Ising language) the Renormalization-Group map maps the Ising Gibbs measure on a Gibbs measure for an in principle computable interaction. They also found, both at phase transitions and near areas of the phase diagram where first order transitions occur, that there are regimes where the formal expression for renormalized interactions behaved in a peculiar way. These “peculiarities” were found for decimation, Kadanoff, and majority rule transformations. The problem underlying the peculiarities is the occurrence of phase transitions in the system, once it is constrained (or conditioned) by prescribing some particular, rather atypical, block-spin configuration. This means that there can exist for these block-spin configurations long-range correlations in the presumed “short-wavelength degrees of freedom”- or “internal spins”-, which are to be integrated out in a Renormalization-Group map.

In their paper Griffiths and Pearce discuss various possible explanations of these “peculiarities”:

P1) The renormalized interaction might not exist,

P2) it might exist but be a singular function of the original interaction,

P3) it might be non-quasilocal,

or

P4) the thermodynamic limit might be problematic.

Shortly after, the problem was studied by Israel [42]. He obtained (very) high-temperature existence results, including approach to trivial fixed points, as well as an analysis of the decimation transformation at low temperature, indicating strongly that in that case the renormalized interaction does not exist. Israel’s results convinced Griffiths that in fact possibility P1)— non-existence of the renormalized interaction— applies [32]; however, it seems that most authors aware of their work interpreted the Griffiths-Pearce peculiarities as singular behaviour of the renormalized interactions (possibility P2)). See for example [39, 11, 41, 96]. Many authors did not even show awareness that the determination of renormalized interactions presented problems beyond mere computational difficulty. See the Appendix for some illustrations of this point.

Deep in the uniqueness regime the Renormalization-Group procedure appeared to be well-defined in some generality.

At that stage, various open questions formulated by Griffiths and Pearce and Israel were still left:

Q1) What is the nature of the “peculiarities”?

Q2) Can one say anything about the critical regime?

Q3) Do different transformations exhibit similar behaviour? For example, are decimation, block-averaging, majority and momentum-space Renormalization-Group transformations similar regarding the occurrence of “peculiarities”?

Q4) As the “peculiarities” are due to rather atypical spin-configurations, can one make the Renormalization-Group enterprise work, by considering only typical configurations, and thus work with appropriate approximations? Or, more generally, is there a framework in which one can implement the whole Renormalization-Group machinery in a mathematically correct way?

## 5 Answered and unanswered questions

About question Q1 —the nature of the peculiarities— we have acquired some more insight. In [103] the Griffiths-Pearce study was taken up and further pursued. We provided a rather extensive discussion of the physical interpretation of these issues, as well as proving a number of varied mathematical results. It was found for example, making use of the above-mentioned variational characterization of Gibbs measures, that the peculiarities could *not* be due to discontinuities in the Renormalization-Group maps. In fact the Renormalization Group map from interactions to interactions is one to at-most-one. The underlying argument is that if one transforms two measures, their relative entropy tends to shrink, for a wide class of transformations. Hence renormalizing two phases (different Gibbs measures for the same interaction, having relative entropy density zero with respect to each other) cannot result in two Gibbs measures for different interactions, for which these relative entropy densities need to be non-zero.

However, in the “peculiar” cases considered by Griffiths and Pearce the renormalized measures all have conditional probabilities with points – spin configurations – of essential discontinuity. That is, they are *non-Gibbsian*. See also [46]. Thus a renormalized Hamiltonian *does not exist*. This despite many attempts to compute these — non-existent — renormalized Hamiltonians, and the various physically plausible and intuitively convincing conclusions, derived from such approximate computations.

In fact, those constrained block-spin configurations, pointed out by Griffiths and Pearce, for which the internal spins have phase transitions are precisely the points of discontinuity – non-quasilocalities – for some conditional probability. The observation that the “peculiarities” were due to the violation of the quasilocality condition was in essence, though somewhat in a slightly implicit way, already made in Israel’s analysis.

At first the non-Gibbsian examples were found at or near phase transitions, at sufficiently low temperatures. Then it was found [101] that decimation applied to many-state Potts models gives non-Gibbsian measures also above the transition temperature.

Somewhat surprisingly, it turned out that even deep in the uniqueness regime, the non-Gibbsianness can occur. This happens for example at low temperature for block-average [103], and majority rule [101] transformations in strong external fields, and it is even possible to devise transformations for which this happens at arbitrarily high temperatures [97]. On the other hand, it turns out that for Gibbs measures well in the uniqueness regime, a repeated application of a decimation transformation, even after composing with another Renormalization-Group map, leads again to a Gibbs measure, although applying the decimation only a few times may result in a non-Gibbsian measure [78, 79]. For related work in this direction see also [4]. As mentioned before, physically this means that one cannot find a “reasonable” (more or less short-range, local) Hamiltonian.

About critical points (question Q2), Haller and Kennedy [38] obtained the first results, proving both for a decimation and a Kadanoff transformation example that a single Renormalization-Group map can map an area including a critical point to a set of renormalized interactions. There are strong indications for similar behaviour for other transformations [45, 3, 12, 86]. The indications are partly numerical, however, and fall short of a rigorous proof. See also the numerical work of [87]. On the other hand, counterexamples where a transformed critical measure is non-Gibbsian also exist [97, 98]

Another critical regime result is the observation of [18], that majority rule scaling limits of critical points above the upper critical dimension (when the critical behaviour is like that of a Gaussian) are non-Gibbsian.

It was found that non-Gibbsian measures can also occur as a result of applying momentum-space transformations [100].

The conclusion of all the above is that different transformations can have very different behaviour. This is a sort of answer to question Q3, although in principle not a very informative one. In fact for applicability, if not for existence, something like this was already expected (compare for example Fisher’s [25] remarks on “aptness” and focusability).

Regarding question Q4) —to find the right setting for implementing Renormalization-Group Theory— the issue is still essentially open. One approach which was stimulated by the late R.L. Dobrushin is related to the observation of Griffiths and Pearce that the block-spin configurations responsible for the peculiarities (the discontinuity points) are rather atypical. By removing them from configuration space, one might hope to be left with a viable theory. Such investigations have led to the notions of “almost”, “intuitively weak” or “weak” Gibbs measures, the study of whose properties is being actively pursued [15, 16, 10, 69, 110, 23, 73, 71, 24, 92, 58, 77, 64, 5, 6, 7, 111]. This approach is somewhat along the lines of Griffiths’ and Pearce’s possibility P3). See also the next section. Whether it is possible to describe Renormalization-Group flows in spaces of such interactions, while keeping a continuous connection between interactions describing a positively and a negatively magnetized state, is not at all clear. A warning sign is that the variational principle can be violated for a weakly Gibbsian situation [51].

As for projections, it is known that on the phase-transition line of the 2-dimensional Ising model the projection to  $Z$  of any Gibbs measure is non-Gibbsian [90]. In the whole uniqueness regime, except possibly at the critical point, this projection results in a Gibbsian measure [74, 62, 63]. In three dimensions the projected measures to two-dimensional planes are again non-Gibbsian on the transition line [24, 75, 70]; however, now, due to the presumably existing surface (layering) transition between different “Basuev states” (states with different layer thicknesses), one expects that the projected measures also in a small field will be non-Gibbsian [62, 63].

The composition of a projection and a decimation in the phase transition region gives rise to a new phenomenon, namely the possibility of a state-dependent result. The transformed plus and minus measures are Gibbs measures for different interactions, while the transformed mixed measures are non-Gibbsian [66]. Their relative entropy densities with respect to each other are positive.

## 6 Further results on non-Gibbsian measures: FK, the colors of Häggström, fuzzy Potts, quenched disorder and non-equilibrium

Further investigations in which non-Gibbsian measures were displayed, have been done about the random-cluster models of Fortuin and Kasteleyn [88, 33, 8, 36, 92], the related coloring model of Häggström, [35], about the Fuzzy Potts image analysis model [76, 37, 34], and about various non-equilibrium models, both in the steady state and in the transient regime – for this last case

it is useful to note that an infinite-temperature Glauber dynamics can be viewed as a single-site stochastic “renormalization-group” map–. [94, 112, 80, 79, 99, 60, 72, 13, 59].

Also, joint quenched measures of disordered systems, have been shown sometimes to be non-Gibbsian [109, 107, 48, 49, 106], affecting the Morita approach to disordered systems [82, 53]. In this last case, the peculiarity can be so strong – and it actually is in the 3-dimensional random field Ising model– as to violate the variational principle. This means in particular that the (weakly Gibbsian) interactions belonging to the plus state and the minus state are different, despite their relative entropy density being zero. Non-Gibbsianness here means that the quenched measure cannot be written as an annealed measure, that is a Gibbs measure on the joint space of spins and disorder variables for some “grand potential”, such as Morita proposed.

The non-Gibbsian character of the various measures considered comes often as an unwelcome surprise. A description in terms of effective, coarse-grained or renormalized potentials is often convenient, and even seems essential for some applications. Thus, the fact that such a description is not available can be a severe drawback. As remarked earlier there have been attempts to tame the non-Gibbsian pathologies, and here we want to give a short comparison of how far one gets with some of those attempts.

1) The fact that the constraints which act as points of discontinuity often involve configurations which are very untypical for the measure under consideration, suggested a notion of *almost* Gibbsian or *weakly* Gibbsian measures. These are measures whose conditional probabilities are either continuous only on a set of full measure or can be written in terms of an interaction which is summable only on a set of full measure. Intuitively, the difference is that in one case the “good” configurations can shield off *all* influences from infinitely far away, and in the other case only *almost all* influences. The weakly Gibbsian approach was first suggested by Dobrushin to various people; his own version was published only later [14, 15, 16]. An early definition of almost Gibbsianness appeared in print in [67], see also [24, 76, 77, 69, 111, 51, 83] for further developments. Some examples of measures which are at the worst almost Gibbsian measures in this sense are decimated or projected Gibbs measures in an external field, random-cluster measures on regular lattices, and low temperature fuzzy Potts measures. In the random-cluster measures one can actually identify explicitly all bond configurations which give rise to the non-quasilocality. They are precisely those configurations in which (possibly after a local change) more than one infinite cluster coexist.

On a tree, because of the possible coexistence of infinitely many infinite clusters with positive probability, the random-cluster measure can violate the weak non-Gibbsianness condition and be strongly non-Gibbsian [36].

Dobrushin [14, 77, 15, 16] showed that for a projected pure phase on the coexistence line of the 2-dimensional Ising model it is possible to find an almost everywhere defined interaction, hence these measures are weakly Gibbsian. His approach, which is via low-temperature expansions, provides a way of obtaining good control for the non-Gibbsian projection. For similar ideas in a Renormalization-Group setting see [10, 58, 65].

For some Renormalization-Group examples an investigation via low temperature expansions into the possibility of recognizing non-Gibbsianness was started in [89].

Another simple counter-example of a strongly non-quasilocal measure, where each configuration can act as a point of discontinuity, is a mixture of two Gibbs measures for different interactions [108].

2) Stability under decimation (and other transformations).

In [78, 79] it was shown how decimating once-renormalized non-Gibbsian measures results in Gibbs measures again after a sufficiently large number of iterations. These often decimated measures are in the high-temperature regime, in which the usually applied Renormalization-Group maps are well-defined (this does not hold true for all maps though, in view of the highly non-linear example given before).



On the other hand, in examples where the non-Gibbsian property is associated with large deviation properties which are not compatible with a Gibbsian character (this holds for example for spin-projected Gaussians, invariant measures of the voter and the Martinelli-Scoppola model [56, 57, 80]) the non-Gibbsian property survives all sort of transformations [79, 108]. The argument is that when some obviously non-Gibbsian measure has a rate function zero with respect to the measure under consideration, this property is generally preserved ([103] Section 3.2 and 3.3).

The family of spin-projected Gaussians include scaling limits for majority-rule transformations in high dimensions [18]. The transformation of those scaling limits is, heuristically, interpreted as making a move from a fixed point in what is usually called a “redundant” direction (cf. Wegner’s contribution to [17]) in some space of Hamiltonians. Here, of course, the whole point is that such Hamiltonians do not exist.

3) The two criteria mentioned above are distinct. A simple one-dimensional example due to J. van den Berg [65] gives a one-dependent measure which has a set of discontinuity points of full measure, but due to the one-dependence the measure becomes after decimation independent and therefore trivially Gibbsian. In the opposite direction, there exist examples of measures whose non-Gibbsianness is robust, although they are weakly, and even almost, Gibbsian [110].

## 7 Conclusions and some further open problems

We have by now managed well to understand the Griffiths-Pearce peculiarities in the sense that we can identify mathematically their nature. However, how to get around them, and make Renormalization-Group theory mathematically respectable, is still a task requiring a lot of further work. Renormalization-Group ideas have of course often been inspirational, also for various rigorous analyses. Renormalization-Group implementations on spin models, which both for numerical and pedagogical convenience have often been treated by Renormalization-Group methods, run into difficulties which seem hard to avoid.

An implementation of Renormalization-Group ideas on contour models looks more promising, at least for the description of first-order phase transitions [27, 9].

On the other hand, having different descriptions and different mathematical objects at high temperatures (lattice spin interactions), at low temperatures (Peierls contour models), and yet again something different at critical temperatures (like possibly in two dimensions SLE, – the Schramm-Loewner Evolutions, see e.g. [93, 54, 91, 44] –) seems not a good way to describe the textbook Renormalization Group flow diagrams, where these three regimes typically are all included. Therefore, although we understand the problems better, solutions seem still to be far out, and results of Renormalization Group in the Mathematical Sciences need to develop a lot further before the situation is satisfactory.

In non-equilibrium statistical mechanics there are still many open questions about the occurrence of non-Gibbsian measures. Whether one can ascribe an effective temperature in a non-equilibrium situation is a topic of considerable interest, (also in the physics literature, see e.g. [85]). The term non-Gibbsian or non-reversible is often used for invariant measures in systems in which there is no detailed balance [61, 20, 19]. It is an open question to what extent such measures are non-Gibbsian in the sense we have described here. It has been conjectured that such measures for which there is no detailed balance are quite generally non-Gibbsian in systems with a stochastic dynamics, see for example [57] or [20], Appendix 1; on the other hand it has been predicted that non-Gibbsian measures are rather exceptional ([61], Open problem IV.7.5, p.224) at least for non-reversible spin-flip processes under the assumptions of rates which are bounded away from zero. The examples we have are for the moment too few to develop a good intuition on this point, but see [59].

We add the remark that sometimes a dynamics description is possible in terms of a Gibbs measure on the space-time histories (in  $d + 1$  dimensions). In such cases, looking at the steady states is considering the  $d$ -dimensional projection of such Gibbs measures.

Recently, a study of non-Gibbsianness in a mean-field setting has started. In this case, the characterization of Gibbs measures as having continuity properties in the product topology breaks down. For these developments, see [50, 37, 52].

Another set of questions where non-Gibbsianness probably plays an important role, but where we don't know much, is in Quantum Statistical Mechanics. A Gibbs state here should be a state on a  $C^*$ -algebra satisfying the KMS condition for a  $C^*$ -automorphism. It seems that the finite-energy property corresponds with the state having a cyclic and separating vector, implying the Tomita-Takesaki structure. This means it has the KMS structure, but on the level of the von Neumann algebra which is the weak closure of the  $C^*$ -algebra [105]. We do not have a good criterion as yet of how to establish that a given quantum state is non-Gibbsian in this sense, but one properly quantum mechanical non-Gibbsian example of a nonequilibrium steady state has been analyzed in [81].

Non-Gibbsian measures thus occur in quite different areas of statistical mechanics, besides renormalization group, and can have quite different properties. By now it seems somewhat surprising that it took so long to appreciate the fact that the Gibbs property is rather special, in particular in view of Israel's [43] result that in the set of all translation invariant (ergodic, nonnull) measures, Gibbs measures are exceptional.

## 8 Appendix: Some quotes from the literature

As an illustration that the heuristic character of Renormalization-Group theory, and in particular the need to consider the existence problem of (approximately) local renormalized interactions was recognized by various practitioners I mention the following quotes:

“One cannot write a renormalization cook book” (K.G. Wilson, cited by Niemeijer and van Leeuwen [17]).

“The notion of renormalization group is not well-defined” ([2], opening sentence).

“It is dangerous to proceed without thinking about the physics” [29].

“...the locality [of the renormalized interactions] is a non-trivial problem which will not be discussed further” [113].

“A Renormalization Group for a space of Hamiltonians should satisfy the following [25]:

A) Existence in the thermodynamic limit,...

B) .....

C) Spatial locality...”

Or, in the words of Lebowitz: “On the cautionary side one should remember that there are still some serious open problems concerning the nature of the RG transformation of Hamiltonians for statistical mechanical systems, i.e. for critical phenomena. A lot of mathematical work remains to be done to make it into a well-defined theory of phases transitions” [55].

As an illustration that on the other hand the occurrence of the above-mentioned problems has not been generally recognized, leading to some incorrect or at least misleading statements, I quote:

“The renormalization-group operator ... transforms an arbitrary system in the [interaction] space ... into another system in the space ...” [96].

“[the set of coupling constants gives rise to]..the most general form of the Hamiltonian...” [40].

“Further iterations of the renormalization group will generate long-range and multi-spin interactions of arbitrary complexity” [114].

“...the space of Ising Hamiltonians in zero field may be specified by the set of all possible spin-coupling constants, and is *closed* [Fisher’s emphasis] under the decimation transformation” [26].

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