The Circle Criterion and Input-to-State Stability for Infinite-Dimensional Systems

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The Circle Criterion and Input-to-State Stability for Infinite-Dimensional Systems*

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1 Introduction

In this paper, the focus is on absolute stability and input-to-state stability of the feedback interconnection of an infinite-dimensional linear system Σ and a nonlinearity $\Phi : \mathrm{dom}(\Phi) \subset L^2_{\mathrm{loc}}(\mathbb{R}_+, Y) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$, where $\mathrm{dom}(\Phi)$ denotes the domain of Φ and U and Y (Hilbert spaces) denote the input and output spaces of Σ, respectively (see Figure 1, wherein v is an essentially bounded input signal). The system Σ is assumed to belong to the rather general class of well-posed systems (see, for example, [11, 13] and the references therein) and the nonlinearity is assumed to satisfy a (generalized) sector condition.

In the literature on the circle criterion for infinite-dimensional systems (see, for example, [3, 4, 5, 7, 9, 12], and the references therein), the emphasis is usually on L^2- or L^∞-stability and global asymptotic or global exponential stability (or some variants thereof) of feedback systems of the type shown in Figure 1, with a static sector-bounded nonlinearity Φ in the feedback path. The new contribution of this paper as compared to the previous literature is twofold.

(i) In addition to static nonlinearities, we include a class of dynamic nonlinearities which may exhibit bias, but still satisfy a generalized pointwise sector condition.

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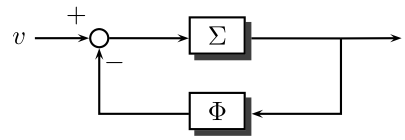

Figure 1. *Feedback interconnection of linear system Σ and nonlinearity Φ*

As specific subclasses, the class of nonlinearities encompasses both static nonlinearities with "negative resistance" and a wide range of hysteretic effects described by so-called Preisach operators.

(ii) The main results of the paper guarantee input-to-state-stability with "bias" (and "standard" input-to-state-stability if the nonlinearity is unbiased), thereby making contact with the important and rapidly developing input-to-state-stability theory in (finite-dimensional) nonlinear control.

As in the classical theory of absolute stability and circle criteria, the methodology involves a "symbiosis" of (generalized) sector data relating to the nonlinearity Φ and properties of the transfer function of the linear system Σ to conclude stability properties of the feedback interconnection. We mention that the viewpoint of this paper is similar in spirit to that of [1]: however, the class of feedback systems considered here is very different to that in [1] as is the methodology adopted.

For sake of brevity, this paper does not contain any proofs: for these we refer to [6].

Notation and terminology. For $\alpha \in \mathbb{R}$, set $\mathbb{C}_\alpha := \{s \in \mathbb{C} : \operatorname{Re} s > \alpha\}$. If S is a non-empty subset of $\mathbb{C}$, then a set $R \subset S$ is said to be *discrete* in S, if, for every $s \in S$, there exists a neighbourhood N of s such that $N \cap R$ is finite. For Hilbert spaces U and Y, let $\mathcal{B}(U,Y)$ denote the space of all linear bounded operators mapping U to Y. We write $\mathcal{B}(U)$ for $\mathcal{B}(U,U)$. For $T \in \mathcal{B}(U)$, we define

$$\operatorname{Re} T := \frac{1}{2}(T + T^*) \in \mathcal{B}(U).$$

The space of all holomorphic and bounded functions $\mathbb{C}_\alpha \to \mathcal{B}(U,Y)$ is denoted by $H^\infty_\alpha(\mathcal{B}(U,Y))$. We write $H^\infty(\mathcal{B}(U,Y))$ for $H^\infty_0(\mathcal{B}(U,Y))$. Moreover, in the scalar case (that is $U = Y = \mathbb{C}$), we simply write H^∞_α, or, if $\alpha = 0$, H^∞ for $H^\infty_\alpha(\mathcal{B}(U,Y))$ and $H^\infty(\mathcal{B}(U,Y))$, respectively. For $\alpha \in \mathbb{R}$, we define the exponentially weighted L^p-space $L^p_\alpha(\mathbb{R}_+, X) := \{f \in L^p_{\text{loc}}(\mathbb{R}_+, U) : f(\cdot)\exp(-\alpha\,\cdot) \in L^p(\mathbb{R}_+, U)\}$. The Laplace transform is denoted by $\mathfrak{L}$.

2 Well-posed linear systems with nonlinear feedback

There are a number of equivalent definitions of well-posed systems, see, for example, [11, 13] and the references therein. We will be brief in the following and refer the reader to the literature for more details. Throughout, we shall be considering a well-posed system Σ with state-space X, input space U and output space Y, generating operators (A, B, C), input-output operator G and transfer function $\mathbf{G}$. Here X, U and Y are separable (complex) Hilbert spaces, A is the generator of a strongly continuous semigroup $\mathbf{T} = (\mathbf{T}_t)_{t\geq 0}$ on X and $B \in \mathcal{B}(U, X_{-1})$ and $C \in \mathcal{B}(X_1, Y)$, respectively, are admissible control and observations for $\mathbf{T}$. The spaces X_1 and X_{-1}, respectively, are interpolation and extrapolation spaces associated with X: $X_1 = \mathrm{dom}(A)$ (the domain of A), endowed with the graph norm of A, whilst X_{-1} denotes the completion of X with respect to the norm $\|x\|_{-1} = \|(\xi I - A)^{-1}x\|$, where $\xi \in \varrho(A)$, the resolvent set of A (different choices of ξ lead to equivalent norms) and $\|\cdot\|$ denotes the norm on X. The control operator B is said to be *bounded* if it is so as a map from the input space U to the state space X, otherwise is said to be *unbounded*; the observation operator C is said to be *bounded* if it can be extended continuously to X, otherwise, C is said to be *unbounded*.

The so-called Λ-*extension* C_Λ of C is defined by

$$C_\Lambda z = \lim_{s\to\infty,\, s\in\mathbb{R}} Cs(sI - A)^{-1}z,$$

with $\mathrm{dom}(C_\Lambda)$ (the domain of C_Λ) consisting of all $z \in X$ for which the above limit exists. The transfer function $\mathbf{G}$ has the property that $\mathbf{G} \in H^\infty_\omega(\mathcal{B}(U, Y))$ for every $\omega > \omega(\mathbf{T})$, where $\omega(\mathbf{T})$ denotes the exponential growth constant of $\mathbf{T}$. Moreover, the input-output operator $G : L^2_{\mathrm{loc}}(\mathbb{R}_+, U) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$ is continuous and shift-invariant; for every $\omega > \omega(\mathbf{T})$, $G \in \mathcal{B}(L^2_\omega(\mathbb{R}_+, U), L^2_\omega(\mathbb{R}_+, Y))$ and

$$(\mathfrak{L}(Gu))(s) = \mathbf{G}(s)(\mathfrak{L}(u))(s), \quad \forall\, s \in \mathbb{C}_\omega,\ \forall\, u \in L^2_\omega(\mathbb{R}_+, U).$$

In the following, let $s_0 \in \mathbb{C}_{\omega(\mathbf{T})}$ be fixed, but arbitrary. For $x^0 \in X$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$, let x and y denote the state and output functions of Σ, respectively, corresponding to the initial condition $x(0) = x^0 \in X$ and the input function u. Then $x(t) = \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-\tau} Bu(\tau)d\tau$ for all $t \in \mathbb{R}_+$, $x(t) - (s_0 I - A)^{-1}Bu(t) \in \mathrm{dom}(C_\Lambda)$ for a.e. $t \in \mathbb{R}_+$ and

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x^0, \quad \text{a.e. } t \in \mathbb{R}_+,\\ &y(t) = C_\Lambda\big(x(t) - (s_0 I - A)^{-1}Bu(t)\big) + \mathbf{G}(s_0)u(t), \quad \text{a.e. } t \geq 0. \end{aligned} \tag{1}$$

Of course, the differential equation in (1) has to be interpreted in X_{-1}. In the following, we identify Σ and (1) and refer to (1) as a well-posed system.

We say that (1) is *exponentially stable* if $\omega(\mathbf{T}) < 0$ and we say that (1) is *input-output stable* if $\mathbf{G} \in H^\infty(\mathcal{B}(U, Y))$ or, equivalently, if $G \in \mathcal{B}(L^2(\mathbb{R}_+, U), L^2(\mathbb{R}_+, Y))$. Furthermore, (1) is said to be *optimizable*, if for every x^0, there exists $u \in L^2(\mathbb{R}_+, U)$ such that the function $t \mapsto \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-\tau}Bu(\tau)d\tau$ is in $L^2(\mathbb{R}_+, X)$. Writing $X^*_{-1} := (X^*)_{-1}$, we have that $X^*_{-1} = (X_1)^*$ and $C^* \in \mathcal{B}(Y, X^*_{-1})$ is an admissible control operator for the adjoint semigroup $\mathbf{T}^* = (\mathbf{T}^*_t)_{t\geq 0}$. We say that (1) is *estimatable* if for every x^0, there exists $u^* \in L^2(\mathbb{R}_+, Y)$ such the function $t \mapsto \mathbf{T}^*_t x^0 + \int_0^t \mathbf{T}^*_{t-\tau} C^* u^*(\tau)d\tau$ is in $L^2(\mathbb{R}_+, X)$.

In the following, we will consider the closed-loop system obtained by applying the nonlinear feedback

$$u = v - \Phi(y) \tag{2}$$

to the well-posed linear system (1), where $v \in L^\infty(\mathbb{R}_+, U)$ and the nonlinear operator $\Phi : \mathrm{dom}(\Phi) \subset L^2_{\mathrm{loc}}(\mathbb{R}_+, Y) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ is causal. To define the concept of a (local) solution of the feedback system given by (1) and (2), we first need to show that Φ can be "localized" in the sense that it can be "extended" to spaces of functions with a finite time horizon. To this end, let $0 < \sigma \leq \infty$ be arbitrary and set

$$\mathrm{dom}_\sigma(\Phi) := \{w \in L^2_{\mathrm{loc}}([0,\sigma), Y) : \forall\, \tau \in (0,\sigma)\, \exists\, w_\tau \in \mathrm{dom}(\Phi) \text{ s.t. } w = w_\tau \text{ on } [0,\tau]\}\,.$$

Trivially, $\mathrm{dom}_\infty(\Phi) = \mathrm{dom}(\Phi)$. For $w \in \mathrm{dom}_\sigma(\Phi)$ with $\sigma < \infty$, we define $\Phi(w)$ by

$$(\Phi(w))(t) = (\Phi(w_\tau))(t), \quad 0 \leq t \leq \tau < \sigma\,,$$

where $w_\tau \in \mathrm{dom}(\Phi)$ such that $w = w_\tau$ on $[0,\tau]$. By causality of Φ, this definition does not depend on the choice of τ and thus $\Phi(w)$ is a well-defined element in $L^2_{\mathrm{loc}}([0,\sigma), U)$.

A *solution* on $[0,\sigma)$ (where $0 < \sigma \leq \infty$) of the feedback system given by (1) and (2) is a pair $(x,y) \in C([0,\sigma), X) \times \mathrm{dom}_\sigma(\Phi)$ such that, with u given by (2),

$$x(t) = \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-\tau} B u(\tau)d\tau, \quad \forall\, t \in [0,\sigma), \tag{3}$$

$$y(t) = C_\Lambda \left(x(t) - (s_0 I - A)^{-1} B u(t)\right) + \mathbf{G}(s_0)u(t), \quad \text{a.e. } t \in [0,\sigma). \tag{4}$$

If $\sigma = \infty$, then we say that (x,y) is a *global solution*. Let $\mathcal{S}$ denote the set of all $(x^0, v) \in X \times L^\infty(\mathbb{R}_+, U)$ for which the feedback system given by (1) and (2) has at least one global solution. If $(x^0, v) \in \mathcal{S}$, then the notation $(x(\cdot\,; x^0, v), y(\cdot\,; x^0, v))$ is used to denote any global solution corresponding to the initial condition x^0 and the closed-loop input v. Furthermore, a routine argument based on Zorn's lemma shows that every solution (x,y) can be extended to a *maximal solution*, that is, to a maximally defined solution which cannot be extended any further. The interval on which a maximal solution is defined is called the *maximal interval of existence* of the solution. We say that the feedback system given by (1) and (2) has the *blow-up property* if for every maximal solution (x,y) defined on a finite maximal interval of existence $[0,\sigma)$, the L^2-norm of y blows up, that is, $\|y\|_{L^2(0,\tau)} \to \infty$ as $\tau \uparrow \sigma$. In this paper, we are mainly concerned with stability properties of the feedback system given by (1) and (2): whilst of fundamental importance, the question of existence of solutions is not the main concern here; this question requires addressing on a less general basis, taking into account relevant features of the particular system or subclass of systems under consideration (see [6] for further comments in this context).

3 The sector condition and input-to-state stability

First, we introduce a sector condition on the class of nonlinearities (in due course, this condition will be weakened to a generalized sector condition).

Definition 1. A nonlinearity $\Phi : \mathrm{dom}(\Phi) \subset L^2_{\mathrm{loc}}(\mathbb{R}_+, Y) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ satisfies a *sector condition* if there exist operators $K_1, K_2 \in \mathcal{B}(Y, U)$ such that

$$\mathrm{Re}\,\langle (\Phi(w))(t) - K_1 w(t), (\Phi(w))(t) - K_2 w(t)\rangle \leq 0, \quad \forall\, w \in \mathrm{dom}(\Phi),\ \text{a.e. } t \in \mathbb{R}_+. \tag{5}$$

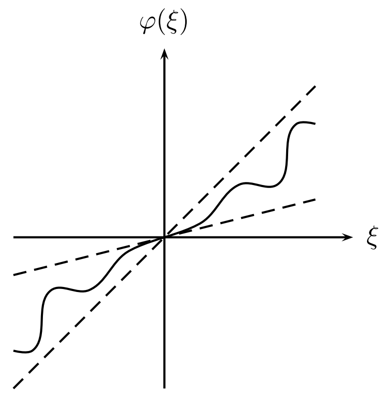

Figure 2. *Sector-bounded static nonlinearity* φ

Example 2 (Static nonlinearities). Let $\varphi : Y \to U$ be continuous and assume that there exist $K_1, K_2 \in \mathcal{B}(Y, U)$ such that

$$\mathrm{Re}\langle \varphi(\xi) - K_1\xi \, , \, \varphi(\xi) - K_2\xi \rangle_U \leq 0 \quad \forall \, \xi \in Y. \tag{6}$$

With φ we may associate the Němyckiĭ operator $\Phi : L^2_{\mathrm{loc}}(\mathbb{R}_+, Y) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$, defined by $\Phi(w) := \varphi \circ w$. This operator satisfies the sector condition (5). Such operators provide a simple prototype class for the general nonlinearities considered in this section: at the simplest illustrative level, static sector-bounded scalar nonlinearities $\varphi : \mathbb{R} \to \mathbb{R}$ of the type shown in Figure 2 (ubiquitous in the literature on the classical circle criterion) are subsumed by the formulation. This observation extends *mutatis mutandis* to encompass time-dependent static nonlinearities $\varphi \colon \mathbb{R}_+ \times Y \to U$. $\diamond$

Anticipating Sections 4 and 5 below, we will also consider static nonlinearities for which the inequality in (6) is assumed to hold only outside some bounded set $E \subset Y$ (see Figure 3). To accommodate these and more general nonlinearities, in Section 4 we will introduce a generalized sector condition and remark here that the generalized formulation encompasses a large class of hysteresis operators, including hysteresis of Preisach type.

Let $K_1, K_2 \in \mathcal{B}(Y, U)$ and define

$$K := \frac{1}{2}\big(K_1 + K_2\big), \quad \kappa := \|K_2 - K_1\|^2. \tag{7}$$

We assemble the following hypotheses on the transfer function $\mathbf{G}$ of (1) which will be variously invoked in the theory presented below.

(H1) There exists $\alpha < 0$ and an open set $\Omega \subset \mathbb{C}_\alpha$ such that $\mathbb{C}_\alpha \setminus \Omega$ is discrete in

$\mathbb{C}_\alpha$ and $\mathbf{G}$ is holomorphic on Ω, the frequency-domain condition

$$\mathbf{G}^*(i\omega)\Big[\frac{\kappa+\delta}{4}I - K^*K\Big]\mathbf{G}(i\omega) \le I + 2\,\mathrm{Re}\big(K\mathbf{G}(i\omega)\big), \quad \text{a.e. } \omega \in \mathbb{R}. \tag{8}$$

holds for some $\delta > 0$ and $\mathbf{G}(I + K\mathbf{G})^{-1} \in H^\infty(\mathcal{B}(U, Y))$,

(H2) $\mathbf{G} \in H^\infty(\mathcal{B}(U, Y))$ and there exist $\delta > 0$ and $\rho < 1$ such that (8) holds and

$$\mathbf{G}^*(i\omega)\Big[\frac{\kappa+\delta}{4}I - K^*K\Big]\mathbf{G}(i\omega) \ge -\rho I, \quad \text{a.e. } \omega \in \mathbb{R}. \tag{9}$$

(H3) There exists an open set $\Omega \subset \mathbb{C}_0$ such that $\mathbb{C}_0 \setminus \Omega$ is discrete in $\mathbb{C}_0$ and $\mathbf{G}$ is holomorphic on Ω, $I + K\mathbf{G}(s)$ is invertible for all $s \in \Omega$ and the frequency-domain condition

$$\mathbf{G}^*(s)\Big[\frac{\kappa+\delta}{4}I - K^*K\Big]\mathbf{G}(s) \le I + 2\,\mathrm{Re}\big(K\mathbf{G}(s)\big), \quad \forall\, s \in \Omega \tag{10}$$

holds for some $\delta > 0$.

(H4) There exists an open set $\Omega \subset \mathbb{C}_0$ such that $\mathbb{C}_0 \setminus \Omega$ is discrete in $\mathbb{C}_0$ and $\mathbf{G}$ is holomorphic on Ω, $K\mathbf{G}(s)$ is compact for all $s \in \Omega$ and the frequency-domain condition (10) holds for some $\delta > 0$.

Remark 3. **(a)** In the case of scalar "sector data", that is $U = Y$ and there exist $k_1, k_2 \in \mathbb{C}$ such that $K_1 = k_1 I$ and $K_2 = k_2 I$, the term

$$\frac{\kappa+\delta}{4}I - K^*K$$

appearing on the left-hand sides of (8)-(10) simplifies to $(\delta/4 - \mathrm{Re}(\bar{k}_1 k_2))I$.

(b) Assume that one of the operators K_1 and K_2 is the zero operator and that the other is a scalar multiple of an isometry. Then it is not difficult to show that (H2) is satisfied, provided that $\mathbf{G} \in H^\infty(\mathcal{B}(U, Y))$ and the positive-real condition

$$\varepsilon I \le I + 2\,\mathrm{Re}\big(K\mathbf{G}(i\omega)\big), \quad \text{a.e. } \omega \in \mathbb{R}$$

holds for some $\varepsilon > 0$.

We are now in the position to state the main result of this section.

Theorem 4. *Assume that* (1) *is optimizable and estimatable and that there exist operators $K_1, K_2 \in \mathcal{B}(Y, U)$ such that Φ satisfies the sector condition* (5). *Let $K \in \mathcal{B}(Y, U)$ and $\kappa \ge 0$ be given by* (7). *If at least one of hypotheses* (H1)–(H4) *holds, then there exist positive constants Γ and γ, such that, for each $(x^0, v) \in \mathcal{S}$,*

$$\|x(t; x^0, v)\| \le \Gamma\left(\exp(-\gamma t)\|x^0\| + \|v\|_{L^\infty}\right), \quad \forall\, t \in \mathbb{R}_+. \tag{11}$$

For the above theorem to be non-vacuous, $\mathcal{S}$ should be non-empty: thus, there is a tacit assumption of global existence of solutions. However, if the feedback system given by (1) and (2) has the blow-up property, then it can be shown that the assumptions of Theorem 4 imply that every (local) solution can be extended to a global solution. Furthermore, we emphasize that (11) implies in particular that the feedback system is input-to-state stable in the sense of Sontag (see [10] for a recent survey of the theory of input-to-state stability).

Theorem 4 can be considered as a generalization and refinement of the circle criterion (see, for example, [4, 12]): in particular, it shows that, under the standard assumptions of the circle criterion (see also Corollaries 5 and 6 below), input-to-state stability is guaranteed. The proof of Theorem 4 (see [6]) is based on a well-known exponential weighting technique which has been used to prove stability results of input-output type (see [4, Section V.3] and the references therein). The application of this technique in an input-to-state stability context seems to be new (even in the finite-dimensional case). In particular, whilst the standard text-book version of the circle criterion for finite-dimensional state-space systems is usually proved using Lyapunov techniques combined with the positive-real lemma (see, for example, [12, p. 227]), the approach based on the exponential weighting technique provides a more elementary alternative.

The following corollary considers the case of scalar "sector data".

Corollary 5. *Assume that* (1) *is optimizable and estimatable,* $U = Y$ *and that there exists an open set* $\Omega \subset \mathbb{C}_0$ *such that* $\mathbb{C}_0 \setminus \Omega$ *is discrete in* $\mathbb{C}_0$ *and* $\mathbf{G}$ *is holomorphic on* Ω*. Furthermore, assume that there exist* $k_1, k_2 \in \mathbb{C}$ *and* $\varepsilon > 0$ *such that* Φ *satisfies* (5) *with* $K_1 = k_1 I$ *and* $K_2 = k_2 I$*,* $I + k_1\mathbf{G}(s)$ *is invertible for every* $s \in \Omega$ *and*

$$\operatorname{Re}\big[\big(I + k_2\mathbf{G}(s)\big)\big(I + k_1\mathbf{G}(s)\big)^{-1}\big] \geq \varepsilon I, \quad \forall\, s \in \Omega. \tag{12}$$

Then there exist positive constants Γ *and* γ*, such that, for each* $(x^0, v) \in \mathcal{S}$*,* (11) *holds.*

For non-zero real numbers k_1 and k_2, we define

$$\Delta(k_1, k_2) := \text{open disk in } \mathbb{C} \text{ with centre in } \mathbb{R} \text{ and } -\frac{1}{k_1} \text{ and } -\frac{1}{k_2} \text{ in its boundary.}$$

The next corollary focuses on the single-input-single-output case. In particular, the classical circle criterion is recovered.

Corollary 6. *Assume that* (1) *is optimizable and estimatable,* $U = Y = \mathbb{R}$ *and there exist real numbers* $k_1 < k_2$ *such that*

$$\big((\Phi(w))(t) - k_1 w(t)\big)\big((\Phi(w))(t) - k_2 w(t)\big) \leq 0, \quad \forall\, w \in \operatorname{dom}(\Phi), \text{ a.e. } t \in \mathbb{R}_+. \tag{13}$$

Then there exist positive constants Γ *and* γ*, such that, for each* $(x^0, v) \in \mathcal{S}$*,* (11) *holds, provided that one of the following conditions is satisfied:*

(C1) $0 < k_1 < k_2$*,* $\mathbf{G}/\big(1 + [(k_1 + k_2)/2]\mathbf{G}\big) \in H^\infty$*,* $\mathbf{G}(i\omega)$ *is bounded away from* $\Delta(k_1, k_2)$ *for all* $\omega \in \mathbb{R}$ *for which* $i\omega$ *is not a pole of* $\mathbf{G}$*;*

(C2) $0 = k_1 < k_2$*,* $\mathbf{G} \in H^\infty$ *and there exists* $\delta > 0$ *such that* $1 + k_2 \operatorname{Re}\mathbf{G}(i\omega) \geq \delta$ *for all* $\omega \in \mathbb{R}$*;*

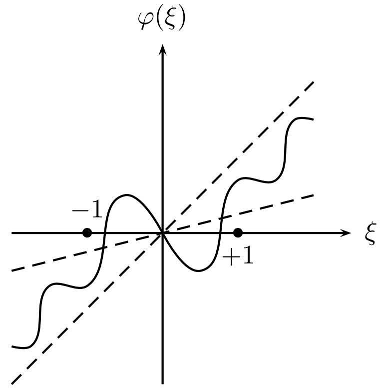

Figure 3. *Static nonlinearity φ satisfying a generalized sector condition*

(C3) *$k_1 < 0 < k_2$, $\mathbf{G} \in H^\infty$, $\mathbf{G}(i\omega) \in \Delta(k_1, k_2)$ for all $\omega \in \mathbb{R}$ and $\mathbf{G}(i\omega)$ is bounded away from $\partial\Delta(k_1, k_2)$ for all $\omega \in \mathbb{R}$.*

Observe that, in this single-input-single-output setting, the sector condition (13) can be expressed in the equivalent form:

$$k_1 w^2(t) \leq (\Phi(w))(t)w(t) \leq k_2 w^2(t), \quad \forall\, w \in \mathrm{dom}(\Phi), \text{ a.e. } t \in \mathbb{R}_+. \tag{14}$$

In many situations, the input-output stability condition $\mathbf{G}/\big(1 + [(k_1 + k_2)/2]\mathbf{G}\big) \in H^\infty$ (imposed in (C1)) is satisfied, provided that the number of anticlockwise encirclements of $\Delta(k_1, k_2)$ by the Nyquist diagram of $\mathbf{G}$ is equal to the number of poles of $\mathbf{G}$ in $\mathbb{C}_0$, see, for example, [4, 12].

4 Generalized sector condition and input-to-state stability with bias

Next, we seek to relax the condition (5) to a *generalized sector condition*. Loosely speaking, we wish to impose the (pointwise) inequality in (5) only when $t \in \mathbb{R}_+$ and $w \in \mathrm{dom}(\Phi)$ are such that $w(t) \in Y \backslash E$, where E (the exceptional set) is some bounded subset of Y. A prototype to bear in mind is the case wherein Φ is the Němyckiĭ operator, given by $\Phi(w) := \varphi \circ w$, associated with a static nonlinearity $\varphi : \mathbb{R} \to \mathbb{R}$, of the form shown in Figure 3 (a nonlinearity with negative resistance), satisfying a sector condition outside the interval $E = [-1, 1]$. Extrapolating this prototype to our abstract setting requires care. The issue is to circumvent the technical difficulty engendered by the fact that the general operator Φ has domain $\mathrm{dom}(\Phi) \subset L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$ and so Φ acts on equivalence classes of functions $\mathbb{R}_+ \to Y$. Let $w \in L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$ and $Z \subset Y$ be arbitrary. Let $w_r : \mathbb{R}_+ \to Y$ be any representative of w and denote the preimage of Z under w_r by $w_r^{-1}(Z) := \{t \in \mathbb{R}_+ : w_r(t) \in Z\}$. Let $\mathbb{I}_{w_r^{-1}(Z)}$ be the indicator or characteristic function of the set $w_r^{-1}(Z)$ and define $\chi_Z(w) \in L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$ to be the equivalence class of this function, that is,

$$\chi_Z(w) := \left[\mathbb{I}_{w_r^{-1}(Z)}\right].$$

Every choice of representative w_r of w yields the same equivalence class $\left[\mathbb{I}_{w_r^{-1}(Z)}\right]$ and so $\chi_Z(w)$ is a well-defined element of $L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$ for all $w \in L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$. We are now in a position to define the requisite generalized sector condition.

Definition 7. A nonlinearity $\Phi : \mathrm{dom}(\Phi) \subset L^2_{\mathrm{loc}}(\mathbb{R}_+, Y) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ satisfies a *generalized sector condition* if there exist operators $K_1, K_2 \in \mathcal{B}(Y, U)$, a bounded set $E \subset Y$ and a constant $b \geq 0$ such that, for all $w \in \mathrm{dom}(\Phi)$ and a.e. $t \in \mathbb{R}_+$,

$$\mathrm{Re}\,\langle (\Phi(w))(t) - K_1 w(t), (\Phi(w))(t) - K_2 w(t)\rangle (\chi_{Y\setminus E}(w))(t) \leq 0 \tag{15}$$

and

$$\|(\Phi(w))(t)\| (\chi_E(w))(t) \leq b. \tag{16}$$

The following result generalizes Theorem 4.

Corollary 8. *Assume that* (1) *is optimizable and estimatable and that there exist operators* $K_1, K_2 \in \mathcal{B}(Y, U)$, $b \geq 0$ *and a bounded set* $E \subset Y$ *such that* Φ *satisfies* (15) *and* (16) *for all* $w \in \mathrm{dom}(\Phi)$ *and a.e.* $t \in \mathbb{R}_+$. *Let* $K \in \mathcal{B}(Y, U)$ *and* $\kappa \geq 0$ *be given by* (7). *If at least one of hypotheses* (H1)–(H4) *holds, then there exist positive constants* Γ *and* γ *such that, for each* $(x^0, v) \in \mathcal{S}$,

$$\|x(t; x^0, v)\| \leq \Gamma\left(\exp(-\gamma t)\|x^0\| + \|v\|_{L^\infty} + \beta\right), \quad \forall\, t \in \mathbb{R}_+, \tag{17}$$

where

$$\beta := \sup\left\{\|(\Phi(w) - Kw)\chi_E(w)\|_{L^\infty} : w \in \mathrm{dom}(\Phi)\right\} \leq b + \sup_{\xi \in E}\|K\xi\|, \tag{18}$$

In particular, (17) provides an input-to-state stability estimate with bias β (input-to-state stability with bias β). Under the additional assumption that the feedback system given by (1) and (2) has the blow-up property, it can be shown that the hypotheses of Corollary 8 imply that every maximal solution is global, so that every (local) solution can be extended to a global solution (to which then the stability conclusions of Corollary 8 apply).

The following results are generalizations of Corollaries 5 and 6.

Corollary 9. *Assume that* (1) *is optimizable and estimatable,* $U = Y$ *and that there exists an open set* $\Omega \subset \mathbb{C}_0$ *such that* $\mathbb{C}_0 \setminus \Omega$ *is discrete in* $\mathbb{C}_0$ *and* $\mathbf{G}$ *is holomorphic on* Ω. *Furthermore, assume that there exist* $k_1, k_2 \in \mathbb{C}$, *a bounded set* $E \subset Y$ *and constants* $b \geq 0$ *and* $\varepsilon > 0$ *such that, for all* $w \in \mathrm{dom}(\Phi)$ *and a.e.* $t \in \mathbb{R}_+$. Φ *satisfies* (15) *and* (16) (*with* $K_1 = k_1 I$ *and* $K_2 = k_2 I$), $I + k_1\mathbf{G}(s)$ *is invertible for every* $s \in \Omega$ *and the positive-real condition*

$$\mathrm{Re}\left[\left(I + k_2\mathbf{G}(s)\right)\left(I + k_1\mathbf{G}(s)\right)^{-1}\right] \geq \varepsilon I, \quad \forall\, s \in \Omega$$

holds. Then there exist constants $\Gamma > 0$ and $\gamma > 0$ such that, for each $(x^0, v) \in \mathcal{S}$, (17) holds, where $\beta \geq 0$ is given by (18).

Corollary 10. *Assume that (1) is optimizable and estimatable, $U = Y = \mathbb{R}$ and there exist real numbers $k_1 < k_2$, a bounded set $E \subset \mathbb{R}$ and $b \geq 0$ such that*

$$\big((\Phi(w))(t) - k_1 w(t)\big)\big((\Phi(w))(t) - k_2 w(t)\big)(\chi_{Y \setminus E}(w))(t) \leq 0, \ \ \forall\, w \in \mathrm{dom}(\Phi), \ \text{a.e. } t \in \mathbb{R}_+$$

and

$$|(\Phi(w))(t)|(\chi_E(w))(t) \leq b, \quad \forall\, w \in \mathrm{dom}(\Phi), \ \text{a.e. } t \in \mathbb{R}_+ .$$

If at least one of the conditions (C1)–(C3) *of* Corollary 6 *is satisfied, then there exist $\Gamma > 0$ and $\gamma > 0$ such that, for each $(x^0, v) \in \mathcal{S}$, (17) holds, where*

$$\beta := \sup\big\{\|(\Phi(w) - (k_1 + k_2)w/2)\chi_E(w)\|_{L^\infty} : w \in \mathrm{dom}(\Phi)\big\} \leq b + |k_1 + k_2| \sup_{\xi \in E} |\xi|/2, \tag{19}$$

5 Hysteretic feedback systems

Consider again the feedback interconnection of Figure 1, but now in a single-input ($U = \mathbb{R}$), single-output ($Y = \mathbb{R}$) setting and with a hysteresis operator Φ in the feedback path. An operator $\Phi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ is a *hysteresis operator* if it is causal and rate independent. Here *rate independence* means that $\Phi(w \circ \zeta) = (\Phi w) \circ \zeta$ for every $w \in C(\mathbb{R}_+)$ and every time transformation ζ, where $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a *time transformation* if it is continuous, non-decreasing and surjective.

For simplicity of presentation, henceforth we restrict attention to the class of Preisach hysteresis operators which model complex hysteresis effects: for example, nested loops in input-output characteristics. A basic building block for the Preisach operator is the hysteresis operator $\mathcal{B}_{\sigma,\,\xi}$, the so-called *backlash* operator with width $\sigma \geq 0$ and "initial condition" $\xi \in \mathbb{R}$. A discussion of the backlash operator (also called *play* operator) can be found in a number of references, see for example, [2] and [8].

Let $\xi : \mathbb{R}_+ \to \mathbb{R}$ be a compactly supported and globally Lipschitz function with Lipschitz constant 1. Let μ be a regular signed Borel measure on $\mathbb{R}_+$. Denoting Lebesgue measure on $\mathbb{R}$ by μ_L, let $f : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ be a locally $(\mu_L \otimes \mu)$-integrable function and let $f_0 \in \mathbb{R}$. The operator $\mathcal{P}_\xi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ defined by

$$(\mathcal{P}_\xi(w))(t) = \int_0^\infty \int_0^{(\mathcal{B}_{\sigma,\,\xi(\sigma)}(w))(t)} f(s,\sigma)\mu_L(ds)\mu(d\sigma) + f_0 \quad \forall\, w \in C(\mathbb{R}_+)\,, \ \ \forall\, t \in \mathbb{R}_+\,, \tag{20}$$

is called a *Preisach* operator, cf. [2, p. 55]. It is well-known that $\mathcal{P}_\xi$ is a hysteresis operator (this follows from the fact that $\mathcal{B}_{\sigma,\,\xi(\sigma)}$ is a hysteresis operator for every $\sigma \geq 0$).

Setting $f(\cdot,\cdot) = 1$ and $f_0 = 0$ in (20), we obtain the *Prandtl* operator $\mathcal{P}_\xi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ defined by

$$\mathcal{P}_\xi(w)(t) = \int_0^\infty (\mathcal{B}_{\sigma,\,\xi(\sigma)}(w))(t)\mu(d\sigma) \quad \forall\, w \in C(\mathbb{R}_+)\,, \ \ \forall\, t \in \mathbb{R}_+\,. \tag{21}$$

For $\xi(\cdot) = 0$ and μ given by $\mu(S) = \int_S \mathbb{I}_{[0,5]}(\sigma)d\sigma$ (where $\mathbb{I}_{[0,5]}$ denotes the indicator function of the interval $[0, 5]$), the Prandtl operator is illustrated in Figure 4.

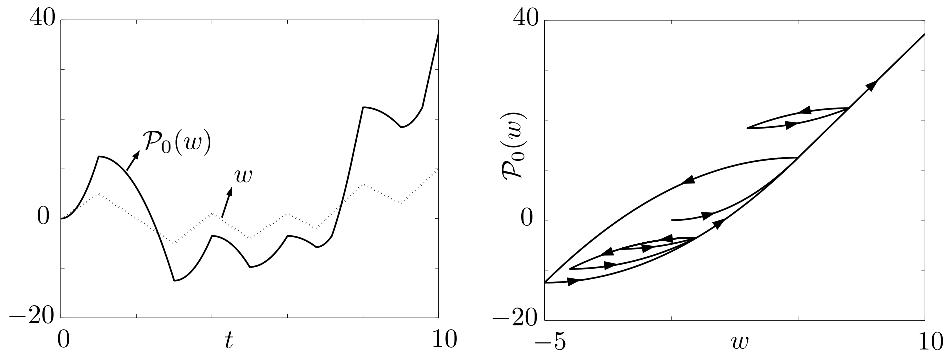

Figure 4. *Example of Prandtl hysteresis*

The next proposition identifies (rather "mild") conditions under which the Preisach operator (20) satisfies a generalized sector bound and hence fits into the theory developed in Section 4. For simplicity, we assume that the measure μ and the function f are non-negative (an important case in applications), although the proposition can be extended to signed measures μ and sign-indefinite functions f.

Proposition 11. *Let $\mathcal{P}_\xi$ be the Preisach operator defined in* (20). *Assume that the measure μ is non-negative, $a_1 := \mu(\mathbb{R}_+) < \infty$ and $a_2 := \int_0^\infty \sigma\mu(d\sigma) < \infty$. Furthermore, assume that*

$$b_1 := \operatorname{ess\,inf}_{(s,\sigma)\in\mathbb{R}\times\mathbb{R}_+} f(s,\sigma) \geq 0\,, \quad b_2 := \operatorname{ess\,sup}_{(s,\sigma)\in\mathbb{R}\times\mathbb{R}_+} f(s,\sigma) < \infty$$

and set

$$a_\mathcal{P} := a_1 b_1, \quad b_\mathcal{P} := a_1 b_2\,, \quad c_\mathcal{P} := a_2 b_2 + |f_0|\,. \tag{22}$$

Then, for all $w \in C(\mathbb{R}_+)$ and all $t \in \mathbb{R}_+$,

$$w(t) \geq 0 \implies a_\mathcal{P} w(t) - c_\mathcal{P} \leq (\mathcal{P}_\xi(w))(t) \leq b_\mathcal{P} w(t) + c_\mathcal{P}\,, \tag{23}$$

$$w(t) \leq 0 \implies b_\mathcal{P} w(t) - c_\mathcal{P} \leq (\mathcal{P}_\xi(w))(t) \leq a_\mathcal{P} w(t) + c_\mathcal{P}\,, \tag{24}$$

and, furthermore, for every $\eta > 0$,

$$|w(t)| \geq c_\mathcal{P}/\eta \implies (a_\mathcal{P} - \eta)w^2(t) \leq (\mathcal{P}_\xi(w))(t)y(t) \leq (b_\mathcal{P} + \eta)w^2(t)\,. \tag{25}$$

In particular, for every $\eta > 0$, the generalized sector conditions (15) *and* (16) *hold with $U = \mathbb{R} = Y$, $E = [-c_\mathcal{P}/\eta\,,\, c_\mathcal{P}/\eta]$, $K_1 = (a_\mathcal{P} - \eta)I$, $K_2 = (b_\mathcal{P} + \eta)I$, and $b = (b_\mathcal{P}/\eta + 1)c_\mathcal{P}$.*

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