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Maximally supersymmetric G-backgrounds of IIB supergravity

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Abstract

We classify the geometry of all supersymmetric IIB backgrounds which admit the maximal number of G-invariant Killing spinors. For compact stability subgroups $G = G_2, SU(3)$ and $SU(2)$, the spacetime is locally isometric to a product $X_n \times Y_{10-n}$ with $n = 3, 4, 6$, where X_n is a maximally supersymmetric solution of a n-dimensional supergravity theory and Y_{10-n} is a Riemannian manifold with holonomy G. For non-compact stability subgroups, $G = K \ltimes \mathbb{R}^8$, $K = Spin(7)$, $SU(4)$, $Sp(2)$, $SU(2) \times SU(2)$ and $\{1\}$, the spacetime is a pp-wave propagating in an eight-dimensional manifold with holonomy K. We find new supersymmetric pp-wave solutions of IIB supergravity.

1. Introduction

Supersymmetric backgrounds in supergravity theories can be characterized by the number of Killing spinors N and their stability subgroup G in an appropriate spin group [1]. For a given stability subgroup G, it has been shown in [2,3] that the Killing spinor equations of IIB supergravity [4–6] simplify for two classes of backgrounds: (i) the backgrounds that admit the maximal number of G-invariant Killing spinors, and (ii) the backgrounds that admit half the maximal number of G-invariant Killing spinors. In particular the Killing spinors for the former case, the maximally supersymmetric G-backgrounds which can be thought of as the vacua of IIB strings, can

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be written as

$$\epsilon_i = \sum_{j=1}^{N} f_{ij}\eta_j, \quad j = 1, \dots, N = 2m, \tag{1.1}$$

where η_p, $p = 1, \dots, m$ is a basis of G-invariant Majorana–Weyl spinors, $\eta_{m+p} = i\eta_p$, and (f_{ij}) is a $N \times N$ matrix with real spacetime functions as entries. In addition, the Killing spinor equations and their integrability conditions factorize, see also Appendix A.

The IIB Killing spinors are invariant under the stability subgroups $Spin(7) \ltimes \mathbb{R}^8$ $(N = 2)$, $SU(4) \ltimes \mathbb{R}^8$ $(N = 4)$, $Sp(2) \ltimes \mathbb{R}^8$ $(N = 6)$, $(SU(2) \times SU(2)) \ltimes \mathbb{R}^8$ $(N = 8)$, $\mathbb{R}^8$ $(N = 16)$, G_2 $(N = 4)$, $SU(3)$ $(N = 8)$, $SU(2)$ $(N = 16)$ and $\{1\}$ $(N = 32)$, where N denotes the (maximal) number of invariant spinors in each case. The maximally supersymmetric IIB backgrounds, $\{1\}(N = 32)$, have been classified in [7], where it was found that they are locally isometric to Minkowski spacetime $\mathbb{R}^{9,1}$, $AdS_5 \times S^5$ [5] and the maximally supersymmetric Hpp-wave [8]. In addition, the geometry of the maximally supersymmetric $Spin(7) \ltimes \mathbb{R}^8$-, $SU(4) \ltimes \mathbb{R}^8$- and G_2-backgrounds has already been investigated [2,3] using the spinorial geometry method of [10]. Here we shall use the same method to investigate the remaining cases. There are two classes of maximally supersymmetric G-backgrounds depending on whether G is a compact or non-compact subgroup of $Spin(9,1)$. The geometry of the backgrounds in the two cases is distinct. To outline our results we denote with $ds^2(S^k)$ the metric of the round k-dimensional sphere S^k, with $ds^2(AdS_k)$ the metric of k-dimensional anti-de Sitter space AdS_k and with $ds^2(CW_k(A))$ the metric[1] of the k-dimensional Cahen–Wallach space $CW_k(A)$ associated with the (constant) quadratic form A. The metric and fluxes are expressed in terms of orthonormal or null frame bases which arise from the description of the spinors in terms of forms. Our spinor conventions can be found in [3].

1.1. Backgrounds with compact stability subgroups

The geometry of the maximally supersymmetric G-backgrounds, where G is a compact subgroup of $Spin(9,1)$, is as follows:

- G_2: The spacetime is locally isometric to the product $\mathbb{R}^{2,1} \times Y_7$, where Y_7 is a G_2 holonomy manifold. The metric and fluxes are

$$ds^2(M) = ds^2(\mathbb{R}^{2,1}) + ds^2(Y_7), \qquad G = P = F = 0, \tag{1.2}$$

i.e. the fluxes vanish.

- $SU(3)$: The spacetime M is locally isometric to a product of a four-dimensional symmetric Lorentzian space and a six-dimensional Calabi–Yau manifold Y_6. In particular, the spacetime is
  - $M = AdS_2 \times S^2 \times Y_6$, and the metric and fluxes are

$$\begin{aligned}
&ds^2(M) = ds^2(AdS_2) + ds^2(S^2) + ds^2(Y_6),\\
&ds^2(AdS_2) = -(e^0)^2 + (e^1)^2, \qquad ds^2(S^2) = (e^5)^2 + (e^6)^2,\\
&F = \frac{1}{2\sqrt{2}}\left[H^1 \wedge \operatorname{Re}\chi - H^2 \wedge \operatorname{Im}\chi\right], \quad \chi = (e^2 + ie^7) \wedge (e^3 + ie^8) \wedge (e^4 + ie^9),
\end{aligned}$$

[1] The metric is $ds^2(CW_k(A)) = 2dx^-(dx^+ + \frac{1}{2}A_{ij}x^i x^j\, dx^-) + (dx^i)^2$, see [9].

$$H^1 = \lambda_1 e^0 \wedge e^1 + \lambda_2 e^5 \wedge e^6, \qquad H^2 = -\lambda_1 e^5 \wedge e^6 + \lambda_2 e^0 \wedge e^1,$$
$$G = P = 0, \tag{1.3}$$

where the scalar curvature of AdS_2 and S^2 are $R_{AdS_2} = -R_{S^2} = -4(\lambda_1^2 + \lambda_2^2)$.

– $M = CW_4(-2\mu^2\mathbf{1}) \times Y_6$, and the metric and fluxes are

$$\begin{aligned}
&ds^2(M) = ds^2(CW_4) + ds^2(Y_6),\\
&F = \frac{1}{2\sqrt{2}}\big[H^1 \wedge \operatorname{Re}\chi - H^2 \wedge \operatorname{Im}\chi\big],\\
&H^1 = \mu e^- \wedge e^1, \qquad H^2 = \mu e^- \wedge e^6,\\
&G = P = 0.
\end{aligned} \tag{1.4}$$

– $M = \mathbb{R}^{3,1} \times Y_6$, and the metric and fluxes are

$$\begin{aligned}
&ds^2(M) = ds^2\big(\mathbb{R}^{3,1}\big) + ds^2(Y_6),\\
&F = G = P = 0.
\end{aligned} \tag{1.5}$$

- $SU(2)$: The spacetime M is locally isometric to a product of a six-dimensional symmetric Lorentzian space and a four-dimensional hyper-Kähler manifold Y_4. In particular, the spacetime is

– $M = AdS_3 \times S^3 \times Y_4$, and the metric and fluxes are

$$\begin{aligned}
&ds^2(M) = ds^2(AdS_3) + ds^2\big(S^3\big) + ds^2(Y_4),\\
&ds^2(AdS_3) = -\big(e^0\big)^2 + \big(e^1\big)^2 + \big(e^2\big)^2, \qquad ds^2\big(S^3\big) = \big(e^3\big)^2 + \big(e^4\big)^2 + \big(e^5\big)^2,\\
&F = \frac{1}{4} v \cdot \hat{\omega} \wedge H,\\
&G = \big(v^4 + i v^5\big)H, \quad H = \lambda e^0 \wedge e^1 \wedge e^2 + \lambda e^3 \wedge e^4 \wedge e^5,\\
&P = 0,
\end{aligned} \tag{1.6}$$

where $v \cdot \hat{\omega} = v^1\hat{\omega}_I + v^2\hat{\omega}_J + v^3\hat{\omega}_K$ is a linear superposition of the Kähler forms $\hat{\omega}_I$, $\hat{\omega}_J$ and $\hat{\omega}_K$ of the hyper-Kähler manifold Y_4, $v^2 = 1$ and the scalar curvature $R_{AdS_3} = -R_{S^3} = -\frac{3}{2}\lambda^2$.

– $M = CW_6(-\frac{1}{4}\mu^2\mathbf{1}) \times Y_4$, and the metric and fluxes are

$$\begin{aligned}
&ds^2(M) = ds^2(CW_6) + ds^2(Y_4),\\
&F = \frac{1}{4} v \cdot \hat{\omega} \wedge H,\\
&G = \big(v^4 + i v^5\big)H, \quad H = \mu e^- \wedge e^1 \wedge e^2 - \mu e^- \wedge e^6 \wedge e^7,\\
&P = 0.
\end{aligned} \tag{1.7}$$

– $M = \mathbb{R}^{5,1} \times Y_4$, and the metric and fluxes are

$$\begin{aligned}
&ds^2(M) = ds^2\big(\mathbb{R}^{5,1}\big) + ds^2(Y_4),\\
&F = G = P = 0.
\end{aligned} \tag{1.8}$$

Therefore, we have shown that the maximally supersymmetric G_2-, $SU(3)$-, and $SU(2)$-backgrounds for G compact are the maximally supersymmetric solutions of $\mathcal{N} = 1$, $\mathcal{N} = 2$ and $(2,0)$-supergravities in three, four and six dimensions, respectively, lifted to IIB supergravity.

The maximally supersymmetric solutions for the $\mathcal{N}=2$ four-dimensional supergravity have been found in [11], see [12] for a more recent account. In six dimensions, the maximally supersymmetric solutions of $(1,0)$ supergravity have been classified in [13] and of the $(2,0)$ supergravity in [14]. In three dimensions, it is straightforward to show that the only maximally supersymmetric solution is locally isometric to Minkowski spacetime.

1.2. Backgrounds with non-compact stability subgroups

Next we turn to investigate the geometry of maximally supersymmetric $G = K \ltimes \mathbb{R}^8$-backgrounds for $K = Spin(7), SU(4), Sp(2), SU(2)\times SU(2)$ and $\{1\}$. It turns out that the spacetime M always admits a null parallel vector field X and the holonomy of the Levi-Civita connection of spacetime is contained in $K \ltimes \mathbb{R}^8$, i.e.

$$\nabla_A X = 0, \quad \mathrm{hol}(\nabla) \subseteq K \ltimes \mathbb{R}^8. \tag{1.9}$$

Therefore, the spacetime is a pp-wave propagating in an eight-dimensional Riemannian manifold Y_8 such that $\mathrm{hol}(\tilde{\nabla}) \subseteq K$, where $\tilde{\nabla}$ is the Levi-Civita connection of Y_8. Alternatively, the spacetime is a two-parameter Lorentzian deformation family of Y_8. Adapting coordinates along the parallel vector field $X = \partial/\partial u$, the metric can be written as

$$ds^2 = 2\,dv\,(du + V\,dv + n) + ds^2(Y_8) = 2\,dv\,(du + V\,dv + n) + \gamma_{IJ}\,dy^I\,dy^J, \tag{1.10}$$

where the metric $\gamma_{IJ} = \delta_{ij} e^i_I e^j_J$ of Y_8 may also depend on the coordinate v. The requirement that $\mathrm{hol}(\tilde{\nabla}) \subseteq K$ implies that the components $e^A \Omega_{A,ij}$ of the connection one-form take values in the Lie algebra of K, $\mathfrak{k}$.

In all cases, the fluxes are null, i.e.

$$P = P_-(v)e^-, \qquad G = e^- \wedge L, \qquad F = e^- \wedge M, \tag{1.11}$$

and the Bianchi identities give $dP = dG = dF = 0$, where L and M are a two- and a self-dual four-form, respectively, of Y_8. In particular, one finds that $P_- = P_-(v)$. The most convenient way to give the conditions that the Killing spinor equations impose on the fluxes is to decompose $L \in \Lambda^2(\mathbb{R}^8)\otimes\mathbb{C}$ and $M \in \Lambda^{4+}(\mathbb{R}^8)$ in irreducible representations of K. In particular, one finds that

$$L = L^{\mathfrak{k}} + L^{\mathrm{inv}}, \qquad M = M^{\mathrm{inv}} + \tilde{M}, \tag{1.12}$$

where $L^{\mathfrak{k}}$ is the Lie algebra valued component of L in the decomposition $\Lambda^2(\mathbb{R}^8) = \mathfrak{k} + \mathfrak{k}^\perp$, and L^{inv} and M^{inv} are K-invariant two- and four-forms, respectively. M^{inv} decomposes further as $M^{\mathrm{inv}} = m^0 + \hat{M}^{\mathrm{inv}}$, where m^0 has the property that the associated Clifford algebra element satisfies $m^0\epsilon = g\epsilon$, $g \neq 0$ a spacetime function, for all Killing spinors ϵ. In a particular gauge, the Killing spinor equations imply that g is proportional to Q_- and restrict the spacetime dependence of L^{inv} and M^{inv}. Furthermore, $\tilde{M}$ takes values in a representation of K in $\Lambda^{4+}(\mathbb{R}^8)$ with the property that the associated Clifford algebra element satisfies $\tilde{M}\epsilon = 0$ for all Killing spinors ϵ. $L^{\mathfrak{k}}$ and $\tilde{M}$ are not determined by the Killing spinor equations. In particular, one finds[2] the following:

[2] To solve all conditions that arise from the Killing spinor equations, we present our results in a particular gauge.

- $Spin(7) \ltimes \mathbb{R}^8$:

$$G = e^- \wedge L^{\mathfrak{spin}(7)}, \qquad F = e^- \wedge \left(\frac{1}{14} Q_-(v)\psi + M^{\mathbf{27}}\right), \tag{1.13}$$

where ψ is the invariant $Spin(7)$ four-form, Q_- depends only on v, and $L^{\mathfrak{spin}(7)}$ and $\tilde{M} = M^{\mathbf{27}}$ are not determined in terms of the geometry.

- $SU(4) \ltimes \mathbb{R}^8$:

$$G = e^- \wedge \left(L^{\mathfrak{su}(4)} + \ell(v)\omega\right),$$
$$F = e^- \wedge \left(-\frac{1}{12} Q_-(v)\omega \wedge \omega + \mathrm{Re}\big(m(v)\chi\big) + \tilde{M}^{2,2}\right), \tag{1.14}$$

where χ is the $SU(4)$-invariant $(4,0)$-form, ℓ, m and Q_- depend only on v as indicated, and $\tilde{M} = \tilde{M}^{2,2}$ is a traceless $(2,2)$-form.

- $Sp(2) \ltimes \mathbb{R}^8$:

$$G = e^- \wedge \left(L^{\mathfrak{sp}(2)} + \ell^r(v)\omega_r\right),$$
$$F = e^- \wedge \left(-\frac{1}{20} Q_-(v)\psi + m^{rs}(v)\omega_r \wedge \omega_s + M^{\mathbf{14}}\right), \tag{1.15}$$

where $\omega_I = \omega_1$, $\omega_J = \omega_2$ and $\omega_K = \omega_3$ are the Hermitian forms of the quaternionic endomorphisms I, J and K, $\psi = \sum_{r=1}^{3} \omega_r \wedge \omega_r$, m^{rs} is a symmetric traceless 3×3-matrix that depends only on v, $\ell^r = \ell(v)$, and $\tilde{M} = M^{\mathbf{14}}$.

- $(SU(2) \times SU(2)) \ltimes \mathbb{R}^8$:

$$G = e^- \wedge \big(L^{\mathfrak{su}(2)\oplus\mathfrak{su}(2)} + \ell^1(v)\omega_1 + \ell^2(v)\omega_2 + \ell^3(v)\chi_1$$
$$+ \ell^4(v)\chi_2 + \ell^5(v)\bar{\chi}_1 + \ell^6(v)\bar{\chi}_2\big),$$
$$F = e^- \wedge \Big(-\frac{1}{4} Q_-(v)[\omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2] + m^1(v)\omega_1 \wedge \omega_2 + \mathrm{Re}\big[m^2(v)\omega_1 \wedge \chi_2$$
$$+ m^3(v)\omega_2 \wedge \chi_1 + m^4(v)\chi_1 \wedge \chi_2 + m^5(v)\chi_1 \wedge \bar{\chi}_2\big] + M^{(\mathbf{3},\mathbf{3})}\Big), \tag{1.16}$$

where the pairs (ω_1, χ_1) and (ω_2, χ_2) are the Hermitian $(1,1)$- and holomorphic $(2,0)$-forms associated with the $(SU(2) \times SU(2)) \ltimes \mathbb{R}^8$-structure, ℓ, m depend only on v, and $\tilde{M} = M^{(\mathbf{3},\mathbf{3})}$.

- $\mathbb{R}^8$:

$$G = e^- \wedge L(v), \qquad F = e^- \wedge M(v), \tag{1.17}$$

where L and M are a two- and a self-dual four-form on $\mathbb{R}^8$, respectively, and depend only on v.

The integrability conditions of the Killing spinor equations and the Bianchi identities imply that all field equations are satisfied provided that $E_{--} = 0$, where E_{--} denotes the '$--$' component of the Einstein equations. This in turn gives

$$-\big(\partial^i + \Omega_{j,}{}^{ji}\big)\big(\partial_i V - \partial_v n_I e^I{}_i\big) + \frac{1}{4}(dn)_{ij}(dn)^{ij} - \frac{1}{2}\gamma^{IJ}\partial_v{}^2\gamma_{IJ} - \frac{1}{4}\partial_v\gamma^{IJ}\partial_v\gamma_{IJ}$$
$$- \frac{1}{6}F_{-i_1\ldots i_4}F_-{}^{i_1\ldots i_4} - \frac{1}{4}G_-{}^{i_1 i_2}G^*_{-i_1 i_2} - 2P_-P^*_- = 0, \tag{1.18}$$

where γ^{IJ} is the inverse of the metric γ_{IJ} defined in (1.10). For the special case of fields independent of v, this equation becomes

$$-\Box_8 V + \frac{1}{4}(dn)_{ij}(dn)^{ij} - \frac{1}{6}F_{-i_1\dots i_4}F_-{}^{i_1\dots i_4} - \frac{1}{4}G_-{}^{i_1 i_2}G^*_{-i_1 i_2} - 2P_- P^*_- = 0, \tag{1.19}$$

where $\Box_8$ is the Laplacian on the eight-dimensional space Y_8 and dn takes values in $\mathfrak{k}$.

The backgrounds that we have found can be thought of as vacua of IIB string theory. This particulary applies to compact stability subgroups. The backgrounds $\mathbb{R}^{9-n,1} \times Y_n$ are vacua of IIB compactifications on G_2 for $n = 7$, and on Calabi–Yau manifolds for $n = 6$ and $n = 4$. The backgrounds $AdS_{5-n/2} \times S^{5-n/2} \times Y_n$ can be thought either of as the vacua of the Calabi–Yau or $S^{5-n/2} \times Y_n$ compactifications with fluxes. For a recent application of the latter see [19]. For non-compact stability subgroups, the situation is different. If one views the solutions as vacua of compactifications and so insists to be invariant under lower-dimensional Poincaré symmetry, then the only solutions are $\mathbb{R}^{9-n,1} \times Y_n$. In particular all the fluxes vanish because of the field equations.

This paper is organized as follows: In Sections 2 and 3, we describe the geometry of maximally supersymmetric $SU(3)$- and $SU(2)$-backgrounds, respectively. In Sections 4–6, we give present the maximally supersymmetric $Sp(2) \ltimes \mathbb{R}^8$-, $(SU(2) \times SU(2)) \ltimes \mathbb{R}^8$- and $\mathbb{R}^8$-backgrounds, respectively. In Section 7, we describe solutions of maximally supersymmetric G-backgrounds, for a non-compact G. In Appendix A, we summarize the Killing spinor equations and some of their integrability conditions.

2. Maximal $SU(3)$-backgrounds

2.1. Supersymmetry conditions

As we have mentioned in the introduction to solve the Killing spinor equations and the integrability conditions of maximally supersymmetric $SU(3)$-backgrounds, one may use a basis in the Majorana–Weyl $SU(3)$-invariant spinors of IIB supergravity. Such a basis[3] is

$$\begin{aligned} &\eta_1 = 1 + e_{1234}, \qquad \eta_2 = i(1 - e_{1234}), \\ &\eta_3 = e_{15} + e_{2345}, \qquad \eta_4 = i(e_{15} - e_{2345}). \end{aligned} \tag{2.1}$$

To proceed, it is convenient to introduce the notation $A = (a, m)$. Here $a = (\alpha, \bar\alpha)$, $\alpha = (-, 1)$ and $\bar\alpha = (+, \bar 1)$ are the 'world-volume' labels and $m = (\mu, \bar\mu)$, $\mu = (2, 3, 4)$ and $\bar\mu = (\bar 2, \bar 3, \bar 4)$ denote those of the 'transverse space'. Due to the null directions, $X^{\bar\alpha} \neq (X^\alpha)^*$ for a real vector field X.

The algebraic Killing spinor equations (A.2) imply that all components of the P-flux vanish. In addition, the same equation requires that

$$\begin{aligned} &G_{\mu_1\mu_2\mu_3} = G_{\mu_1\mu_2}{}^{\mu_2} = G_{\bar\mu_1\mu_2}{}^{\mu_2} = G_{\bar\mu_1\bar\mu_2\bar\mu_3} = 0, \\ &G_{\alpha\mu_1\mu_2} = G_{\alpha_1\mu}{}^{\mu} - G_{\alpha_1\alpha_2}{}^{\alpha_2} = G_{\alpha\bar\mu_1\bar\mu_2} = 0, \\ &G_{\bar\alpha\mu_1\mu_2} = G_{\bar\alpha_1\mu}{}^{\mu} + G_{\bar\alpha_1\alpha_2}{}^{\alpha_2} = G_{\bar\alpha\bar\mu_1\bar\mu_2} = 0, \\ &G_{\alpha_1\bar\alpha_2\mu} - \frac{1}{2}g_{\alpha_1\bar\alpha_2}G_{\mu\alpha_3}{}^{\alpha_3} = G_{\alpha_1\alpha_2\bar\mu} = G_{\bar\mu\alpha}{}^{\alpha} = G_{\bar\alpha_1\bar\alpha_2\bar\mu} = 0. \end{aligned} \tag{2.2}$$

[3] Note that the $SU(3)$-invariant spinors are annihilated by $\Gamma^{\mu_1\bar\mu_2}$, where $\mu_1 \neq \mu_2$: indeed this gives rise to two independent projection operators, allowing for eight supersymmetries.

The gravitino Killing spinor equations (A.3) involving G imply

$$G_{Abm} = G_{A\mu_1\mu_2} = G_{A\bar{\mu}_1\bar{\mu}_1} = 0. \tag{2.3}$$

Combining the above results from the gravitino and algebraic Killing spinor equations, one finds that

$$P = G = 0, \tag{2.4}$$

i.e. all the P and G fluxes vanish.

The gravitino Killing spinor equations require that F satisfies

$$\begin{aligned}
&F_{A\mu_1\mu_2\mu_3}{}^{\mu_3} = 0,\\
&F_{A\alpha_1\mu_1\alpha_2}{}^{\alpha_2} - F_{A\alpha_1\mu_1\mu_2}{}^{\mu_2} = F_{A\bar{\alpha}_1\mu_1\alpha_2}{}^{\alpha_2} + F_{A\bar{\alpha}_1\mu_1\mu_2}{}^{\mu_2} = 0,\\
&F_{A\mu_1\mu_2\alpha_1\bar{\alpha}_2} - \frac{1}{2} g_{\alpha_1\bar{\alpha}_2} F_{A\mu_1\mu_2\alpha_3}{}^{\alpha_3} = 0,
\end{aligned} \tag{2.5}$$

from which follows that

$$F_{\mu_1\mu_2\mu_3\bar{\mu}_4\bar{\mu}_5} = F_{a\mu_1\mu_2\mu_3\bar{\mu}_4} = F_{a_1a_2a_3\mu_1\mu_2} = 0. \tag{2.6}$$

Subsequently, the self-duality constraint on F implies that

$$F_{a\mu_1\mu_2\bar{\mu}_3\bar{\mu}_4} = F_{a_1a_2\mu_1\mu_2\bar{\mu}_3} = F_{a_1a_2a_3\mu_1\bar{\mu}_2} = F_{a_1\dots a_4\mu} = 0. \tag{2.7}$$

Therefore the non-vanishing components of F are

$$F_{\alpha_1\alpha_2 234}, \quad F_\alpha{}^\alpha{}_{234}, \quad F_{\bar{\alpha}_1\bar{\alpha}_2 234}, \quad \tilde{F}_{\alpha_1\bar{\alpha}_2\bar{2}\bar{3}\bar{4}}, \tag{2.8}$$

and their complex conjugates, where tilde denotes the traceless part. These are all singlets under self-duality.

Next turn to the conditions on the geometry, Eq. (A.3) implies the constraints

$$\Omega_{A,bm} = \Omega_{A,\mu_1\mu_2} = 0, \tag{2.9}$$

for the spin connection.

The remaining components of the spin connection and fluxes give rise to the following parallel transport equation:

$$\begin{aligned}
&\partial_A\epsilon - \frac{1}{2} i Q_A\epsilon + \frac{1}{2}\Omega_{A,\mu}{}^{\mu}\Gamma^{2\bar{2}}\epsilon + \frac{1}{4}\Omega_{A,b_1b_2}\Gamma^{b_1b_2}\epsilon + \frac{1}{2} i F_{Ab234}\Gamma^{b234}\epsilon\\
&\quad + \frac{1}{2} i F_{Ab\bar{2}\bar{3}\bar{4}}\Gamma^{b\bar{2}\bar{3}\bar{4}}\epsilon = 0.
\end{aligned} \tag{2.10}$$

The generators 1, $\Gamma^{2\bar{2}}$ and $\Gamma^{b_1b_2}$ span a $\mathfrak{u}(1)^2 \oplus \mathfrak{so}(3,1)$ algebra inside $\mathfrak{u}(1) \oplus \mathfrak{spin}(9,1)$. However, for $A = a$ there are also the generators $i\Gamma^{b234}$ and $i\Gamma^{b\bar{2}\bar{3}\bar{4}}$ in this connection due to the non-vanishing flux components (2.8). Note that these generators satisfy the same algebra as 1, $\Gamma^{2\bar{2}}$, $\Gamma^{b_1b_2}$, Γ^{b2} and $\Gamma^{b\bar{2}}$; therefore the connection in (2.10) takes values in a $\mathfrak{u}(1) \oplus \mathfrak{so}(5,1) \equiv \mathfrak{u}(1) \oplus \mathfrak{sl}(2,\mathbb{H})$ algebra[4] which is not embedded in the $\mathfrak{u}(1) \oplus \mathfrak{spin}(9,1)$ gauge

[4] Note that the holonomy of the supercovariant connection of $\mathcal{N} = 2$ ungauged supergravity in four dimensions is $SL(2,\mathbb{H})$ [16].

symmetry. Because of this, one cannot set the connection to zero by a suitable gauge transformation. Observe that the traceless part of $\Omega_{A,\mu_1\bar{\mu}_2}$ does not appear in the parallel transport equations.

The vanishing of the curvature of the connection appearing in (2.10) gives rise to the following equations:

$$\begin{aligned}&\partial_{[A}Q_{B]}=0, \qquad R_{AB,\mu}{}^{\mu}-8F_{[A|c234}F_{B]}{}^{c}{}_{\bar{2}\bar{3}\bar{4}}=0,\\&R_{AB,c_1c_2}-8F_{[A|c_1234}F_{B]c_2\bar{2}\bar{3}\bar{4}}+8F_{[A|c_2234}F_{B]c_1\bar{2}\bar{3}\bar{4}}=0, \qquad \nabla_{[A}F_{B]c234}=0.\end{aligned} \tag{2.11}$$

It will be important in the following that the flux bilinear terms in the first line vanish due to the conditions (2.8) on F. The conditions (2.4), (2.9), (2.8) and (2.11) impose restrictions on the geometry of spacetime which we shall investigate.

2.2. Geometry of spacetime

We write the spacetime metric as $ds^2=\eta_{ab}e^ae^b+\delta_{mn}e^me^n$. The torsion free condition for the frame e^a, e^m and the condition $\Omega_{A,bm}=0$ in (2.9) imply that the spacetime admits an integrable bi-distribution of co-dimensions four or six, i.e. both $\{e^a\}$ and $\{e^m\}$ span an integrable distribution. Therefore the spacetime M is locally a topological product, $M=X_4\times Y_6$. Furthermore, $\Omega_{A,bm}=0$ in (2.9) implies that the metric compatible product structure $\pi=\eta_{ab}e^ae^b-\delta_{mn}e^me^n$ is parallel with respect to the Levi-Civita connection. This in turn implies that π is integrable and in the coordinate system that π is diagonal, the metric is a product. In particular,

$$\begin{aligned}&ds^2(M)=ds^2(X_4)+ds^2(Y_6), \qquad ds^2(X_4)=\eta_{ab}e^ae^b,\\&ds^2(Y_6)=\delta_{mn}e^me^n,\end{aligned} \tag{2.12}$$

i.e. $ds^2(X_4)$ does not depend on the coordinates of Y_6 and vice versa. The geometry of X_4 and Y_6 can be separately investigated. First consider the geometry of Y_6. The condition $\Omega_{m,\mu_1\mu_2}=0$ in (2.9) and $\Omega_{m,\mu}{}^{\mu}=0$, which can be easily derived from (2.11) after a suitable choice of gauge, imply that Y_6 is Calabi–Yau. There are no additional conditions on Y_6.

Next let us turn to investigate the geometry of X_4. For this, observe that the five form can be written as

$$F=\frac{1}{2\sqrt{2}}\left[H^1\wedge\operatorname{Re}\chi-H^2\wedge\operatorname{Im}\chi\right], \tag{2.13}$$

where χ is the parallel (3,0)-form on the Calabi–Yau manifold Y_6, and H^1 and H^2 are two-forms on Y_4. In addition the Bianchi identity of F together with the last equation of (2.11) imply that H^1, H^2 are independent of the coordinates of X_6 and are parallel forms on X_4. The remaining conditions can now be written as restrictions on the geometry of X_4. In particular, one has

$$\begin{aligned}&R_{a_1a_2,b_1b_2}-4H^1_{[a_1|b_1|}H^1_{a_2]b_2}-4H^2_{[a_1|b_1|}H^2_{a_2]b_2}=0,\\&\nabla_aH^1_{bc}=0, \qquad \nabla_aH^2_{bc}=0, \qquad H^1_{[a|c|}H^2_{b]}{}^{c}=0, \qquad \star H^1=H^2,\end{aligned} \tag{2.14}$$

where the last condition is implied by the self-duality of F. Since H^1 and H^2 are parallel, the first equation implies that the Riemann curvature R of X_4 is also parallel. Therefore X_4 is a Lorentzian symmetric space. The fields H^1 and H^2 are uniquely determined by their values at the origin of the symmetric space up to rigid $SO(3,1)$ transformations. Since H^1 and H^2 are related by the Hodge star operator in X_4, it suffices to find H^1. It turns out that H^1 can be chosen as, see, e.g., [7,17],

$$\lambda_1 e^0 \wedge e^1 + \lambda_2 e^5 \wedge e^6, \qquad \mu e^- \wedge e^1, \tag{2.15}$$

and so H^2 is

$$-\lambda_1 e^5 \wedge e^6 + \lambda_2 e^0 \wedge e^1, \qquad \mu e^- \wedge e^6, \tag{2.16}$$

where μ, λ_1 and λ_2 are real constants. Therefore H^1 defines a two-plane at the origin of the symmetric space Y_4 which is either time-like and/or space-like, or null. Moreover H^1 commutes with H^2. It is straightforward to see that in the case of the time-like and/or spacelike plane, $X_4 = AdS_2 \times S^2$, where both factors have the same radius and scalar curvature $R_{AdS_2} = -4(\lambda_1^2 + \lambda_2^2)$ and $R_{S^2} = 4(\lambda_1^2 + \lambda_2^2)$, respectively. In the case that the plane is null $X_4 = CW_4(-2\mu^2\mathbf{1})$. Note that the three different geometries of X_4 are related by Penrose limits of $AdS_2 \times S^2$ [18]. These are the maximally supersymmetric solutions of four-dimensional $\mathcal{N} = 2$ supergravity [11]. This completes the proof for the maximally supersymmetric $SU(3)$-backgrounds. The result is summarized in the introduction.

3. Maximal $SU(2)$-backgrounds

3.1. Supersymmetry conditions

A basis in the space of the $SU(2)$-invariant Majorana–Weyl spinors is

$$\begin{aligned} &\eta_1 = 1 + e_{1234}, && \eta_2 = i(1 - e_{1234}), && \eta_3 = e_{12} - e_{34}, && \eta_4 = i(e_{12} + e_{34}), \\ &\eta_5 = e_{15} + e_{2345}, && \eta_6 = i(e_{15} - e_{2345}), && \eta_7 = e_{52} + e_{1345}, && \eta_8 = i(e_{52} - e_{1345}). \end{aligned} \tag{3.1}$$

To find the conditions that the Killing spinor equations of Appendix A impose on the geometry of spacetime, it is convenient to split up the ten-dimensional frame indices into $A = (a, m)$, where $a = (\alpha, \bar{\alpha})$, with $\alpha = (-, 1, 2)$ and $\bar{\alpha} = (+, \bar{1}, \bar{2})$, and $m = (\mu, \bar{\mu})$, with $\mu = (3, 4)$ and $\bar{\mu} = (\bar{3}, \bar{4})$.

The algebraic Killing spinor equations (A.2) imply that

$$P = 0. \tag{3.2}$$

In addition, G is constrained as

$$\begin{aligned} &G_{a_1 a_2 \mu} = G_{a_1 a_2 \bar{\mu}} = G_{a \mu_1 \mu_2} = G_{a\mu}{}^{\mu} = G_{a \bar{\mu}_1 \bar{\mu}_2} = G_{\mu_1 \mu_2 \bar{\mu}_3} = G_{\mu_1 \bar{\mu}_2 \bar{\mu}_3} = 0, \\ &\tilde{G}_{\alpha_1 \alpha_2 \bar{\alpha}_3} = G_{\bar{\alpha}_1 \alpha_2}{}^{\alpha_2} = G_{\bar{\alpha}_1 \bar{\alpha}_2 \bar{\alpha}_3} = 0, \end{aligned} \tag{3.3}$$

where tilde denotes the traceless component. The gravitino Killing spinor equations (A.3) imply that

$$G_{Abm} = 0. \tag{3.4}$$

Due to these constraints, the components $G_{a \mu_1 \bar{\mu}_2}$ also vanish and one is only left with $G_{a_1 a_2 a_3}$ components, subject to (3.3). Incidentally, G^* satisfy the same conditions as it can be seen by taking the complex conjugate of those for G.

The gravitino Killing spinor equations (A.3) together with the self-duality of F imply that the only non-vanishing components are

$$F_{b_1 b_2 b_3 \mu_1 \mu_2}, \quad F_{b_1 b_2 b_3 \mu}{}^{\mu}, \quad F_{b_1 b_2 b_3 \bar{\mu}_1 \bar{\mu}_2}, \tag{3.5}$$

subject to the conditions

$$\tilde{F}_{\alpha_1\alpha_2\bar{\alpha}_3\mu_1\mu_2} = F_{\bar{\alpha}_1\alpha_2}{}^{\alpha_2}{}_{\mu_1\mu_2} = F_{\bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3\mu_1\mu_2} = 0 \tag{3.6}$$

and similarly for the remaining two components. In addition, (A.3) requires that the component

$$\Omega_{A,bm} = 0, \tag{3.7}$$

of the spin connection.

Using the above conditions on the fluxes, the parallel transport equation becomes

$$\begin{aligned} &\partial_A\epsilon - \frac{1}{2}iQ_A\epsilon + \frac{1}{4}\Omega_{A,b_1b_2}\Gamma^{b_1b_2}\epsilon + \frac{1}{2}\Omega_{A,34}\Gamma^{34}\epsilon + \frac{1}{2}\Omega_{A,\mu}{}^{\mu}\Gamma^{3\bar{3}}\epsilon + \frac{1}{2}\Omega_{A,\bar{3}\bar{4}}\Gamma^{\bar{3}\bar{4}}\epsilon \\ &\quad + \frac{i}{8}F_{Ab_1b_2m_1m_2}\Gamma^{b_1b_2m_1m_2}\epsilon + \frac{1}{8}G_{Ab_1b_2}\Gamma^{b_1b_2}C*\epsilon = 0. \end{aligned} \tag{3.8}$$

A necessary condition for the existence of solutions to this parallel transport equation is the vanishing of the curvature. This leads to the conditions

$$\begin{aligned} &\partial_{[A}Q_{B]} - \frac{1}{16}iG_{[A|c_1c_2}G^*_{B]}{}^{c_1c_2} = 0, \\ &R_{AB,34} - F_{[A|b_1b_23Q}F_{B]}{}^{b_1b_2}{}_4{}^Q = 0, \\ &R_{AB,\mu}{}^{\mu} - F_{[A|b_1b_2\mu n}F_{B]}{}^{b_1b_2\mu n} = 0, \\ &R_{AB,b_1b_2} - \frac{1}{4}G_{[A|b_1c}G^*_{B]b_2}{}^c + \frac{1}{4}G_{[A|b_2c}G^*_{B]b_1}{}^c - 2F_{[A|b_1cm_1m_2}F_{B]b_2}{}^{cm_1m_2} = 0, \\ &\nabla_{[A}F_{B]b_1b_2m_1m_2} = (\nabla_{[A} - iQ_{[A})G_{B]b_1b_2} = 0, \\ &F^{[A}{}_{m_1n[b_1b_2}F^{B]}{}_{b_3b_4]m_2}{}^n - F^{[A}{}_{m_2n[b_1b_2}F^{B]}{}_{b_3b_4]m_1}{}^n = 0, \\ &F^{[A}{}_{m_1m_2[b_1b_2}G^{B]}{}_{b_3b_4]} = G^{[A}{}_{[b_1b_2}(G^*)^{B]}{}_{b_3b_4]} = 0. \end{aligned} \tag{3.9}$$

The flux bilinear terms in the first three lines vanish due to the conditions (3.3) and (3.6). It remains to solve these conditions and find the geometry of spacetime.

3.2. Geometry of spacetime

The metric of the spacetime can be written as $ds^2 = \eta_{ab}e^ae^b + \delta_{mn}e^me^n$. In addition (3.7) implies that the spacetime M admits an integrable bi-distribution of co-dimension six and a metric compatible parallel product structure π. As in the $SU(3)$ case previously, $M = X_6 \times Y_4$, where X_6 is a Lorentzian manifold and Y_4 is a Riemannian manifold. In addition, the metric is a product, i.e.

$$ds^2(M) = ds^2(X_6) + ds^2(Y_4), \qquad ds^2(X_6) = \eta_{ab}e^ae^b, \qquad ds^2(Y_4) = \delta_{mn}e^me^n, \tag{3.10}$$

where $ds^2(X_6)$ does not dependent on the coordinates of Y_4 and vice versa. First let us examine the geometry of Y_4. It is straightforward to observe from (3.9) that the components $R_{mn,\mu}{}^{\mu}$ and $R_{mn,34}$ of the Riemann curvature vanish. These curvature components span an $\mathfrak{su}(2)$ subalgebra in $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \subset \mathfrak{spin}(9,1)$. This implies that the holonomy of the Levi-Civita connection of Y_4 is contained in $SU(2)$ and so Y_4 is hyper-Kähler.

Next let us turn to examine the geometry of X_6. Using (3.9), one can see that the Riemann curvature of X_6 is

$$R_{a_1a_2,b_1b_2} = \frac{1}{4}G_{[a_1|b_1c}G^*_{a_2]b_2}{}^c - \frac{1}{4}G_{[a_1|b_2c}G^*_{a_2]b_1}{}^c + 2F_{[a_1|b_1cm_1m_2}F_{a_2]b_2}{}^{cm_1m_2}. \tag{3.11}$$

Moreover, (3.9) and the Bianchi identities imply that F and G are parallel

$$\nabla_A F_{b_1b_2b_3m_1m_2} = \nabla_A G_{b_1b_2b_3} = 0. \tag{3.12}$$

This in particular implies that the curvature of X_6 is parallel and so X_6 is a symmetric space. Next observe that the fluxes can be written as

$$F = \frac{1}{4}\big[H^1 \wedge \hat{\omega}_I + H^2 \wedge \hat{\omega}_J + H^3 \wedge \hat{\omega}_K\big],$$
$$G = \operatorname{Re} G + i \operatorname{Im} G = H^4 + i H^5, \tag{3.13}$$

where H^s, $s = 1, \dots, 5$, are parallel 3-forms on X_6 and $\hat{\omega}_I$, $\hat{\omega}_J$ and $\hat{\omega}_K$ are the Kähler forms associated with the hyper-complex structure on Y_4. Furthermore, the conditions (3.3) and (3.6) imply that H^s are anti-self-dual three-forms on X_6. The remaining conditions conditions in terms of H^s can now be written as

$$R_{a_1a_2,a_3a_4} - \frac{1}{2}\sum_s H^s_{[a_1|a_3b|}H^s_{a_2]a_4}{}^b = 0, \qquad \nabla_{a_1} H^s_{a_2a_3a_4} = H^{[s}_{a_1[b_1b_2}H^{r]}_{b_3b_4]a_2} = 0. \tag{3.14}$$

These conditions are precisely those that one finds for the maximally supersymmetric solutions of (2, 0) supergravity in six dimensions [14] and the $SU(2)$-invariant Killing spinor case of the heterotic string [15]. In particular X_6 is a six-dimensional Lorentzian Lie group with anti-self-dual structure constants. These groups have been classified in [14] and they are locally isometric to $\mathbb{R}^{5,1}$, $AdS_3 \times S^3$ and $CW_6(\lambda\mathbf{1})$, and

$$H^s = v^s H, \tag{3.15}$$

where H are the structure constants of X_6 and v, $v^2 = 1$, is a constant vector. The maximally supersymmetric IIB $SU(2)$-backgrounds have been summarized in the introduction.

4. Maximal $Sp(2) \ltimes \mathbb{R}^8$-backgrounds

4.1. Supersymmetry conditions

A basis in the space of the $Sp(2) \ltimes \mathbb{R}^8$-invariant Majorana–Weyl spinors is

$$\eta_1 = 1 + e_{1234}, \qquad \eta_2 = i(1 - e_{1234}), \qquad \eta_3 = i(e_{12} + e_{34}). \tag{4.1}$$

To find the conditions that the Killing spinor equations of Appendix A impose on the geometry of spacetime, it is convenient to split up the ten-dimensional frame indices into $A = (-, +, i)$, where $i = (\alpha, \bar{\alpha})$ and $\alpha = (1, \dots, 4)$.

The algebraic Killing spinor equations (A.2) imply that

$$P_+ = P_i = 0, \tag{4.2}$$

i.e. only P_- is non-vanishing. In addition, the algebraic and the gravitino Killing spinor equations imply that

$$G_{-+i} = G_{+ij} = G_{ijk} = 0, \tag{4.3}$$

i.e. the only non-vanishing components are $G = e^- \wedge L$, where $L = \frac{1}{2}L_{ij}e^i \wedge e^j$. These components are in addition constrained as

$$L^{\mathbf{5}} = 0, \tag{4.4}$$

where we have used the decomposition of the space of two-forms, $\Lambda^2(\mathbb{R}^8) = \mathfrak{sp}(2) \oplus 3\Lambda^2_{\mathbf{5}} \oplus 3\Lambda^2_{\mathbf{1}}$, under $Sp(2) = Spin(5)$. Therefore, one can write that

$$G = e^- \wedge \left(L^{\mathfrak{sp}(2)} + \ell^r \omega_r\right), \tag{4.5}$$

where

$$\begin{aligned}&\omega_1 = \omega_I = -i\delta_{\alpha\bar\beta} e^\alpha \wedge e^{\bar\beta},\\ &\omega_2 = \omega_J = \mathrm{Re}\left(\epsilon_{\alpha\beta} e^\alpha \wedge e^\beta\right), \qquad \omega_3 = \omega_K = -\mathrm{Im}\left(\epsilon_{\alpha\beta} e^\alpha \wedge e^\beta\right),\end{aligned} \tag{4.6}$$

are the Hermitian forms generated by the quaternionic endomorphisms I, J and K, and ℓ^r are spacetime functions. We follow the notation of [15].

Next let us turn to the conditions on the F fluxes. The gravitino Killing spinor equations (A.3) together with the self-duality of F imply that

$$F_{i_1\ldots i_5} = F_{+i_1\ldots i_4} = F_{-+i_1 i_2 i_3} = 0. \tag{4.7}$$

Therefore one can write

$$F = e^- \wedge M, \quad M = \frac{1}{4!} M_{i_1\ldots i_4} e^{i_1} \wedge \cdots \wedge e^{i_4}. \tag{4.8}$$

In addition, the Killing spinor equations imply that

$$M^{\mathbf{5}} = 0, \tag{4.9}$$

where we have used the decomposition of self-dual 4-forms, $\Lambda^{4+}(\mathbb{R}^8) = \Lambda^{4+}_{\mathbf{14}} \oplus 3\Lambda^{4+}_{\mathbf{5}} \oplus 6\Lambda^{4+}_{\mathbf{1}}$, under $Sp(2)$ representations.[5] Therefore, one can write

$$F = e^- \wedge \left(M^{\mathbf{14}} + m^{rs}\omega_r \wedge \omega_s\right), \tag{4.10}$$

where (m^{rs}) is a symmetric matrix of spacetime functions.

Furthermore, the gravitino Killing spinor equation (A.3) imposes the conditions

$$\Omega_{A,+i} = 0, \qquad \Omega^{\mathbf{5}}_{A,ij} = 0, \tag{4.11}$$

on the geometry of spacetime, where the restriction to the five-dimensional $Sp(2)$ representation is made in the i, j indices. Therefore, one can write that

$$\Omega_{A,ij} = \Omega^{\mathfrak{sp}(2)}_{A,ij} + \Omega^r_A(\omega_r)_{ij}. \tag{4.12}$$

Using the above expressions for the fluxes and the geometry, the parallel transport equation becomes

$$\begin{aligned}&\partial_A\epsilon - \frac{1}{2}iQ_A\epsilon + \frac{1}{2}\Omega_{A,-+}\epsilon + \frac{1}{4}\Omega^r_A(\omega_r)_{ij}\Gamma^{ij}\epsilon = 0, \quad A \neq -,\\ &\partial_-\epsilon - \frac{1}{2}iQ_-\epsilon + \frac{1}{2}\Omega_{-,-+}\epsilon + \frac{1}{4}\Omega^r_-(\omega_r)_{ij}\Gamma^{ij}\epsilon + \frac{i}{8}m^{rs}(\omega_r)_{ij}(\omega_s)_{kl}\Gamma^{ijkl}\epsilon\\ &\quad + \frac{1}{8}\ell^r(\omega_r)_{ij}\Gamma^{ij}C^*\epsilon = 0.\end{aligned} \tag{4.13}$$

The components $L^{\mathfrak{sp}(2)}$, $\Omega^{\mathfrak{sp}(2)}_A$ and $M^{\mathbf{14}}$ do not appear in the parallel transport equations and so the Killing spinor equations do not constrain them further. The integrability condition of (4.13)

5 Using $\mathfrak{sp}(2) = \mathfrak{so}(5)$, $\Lambda_{\mathbf{14}}$ can be identified with the traceless symmetric representation $\tilde S^2(\mathbb{R}^5)$.

is the vanishing of the curvature of the associated connection which depends on the fluxes. This leads to the flatness conditions

$$\partial_{[A}\Omega_{B],-+}=0, \qquad R^r_{AB}=0, \qquad \partial_{[A}\hat{Q}_{B]}=0,$$
$$\hat{\nabla}_A\ell^r=0, \qquad \hat{\nabla}_A\left(m^{rs}-\frac{1}{3}\delta^{rs}\,\mathrm{tr}\,m\right)=0, \tag{4.14}$$

where $\hat{\nabla}$ is the connection and R^r_{AB} is the curvature of the $\mathfrak{sp}(1)$ connection Ω^r, respectively, and

$$\hat{Q}_A=Q_A, \quad A\neq -, \qquad \hat{Q}_-=Q_-+\frac{20}{3}\,\mathrm{tr}\,m. \tag{4.15}$$

Notice that in this case $m^0=\frac{1}{3}\,\mathrm{tr}\,m\sum_{r=1}^{3}\omega_r\wedge\omega_r$, i.e. it is proportional to the $Sp(2)\cdot Sp(1)$-invariant four-form. It turns out that the components $\Omega_{A,-+}$, Ω^r_A, $\hat{Q}_A$ of the connection can be set to zero with a gauge transformation in $U(1)\times SO(1,1)\times Sp(1)\subset U(1)\times Spin(9,1)$. In this gauge, one finds that the remaining conditions of (4.13) together with $dP=0$ imply that

$$\ell^r=\ell^r(v), \qquad m^{rs}=m^{rs}(v), \qquad \mathrm{tr}\,m=-\frac{3}{20}Q_-(v). \tag{4.16}$$

The expressions for the fluxes are summarized in the introduction.

4.2. Geometry and field equations

In the lightcone frame (e^-,e^+,e^i) which arises from the description of spinors in terms of forms, the spacetime metric can be written as $ds^2=2e^-e^++\delta_{ij}e^ie^j$. Choosing the gauge $\Omega_{A,+-}=0$ and using the conditions (4.11), one finds that $\Omega_{A,+B}=0$. So the null vector field $X=e_+$ is parallel[6]

$$\nabla X=0. \tag{4.17}$$

The conditions (4.11), (4.12) and (4.14) imply that the holonomy of the Levi-Civita connection of the spacetime is

$$\mathrm{hol}(\nabla)\subseteq Sp(2)\ltimes\mathbb{R}^8. \tag{4.18}$$

Adapting coordinates along $X=\frac{\partial}{\partial u}$ and using that X is rotation free, the spacetime metric can be written as

$$ds^2=2\,dv\left(du+V\,dv+n_ie^i\right)+\delta_{ij}e^ie^j, \qquad e^-=dv,$$
$$e^+=du+V\,dv+n_ie^i, \tag{4.19}$$

where all the components of the metric are independent of u but they may depend on v and the remaining coordinates. Clearly the spacetime is a pp-wave propagating on an eight-dimensional manifold Y_8 given by $u,v=\text{const}$. The metric of Y_8 is $d\tilde{s}^2=\delta_{ij}e^ie^j$. It is straightforward to see that the conditions on the geometry imply that the holonomy of the Levi-Civita connection, $\tilde{\nabla}$, of Y_8 is contained in $Sp(2)$, $\mathrm{hol}(\tilde{\nabla})\subseteq Sp(2)$, i.e. Y_8 is a hyper-Kähler manifold. Observe that

[6] There is a parallel null vector field independent of the choice of gauge, i.e. if $\Omega_{A,+-}=\partial_Af$, then $X=e^fe_+$ is parallel.

the metric of Y_8 depends on v and so v can be thought of as a deformation parameter of the $Sp(2)$-structure.

Furthermore, one can use the torsion free conditions to compute the Levi-Civita connection of (4.19). The result has been presented in (7.1). In this case, the conditions on the geometry imply that $\Omega_{-,ij}$ take values in $\mathfrak{sp}(2)$. The fluxes and conditions on the geometry are summarized in the introduction. The remaining cases with non-compact stability subgroup can be analyzed in a similar way. Because of this, we shall not present all the details.

It is well known that the Killing spinor equations impose some of the supergravity field equations. So it remains to find the field equations that are not satisfied as consequence of the Killing spinor equations. Since the fluxes are null, the Bianchi identities reduce to $dP = dG = dF = 0$. In addition after some investigation of the integrability equations of Appendix A, one finds that if

$$E_{--} = 0, \tag{4.20}$$

then all the field equations are satisfied. This is the case for all maximally supersymmetric G-backgrounds for G non-compact. Because of this, we shall not repeat this analysis in the other cases.

5. Maximal $(SU(2) \times SU(2)) \ltimes \mathbb{R}^8$-backgrounds

A basis in the space of the $(SU(2) \times SU(2)) \ltimes \mathbb{R}^8$-invariant Majorana–Weyl spinors is

$$\begin{aligned} &\eta_1 = 1 + e_{1234}, && \eta_2 = i(1 - e_{1234}), \\ &\eta_3 = e_{12} - e_{34}, && \eta_4 = i(e_{12} + e_{34}). \end{aligned} \tag{5.1}$$

To find the conditions that the Killing spinor equations of Appendix A impose on the geometry of spacetime, it is convenient to use light-cone frame indices $A = (-, +, i)$ and split up $i = (a, m)$ according to embedding $SO(4) \times SO(4) \subset SO(8)$. In addition, we use holomorphic and anti-holomorphic indices, $U(2) \times U(2) \subset SO(4) \times SO(4)$, as $a = (\alpha, \bar{\alpha})$, with $\alpha = (1, 2)$, and $m = (\mu, \bar{\mu})$, with $\mu = (3, 4)$.

The algebraic Killing spinor equations (A.2) imply that

$$P_+ = P_i = 0, \tag{5.2}$$

i.e. only P_- is non-vanishing. In addition, the algebraic (A.2) and gravitino (A.3) Killing spinor equations imply that

$$G_{+A_1A_2} = G_{ijk} = 0. \tag{5.3}$$

Therefore, the non-vanishing components of G are

$$G = e^- \wedge L, \quad L = \frac{1}{2} L_{ij} e^i \wedge e^j. \tag{5.4}$$

The Killing spinor equations imply that

$$G_{-am} = 0. \tag{5.5}$$

Thus we find that

$$L = \frac{1}{2}\left(L_{ab} e^a \wedge e^b + L_{mn} e^m \wedge e^n\right). \tag{5.6}$$

Each of these components decomposes further under $SU(2) \subset SO(4)$ as $\Lambda^2(\mathbb{R}^4) = 3\Lambda^2_{\mathbf{1}} \oplus \mathfrak{su}(2)$. Therefore L can be written as

$$
\begin{aligned}
& L = L^{\mathfrak{su}(2)\oplus\mathfrak{su}(2)} + L^{\text{inv}}, \\
& L^{\text{inv}} = \ell^1\omega_1 + \ell^2\omega_2 + \ell^3\chi_1 + \ell^4\chi_2 + \ell^5\bar{\chi}_1 + \ell^6\bar{\chi}_2,
\end{aligned} \tag{5.7}
$$

where $\omega_1 = -ie^1 \wedge e^{\bar{1}} - ie^2 \wedge e^{\bar{2}}$ and $\chi = 2e^1 \wedge e^2$ are the Hermitian and holomorphic volume forms associated with $SU(2) \times \{1\} \subset SU(2) \times SU(2)$, respectively, and similarly for ω_2 and χ_2. Furthermore, $\ell^1, \dots, \ell^6$ are spacetime functions and the first component of L takes values in $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ as indicated.

Next, let us turn to the conditions on the F fluxes. Again, one can show using the Killing spinor equations that the non-vanishing components of F can be written as

$$
F = e^- \wedge M, \quad M = \frac{1}{4!} M_{ijkl} e^i \wedge e^j \wedge e^k \wedge e^l. \tag{5.8}
$$

The gravitino Killing spinor equations (A.3) together with the self-duality of F imply additional conditions on M. It turns out that M can be written as

$$
M = m^0[\omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2] + \frac{1}{4} M_{a_1a_2m_1m_2} e^{a_1} \wedge e^{a_2} \wedge e^{m_1} \wedge e^{m_2}. \tag{5.9}
$$

The last component is further restricted. Decomposing the last components of M in $SU(2) \times SU(2)$ representations, one can write that

$$
\begin{aligned}
& M = m^0[\omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2] + \hat{M}^{\text{inv}} + M^{(\mathbf{3},\mathbf{3})}, \\
& \hat{M}^{\text{inv}} = m^1\omega_1 \wedge \omega_2 + \operatorname{Re}\big[m^2\omega_1 \wedge \chi_2 + m^3\omega_2 \wedge \chi_1 + m^4\chi_1 \wedge \chi_2 + m^5\chi_1 \wedge \bar{\chi}_2\big], \\
& M^{(\mathbf{3},\mathbf{3})} = \frac{1}{4}\tilde{M}_{\alpha\bar{\beta}\mu\bar{\nu}} e^\alpha \wedge e^{\bar{\beta}} \wedge e^\mu \wedge e^{\bar{\nu}},
\end{aligned} \tag{5.10}
$$

where we have used the decomposition $\Lambda^2(\mathbb{R}^4) \otimes \Lambda^2(\mathbb{R}^4) = 9\Lambda_{(\mathbf{1},\mathbf{1})} \oplus 3\Lambda_{(\mathbf{1},\mathbf{3})} \oplus 3\Lambda_{(\mathbf{3},\mathbf{1})} \oplus \Lambda_{(\mathbf{3},\mathbf{3})}$ under $SU(2) \times SU(2)$, and $\tilde{M}$ traceless. Furthermore m^0 and m^1 are real and $m^2, \dots, m^5$ are complex functions of spacetime, respectively.

The Killing spinor equation (A.3) also restricts the geometry of spacetime. In particular, one finds that

$$
\Omega_{A,bm} = \Omega_{A,+i} = 0. \tag{5.11}
$$

The spin connection can be written as as

$$
\Omega_{A,ij} = \Omega^{\mathfrak{su}(2)\oplus\mathfrak{su}(2)}_{A,ij} + \Omega^{\text{inv}}_{A,ij} \tag{5.12}
$$

in analogy with (5.7), where the decomposition is only in the i, j indices. Using this, the parallel transport equation can be written as

$$
\begin{aligned}
& \partial_A\epsilon - \frac{1}{2}iQ_A\epsilon + \frac{1}{2}\Omega_{A,-+}\epsilon + \frac{1}{4}\Omega^{\text{inv}}_{A,ij}\Gamma^{ij}\epsilon = 0, \quad A \neq -, \\
& \partial_-\epsilon - \frac{1}{2}iQ_-\epsilon + \frac{1}{2}\Omega_{-,-+}\epsilon + \frac{1}{4}\Omega^{\text{inv}}_{-,ij}\Gamma^{ij}\epsilon \\
& \quad - 2im^0\epsilon + \frac{1}{48}i\hat{M}^{\text{inv}}_{ijkl}\Gamma^{ijkl}\epsilon + \frac{1}{8}L^{\text{inv}}_{ij}\Gamma^{ij}(C*)\epsilon = 0.
\end{aligned} \tag{5.13}
$$

These parallel transport equations are independent of $\Omega^{\mathfrak{su}(2)\oplus\mathfrak{su}(2)}$, $L^{\mathfrak{su}(2)\oplus\mathfrak{su}(2)}$ and $M^{(\mathbf{3},\mathbf{3})}$. So there are no further conditions on these components imposed by the Killing spinor equations. It remains to solve the above parallel transport equations. For this observe that the connection Ω^{inv} takes values in $\mathfrak{su}(2)^{\perp} \oplus \mathfrak{su}(2)^{\perp} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. This is because $\mathfrak{so}(4) = \Lambda^2(\mathbb{R}^4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. The vanishing of the curvature implies that

$$\begin{aligned} &\partial_{[A}\Omega_{B],-+} = 0, \qquad R^{\mathrm{inv}} = 0, \qquad \partial_{[A}\hat{Q}_{B]} = 0, \\ &\nabla^{\mathrm{inv}}_A \hat{M}^{\mathrm{inv}} = \nabla^{\mathrm{inv}}_A L^{\mathrm{inv}} = 0, \quad A \neq -, \end{aligned} \tag{5.14}$$

where

$$\hat{Q}_A = Q_A, \quad A \neq -, \qquad \hat{Q}_- = Q_- + 4m^0, \tag{5.15}$$

and ∇^{inv} is the covariant derivative and R^{inv} is the curvature of the connection Ω^{inv}, respectively. As in the previous case, there is a local $U(1) \times Spin(9,1)$ transformation to set $\Omega_{A,-+} = \hat{Q}_A = \Omega^{\mathrm{inv}}_A = 0$. In this gauge and using $dP = 0$, we find that (5.14) imply that

$$m^0 = -\frac{1}{4}Q_-(v), \tag{5.16}$$

and that the spacetime functions in (5.7) and (5.10) that determine L^{inv} and M^{inv} depend only on the v coordinate. The description of the geometry of spacetime is similar to that of the $Sp(2) \ltimes \mathbb{R}^8$ case we have already investigated. In particular, there is a null parallel vector field X and the holonomy of the Levi-Civita connection is contained in $(SU(2) \times SU(2)) \ltimes \mathbb{R}^8$. Therefore the spacetime is a pp-wave propagating in an eight-dimensional space Y_8 which has holonomy[7] $Spin(4) = SU(2) \times SU(2)$. The results of our analysis have been summarized in the introduction.

6. Maximal $\mathbb{R}^8$-backgrounds

To investigate the Killing spinor equations and the integrability conditions of the maximally supersymmetric $\mathbb{R}^8$-backgrounds, one needs the Majorana $\mathbb{R}^8$-invariant spinors of IIB supergravity. A basis of the $\mathbb{R}^8$-invariant spinors is

$$\begin{aligned} &\eta_1 = 1 + e_{1234}, \qquad \eta_2 = i(1 - e_{1234}), \\ &\eta_3 = e_{12} - e_{34}, \qquad \eta_4 = i(e_{12} + e_{34}), \\ &\eta_5 = e_{13} + e_{24}, \qquad \eta_6 = i(e_{13} - e_{24}), \\ &\eta_7 = e_{23} - e_{14}, \qquad \eta_8 = i(e_{23} + e_{14}). \end{aligned} \tag{6.1}$$

Observe that these spinors are characterized by the condition

$$\Gamma^{-}\eta = 0. \tag{6.2}$$

In this section we shall again use the the light-cone decomposition of the frame indices $A = (-, +, i)$. The algebraic Killing spinor equations (A.2) and (A.3) imply that the non-vanishing components of P and G are

$$P = P_- e^-, \qquad G = e^- \wedge L, \quad L = \frac{1}{2}L_{ij}e^i \wedge e^j. \tag{6.3}$$

[7] If Y_8 is compact and simply connected, then it is a product $Y_8 = M_1 \times M_2$, where M_1 and M_2 are four-dimensional hyper-Kähler manifolds.

There are no further restrictions on L. Similarly, (A.3) implies that the non-vanishing components of F are

$$F = e^- \wedge M, \quad M = \frac{1}{4!} M_{ijkl} e^i \wedge e^j \wedge e^k \wedge e^l. \tag{6.4}$$

There are no further restrictions on M. The condition on the geometry in this case is

$$\Omega_{A,+i} = 0 \tag{6.5}$$

together with the parallel transport equations. The parallel transport equation for f now reads as follows. For $A \neq -$ we have

$$\begin{aligned}
&\partial_A \epsilon - \frac{1}{2} i Q_A \epsilon + \frac{1}{2} \Omega_{A,-+} \epsilon + \frac{1}{4} \Omega_{A,ij} \Gamma^{ij} \epsilon = 0, \quad A \neq -, \\
&\partial_- \epsilon - \frac{1}{2} i Q_- \epsilon + \frac{1}{2} \Omega_{-,-+} \epsilon + \frac{1}{4} \Omega_{-,ij} \Gamma^{ij} \epsilon + \frac{1}{8} L_{ij} \Gamma^{ij} C * \epsilon + \frac{1}{48} i M_{ijkl} \Gamma^{ijkl} \epsilon = 0.
\end{aligned} \tag{6.6}$$

The connection C, see Appendix A, takes values in $\mathfrak{gl}(16, \mathbb{R}) = \mathfrak{gl}(8, \mathbb{R}) \otimes \mathbb{H}$. The integrability conditions of the above parallel transport equations imply that

$$\begin{aligned}
&\partial_{[A} \Omega_{B],-+} = 0, \qquad \partial_{[A} Q_{B]} = 0, \qquad R^{ij}_{AB} = 0, \\
&\hat{\nabla}_A L_{ij} = 0, \qquad \hat{\nabla}_A M_{ijkl} = 0, \quad A \neq -,
\end{aligned} \tag{6.7}$$

where $\hat{\nabla}$ and R^{ij} is the covariant derivative and the curvature of $\Omega_{A,ij}$, respectively. A similar analysis to the previous case reveals that in the gauge $Q_A = \Omega_{A,-+} = \Omega_{A,ij} = 0$, L and M depend only on v. Our solutions generalize those of [20] since they contain both G and F fluxes. Compare also our result with the eleven-dimensional supergravity pp-wave solution of [21]. Generic backgrounds preserve sixteen supersymmetries. However, for special choices of fluxes the supersymmetry can be enhanced [8,22,23,25]. The results have been summarized in the introduction.

7. pp-wave solutions with fluxes

We have identified all maximally supersymmetric G-backgrounds, for G compact, up to a local isometry. It remains to extend this to the cases where G is non-compact. The torsion free condition implies that

$$\begin{aligned}
&\Omega_{i,j-} = e^I{}_{(i} \partial_v e_{j)I} + \frac{1}{2} (dn)_{ij}, \qquad \Omega_{-,-i} = \partial_i V - \partial_v n_I e^I{}_i, \\
&\Omega_{-,ij} = e^I{}_{[i} \partial_v e_{j]I} - \frac{1}{2} (dn)_{ij}.
\end{aligned} \tag{7.1}$$

So to find the solutions in the non-compact case, one has to find the most general solution of (1.18) and restrict $e^A \Omega_{A,ij}$ to $\mathfrak{k}$. This is a rather challenging problem in the case that the fields depend on the coordinate v. However, the problem is considerably simplified provided that the fields are taken to be independent of v. In such a case, the field equation reduces to (1.19) and dn is required to take values in $\mathfrak{k}$. This equation is a Laplacian equation on the eight-dimensional transverse space Y_8 for the function[8] V with a source term reminiscent of that of resolved

[8] The function α can remain an arbitrary function of v.

branes in [24]. The source term depends on the fluxes and a rotation term depending on $d\beta$. The simplest case is whenever the fluxes $F = G = 0$ and $dn = 0$. In this case, V is a harmonic function of Y_8, $\Box_8 V = 0$. These are the standard type of pp-waves propagating on manifolds of holonomy K. Many such solutions have been found by solving for α. In particular, in the case $Y_8 = \mathbb{R}^8$, $V = \mu_0 + \sum_i \frac{\mu_i}{|y-y_i|^6}$. A generalization of these solutions is to allow for the presence of fluxes. In particular, one can take $L^{\mathfrak{k}} = \tilde{M} = 0$ but $L^{\mathrm{inv}} = M^{\mathrm{inv}} \neq 0$. In this case, the equation for α becomes

$$\Box_8 V = -2\lambda^2, \tag{7.2}$$

where λ is a constant that depends on the coefficients of the invariant terms. This equation can be solved in a variety of cases. For example if $Y_8 = \mathbb{R}^8$, then one can write

$$V = -A_{ij} y^i y^j + B_i y^i + \mu_0 + \sum_i \frac{\mu_i}{|y - y_i|^6}, \qquad \operatorname{tr} A = \lambda^2. \tag{7.3}$$

The additional term modifies the asymptotic behavior of the solution as $|y| \to \infty$ which is now a plane wave instead of flat space.

One can also construct examples with $dn \neq 0$. In all these cases, dn takes values in $\mathfrak{k}$. Solutions to these conditions are known in many cases. For example for $Y_8 = \mathbb{R}^8$, some solutions have been summarized in [27].

It is also possible to obtain under certain conditions smooth solutions for Y_8 compact without boundary. Integrating (1.19) by parts and using (1.11), we find that

$$\int_{Y_8} d\,\mathrm{vol}\big[\|dn\|^2 - 8\|M\|^2 - \|L\|^2 - 4\|P\|^2\big] = 0. \tag{7.4}$$

This equation can be read as a condition for the cancelation of field fluxes against angular momentum associated to the spacetime. If $dn = 0$ the above condition cannot be satisfied and smooth solutions do not exist. The above condition can be written in various ways. In particular using (1.12) and the orthogonality in the decomposition of the fluxes, one finds that

$$\int_{Y_8} d\,\mathrm{vol}\big[\|dn\|^2 - 8\big(\|M^{\mathrm{inv}}\|^2 + \|\tilde{M}\|^2\big) - \big(\|L^{\mathfrak{k}}\|^2 + \|L^{\mathrm{inv}}\|^2\big) - 4\|P\|^2\big] = 0. \tag{7.5}$$

In addition, in many cases (7.5) depends on the cohomology class $[dn] \in H^2(Y_8, \mathbb{R})$ and not on the representative chosen. For example, in the Calabi–Yau case (1.14), the condition (7.5) can be written as

$$\int_{Y_8} \left[-\frac{1}{2} dn \wedge dn \wedge \omega^2 - 8\big(M^{\mathrm{inv}} \wedge M^{\mathrm{inv}} + \tilde{M} \wedge \tilde{M}\big) + \frac{1}{2}\bar{L}^{\mathfrak{k}} \wedge L^{\mathfrak{k}} \wedge \omega^2\right]$$
$$- [4\ell^*\ell + 4P_-^* P_-]\,\mathrm{Vol}(Y_8) = 0. \tag{7.6}$$

To find a solution, it remains to specify dn, $\tilde{M}$ and $L^{\mathfrak{k}}$. The existence of these require additional conditions, see, e.g., [26]. For example, in the Calabi–Yau case, the existence of dn and $L^{\mathfrak{k}}$ requires that

$$\int_{Y_8} dn \wedge \omega^3 = 0, \qquad \int_{Y_8} L^{\mathfrak{su}(4)} \wedge \omega^3 = 0. \tag{7.7}$$

It is likely that similar conditions are required for the remaining cases. Many examples can be constructed for Y_8 non-compact. However, this may require case by case investigation.

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Appendix A. Killing spinor and integrability conditions for maximal G-backgrounds

The Killing spinors of maximally supersymmetric G-backgrounds can be written as

$$\epsilon_i = \sum_j f_{ij}\eta_j, \quad i, j = 1, \ldots, N_{\max}, \tag{A.1}$$

where η_p, $p \leqslant m$, are G-invariant Majorana spinors and $\eta_{m+p} = i\eta_p$, $N_{\max} = 2m$, and $f = (f_{ij})$ is a $N_{\max} \times N_{\max}$ invertible matrix with entries real spacetime functions. It has been shown in [2,3] that the algebraic Killing spinor equations of IIB supergravity for the maximally supersymmetric G-backgrounds can be written as

$$\begin{aligned}
&P_A\Gamma^A\eta_p = 0, \quad p = 1, \ldots, m,\\
&\Gamma^{ABC}G_{ABC}\eta_p = 0, \quad p = 1, \ldots, m.
\end{aligned} \tag{A.2}$$

Similarly, the gravitino Killing spinor equation can be expressed as

$$\begin{aligned}
&\frac{1}{2}\left[\sum_{j=1}^{N}\left(f^{-1}D_M f\right)_{pj}\eta_j - i\sum_{j=1}^{N}\left(f^{-1}D_M f\right)_{m+pj}\eta_j\right] + \nabla_M\eta_p\\
&\quad + \frac{i}{48}\Gamma^{N_1\ldots N_4}\eta_p F_{N_1\ldots N_4 M} = 0,\\
&\sum_{j=1}^{N}\left(f^{-1}D_M f\right)_{pj}\eta_j + i\sum_{j=1}^{N}\left(f^{-1}D_M f\right)_{m+pj}\eta_j + \frac{1}{4}G_{MBC}\Gamma^{BC}\eta_p = 0,
\end{aligned} \tag{A.3}$$

where we have set $N = N_{\max}$ for simplicity. In turn, these equations can be rewritten as a set of algebraic conditions on the fluxes and a parallel transport equation associated with the restriction of the supercovariant derivative along the bundle of Killing spinors. The latter condition can be written as $f^{-1}df + C = 0$. This gives rise to the integrability condition $dC - C \wedge C = 0$.

Sometimes it is helpful to express (A.3) in terms of the Killing spinors ϵ. This gives

$$\begin{aligned}
&\partial_A\epsilon - \frac{1}{2}iQ_A\epsilon + \frac{1}{4}\Omega_{A,B_1B_2}\Gamma^{B_1B_2}\epsilon + \frac{1}{48}iF_{AB_1\ldots B_4}\Gamma^{B_1\ldots B_4}\epsilon\\
&\quad + \frac{1}{8}G_{AB_1B_2}\Gamma^{B_1B_2}C * \epsilon = 0.
\end{aligned} \tag{A.4}$$

However in this form, the various terms that arise with different powers of gamma matrices are not linearly independent. The integrability condition is

$$
\begin{aligned}
&-\frac{1}{2}i\left(\partial_{[A}Q_{B]}-\frac{1}{16}iG_{[A|D_1D_2}G^{*}_{B]}{}^{D_1D_2}\right)\epsilon\\
&\quad+\frac{1}{2}\left(\frac{1}{4}R_{ABC_1C_2}-\frac{1}{12}F_{[A|C_1D_1\dots D_3}F_{B]C_2}{}^{D_1\dots D_3}-\frac{1}{8}G_{[A|C_1D}G^{*}_{B]C_2}{}^{D}\right)\Gamma^{C_1C_2}\epsilon\\
&\quad+\frac{1}{8}\left(\nabla_{[A}G_{B]C_1C_2}-iQ_{[A}G_{B]C_1C_2}-\frac{1}{2}iF_{[A|C_1C_2D_1D_2}G_{B]}{}^{D_1D_2}\right)\Gamma^{C_1C_2}C*\epsilon\\
&\quad+\frac{1}{48}i\left(\nabla_{[A}F_{B]C_1\dots C_4}-\frac{3}{4}iG_{[A|C_1C_2}G^{*}_{B]C_3C_4}\right)\Gamma^{C_1\dots C_4}\epsilon\\
&\quad+\frac{1}{144}F_{[A|C_1\dots C_3D}F_{B]C_4\dots C_6}{}^{D}\Gamma^{C_1\dots C_6}\epsilon\\
&\quad+\frac{1}{192}iF_{[A|C_1\dots C_4}G_{B]C_5C_5}\Gamma^{C_1\dots C_6}C*\epsilon=0.
\end{aligned}
\tag{A.5}
$$

As we have already mentioned, the linear system that determines the components of the field equations that are implied from the Killing spinor equations simplifies for maximally supersymmetric G-backgrounds [3]. In particular, one finds that

$$
\begin{aligned}
&\left[\frac{1}{2}\Gamma^{B}E_{AB}-i\Gamma^{B_1B_2B_3}LF_{AB_1B_2B_3}\right]\eta_p=0,\\
&\left[\Gamma^{B}LG_{AB}-\Gamma_A{}^{B_1\dots B_4}BG_{B_1\dots B_4}\right]\eta_p=0,\\
&\left[\frac{1}{2}\Gamma^{AB}LG_{AB}+\Gamma^{A_1\dots A_4}BG_{A_1\dots A_4}\right]\eta_p=0,\\
&\left[LP+\Gamma^{AB}BP_{AB}\right]\eta_p=0,\quad p=1,\dots,m,
\end{aligned}
\tag{A.6}
$$

where the expressions for the field equations and our notation is explained in [3]. We use this linear system to find the field equations that must be imposed in addition to the Killing spinor equations for a supersymmetric configuration to be a solution of the supergravity theory.

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