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Scherpen, J.M.A.; Schaft, A.J. van der

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A structure preserving minimal representation of a nonlinear port-Hamiltonian system

Jacquelien M.A. Scherpen
Industrial Technology and Management
Fac. Math. & Natural Sciences
University of Groningen
Nijenborgh 4, 9747 AG Groningen
j.m.a.scherpen@rug.nl

Arjan J. van der Schaft
Institute of Mathematics and Computing Science
Fac. Math. and Natural Sciences
University of Groningen
Nijenborgh 9, 9747 AG Groningen
a.j.van.der.schaft@math.rug.nl

Abstract— In this paper an approach to reduce nonlinear non-observable and non-strongly accessible port-Hamiltonian systems to an observable and strongly accessible port-Hamiltonian system, respectively, is treated. A local state decomposition (the nonlinear version of the Kalman decomposition) is instrumental for the approach that preserves the port-Hamiltonian structure. The strongly accessible reduction scheme goes along similar lines as the linear scheme. However, the observable reduction scheme is somewhat more involved. Under some additional assumptions, the reduction can be performed along the lines of the linear scheme. If these assumptions are not fulfilled, a reduction scheme for a zero-observable representation using duality in the co-energy coordinates is developed. Finally, the possibilities to apply the approaches of this paper to approximate order reduction by e.g., use of balancing procedures, is discussed.

I. INTRODUCTION

The problem of determining the minimal state-space representation is a fundamental problem for control systems. It connects to many other topics in realization theory, like controllability and observability properties, similarity invariants, balanced realizations and model reduction. Recently, additional properties, like structure preservation, for obtaining a minimal realization or an approximate reduced order model have received interest, e.g., [1], [5], [10]. Here, we take an analysis and control perspective, which motivates our interest in obtaining a minimal representation of a non-minimal port-Hamiltonian (PH) system that preserves the PH structure.

For linear systems the latter structure preserving reduction problem is treated in [7], where it is also used for new structure preserving approximate model reduction schemes. In the linear case, minimality is equivalent with observability and controllability, and the corresponding Kalman decomposition of the linear system becomes a very useful tool for structure preserving order reduction.

For nonlinear PH systems we use the insights obtained from the linear case, and extend this to the reduction of a non-strongly accessible PH system to a strongly accessible PH system. In this, we use the nonlinear extension of the Kalman decomposition, e.g., [4], [6]. For non-observable PH systems we are able to use the non-linear Kalman decomposition, but we have to impose additional assumptions on the system in order to be able to preserve the structure in

obtaining an observable PH system. If these assumptions do not hold, we investigate the possibility to use a duality notion from [2] for obtaining a zero observable PH system. Finally, we present the use for approximate structure preserving model order reduction, where almost non-minimality based on balanced realizations is a tool for model order reduction.

II. LINEAR PORT-HAMILTONIAN SYSTEMS

In this section we summarize how a linear uncontrollable and/or unobservable port-Hamiltonian system is reduced to a controllable/observable system that is again a port-Hamiltonian system, [7].

A. Reduction to a controllable port-Hamiltonian system

In the linear case, and in the absence of algebraic constraints, linear port-Hamiltonian systems take the form

$$
\begin{aligned}
\dot{x} &= (J - R)Qx + Bu, & J = -J^T, \ R = R^T \geq 0 \\
y &= B^T Qx, & Q = Q^T \geq 0
\end{aligned}
\tag{1}
$$

with $x \in \mathcal{X} \subset \mathbb{R}^n$, $u, y \in \mathbb{R}^m$, $H(x) = \frac{1}{2}x^T Qx$ the total energy (Hamiltonian) and R the dissipation matrix. The matrices J and B specify the interconnection structure. Define $F := J - R$.

Consider a linear port-Hamiltonian system which is not controllable. Take linear coordinates $x = (x^1, x^2)$ such that the upper part of

$$
\begin{aligned}
\begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \end{pmatrix} &= \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \\
&\quad + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u
\end{aligned}
\tag{2}
$$

$$
y = \begin{pmatrix} B_1^T & B_2^T \end{pmatrix} \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}
$$

is the reachability subspace $\mathcal{R}$. By invariance of $\mathcal{R}$ this implies

$$
F_{21}Q_{11} + F_{22}Q_{21} = 0, \quad B_2 = 0
\tag{3}
$$

It follows that the dynamics restricted to $\mathcal{R}$ is given as

$$
\begin{aligned}
\dot{x}^1 &= (F_{11}Q_{11} + F_{12}Q_{21})x^1 + B_1 u \\
y &= B_1^T Q_{11} x^1
\end{aligned}
\tag{4}
$$

Now assume that F_{22} is invertible. Then we obtain from (3) that $Q_{21} = -F_{22}^{-1}F_{21}Q_{11}$. Substitution in (4) yields

$$
\begin{aligned}
\dot{x}^1 &= (F_{11} - F_{12}F_{22}^{-1}F_{21})Q_{11}x^1 + B_1 u \\
y &= B_1^T Q_{11} x^1
\end{aligned} \tag{5}
$$

which is again a port-Hamiltonian system. Indeed, $F + F^T \leq 0$ implies that the Schur complement $\bar{F} := F_{11} - F_{12}F_{22}^{-1}F_{21}$ also satisfies $\bar{F} + \bar{F}^T \leq 0$.

B. Reduction to an observable port-Hamiltonian system

Consider a linear port-Hamiltonian system (1) and suppose the system is not observable. Then there exist coordinates $x = (x^1, x^2)$ such that the *lower* part of (2) is the unobservability subspace $\mathcal{N}$. By invariance of $\mathcal{N}$ it follows that

$$
F_{11}Q_{12} + F_{12}Q_{22} = 0, \quad B_1^T Q_{12} + B_2^T Q_{22} = 0 \tag{6}
$$

Then the dynamics on the quotient space $\mathcal{X}/\mathcal{N}$ is

$$
\begin{aligned}
\dot{x}^1 &= (F_{11}Q_{11} + F_{12}Q_{21})x^1 + B_1 u \\
y &= B_1^T Q_{11} x^1 + B_2^T Q_{21} x^1
\end{aligned} \tag{7}
$$

Assuming invertibility of Q_{22} it follows from (6) that $F_{12} = -F_{11}Q_{12}Q_{22}^{-1}$ and $B_2^T = -B_1^T Q_{12}Q_{22}^{-1}$. Substitution in (7)

$$
\begin{aligned}
\dot{x}^1 &= F_{11}(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x^1 + B_1 u \\
y &= B_1^T (Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x^1
\end{aligned} \tag{8}
$$

which is again a port-Hamiltonian system with Hamiltonian $\bar{H} = \frac{1}{2}x^{1T}(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x^1$, since the Schur complement $(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})$ is again symmetric (and ≥ 0 if $Q \geq 0$).

III. NONLINEAR SYSTEMS AND MINIMALITY

Consider a smooth, i.e., C^∞, nonlinear system of the form

$$
\dot{x} = f(x) + g(x)u, \quad y = h(x) \tag{9}
$$

where $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$, $y = (y_1, \ldots, y_p) \in \mathbb{R}^p$ and $x = (x_1, \ldots, x_n)$ are local coordinates for a smooth state space manifold denoted by M. Throughout we assume that the system has an equilibrium. Without loss of generality we take this equilibrium to be at 0, i.e. $f(0) = 0$, and we also take $h(0) = 0$. For the analysis in this paper the definitions of local *reachability, (strong) accessibility*, and *observability* are needed. We refer to standard references like [3], [4], [6], [11]. For clarity we mention a special case of observability, though also well-known, it is less standard, namely, *zero observability*. The system (9) is *zero observable* if it is observable for $u \equiv 0$. We say that the system (9) is *locally zero observable*, if there exists a neighborhood W of 0 where the system is zero observable.

A. The nonlinear Kalman decomposition

It is well-known, e.g. [6], that for the accessibility distribution, C, the strong accessibility distribution, C_0, and the observation space, $\mathcal{O}$, with its corresponding co-distribution, $d\mathcal{O}$, there exist rank conditions implying local (strong) accessibility and local observability. The following result relates minimality of an analytic realization for a formal power series (Chen-Fliess functional expansion) with the observability and accessibility rank conditions.

Theorem 3.1: [4] An analytic realization (f, g, h) about x_0 of a formal power series is minimal if and only if $\dim C(x_0) = n$ and $\dim d\mathcal{O}(x_0) = n$. $\square$

If the system is not locally observable, and/or not locally strongly accessible, there exists a nonlinear version of the Kalman decomposition, e.g., [6]. Note that for the above characterization of minimality we consider accessibility, whereas for the Kalman decomposition we use the stronger notion of strong accessibility.

Theorem 3.2: Assume that the distributions C_0, $\ker d\mathcal{O}$ and $C_0 + \ker d\mathcal{O}$ all have constant dimension and that $C_0 + \ker d\mathcal{O}$ is involutive. Then one can find local coordinates $x = (x^1, x^2, x^3, x^4)$ such that $C_0 = \text{span}\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\}$ and $\ker d\mathcal{O} = \text{span}\{\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^4}\}$. The system (9) takes the form

$$
\dot{x}^1 = f^1(x^1, x^3) + \sum_{j=1}^{m} g_j^1(x^1, x^3)u_j \tag{10}
$$

$$
\dot{x}^2 = f^2(x^1, x^2, x^3, x^4) + \sum_{j=1}^{m} g_j^2(x^1, x^2, x^3, x^4)u_j \tag{11}
$$

$$
\dot{x}^3 = f^3(x^3) \tag{12}
$$

$$
\dot{x}^4 = f^4(x^3, x^4) \tag{13}
$$

$$
y = h(x^1, x^3). \tag{14}
$$

$\square$

For local zero observability a similar rank condition as for observability and (strong) accessibility exists with the zero observation space $\mathcal{O}_0$ defined by the linear space of functions on M containing $h_1, \ldots, h_p$ and all repeated Lie derivatives $L_f^k h_j$, $j \in 1, \ldots, p$, $k = 1, 2, \ldots$. As a consequence, local zero observability implies local observability at 0. In the case of zero observability, we can apply the Kalman decomposition as well, with the difference that the input vector field in equation (10) becomes $g_j^1(x^1, x^2, x^3, x^4)$, $j = 1, \ldots, m$, see [9].

B. Energy functions

We can relate the following energy functions with system (9), (e.g., [8]).

Definition 3.3: The *controllability* and *observability functions* of a nonlinear system (9) are given by

$$
L_c(x_0) = \min_{\substack{u \in L_2^- \\ x(-\infty) = 0, \\ x(0) = x_0}} \frac{1}{2}\int_{-\infty}^{0} \| u(t) \|^2 \, dt, \tag{15}
$$

and

$$
L_o(x_0) = \frac{1}{2}\int_{0}^{\infty} \| y(t) \|^2 \, dt, \tag{16}
$$

$$
x(0) = x_0, \quad u(t) \equiv 0, \quad 0 \leq t < \infty,
$$

respectively.   □

The above energy functions can be characterized by Hamilton-Jacobi-Bellman type of equations, stemming from Optimal Control theory, [8]. The following theorem is closely related to results that appear in [3], [11]. It reveals an important relationship between zero observability and positive definiteness of the observability function.

Theorem 3.4: [9] Assume $f(x)$ is asymptotically stable on a neighborhood W of 0. If the system (9) is zero observable on W, then $L_o(x) > 0$, $\forall x \in W$, $x \neq 0$.   □

To this end, we state a result of [9] that relates positivity of L_o with the zero-observability rank condition.

Theorem 3.5: ([9]) Assume that the zero-observability co-distribution $d\mathcal{O}_0$ has constant dimension about 0. If the observability function (16) is smooth, finite and satisfies $L_o(x) > 0$, $x \in W$, $x \neq 0$, then $\dim d\mathcal{O}_0(0) = n$.   □

In [2], a duality characterization with help of the above functions was given. We will use the following result in our observability study.

Proposition 3.6: ([2]) Consider the smooth state space system (9), with $f(0) = 0$ and $h(0) = 0$. Factorize $f(x) = A(x)x$, and $h(x) = C(x)x$ to obtain $A(x)$ and $C(x)$. Assume that 0 is an asymptotically stable equilibrium of $\dot{x} = f(x)$ and that $L_o(x)$ and $L_c(x)$ exist and are smooth. Consider the system

$$\begin{cases} \dot{p} &= A^T(\phi_i(p))p + C^T(\phi_i(p))u_a \\ y_a &= g^T(\phi_i(p))p \end{cases} \tag{17}$$

with the subscript $i \in \{c, o\}$. Let $x = \phi_c(p)$ denote the inverse mapping of $p = (\partial L_c(x)/\partial x)^T$. Suppose that (17) has observability function $\tilde{L}_o(p)$ and that $i = c$. Then $\tilde{L}_o(p)$ is given by the Legendre transformation

$$\tilde{L}_o(p) = -L_c(x) + p^T x. \tag{18}$$

Let $x = \phi_o(p)$ denote the inverse mapping of $p = (\partial L_o(x)/\partial x)^T$. Suppose that (17) has controllability function $\tilde{L}_c(p)$ and that $i = o$. Then $\tilde{L}_c(p)$ is given by the Legendre transformation

$$\tilde{L}_c(p) = -L_o(x) + p^T x. \tag{19}$$

   □

IV. REDUCTION TO A STRONGLY ACCESSIBLE PH SYSTEM

Consider a port-Hamiltonian (PH) system of the form

$$\dot{x} = (J(x) - R(x))\frac{\partial H}{\partial x}(x) + g(x)u$$
$$y = g(x)^T\frac{\partial H}{\partial x}(x) \tag{20}$$

where $u = (u_1, \ldots, u_m)^T \in \mathbb{R}^m$, $y = (y_1, \ldots, y_p)^T \in \mathbb{R}^p$, and $x = (x_1, \ldots, x_n)^T$ are local coordinates for a smooth state space manifold denoted by M. Furthermore, $J(x) = -J(x)^T$ and $R(x) = R(x)^T \geq 0$. We assume that there exists an equilibrium point in $x = 0$.

Define $F(x) := J(x) - R(x)$. Similar to the linear case, [7], we have that

$$F(x) + F^T(x) \leq 0, \tag{21}$$

while conversely any $F(x)$ satisfying (21) can be written as $J(x) - R(x)$ as above by decomposing $F(x)$ into its skew-symmetric and symmetric part, i.e.,

$$\begin{aligned} J(x) &= \tfrac{1}{2}(F(x) - F^T(x)) \\ R(x) &= \tfrac{1}{2}(F(x) + F^T(x)) \end{aligned} \tag{22}$$

Assume that the strong accessibility distribution C_0 has constant dimension. Then there exist local coordinates such that $C_0 = \text{span}\{\frac{\partial}{\partial x^1}\}$ (see Theorem 3.2). In these coordinates, we can write system (20) as

$$\begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \end{pmatrix} = \begin{pmatrix} F_{11}(x) & F_{12}(x) \\ F_{21}(x) & F_{22}(x) \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial x^1}(x) \\ \frac{\partial H}{\partial x^2}(x) \end{pmatrix} + \begin{pmatrix} g_1(x) \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} g_1(x)^T & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial x^1}(x) \\ \frac{\partial H}{\partial x^2}(x) \end{pmatrix} \tag{23}$$

Theorem 4.1: Assume that system (23) is in the local coordinates such that $C_0 = \text{span}\{\frac{\partial}{\partial x^1}\}$, and that $F_{22}(x^1, 0)$ is invertible for all x^1. Then the PH dynamics restricted to C_0 can be written as

$$\begin{aligned} \dot{x}^1 &= \left(\bar{F}_{11}(x^1) - \bar{F}_{12}(x^1)\bar{F}_{22}^{-1}(x^1)\bar{F}_{21}(x^1)\right)\frac{\partial H}{\partial x^1}(x^1, 0) \\ &\quad + \bar{g}_1(x^1)u \end{aligned} \tag{24}$$
$$y = \bar{g}_1(x^1)\frac{\partial H}{\partial x^1}(x^1, 0)$$

where $\bar{F}_{ij} = F_{ij}(x^1, 0)$ for $i, j = 1, 2$, $\bar{g}_1(x^1) = g_1(x^1, 0)$, which is again a PH system.

Proof

Since $C_0 = \text{span}\{\frac{\partial}{\partial x^1}\}$, we conclude from Theorem 3.2 that

$$F_{21}(x)\frac{\partial H}{\partial x^1}(x) + F_{22}(x)\frac{\partial H}{\partial x^2}(x) = f(x^2).$$

Since for any value of x^2 the x^1 sub-system is strongly accessible, we consider $x^2 = 0$. Since $f(0) = 0$, we obtain

$$F_{21}(x^1, 0)\frac{\partial H}{\partial x^1}(x^1, 0) + F_{22}(x^1, 0)\frac{\partial H}{\partial x^2}(x^1, 0) = 0$$
$$\Leftrightarrow$$
$$\frac{\partial H}{\partial x^2}(x^1, 0) = -F_{22}^{-1}(x^1, 0)F_{21}(x^1, 0)\frac{\partial H}{\partial x^1}(x^1, 0).$$

Substituting the latter into (23) we obtain (24). Given that $F(x^1, 0) + F^T(x^1, 0) \leq 0$ implies that the Schur complement $\tilde{F}(x^1) = \bar{F}_{11}(x^1) - \bar{F}_{12}(x^1)\bar{F}_{22}^{-1}(x^1)\bar{F}_{21}(x^1)$ satisfies $\tilde{F}(x^1) + \tilde{F}^T(x^1) \leq 0$. Thus, (24) is a PH system.   □

V. REDUCTION TO AN OBSERVABLE PH SYSTEM

The observability case is somewhat more complex. However, under the extra assumption that part of the matrices $F(x)$ and $g(x)$ do not depend on the observable coordinates the nonlinear Kalman decomposition is immediately seen to lead to an observable system that is again port-Hamiltonian. Furthermore, the form of this port-Hamiltonian system is somewhat *dual* to the reduced port-Hamiltonian system found in the previous section considering its strong accessibility properties. In Section V-B we will discuss an alternative route for reduction to an observable system that is again in port-Hamiltonian form, by transforming the system into co-energy variables.

A. Special F and g

If we assume that the observability co-distribution $d\mathcal{O}$ has constant dimension, then there exist local coordinates (x^1, x^2) such that ker $d\mathcal{O} =$ span$\{\frac{\partial}{\partial x^2}\}$ (see Theorem 3.2). Now assume throughout Section V-A that

- $d\mathcal{O}$ has constant dimension.
- F and g are in a form such that F_{11}, F_{12}, g_1, and g_2 only depend on x^1.

Then, the PH system (20) takes the form

$$\begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \end{pmatrix} = \begin{pmatrix} F_{11}(x^1) & F_{12}(x^1) \\ F_{21}(x) & F_{22}(x) \end{pmatrix} \begin{pmatrix} \dfrac{\partial H}{\partial x^1}(x) \\ \dfrac{\partial H}{\partial x^2}(x) \end{pmatrix}$$
$$+ \begin{pmatrix} g_1(x^1) \\ g_2(x^1) \end{pmatrix} u \qquad (25)$$
$$y = \begin{pmatrix} g_1^T(x^1) & g_2^T(x^1) \end{pmatrix} \begin{pmatrix} \dfrac{\partial H}{\partial x^1}(x) \\ \dfrac{\partial H}{\partial x^2}(x) \end{pmatrix}$$

where

$$F_{11}(x^1)\frac{\partial H}{\partial x^1}(x) + F_{12}(x^1)\frac{\partial H}{\partial x^2}(x) = f(x^1) \quad (26)$$
$$g_1^T(x^1)\frac{\partial H}{\partial x^1}(x) + g_2^T(x^1)\frac{\partial H}{\partial x^2}(x) = h(x^1) \quad (27)$$

Under the standing assumption differentiation of (26) and (27) with respect to x^2 yields

$$F_{11}(x^1)\frac{\partial^2 H}{\partial x^1 \partial x^2}(x) + F_{12}(x^1)\frac{\partial^2 H}{\partial x^{2^2}}(x) = 0, \quad (28)$$
$$g_1^T(x^1)\frac{\partial^2 H}{\partial x^1 \partial x^2}(x) + g_2^T(x^1)\frac{\partial^2 H}{\partial x^{2^2}}(x) = 0. \quad (29)$$

Theorem 5.1: Assume that system (23) is in the local coordinates such that ker $d\mathcal{O} =$ span$\{\frac{\partial}{\partial x^2}\}$, and that $\frac{\partial^2 H}{(\partial x^2)^2}(x)$ is invertible for all x. Then the equation

$$\frac{\partial H}{\partial x^2}(x^1, x^2) = 0 \qquad (30)$$

can be solved (at least locally) for x^2 as a function $x^2(x^1)$. Define the restricted Hamiltonian $\widetilde{H}(x^1) := H(x^1, x^2(x^1))$. Then the PH system restricted to its observable part can be written as the PH system

$$\dot{x}^1 = F_{11}(x^1)\frac{\partial \widetilde{H}}{\partial x^1}(x^1) + g_1(x^1)u \qquad (31)$$

$$y = g_1(x^1)^T \frac{\partial \widetilde{H}}{\partial x^1}(x^1) \qquad (32)$$

Proof

From (28) and (29) and the invertibility of $\frac{\partial^2 H}{\partial x^{2^2}}(x)$ we obtain

$$F_{12}(x^1) = -F_{11}(x^1)\frac{\partial^2 H}{\partial x^1 \partial x^2}(x)\left(\frac{\partial^2 H}{(\partial x^2)^2}(x)\right)^{-1},$$
$$g_2^T(x^1) = -g_1^T(x^1)\frac{\partial^2 H}{\partial x^1 \partial x^2}(x)\left(\frac{\partial^2 H}{(\partial x^2)^2}(x)\right)^{-1}.$$

Substituting this into the x^1 equation of (25), we obtain

$$\dot{x}^1 = F_{11}(x^1)\left(\frac{\partial H}{\partial x^1}(x) \right.$$
$$\left. -\frac{\partial^2 H}{\partial x^1 \partial x^2}(x)\left(\frac{\partial^2 H}{(\partial x^2)^2}(x)\right)^{-1}\frac{\partial H}{\partial x^2}(x)\right) + g_1(x^1)u$$
$$y = g_1(x^1)^T\left(\frac{\partial H}{\partial x^1}(x) \right.$$
$$\left. -\frac{\partial^2 H}{\partial x^1 \partial x^2}(x)\left(\frac{\partial^2 H}{(\partial x^2)^2}(x)\right)^{-1}\frac{\partial H}{\partial x^2}(x)\right)$$

On the other hand, we have

$$\frac{\partial \widetilde{H}}{\partial x^1}(x^1) = \frac{\partial H}{\partial x^1}(x^1, x^2(x^1)) + \frac{\partial x^2(x^1)}{\partial x^1}\frac{\partial H}{\partial x^2}(x^1, x^2(x^1)) \quad (33)$$

Furthermore, differentiation of (30) yields

$$\frac{\partial^2 H}{\partial x^1 \partial x^2}(x^1, x^2(x^1)) + \frac{\partial x^2(x^1)}{\partial x^1}\frac{\partial H}{\partial x^2}(x^1, x^2(x^1)) = 0 \quad (34)$$

which upon substitution in (33) yields

$$\frac{\partial \widetilde{H}}{\partial x^1}(x^1) = \frac{\partial H}{\partial x^1}(x^1, x^2(x^1))$$
$$-\frac{\partial^2 H}{\partial x^1 \partial x^2}(x^1, x^2(x^1))\left(\frac{\partial^2 H}{(\partial x^2)^2}(x^1, x^2(x^1))\right)^{-1}\frac{\partial H}{\partial x^2}(x^1, x^2(x^1))$$

resulting in the observable PH system (31), (32). $\qquad \square$

B. Zero observability in the co-energy variables

If we allow $F(x)$ and $g(x)$ to also depend on x^2, it is clear that equations (28) and (29) are not valid anymore. Therefore, we consider the co-energy variable representation, i.e., consider the PH system (20), and transform the system into the coordinates

$$z = \frac{\partial H}{\partial x}(x) =: \gamma(x),$$

under the assumption that the transformation is non-singular. Take $\widehat{H}(z)$ as the full Legendre transform of $H(x)$, i.e.,

$$\widehat{H}(z) = x^T z - H(x),$$

then $x = \frac{\partial \widehat{H}}{\partial z}(z) = \gamma^{-1}(z)$. System (20) transforms into

$$\dot{z} = \left(\frac{\partial^2 \widehat{H}}{\partial z^2}(z)\right)^{-1}\left(\widehat{J}(z) - \widehat{R}(z)\right)z$$
$$+ \left(\frac{\partial^2 \widehat{H}}{\partial z^2}(z)\right)^{-1}\widehat{g}(z)u \qquad (35)$$
$$y = \widehat{g}(z)^T z$$

with $\widehat{J}(z) := J(\gamma^{-1}(z))$, $\widehat{R}(z) := R(\gamma^{-1}(z))$, and $\widehat{g}(z) := g(\gamma^{-1}(z))$. Define

$$Q(z) := \left(\frac{\partial^2 \widehat{H}}{\partial z^2}(z) \right)^{-1} = \frac{\partial^2 H}{\partial x^2}(\gamma^{-1}(z)),$$

$$\text{and} \quad \widehat{F}(z) := \widehat{J}(z) - \widehat{R}(z).$$

Since we like to apply the duality result of Proposition 3.6, we study the zero observability co-distribution, rather than the observability co-distribution. Assume throughout Section V-B that zero observability co-distribution $d\mathcal{O}_0$ has constant dimension, then there exists local coordinates (z^1, z^2) such that $\ker d\mathcal{O}_0 = \text{span}\{\frac{\partial}{\partial z^2}\}$ (see e.g., [9]). Then the co-energy variable system takes the form

$$\begin{pmatrix} \dot{z}^1 \\ \dot{z}^2 \end{pmatrix} = \begin{pmatrix} Q_{11}(z) & Q_{12}(z) \\ Q_{21}(z) & Q_{22}(z) \end{pmatrix} \begin{pmatrix} \widehat{F}_{11}(z) & \widehat{F}_{12}(z) \\ \widehat{F}_{21}(z) & \widehat{F}_{22}(z) \end{pmatrix} \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}$$

$$+ \begin{pmatrix} Q_{11}(z) & Q_{12}(z) \\ Q_{21}(z) & Q_{22}(z) \end{pmatrix} \begin{pmatrix} \widehat{g}_1(z) \\ \widehat{g}_2(z) \end{pmatrix} u \qquad (36)$$

$$y = \begin{pmatrix} \widehat{g}_1^T(z) & \widehat{g}_2^T(z) \end{pmatrix} \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \qquad \text{with}$$

$$Q_{11}(z)\widehat{F}_{11}(z)z^1 + Q_{12}\widehat{F}_{21}(z)z^1 + Q_{11}(z)\widehat{F}_{12}(z)z^2$$
$$+ Q_{12}(z)\widehat{F}_{22}(z)z^2 = a(z^1) \qquad (37)$$
$$\widehat{g}_1(z)^T z^1 + \widehat{g}_2(z)^T z^2 = c(z^1) \qquad (38)$$

Assume that $F_{22}(z)$ is invertible, and that L_o is smooth and exists. Clearly, in these coordinates it holds that $L_o(0, z^2) = 0$ (see [9]). Additionally, assume that we can bring the observability function $L_o(z)$ in a form so that it only depends on z^1, i.e., $L_o(z^1)$.

Theorem 5.2: Under the above assumptions there exists a coordinate transformation $z = \xi(\bar{z})$ such that the co-energy variable dynamics restricted to its zero observable part can be written as

$$\dot{\bar{z}} = \bar{Q}_{11}(\bar{z}) \left(\bar{F}_{11}(\bar{z}) - \bar{F}_{12}(\bar{z}) \left(\bar{F}_{22}(\bar{z}) \right)^{-1} \bar{F}_{21}(\bar{z}) \right) \bar{z}$$

$$+ \bar{Q}_{11}(\bar{z})\bar{g}_1(\bar{z})u \qquad (39)$$

$$y = \bar{g}_1^T(\bar{z})\bar{z}$$

with $\bar{Q}_{11}(\bar{z}) = Q_{11}(\xi(\bar{z})) = \frac{\partial^2 H}{(\partial x^1)^2}(\gamma^{-1}(\xi(\bar{z}))$, $\bar{F}_{ij}(\bar{z}) = F_{ij}(\xi(\bar{z}))$, $i, j = 1, 2$. As in Theorem 4.1 $\bar{F}(\bar{z}) := \bar{F}_{11}(\bar{z}) - \bar{F}_{12}(\bar{z}) \left(\bar{F}_{22}(\bar{z}) \right)^{-1} \bar{F}_{21}(\bar{z})$ is such that $\bar{F}(\bar{z}) + \bar{F}^T(\bar{z}) \leq 0$.
Proof
Consider the dual system of (36) along the lines of Proposition 3.6, i.e.,

$$\begin{pmatrix} \dot{p}^1 \\ \dot{p}^2 \end{pmatrix} = \begin{pmatrix} \widehat{F}_{11}^T(z) & \widehat{F}_{21}^T(z) \\ \widehat{F}_{12}^T(z) & \widehat{F}_{22}^T(z) \end{pmatrix} \begin{pmatrix} Q_{11}^T(z) & Q_{21}^T(z) \\ Q_{12}^T(z) & Q_{22}^T(z) \end{pmatrix} \begin{pmatrix} p^1 \\ p^2 \end{pmatrix}$$

$$+ \begin{pmatrix} \widehat{g}_1(z) \\ \widehat{g}_2(z) \end{pmatrix} u_d \qquad (40)$$

$$y_a = \begin{pmatrix} \widehat{g}_1^T(z) & \widehat{g}_2^T(z) \end{pmatrix} \begin{pmatrix} Q_{11}^T(z) & Q_{21}^T(z) \\ Q_{12}^T(z) & Q_{22}^T(z) \end{pmatrix} \begin{pmatrix} p^1 \\ p^2 \end{pmatrix}$$

If we consider the Legendre transform $\widehat{L}_c(p) = p^T z - L_o(z)$ of system (36), we obtain the controllability function $\widehat{L}_c(p)$ of the dual system (40), with $z = \frac{\partial \widehat{L}_c}{\partial p}(p) =: \phi(p)$, and $p = \frac{\partial L_o}{\partial z}(z)$. Hence, $p^2 = \frac{\partial L_o}{\partial z}(z) = 0$ and $\widehat{L}_c(p^1, p^2)$ does not exist for $p^2 \neq 0$, while $\widehat{L}_c(p^1, 0) < \infty$. Thus, p^2 corresponds to the non-asymptotically reachable part of the system, [9]. Since $p^2 = 0$, $\dot{p}^2 = 0$, we obtain similar restrictions as in the proof of Theorem 4.1, i.e.,

$$(\widetilde{F}_{12}^T(p^1)\widetilde{Q}_{11}^T(p^1) + \widetilde{F}_{22}^T(p^1)\widetilde{Q}_{12}^T(p^1))p^1 = 0, \text{ and } \widetilde{g}_2(p) = 0,$$

with $\widetilde{F}_{ij}(p^1) = \widehat{F}_{ij}(\phi(p^1, 0))$, $\widetilde{Q}_{ij}(p^1) = Q_{ij}(\phi(p^1, 0))$, $\widetilde{g}_i(p^1) = \widehat{g}_i(\phi(p^1, 0))$, $i, j = 1, 2$. Since $\widetilde{F}_{22}(p^1, 0)$ is invertible, we obtain

$$\widetilde{Q}_{12}^T(p^1)p^1 = -\widetilde{F}_{22}^{-T}(p^1)\widetilde{F}_{12}^T(p^1)\widetilde{Q}_{11}^T(p^1)p^1$$

Then the strongly accessible p^1 dynamics are

$$\dot{p}^1 = \left(\widetilde{F}_{11}^T(p^1) - \widetilde{F}_{21}^T(p^1)\widetilde{F}_{22}^{-T}(p^1)\widetilde{F}_{12}^T(p^1) \right) \widetilde{Q}_{11}^T(p^1)p^1$$

$$+ \widetilde{g}_1(p^1)u_d \qquad (41)$$

$$y_d = \widetilde{g}_1^T(p^1)\widetilde{Q}_{11}^T(p^1)p^1$$

Now, we can consider the dual system of (41), and the Legendre transform of its controllability function $\widetilde{L}_c(p^1) = \widehat{L}_c(p^1, 0)$, i.e., $\bar{L}_o(\bar{z}) = \bar{z}^T p^1 - \widetilde{L}_c(p^1)$, $p^1 = \frac{\partial \bar{L}_o}{\partial \bar{z}}(\bar{z}) =: \psi(\bar{z})$. The dual zero observable system is now given by (39) with $\xi(\bar{z}) = \phi(\psi(\bar{z}), 0)$. $\qquad \square$

Our zero observable representation (39) corresponds to the linear co-energy variable case presented in [7]. However, to proof that Q_{11} is the Hessian of a Hamiltonian is still an open issue.

VI. APPROXIMATE MODEL REDUCTION

In the preceding two sections we have seen how the nonlinear Kalman decomposition of a port-Hamiltonian system results in a minimal system that is again port-Hamiltonian, and takes either the form (24) (reduction in the non strongly accessible case) or (31) (reduction in the non observable case). Of course, both methods can be combined in case the system is not strongly accessible *as well as* not observable.

These methods for *exact* model reduction (from a non-minimal to a minimal and externally equivalent system) may be taken as starting point for approximate *structure-preserving* model reduction of port-Hamiltonian systems. Thus consider a general port-Hamiltonian system

$$\begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \end{pmatrix} = \begin{pmatrix} F_{11}(x) & F_{12}(x) \\ F_{21}(x) & F_{22}(x) \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial x^1}(x) \\ \frac{\partial H}{\partial x^2}(x) \end{pmatrix} + \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix} u$$

$$y = \begin{pmatrix} g_1(x)^T & g_2(x)^T \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial x^1}(x) \\ \frac{\partial H}{\partial x^2}(x) \end{pmatrix}$$

$$(42)$$

and suppose we regard the second group of state coordinates x^2 as relatively unimportant for the input-output behavior of the PH system (e.g., on the basis of *balancing*).

For approximate structure-preserving model reduction the approach based on the reduction of a non-observable PH system of Section V-A is most easily applicable. In this approach the full-order PH system is approximated by the reduced-order PH system

$$
\begin{aligned}
\dot{x}^1 &= F_{11}(x^1, x^2(x^1))\frac{\partial \widetilde{H}}{\partial x^1}(x^1) + g_1(x^1, x^2(x^1))u \\
y &= g_1(x^1, x^2(x^1))^T \frac{\partial \widetilde{H}}{\partial x^1}(x^1)
\end{aligned}
\tag{43}
$$

where as before $\widetilde{H} = H(x^1, x^2(x^1))$, with $x^2(x^1)$ the solution of the equation $\frac{\partial H}{\partial x^2}(x^1, x^2) = 0$. Note that, contrary to exact reduction for non-observability, we do *not* necessarily assume that F_{11}, F_{12}, g_1, g_2 only depend on x^1.

Alternatively, based on the reduction of a non strongly accessible PH system, the full-order PH system can be reduced as follows. Write the full-order model (42) succinctly as

$$
\begin{aligned}
\begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \end{pmatrix} &= \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \begin{pmatrix} e^1 \\ e^2 \end{pmatrix} + \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} u \\
y &= \begin{pmatrix} g_1^T & g_2^T \end{pmatrix} \begin{pmatrix} e^1 \\ e^2 \end{pmatrix}
\end{aligned}
\tag{44}
$$

where $e^1 = \frac{\partial H}{\partial x^1}(x), e^2 = \frac{\partial H}{\partial x^2}(x)$. Now set $\dot{x}^2$ equal to zero. This yields

$$0 = F_{21}e^1 + F_{22}e^2 + g_2 u$$

which by assuming invertibility of F_{22} yields $e^2 = -F_{22}^{-1}F_{21}e^1 - F_{22}^{-1}g_2 u$. Substitution of this expression, together with $x^2 = 0$, then yields

$$
\begin{aligned}
\dot{x}^1 &= \left(\tilde{F}_{11}(x^1) - \tilde{F}_{12}(x^1)\tilde{F}_{22}^{-1}(x^1)\tilde{F}_{21}(x^1) \right) \frac{\partial H}{\partial x^1}(x^1, 0) \\
&\quad + (g_1(x^1, 0) - \tilde{F}_{12}(x^1)\tilde{F}_{22}^{-1}(x^1)g_2(x^1, 0))u \\
y &= (g_1^T(x^1, 0) - g_2^T(x^1, 0)\tilde{F}_{22}^{-1}(x^1)\tilde{F}_{21}(x^1) \\
&\quad - g_2^T(x^1, 0)\tilde{F}_{22}^{-1}(x^1)g_2(x^1, 0)u
\end{aligned}
\tag{45}
$$

with $\tilde{F}_{ij}(x^1) = F_{ij}(x^1, 0)$, $i, j = 1, 2$, which is again a port-Hamiltonian system (with through-term) provided that $(F_{12}F_{22}^{-1})^T = F_{22}^{-1}F_{21}$.

We will call the first structure-preserving method the Effort-constraint reduction, while the second one will be called the Flow-constraint reduction. The second terminology stems from the fact that the Flow-constraint reduction corresponds to taking the 'flow' $\dot{x}^2$ equal to 0, both in the dynamical equations as well as in the state vector, corresponding to setting $x^2 = 0$ (or possibly another constant value). On the other hand, the Effort-constraint reduction corresponds to taking the 'effort' $\frac{\partial H}{\partial x^2}$ equal to 0, both in the dynamical equations, as well as in the state vector (leading to the new Hamiltonian $\widetilde{H}$). Note that both methods imply that the power $\frac{\partial^T H}{\partial x^2}\dot{x}^2$ through the power-port corresponding to the

flow $\dot{x}^2$ and the effort $\frac{\partial H}{\partial x^2}$ is approximated to be equal to 0.

Remark 6.1: Effort-constraints are in fact quite common in physical system modeling. For example, kinematic constraints are of this type. Kinematic constraints for a mechanical system, represented in Hamiltonian form as

$$\dot{q} = \frac{\partial H}{\partial p}(q, p), \quad \dot{p} = -\frac{\partial H}{\partial q}(q, p)$$

with q denoting the generalized position coordinates and p the corresponding generalized momenta, are of the form

$$0 = A^T(q)\dot{q} = A^T(q)\frac{\partial H}{\partial p}(q, p)$$

for a certain matrix A with entries depending on q, thus constraining the vector of co-energy variables $z = \frac{\partial H}{\partial x}(x)$ with $x = (q, p)$. In many cases such kinematic constraints constitute an idealization, or approximation, of reality. For example, "rolling without slipping constraints" are often an idealization of the case where the physical phenomenon of rolling involves various effects, including dynamical ones, but the modeler takes the decision to reduce and simplify the model by imposing the idealized rolling without slipping constraints. In this sense, the structure-preserving Effort-constraint model reduction method as described above is close to modeling practice. □

VII. CONCLUSIONS

In this paper we have developed schemes to reduce a non-minimal PH system to a minimal PH system. This can be done by first reducing to a strongly accessible PH system, and then to a zero observable system. The reduction methods open new possibilities for developing approximate structure preserving order reduction methods for PH systems.

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