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Minimizing the symmetric difference distance in conic spline approximation

Sunayana Ghosh and Gert Vegter

Abstract. We show that the complexity (the number of elements) of an optimal parabolic or conic spline approximating a smooth curve with non-vanishing curvature to within symmetric difference distance ε is $c_1\,\varepsilon^{-1/4} + O(1)$, if the spline consists of parabolic arcs and $c_2\,\varepsilon^{-1/5} + O(1)$, if it is composed of general conic arcs of varying type. The constants c_1 and c_2 are expressed in the affine curvature of the curve. We define an *equisymmetric* conic arc tangent to a curve at its endpoints, to be the (unique) conic such that the areas of the two moons formed by this conic and the given curve are equal, and show that its complexity is asymptotically equal to the complexity of an optimal conic spline. We also show that the symmetric difference distance between a curve and an equisymmetric conic arc tangent at its endpoints is increasing with affine arc length, provided the affine curvature along the arc is monotone. This property yields a simple and an efficient bisection algorithm for the computation of an optimal parabolic or equisymmetric conic spline.

1. Introduction

In computer aided geometric design one of the central topics of research is the approximation of complex geometric objects with simpler ones. An important part of this field concerns the approximation of plane curves and the asymptotic analysis of the rate of convergence of approximation schemes with respect to different metrics. In Ghosh, Petitjean and Vegter [3] we determined the complexity, i.e., the number of elements, of parabolic and conic splines approximating a smooth planar curve to within a given Hausdorff distance. In this paper we extend this work by focusing on the *symmetric difference distance*. Recall that the symmetric difference distance of two closed curves is the total area of the set-theoretic symmetric difference of the regions enclosed by these curves. The symmetric difference distance of two curves that are not closed, but have common endpoints, is the total area of the regions enclosed by the two curves. See Figure 1.
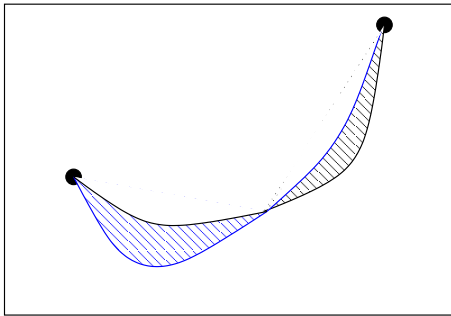
FIGURE 1. The symmetric difference of the two curves is the total area of the shaded regions.

Various error bounds and convergence rates have been obtained for several types of (low-degree) approximation primitives. For the approximation of plane convex curves by polygons with n edges, the order of convergence is $O(n^{-2})$ for several metrics, including the symmetric difference metric [5, 7, 12]. For approximation by a tangent continuous conic spline, the order of convergence, for a strictly convex curve is $O(n^{-5})$, where n is the number of elements of the conic spline, with respect to the Hausdorff distance [3, 10]. Ludwig [6] considers optimal parabolic spline approximation of strictly convex curves having monotone affine curvature with respect to the symmetric difference metric.

In this paper we not only study optimal approximation — with respect to the symmetric difference metric — of a strictly convex smooth curve by parabolic splines, but also by conic splines. As in [6] and our earlier paper [3] we consider convex curves that are *affine spirals*, i.e., curves with monotone affine curvature. (For a definition of affine curvature and an overview of related concepts from differential geometry we refer to Section 2.) We present the first sharp asymptotic bound on the approximation error in terms of the symmetric difference distance (and, consequently a sharp bound on the complexity of the approximation). We implemented the approximation algorithm, and our experiments corroborate this sharp bound for optimal parabolic spline approximation and *near optimal* conic spline approximation. This near-optimal approximation scheme will be explained later in this section.

1.1. Related Work

McClure and Vitale [7] consider the problem of approximating a convex C^2-curve C in the plane by an inscribed $n-$gon with respect to the *symmetric difference metric* δ_S. They prove that, with regard to the symmetric difference distance the optimal $n-$gon P_n, satisfies

$$\delta_S(C, P_n) = \tfrac{1}{12}\Big(\int_0^l \kappa^{1/3}(s)\, ds\Big)^3 \frac{1}{n^2} + O(\frac{1}{n^4}).$$

Ludwig [6] shows that the symmetric difference distance of an optimal parabolic spline with $n-$knots and a convex C^4-curve C in the plane satisfies,

$$\delta_S(C, Q_n) = \frac{1}{240} \Big(\int_0^l |k(u)|^{1/5} \, du \Big)^5 \frac{1}{n^4} + O(\frac{1}{n^5}),$$

where u is the affine arc length parameter of the curve C. Schaback [10] introduces a scheme that yields an interpolating conic spline with tangent continuity for a curve with non-vanishing curvature, and achieves an approximation order of $O(h^5)$ with respect to the Hausdorff distance, where h is the maximal distance of adjacent data points on the curve. The problem of approximating a planar curve by a conic spline has also been studied from a more practical standpoint by Farin [2], Pottmann [9], Yang [13], and Li et al. [4]. The methods employed have some limitations, like the dependence on the specific parametrization of the curve, the large number of conic segments produced or the lack of accuracy and absence of control of the error.

Ghosh, Petitjean and Vegter [3] presents the first sharp asymptotic bounds for an optimal parabolic and conic spline approximation for a sufficiently smooth curve with non-vanishing curvature, with respect to the Hausdorff distance. The complexity of an optimal parabolic spline approximating the curve to within Hausdorff distance ε is shown to be

$$N(\varepsilon) = (128)^{-1/4} \Big(\int_0^L |k(s)|^{1/4} \, \kappa(s)^{5/12} \, ds \Big) \, \varepsilon^{-1/4} \, (1 + O(\varepsilon^{1/4})).$$

Furthermore, the complexity of an optimal conic spline with respect to the Hausdorff distance is

$$N(\varepsilon) = (2000\sqrt{5})^{-1/5} \Big(\int_0^L |k'(s)|^{1/5} \, \kappa(s)^{2/5} \, ds \Big) \, \varepsilon^{-1/5} \, (1 + O(\varepsilon^{1/5})),$$

where $\kappa(s)$ is the Euclidean curvature, $k(s)$ is the affine curvature and s is the arc length parameter. Furthermore, *bitangent conic arcs* of affine spirals, i.e., conic arcs tangent to the affine spiral at both endpoints, have some useful global properties. First, among the *bitangent conic arcs* of an affine spiral there is a unique one minimizing Hausdorff distance. Furthermore, it is shown that the Hausdorff distance between a curve and its optimal bitangent conic arc is a monotone function of arc length. This property gives rise to a simple bisection algorithm for the computation of optimal conic splines.

1.2. Results of this paper

We consider the problem of optimally approximating a convex curve with respect to the symmetric difference distance by parabolic and conic splines. Our derivation of the complexity follows the lines of [3], but requires some new techniques to cope with the peculiarities of the symmetric difference distance.

Complexity of conic approximants. We show that the complexity $N(\varepsilon)$, the number of elements of an optimal *parabolic spline* approximating the curve to within symmetric difference distance ε, is given by

$$N(\varepsilon) = (240)^{-1/4} \Big(\int_0^\varrho |k(r)|^{1/5}\, dr \Big)^{5/4} \varepsilon^{-1/4}\, (1 + O(\varepsilon^{1/4})),$$

where $k(r)$ is the affine curvature of a smooth convex curve γ at $\gamma(r)$ and r is the affine arc length parameter. An optimal *conic spline* approximates the curve to within fifth order, with respect to the symmetric difference distance. More precisely, we show that its complexity is given by

$$N(\varepsilon) = (7680)^{-1/5} \Big(\int_0^\varrho |k'(r)|^{1/6}\, dr \Big)^{6/5} \varepsilon^{-1/5}\, (1 + O(\varepsilon^{1/5})).$$

These bounds on the complexity are obtained by first deriving an expression for the symmetric difference distance of a conic arc that is tangent to a (sufficiently short) curve at its endpoints, and minimizes the symmetric difference distance among all such bitangent conics. We derive explicit constants for the asymptotic expansion of the symmetric difference distance as a function of arc length. Our method for computing the asymptotic error bound of an optimal parabolic spline are different from those of [6], and allow us to determine the optimal asymptotic error bound in case of general conic splines as well. Obviously, our result for parabolic splines match those of Ludwig [6]. Furthermore, for deriving the asymptotic error bounds, we use the relation between affine curvatures of offset curves as proved in [3, Lemma 4.1], and the fact that conics have constant affine curvature.

We *conjecture* that there is a unique bitangent conic which minimizes the symmetric difference distance to a smooth affine spiral. This property would be the equivalent of the unicity of the bitangent conic minimizing the Hausdorff distance to the affine spiral, and would be of paramount importance for the design of an algorithm computing the optimal approximant. However, there is another conic spline achieving the same asymptotic bound on the symmetric difference metric, that exhibits these features. More precisely, we introduce the *equisymmetric bitangent conic* of an affine spiral, which is *uniquely* determined by the fact that the two moons it forms with the affine spiral have equal area. An *equisymmetric conic spline* is a tangent continuous conic spline all of whose elements are equisymmetric bitangent conics of the affine spiral. The equisymmetric conic spline, has the property that all moons formed by this spline and the affine spiral have equal area, and we denote by C_{es} the spline that minimizes the symmetric difference distance to the spiral among all equisymmetric conic splines. Furthermore, the complexity of this equisymmetric conic spline as a function of the symmetric difference distance to the affine spiral is asymptotically equal to the complexity of the optimal conic spline with respect to this error metric. Therefore, we call the computation of the optimal equisymmetric conic spline a *near-optimal approximation scheme.*

Algorithmic issues. We implement the near-optimal approximation scheme for affine spirals. The symmetric difference distance between a section of an affine

spiral and its equisymmetric bitangent conic arc is a monotone function of the arc length of the spiral section. This useful property gives rise to an efficient, bisection based algorithm computing the equisymmetric conic spline. For several curves we compare the theoretical complexity of an optimal conic spline with the computed number of elements in an equisymmetric tangent continuous conic spline, and find that these numbers match almost exactly.

1.3. Paper overview

Section 2 reviews notions from affine differential geometry used in this paper. Section 3 presents the global properties of equisymmetric bitangent arcs of an affine spiral, like the monotonicity of the symmetric difference distance to the spiral as a function of arc length. In Section 4 we derive the complexity of an optimal conic spline with respect to the symmetric difference distance, and show that the optimal equisymmetric conic spline has the same asymptotic complexity. The algorithm is presented in Section 5, together with experimental results corroborating our theoretical complexity bounds.

2. Mathematical Preliminaries

In this section we introduce some key concepts like *affine curvature* and *affine arc length parametrization* from affine differential geometry briefly. These are required in understanding the asymptotic error bound expansion for the symmetric difference distance , which is given for curves parametrized with affine arc length. Straight lines and circular arcs are the only convex curves in the plane with constant Euclidean curvature, whereas conics are the only curves in the plane with constant affine curvature. The latter property is crucial for our approach, so we briefly review some concepts and properties from affine differential geometry of planar curves. See also Blaschke [1].

2.1. Affine Curvature

Recall that a regular curve $\alpha : J \to \mathbb{R}^2$ defined on a closed real interval J, i.e., a curve with non-vanishing tangent vector $T(s) := \alpha'(s)$, is parametrized according to Euclidean arc length if its tangent vector T has unit length. In this case, the derivative of the tangent vector is in the direction of the unit normal vector $N(s)$, and the Euclidean curvature $\kappa(s)$ measures the rate of change of T, i.e., $T'(s) = \kappa(s)\, N(s)$. Euclidean curvature is a differential invariant of regular curves under the group of rigid motions of the plane, i.e., a regular curve is uniquely determined by its Euclidean curvature, upto a rigid motion. The larger group of *equi-affine transformations* of the plane, i.e., linear transformations with determinant one(in other words, area preserving linear transformations), also gives rise to a differential invariant, called the *affine curvature* of the curve. To introduce this invariant, let $I \subset \mathbb{R}$ be an interval, and let $\gamma : I \to \mathbb{R}^2$ be a smooth, regular plane curve. The curve γ is parametrized according to *affine arc length* if

$$[\gamma'(r), \gamma''(r)] = 1. \tag{2.1}$$

Here $[v, w]$ denotes the determinant of the pair of vectors $\{v, w\}$. It follows from (2.1) that γ has a non-zero Euclidean curvature. Conversely, every curve $\alpha : J \subset \mathbb{R} \to \mathbb{R}^2$ with non-zero Euclidean curvature satisfies $[\alpha'(s), \alpha''(s)] \neq 0$, for $s \in J$, so it can be reparametrized according to affine arc length. Note that the property of being parametrized of being parametrized according to affine arc length is an invariant of the curve under equi-affine transformations. If γ is parametrized according to affine arc length, then differentiation of (2.1) yields $[\gamma'(r), \gamma'''(r)] = 0$, so there is a scalar function k such that

$$\gamma'''(r) + k(r)\,\gamma'(r) = 0. \tag{2.2}$$

The quantity $k(r)$ is called the *affine curvature* of the curve γ at $\gamma(r)$. A regular curve is uniquely determined by its affine curvature, up to an equi-affine transformation of the plane. At a point of non-vanishing Euclidean curvature there is a unique conic, called the *osculating conic*, having fourth order contact with the curve at that point (or, in other words, having five coinciding points of intersection with the curve). The affine curvature of this conic is equal to the affine curvature of the curve at the point of contact. Moreover, the contact is of order five if the affine curvature has vanishing derivative at the point of contact. (The curve has to be C^5.) In that case the point of contact is a *sextactic point*. See [1] for further details.

Conics have constant affine curvature. Solving the differential equation (2.2) shows that a curve of constant affine curvature is a conic arc. More precisely, a curve with constant affine curvature is a hyperbolic, parabolic or elliptic arc iff its affine curvature is negative, zero or positive, respectively.

2.2. Affine Frenet-Serret frame

The well known Frenet-Serret identity for the Euclidean frame, has a counterpart in the affine context. More precisely, let γ be a strictly convex curve parametrized by affine arc length. The *affine Frenet-Serret frame* $\{t(r), n(r)\}$ of γ is a moving frame at $\gamma(r)$, defined by $t(r) = \gamma'(r)$ and $n(r) = t'(r)$, respectively. Here the dash indicates differentiation with respect to affine arc length. The vector t is called the affine tangent, and the vector n is called the affine normal of the curve. The affine frame satisfies

$$\alpha' = t, \qquad t' = n, \qquad n' = -k\,t. \tag{2.3}$$

The affine Frenet-Serret identitites as given in equation (2.3) yield the following values for the derivatives with respect to the affine arc length parametrization, of the curve γ, upto order five

$$\gamma' = t, \quad \gamma'' = n, \quad \gamma''' = -k\,t,$$

$$\gamma^{(4)} = -k'\,t - k\,n, \quad \gamma^{(5)} = (k^2 - k'')\,t - 2k'\,n. \tag{2.4}$$

Combining identities given in (2.4) with Taylor series expansion of γ at a given point yields the following affine local canonical form of the curve.

Lemma 2.1. *Let $\gamma : I \to \mathbb{R}^2$ be a regular curve with non-vanishing curvature, and with affine Frenet-Serret frame $\{t, n\}$. Then*

$$\gamma(r_0 + r) = \gamma(r_0) + \left(r - \tfrac{1}{3!}k_0 r^3 - \tfrac{1}{4!}k_0' r^4 + O(r^6)\right)t_0$$

$$+\left(\tfrac{1}{2}r^2 - \tfrac{1}{4!}k_0 r^4 - \tfrac{2}{5!}k_0' r^5 + O(r^6)\right)n_0 \qquad (2.5)$$

where t_0, n_0, k_0 and k_0' are the values of t, n, k and k' at $\alpha(0)$.

3. Near optimal conic approximation of affine spiral arcs

The main result of this section concerns the equisymmetry property and the monotonicity property of the symmetric difference distance. Both properties are *global*, since the *affine spiral* is not necessarily short.

3.1. Intersections of conics and affine spirals

At this point we state some global properties of affine spirals.

Proposition 3.1.   1. *A conic intersects an affine spiral in at most five points, counted with multiplicity.*
  2. *The osculating conics of an affine spiral are disjoint, and do not intersect the spiral arc except at their point of contact.*

A proof of this theorem is given in [8, Chapter 4]. The second part is an exercise in [1, Chapter 1]. A modern proof is given in [11].

Now consider an affine spiral arc $\gamma : [u_0, u_1] \to \mathbb{R}^2$. Let C_u, $u_0 \leqslant u \leqslant u_1$, be the unique conic that is tangent to γ at its endpoints, and intersects it at the point $\gamma(u)$. For $u = u_0$ and $u = u_1$ the conic has a triple intersection with the curve, or, in other words, it has a contact of second order with γ there.

Proposition 3.2.   1. *Two conics C_u and $C_{u'}$, $u \neq u'$, are tangent at $\gamma(u_0)$ and $\gamma(u_1)$, and have no other intersections.*
  2. *Conic C_u intersects arc γ at $\gamma(u_0)$, $\gamma(u)$, and $\gamma(u_1)$, but at no other point.*

For the proof of Proposition 3.2, we refer to [3, Section 3.1].

3.2. Uniqueness of equisymmetric conic

In this section we will concern ourselves with the global result, that given an affine spiral arc $\gamma : [u_0, u_1] \to \mathbb{R}^2$, there is a unique bitangent conic C_σ, in the one parameter family of bitangent conics, such that the areas of the two moons formed by γ and C_σ are equal. Here C_σ is conic arc tangent to γ at $\gamma(u_0)$ and $\gamma(u_1)$, and intersecting it at an interior point $\gamma(\sigma)$. Moreover we show that with respect to the equisymmetry property, the symmetric difference distance is an increasing function of the arc length of the given affine spiral curve γ. Even though we do not show the existence of a unique conic, which minimizes the symmetric difference distance between the curve γ and itself, in the next section we prove that asymptotic error expressions for the symmetric difference distance of a conic minimimizing area
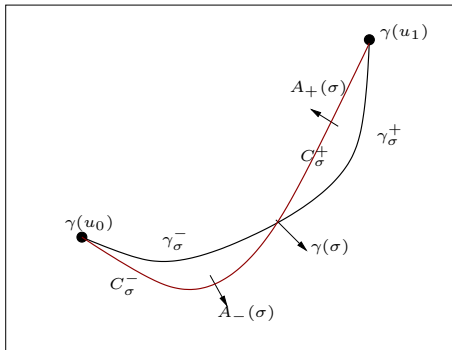
FIGURE 2. Notations for symmetric difference distance for bitangent conics

and an equisymmetric conic are the same upto terms of order 6, in the length of a very short arc. Thus we that the approximation with respect to the equisymmetric conic is very close to the optimal conic approximation. Before we state the main result for this section, let us make some notations clear. Let C_σ be the bitangent conic to γ, intersecting it at an interior point $\gamma(\sigma)$. Let γ_σ^- be the arc of γ, defined over $[u_0, \sigma]$ and γ_σ^+ be the arc defined over $[\sigma, u_1]$. Similarly C_σ^- is part of the conic arc in the interval $[u_0, \sigma]$ and C_σ^+ is part of the conic arc in the interval $[\sigma, u_1]$. Let $A_-(\sigma)$ be the area between γ_σ^- and C_σ^- and let $A_+(\sigma)$ be the area between γ_σ^+ and C_σ^+. Therefore the symmetric difference between γ and C_σ is given by $\delta_S(\gamma, C_\sigma) = A_-(\sigma) + A_+(\sigma)$. Figure 2, makes these notations clear.

Lemma 3.3 (Unicity of the equisymmetric conic). *Given an affine spiral arc $\gamma : [u_0, u_1] \to \mathbb{R}^2$ there is a one parameter family of bitangent conics which intersects γ at $\gamma(\sigma)$ as σ varies in the interval $[u_0, u_1]$. In this family of bitangent conics there is a unique equisymmetric conic C_{σ^*} (i.e., $A_(\sigma^*) = A_+(\sigma^*)$).*

Proof. We are given that $\gamma : [u_0, u_1] \to \mathbb{R}^2$ is an affine spiral curve, thus from Proposition 3.2, we have that the family of bitangent conics are lying side by side, as shown in the figure 3. Therefore it follows that the function A_- is strictly increasing as σ varies in the interval $[u_0, u_1]$, moreover $A_-(u_0) = 0$. A_+ is strictly decreasing as σ varies in the interval $[u_0, u_1]$, moreover $A_+(u_1) = 0$. Therefore, we conclude that there exists a unique σ^* such that $A_-(\sigma^*) = A_+(\sigma^*)$.   □

3.3. Monotonicity of the equisymmetric distance

If one endpoint of the affine spiral moves along the curve γ, the symmetric difference between the affine spiral and its equisymmetric conic arc is *monotone* in the affine arc length of the affine spiral. This result shows that an adaptive method can be used for the computation of a near optimal approximating conic arc. We use this property for the implementation of the algorithm presented in Section 5.
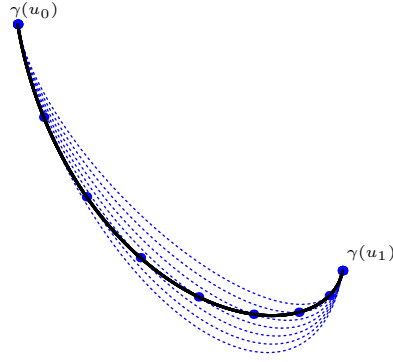
FIGURE 3. Family of bitangent conics, lying side by side, on an affine spiral section

Proposition 3.4 (Monotonicity of symmetric difference along affine spiral arcs). *Let $\gamma : I \to \mathbb{R}^2$ be an affine spiral arc, where I is an open interval containing 0. For $\varrho > 0$ let γ_ϱ be the sub-arc between $\gamma(0)$ and $\gamma(\varrho)$, and let β_ϱ be the (unique) equisymmetric conic arc tangent to γ_ϱ at its endpoints. Then the symmetric difference between γ_ϱ and β_ϱ is a monotonically increasing function of ϱ, for $\varrho \geqslant 0$*

The proof of monotonicity of equisymmetric distance proceeds similar to the monotonicity proof of the Hausdorff distance as given in [3]. Therefore, we omit the proof and refer the reader to [3, Section 3.4], for the details of the proof.

4. Optimal approximation with conics

In this section our goal is to determine the symmetric difference distance of an optimally approximating conic arc of an arc of γ, with affine arc length ϱ. This optimally approximating conic arc is tangent to γ at its endpoints. If the conic is a parabola, these conditions uniquely determine the parabolic arc. If we approximate with a general conic, there is one degree of freedom left, which we use to minimize the symmetric difference distance between the arc of γ and the approximating conic arc β. Moreover, we also give an asymptotic expansion of the symmetric difference distance between the arc of γ and its unique equisymmetric conic arc. We also show that the asymptotic expansion of the symmetric difference distance for an optimal conic spline and an equisymmetric conic spline are equal upto terms of order six in the arc length of the curve.

4.1. Complexity of conic splines

A bitangent parabolic arc, of a regular curve $\gamma : I \to \mathbb{R}^2$, is given by $\beta : I \times I \to \mathbb{R}^2$, and β has the following parametrization

$$\beta(r, \varrho) = \gamma(r) + r^2 (r - \varrho)^2 (P(r, \varrho)t(r) + Q(r, \varrho)\, n(r))$$

A bitangent conic arc, of γ, intersecting γ at an interior point $\gamma(\sigma)$, is given by $\beta : I \times I \times I \to \mathbb{R}^2$, and the bitangent conic arc has the following parametrization

$$\beta(r, \sigma, \varrho) = \gamma(r) + r^2 (r - \varrho)^2 (r - \sigma) (P(r, \varrho)t(r) + Q(r, \varrho) n(r)).$$

Before we give the proof of the Theorem on asymptotic expansion of symmetric difference distance , we state the Lemma 4.1 from [3], relating the affine curvatures of offset curves with the affine curvature of the curve γ itself. This result is useful since, parabolic and conic arcs are offset curves to γ, with constant affine curvature. Throughout in this paper, we consider the curve γ to be smooth, but the theorems can also be proved, where γ is a C^m-curve, for some finite value of m.

Lemma 4.1 (Affine curvature of offset curves). *Let γ be a $C^\infty-$regular curve.*

1. *Let $\beta : I \times I \to \mathbb{R}^2$ be a smooth function, such that, $\beta(., \varrho)$ is a curve tangent to γ at $\gamma(0)$ and $\gamma(\varrho)$, for $\varrho \in I$. There are smooth functions $P, Q : I \times I \to \mathbb{R}$ such that*

$$\beta(r, \varrho) = \gamma(r) + d(r, \varrho) (P(r, \varrho) t(r) + Q(r, \varrho) n(r)),$$

   where $d(r, \varrho) = r^2 (r - \varrho)^2$. Here $t(r)$ and $n(r)$ are the affine tangent and the affine normal of γ, respectively. Furthermore, the affine curvature $k_\beta(r, \varrho)$ of $\beta(\cdot, \varrho)$ at $0 \leqslant r \leqslant \varrho$ is given by

$$k_\beta(r, \varrho) = k(0) + 8 Q(0, 0) + O(\varrho).$$

2. *Let $\beta : I \times I \times I \to \mathbb{R}^2$ be a smooth function, such that, $\beta(\cdot, \sigma, \varrho)$ is a curve tangent to γ at $\gamma(0)$ and $\gamma(\varrho)$ and intersecting γ at $\gamma(\sigma)$, for $\sigma, \varrho \in I$ and $0 \leqslant \sigma \leqslant \varrho$. If β also intersects γ at $\gamma(\sigma)$, with $0 \leqslant \sigma \leqslant \varrho$, then there are smooth functions $P, Q : I \times I \times I \to \mathbb{R}$ such that*

$$\beta(r, \sigma, \varrho) = \gamma(r) + d(r, \sigma, \varrho) (P(r, \sigma, \varrho) t(r) + Q(r, \sigma, \varrho) n(r)).$$

   Furthermore, the affine curvature $k_\beta(r, \sigma, \varrho)$ of $\beta(\cdot, \sigma, \varrho)$ at $0 \leqslant u \leqslant \varrho$ is given by

$$k_\beta(r, \sigma, \varrho) = k(0) + k'(0) r + 8 (5 r - \sigma - 2 \varrho) Q(0, 0, 0) + O(\varrho^2).$$

The asymptotic error bound for the parabolic case has already been computed by Ludwig in [6]. We on the other hand use the general formula of symmetric difference distance given by (4.3) and the property that the affine curvature of the parabolic arc is zero. In fact our method allows us to generalize the result for any general conic by using the fact that conics are the only curves in the plane, with constant affine curvature.

Theorem 4.2 (Error in symmetric difference distance approximation). *Let $\gamma : [0, \varrho] \to \mathbb{R}^2$ be a sufficiently smooth, regular curve with non-vanishing Euclidean curvature.*

1. *Let β be the parabolic arc tangent to γ at the endpoints, the symmetric difference between the two arcs has the following asymptotic expansion*

$$\delta_S(\gamma, \beta) = \frac{1}{240}|k_0|\varrho^5 + O(\varrho^6),$$

where k_0 is the affine curvature of γ at $\gamma(0)$.

2. Let β be a bitangent conic arc, minimizing the symmetric difference, then the symmetric difference between the two arcs has the following asymptotic expansion

$$\delta_S(\gamma, \beta) = \tfrac{1}{7680}|k_0'|\varrho^6 + O(\varrho^7), \tag{4.1}$$

where k_0' is the derivative of the affine curvature of γ at $\gamma(0)$.

3. Let β be the equisymmetric bitangent conic arc of γ, then the asymptotic expansion of the symmetric difference between the two curves is given by (4.1).
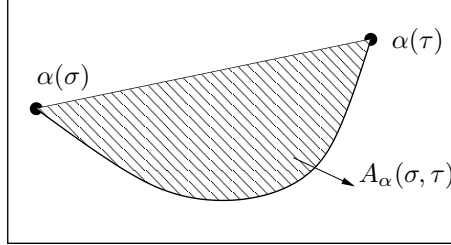

FIGURE 4. The area of the shaded region is the symmetric difference distance between α and chord $\alpha(\sigma)$ and $\alpha(\tau)$.

Proof. First we introduce some notation. The symmetric difference distance between a convex curve α and a chord $\alpha(\sigma)\alpha(\tau)$ is equal to the area of the shaded region in Figure 4, and will be denoted by $A_\alpha(\sigma, \tau)$. Then

$$A_\alpha(\sigma, \tau) = \tfrac{1}{2}\int_\sigma^\tau [\alpha(u) - \alpha(\sigma), \alpha'(u)]\, du, \tag{4.2}$$

and $[v, w]$ denotes the determinant of two vectors v and w in $\mathbb{R}^2$.

1. Consider the case when the approximating curve β is a parabolic arc. The symmetric difference distance between γ and β in the interval $[0, \varrho]$ is given by

$$\delta_S(\beta, \gamma) = |A_\beta(0, \varrho) - A_\gamma(0, \varrho)|. \tag{4.3}$$

Also see Figure 5 (left). Inserting the Taylor series expansion (2.5) of β, into (4.2), we obtain

$$A_\beta(0, \varrho) = \tfrac{1}{12}\varrho^3 + \tfrac{1}{240}\left(-k_0 - 8\,Q(0,0)\right)\varrho^5 + O(\varrho^6),$$

and

$$A_\gamma(0, \varrho) = \tfrac{1}{12}\varrho^3 - \tfrac{1}{240}k_0\,\varrho^5 + O(\varrho^6).$$

Therefore, in view of (??)

$$\delta_S(\gamma, \beta) = |A_\beta(0, \varrho) - A_\gamma(0, \varrho)| = \tfrac{1}{30}|Q(0,0)|\,\varrho^5 + O(\varrho^6). \tag{4.4}$$

Using the relation between affine curvatures of a curve γ and its offset β, given in Lemma 4.1, and the fact that the affine curvature of a parabolic arc is zero everywhere, we obtain $Q(0,0) = -\tfrac{1}{8}k_0 + O(\varrho)$. Substituting this expression into (4.4),
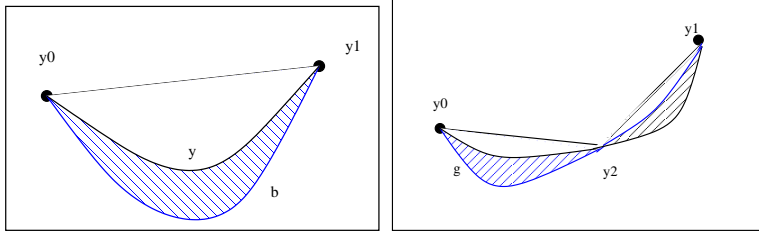
FIGURE 5. Shaded region in the first figure shows symmetric difference distance given by (4.3) and shaded region in the second figure shows symmetric difference distance given by (4.5).

we obtain

$$\delta_S(\gamma, \beta) = \tfrac{1}{240}\, |k_0|\, \varrho^5 + O(\varrho^6).$$

2. The curve γ has a one parameter family of bitangent conic arcs. Our aim is to give an asymptotic expression for the minimal symmetric difference distance . In our case, the symmetric difference distance between γ and any bitangent conic β is given by the equation (4.5), where $\sigma = c\,\varrho + O(\varrho^2)$, and $c \in [0, 1]$, and the bitangent conic β, intersects γ at $\gamma(\sigma)$.

The symmetric difference distance between a given smooth convex curve γ and a bitangent conic arc β, intersecting γ at $\gamma(\sigma)$ is given by

$$\delta_S(\beta, \gamma) = |A_\beta(0, \sigma) - A_\gamma(0, \sigma)| + |A_\gamma(\sigma, \varrho) - A_\beta(\sigma, \varrho)|. \qquad (4.5)$$

Also see Figure 5 (right). Using the Taylor series expansion (2.5) of γ and (4.2) we derive

$$|A_\beta(0, \sigma) - A_\gamma(0, \sigma)| = |\tfrac{1}{60}\,(5\,c^4 - 6\,c^5 + 2\,c^6)||Q(0,0,0)|\,\varrho^6 + O(\varrho^7),$$

and

$$|A_\beta(\sigma, \varrho) - A_\gamma(\sigma, \varrho)| = |\tfrac{1}{60}\,(1 - 2\,c + 5\,c^4 - 6\,c^5 + 2\,c^6)|\,|Q(\sigma, \sigma, \varrho)|\,\varrho^6 + O(\varrho^7).$$

Furthermore, $Q(\sigma, \sigma, \varrho)$ can be written as

$$Q(\sigma, \sigma, \varrho) = Q(0,0,0) + c\,\varrho\,Q_u(0,0,0) + O(\varrho^2),$$

plugging this expression into the expression for $A_\beta(\sigma, \varrho) - A_\gamma(\sigma, \varrho)$, we have

$$|A_\beta(\sigma, \varrho) - A_\gamma(\sigma, \varrho)| = \tfrac{1}{60}|1 - 2\,c + 5\,c^4 - 6\,c^5 + 2\,c^6|\,|Q(0,0,0)|\,\varrho^6 + O(\varrho^7).$$

Using (4.5) we obtain

$$\delta_S(\gamma, \beta) = \tfrac{1}{60}(|1 - 2\,c + 5\,c^4 - 6\,c^5 + 2\,c^6| + |5\,c^4 - 6\,c^5 + 2\,c^6|)|Q(0,0,0)|\,\varrho^6 + O(\varrho^7). \qquad (4.6)$$

Since we want to find the asymptotic error bound for the conic minimizing symmetric difference distance , we minimize (4.6) with respect to c. We conclude that $\delta_S(\gamma, \beta)$ is minimal for $c = \tfrac{1}{2}$. Therefore, equation (4.6) reduces to

$$\delta_S(\gamma, \beta) = \tfrac{1}{192}\, |Q(0,0,0)|\, \varrho^6 + O(\varrho^7).$$

Referring to the Lemma 4.1, relating the affine curvature of offset curves, and the fact that the affine curvatures of conics are constant we have that, $Q(0,0,0) = -\frac{1}{40} k'_0 + O(\varrho)$. Plugging in the expression for $Q(0,0,0)$ into the last expression for $\delta_S(\gamma, \beta)$, we have

$$\delta_S(\gamma, \beta) = \tfrac{1}{7680} |k'_0| \, \varrho^6 + O(\varrho^7).$$

3. The asymptotic expansion of the symmetric difference distance between the given arc of γ and its unique equisymmetric conic arc β, is found by equating $|A_\beta(0,\sigma) - A_\gamma(0,\sigma)|$ to $|A_\beta(\sigma, \varrho) - A_\gamma(\sigma, \varrho)|$, yielding $c = \frac{1}{2}$. Further simplifying we see that $\delta_S(\gamma, \beta)$ is of the same form as given by (4.1).

We therefore conclude that the asymptotic error bounds for a conic minimizing symmetric difference and an equisymmetric conic are the same upto terms of order 6 in ϱ, and therefore we say that the approximation with an equisymmetric conic is near optimal. $\qquad\square$

In the following corollary we give expressions for the symmetric difference between a given convex curve γ and its best approximating parabolic and conic spline. The corollary can be proven using the same techniques as used by McClure and Vitale in [7] and Ludwig in [6].

Corollary 4.3 (Symmetric difference distance for an optimal spline). *Let $\gamma : I \to \mathbb{R}^2$ be a sufficiently smooth convex curve, with strictly increasing or decreasing affine curvature.*

1. *The symmetric difference between γ and a best approximating parabolic spline P_n with n knots is given by*

$$\delta_S(\gamma, P_n) = \tfrac{1}{240} \Big(\int_0^\varrho |k(r)|^{1/5} \, dr \Big)^5 \frac{1}{n^4} + O(\frac{1}{n^5}).$$

2. *The symmetric difference between γ and a best approximating conic spline C_n with n knots is given by*

$$\delta_S(\gamma, C_n) = \tfrac{1}{7680} \Big(\int_0^\varrho |k'(r)|^{1/6} \, dr \Big)^6 \frac{1}{n^5} + O(\frac{1}{n^6}).$$

Here γ is parametrized by the affine arc length parameter r and the affine curvature of the curve γ is denoted by k.

In the following corollary, we give the asymptotic expression for the symmetric difference distance between a given convex curve γ, and its equisymmetric conic spline, with n knots and denoted by ES_n. As the name suggests, an equisymmetric conic spline, is a spline such that every element in it is an equisymmetric conic.

Corollary 4.4 (Symmetric difference distance for an equisymmetric conic spline). *The symmetric difference distance between γ and an equisymmetric conic spline with n knots is given by*

$$\delta_S(\gamma, ES_n) = \tfrac{1}{7680} \Big(\int_0^\varrho |k'(r)|^{1/6} \, dr \Big)^6 \frac{1}{n^5} + O(\frac{1}{n^6}).$$

Remark. The basic idea behind proving Corollary 4.3 or Corollary 4.4 is to define functions called *parabolic content* and *conic content*. Given a sufficiently smooth strictly convex curve $\gamma : [\sigma, \tau] \to \mathbb{R}^2$, its parabolic content is defined by $\lambda_p = \int_\sigma^\tau |k(r)|^{1/5} \, dr$. Similarly the conic content of γ is given by $\lambda_c = \int_\sigma^\tau |k'(r)|^{1/6} \, dr$. These functions are useful in distributing the knots over the curve γ, in such a way, that the symmetric difference distance of all the segments are equal. Here each segment consists of a region bounded by the arc of γ lying between two knots and the bitangent parabolic or the equisymmetric conic arc approximating it. The aim for this kind of approximation is to distribute the knots uniformly over the curve with respect to the parabolic or the conic content. In fact the methods used by McClure and Vitale in [7] and Ludwig in [6] use this notion of content to show that there exists an optimal spline minimizing the symmetric difference distance for a curve with a given number of knots.

The preceding result represents an asymptotic expression for the number of elements of an optimal parabolic or conic spline and also an asymptotic expression for an equisymmetric conic spline in terms of the symmetric difference distance.

Corollary 4.5 (Complexity of parabolic and conic splines). *Let $\gamma : [0, \varrho] \to \mathbb{R}^2$ be a sufficiently smooth regular curve with non-vanishing Euclidean curvature of length ϱ, parametrized by affine arc length, and let $k(r)$ be its affine curvature at $\gamma(r)$.*

1. *The minimial number of arcs in a tangent conitnuous parabolic spline approximating γ to within symmetric difference distance ε is*

$$N(\varepsilon) = (240)^{-1/4} \Big(\int_0^\varrho |k(r)|^{1/5} \, dr \Big)^{5/4} \varepsilon^{-1/4} \, (1 + O(\varepsilon^{1/4})).$$

2. *The minimal number of arcs in a tangent continuous conic spline approximating γ, to within symmetric difference distance ε is*

$$N(\varepsilon) = (7680)^{-1/5} \Big(\int_0^\varrho |k'(r)|^{1/6} \, dr \Big)^{6/5} \varepsilon^{-1/5} \, (1 + O(\varepsilon^{1/5})).$$

The expression for complexity of an equisymmetric conic spline is of the same form as the expression for complexity of an optimal conic spline. The expressions match in the most significant terms, implying that the minimal number of elements in either case differ by a constant for a given value of ε. For all practical cases this difference turned out to be small.

5. Implementation

We implemented an algorithm in `C++` using the symbolic computing library `GiNaC`[1], for the computation of an optimal parabolic or an equisymmetric conic spline, based on the monotonicity property. For computing the optimal parabolic spline, the curve is subdivided into affine spirals. Then for a local (symmetric difference) stopping condition ε_l, the algorithm iteratively computes the optimal parabolic

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[1] http://www.ginac.de

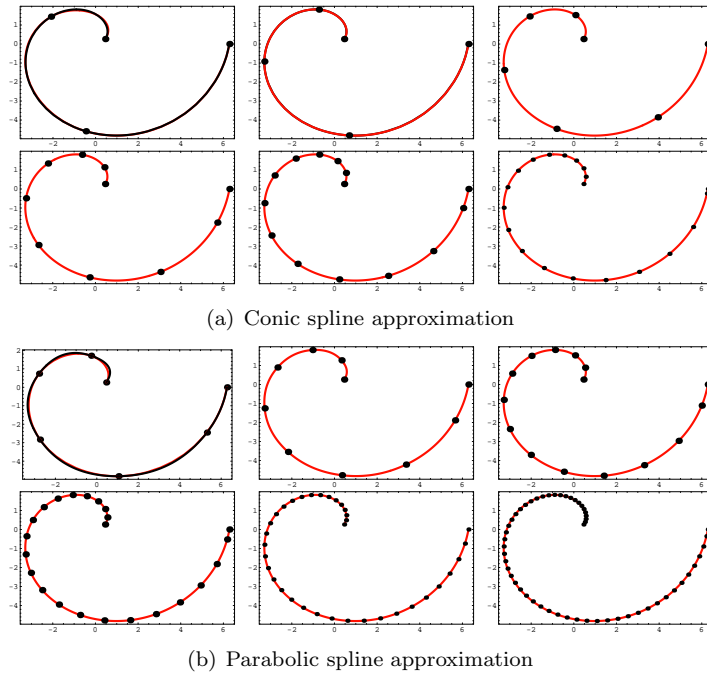(a) Conic spline approximation

(b) Parabolic spline approximation

FIGURE 6. Approximation of the spiral for ε ranging between 10^{-1} to 10^{-6}.

arcs starting at one endpoint. We give a local, stopping condition, since from the theory we have that, for a parabolic spline with symmetric difference distance ε and n-knots, the local ε_l is given by $\varepsilon_l = \frac{\varepsilon}{n}$. Infact our algorithm gives an exact match between the theoretical complexity and the experimental complexity, for sufficiently small values of ε.

Below we present two examples of computations of optimal parabolic and near optimal conic splines.

5.1. A spiral curve

We present the results of our algorithm applied to the *spiral curve*, parameterized by $\alpha(t) = (t\cos(t), t\sin(t))$, with $\frac{1}{6}\pi \leqslant t \leqslant 2\pi$.

Figures 6(a) and 6(b) depict the result of the algorithm applied to the spiral for different values of the local error bound ε, for the approximation by conic arcs and parabolic arcs respectively. There is no visual difference between the curve and its approximating conic, for the values of ε under consideration.

Table 1 gives the number of arcs computed by the algorithm, and the theoretical bounds on the number of arcs for varying values of ε, both for the parabolic and for the conic spline.

ε	Parabolic Exp./ Th.	Conic Exp./ Th.
10^{-1}	9	4
10^{-2}	15	6
10^{-3}	26	9
10^{-4}	46	13
10^{-5}	82	21
10^{-6}	146	33

TABLE 1. The complexity (number of arcs) of the parabolic spline and the conic spline approximating the Spiral Curve. The theoretical complexity matches exactly with the experimental complexity, for various values of the symmetric difference distance ε.
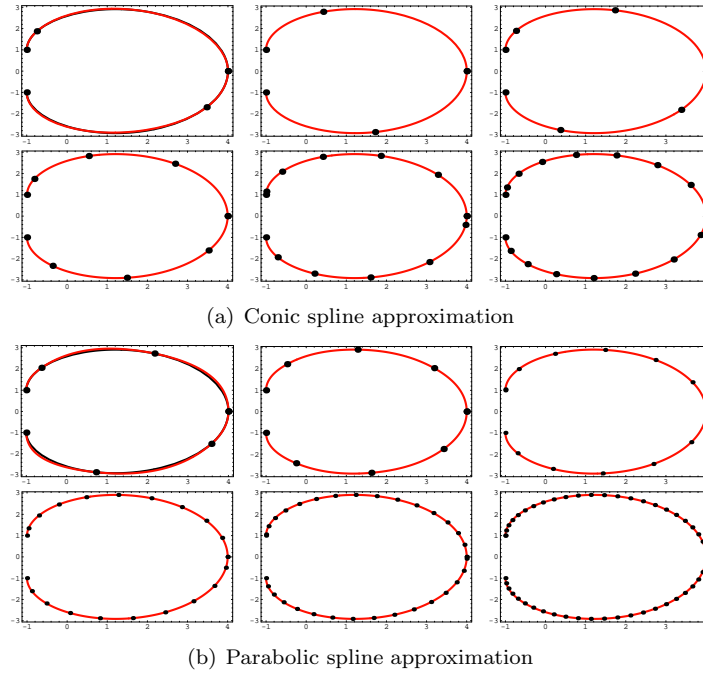

(a) Conic spline approximation

(b) Parabolic spline approximation

FIGURE 7. Plot of the approximations of a part of Cayley's sextic for ε ranging from 10^{-1} to 10^{-6}.

5.2. Cayley's sextic

We present the results of our algorithm applied to the *Cayley's sextic*, the curve parameterized by $\alpha(t) = (4\cos(\frac{t}{3})^3 \cos(t), 4\cos(\frac{t}{3})^3 \sin(t))$, with $-\frac{3}{4}\pi \leqslant t \leqslant \frac{3}{4}\pi$.

This curve has a sextactic point at $t = 0$. For all values of ε we divide the parameter interval into two parts $[-\frac{3}{4}\pi, 0]$ and $[0, \frac{3}{4}\pi]$ each containing the sextactic point as an endpoint, and then approximate with conic arcs using the Incremental Algorithm.

The pictures in Figure 7(a) give the *conic spline* approximation images for Cayley's sextic for different values of ε. The pictures in Figure 7(b) gives only the parabolic spline approximation for Cayley's sextic for different errors, since the original curve and the approximating parabolic spline are not visually distinguishable.

Table 2 gives the number of arcs computed by the algorithm, and the theoretical bounds on the number of arcs for varying values of ε_l, both for the parabolic and for the conic spline.

ε	Parabolic Exp./ Th.	Conic Exp./ Th.
10^{-1}	6	4
10^{-2}	12	4
10^{-3}	20	6
10^{-4}	34	10
10^{-5}	60	16
10^{-6}	108	24

TABLE 2. The complexity of the parabolic spline and the conic spline approximating Cayley's sextic. The theoretical complexity matches exactly with the complexity measured in experiments, for various values of the local symmetric difference distance ε.

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