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New Results on the Equivalence of Rational Representations of Behaviors

Sasanka V. Gottimukkala*, Harry L. Trentelman*, Shaik Fiaz*

Abstract—This article deals with the equivalence of representations of behaviors of linear differential systems. In general, the behavior of a given linear differential system has many different representations. In this paper we restrict ourselves to kernel and image representations. Two kernel representations are called equivalent if they represent one and the same behavior. For kernel representations defined by polynomial matrices, necessary and sufficient conditions for equivalence are well-known. In this paper, we deal with the equivalence of rational representations, i. e. kernel and image representations that are defined in terms of rational matrices. As the main result of this paper, we will derive a new condition for equivalence of rational kernel representations of possibly uncontrollable behaviors. This paper also deals with the equivalence of polynomial as well as rational image representations.

I. INTRODUCTION

An important issue in the behavioral approach to systems and control is the issue of representation. In the behavioral approach, a system is defined in terms of its *behavior*, which is the set of all time trajectories that are compatible with the laws of the system (see [5]). In the context of linear, finite-dimensional, time-invariant systems this leads to the concept of *linear differential system*. A linear differential system is defined to be a system whose behavior is equal to the set of solutions of a finite number of higher order, linear, constant coefficient differential equations. This set of differential equations is then called a representation of the behavior, often called a *kernel representation*. It is well known that the behavior of a given linear differential system admits many different kinds of representations. Apart from higher order linear differential equations, the behavior of a linear differential system can be represented for example in terms of finite-dimensional state space models, possibly (but not necessarily) even distinguishing between inputs and outputs (see [5], [10], [9]). Also, if it is controllable, it can be represented as the *image* of a polynomial differential operator (we then speak of an *image representation*). Traditionally, kernel and image representations of linear differential systems involve *polynomial matrices*. Recently, in [12], the concept of *rational representation* was defined and elaborated, extending the class of representations to kernel, hybrid, and image representations involving *rational matrices*.

As noted above, a given linear differential system admits many different representations. Two representations are

called *equivalent* if they represent the same behavior. The issue of equivalence of representations of behaviors has been studied before, in an input-output framework in [6], [7], [4], [13], [2] and [1], and in a behavioral framework in [5], [10], [8] and [3]. In the present paper, we will study the equivalence of kernel representations and image representation in terms of rational matrices. In particular, we consider the question how the rational matrices appearing in equivalent rational kernel representations and rational image representations are related.

The outline of this article is as follows. In the remainder of this section we will introduce the notation, and review some basic material on polynomial and rational matrices. In Section II we will review linear differential systems and their polynomial and rational kernel and image representations. In Section III we formally state the problem that we are addressing in the current paper. In Section IV we review the problem of equivalence of polynomial kernel representations. We establish new results here, and obtain, for two given polynomial kernel representations, separate conditions under which their controllable parts are equal, and their sets of autonomous parts are equal. Combining these conditions, we reobtain the well-known “classical” result on the equivalence of polynomial kernel representations. In section V we will apply these results to obtain up to now unknown conditions under which rational representations of possibly uncontrollable behaviors are equivalent. In section VI we consider the equivalence of image representations. Due to space limitations, some of the proofs have been omitted. For detailed proofs we refer to the forthcoming journal version of this paper.

As announced, first a few words about the notation and nomenclature used. We use the standard symbols for the fields of real and complex numbers \mathbb{R} and \mathbb{C} . \mathbb{C}^- will denote the open left half complex plane. We use \mathbb{R}^n , $\mathbb{R}^{n \times m}$, etc. for the real linear spaces of vectors and matrices with components in \mathbb{R} . $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^w)$ denotes the set of infinitely often differentiable functions from \mathbb{R} to \mathbb{R}^w .

$\mathbb{R}(\xi)$ will denote the field of real rational functions in the indeterminate ξ . $\mathbb{R}[\xi]$ will denote the ring of polynomials in the indeterminate ξ with real coefficients. We will use $\mathbb{R}(\xi)^n$, $\mathbb{R}(\xi)^{n \times m}$, $\mathbb{R}[\xi]^n$, $\mathbb{R}[\xi]^{n \times m}$, etc. for the spaces of vectors and matrices with components in $\mathbb{R}(\xi)$, and $\mathbb{R}[\xi]$ respectively. If one, or both, dimensions are unspecified, we will use the notation $\mathbb{R}(\xi)^{\bullet \times m}$, $\mathbb{R}(\xi)^{n \times \bullet}$ or $\mathbb{R}(\xi)^{\bullet \times \bullet}$, etc. Elements of $\mathbb{R}(\xi)^{n \times m}$ are called *real rational matrices*, elements of $\mathbb{R}[\xi]^{n \times m}$ are called *real polynomial matrices*.

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II. LINEAR DIFFERENTIAL SYSTEMS

In this section we will review the basic material on linear differential systems and their polynomial and rational representations.

In the behavioral approach to linear systems, a dynamical system is given by a triple $\Sigma = (\mathbb{R}, \mathbb{R}^w, \mathfrak{B})$, where \mathbb{R} is the time axis, \mathbb{R}^w is the signal space, and the *behavior* \mathfrak{B} is a linear subspace of $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^w)$ consisting of all solutions of a set of higher order, linear, constant coefficient differential equations. Such a triple is called a *linear differential system*. The set of all linear differential systems with w variables is denoted by \mathcal{L}^w .

For any linear differential system $\Sigma = (\mathbb{R}, \mathbb{R}^w, \mathfrak{B})$ there exists a real polynomial matrix R with w columns, i. e. $R \in \mathbb{R}[\xi]^{\bullet \times w}$, such that \mathfrak{B} is equal to the space of solutions of

$$R\left(\frac{d}{dt}\right)w = 0. \quad (1)$$

If a behavior \mathfrak{B} is represented by $R\left(\frac{d}{dt}\right)w = 0$ (or: $\mathfrak{B} = \ker(R)$), with $R(\xi)$ a real polynomial matrix, then we call this a *polynomial kernel representation* of \mathfrak{B} . If R has p rows, then the polynomial kernel representation is said to be *minimal* if every polynomial kernel representation of \mathfrak{B} has at least p rows. A given polynomial kernel representation, $\mathfrak{B} = \ker(R)$, is minimal if and only if the polynomial matrix R has full row rank (see [5], Theorem 3. 6. 4). The number of rows in any minimal polynomial kernel representation of \mathfrak{B} , denoted by $p(\mathfrak{B})$, is called the *output cardinality* of \mathfrak{B} . This number corresponds to the number of outputs in any input/output representation of \mathfrak{B} . For a detailed exposition of polynomial representations of behaviors, we refer to [5].

Recently, in [12], representations of linear differential systems using *rational matrices* instead of polynomial matrices were introduced. In [12], a meaning was given to the equation $R\left(\frac{d}{dt}\right)w = 0$, where $R(\xi)$ is a given real *rational* matrix. In order to do this, we need the concept of left coprime factorization over $\mathbb{R}[\xi]$.

Definition 2.1: Let R be a real rational matrix. The pair of real polynomial matrices (P, Q) is called a *left coprime factorization of R over $\mathbb{R}[\xi]$* if

- 1) $\det(P) \neq 0$,
- 2) $R = P^{-1}Q$,
- 3) the matrix $(P(\lambda) \ Q(\lambda))$ has full row rank for all $\lambda \in \mathbb{C}$.

A meaning to the equation

$$R\left(\frac{d}{dt}\right)w = 0, \quad (2)$$

with $R(\xi)$ a real rational matrix is then given as follows: Let (P, Q) be a left coprime factorization of R over $\mathbb{R}[\xi]$. Then we define:

Definition 2.2: Let $w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^w)$. Then we define w to be a solution of (2) if it satisfies the differential equation $Q\left(\frac{d}{dt}\right)w = 0$.

It can be proven that the space of solutions defined in this way is independent of the particular left coprime factorization. Hence (2) represents the linear differential system $\Sigma = (\mathbb{R}, \mathbb{R}^w, \ker(Q)) \in \mathcal{L}^w$.

Since the behavior \mathfrak{B} of the system Σ is the central item, often we will speak about the system $\mathfrak{B} \in \mathcal{L}^w$ (instead of $\Sigma \in \mathcal{L}^w$). If a behavior \mathfrak{B} is represented by $R\left(\frac{d}{dt}\right)w = 0$ (or: $\mathfrak{B} = \ker(R)$), with $R(\xi)$ a real rational matrix, then we call this a *rational kernel representation* of \mathfrak{B} . If R has p rows, then the rational kernel representation is called *minimal* if every rational kernel representation of \mathfrak{B} has at least p rows. It can be shown that a given rational kernel representation $\mathfrak{B} = \ker(R)$ is minimal if and only if the rational matrix R has full row rank. As in the polynomial case, every $\mathfrak{B} \in \mathcal{L}^w$ admits a minimal rational kernel representation. The number of rows in any minimal rational kernel representation of \mathfrak{B} is equal to the number of rows in any minimal polynomial kernel representation of \mathfrak{B} , and therefore equal to $p(\mathfrak{B})$, the output cardinality of \mathfrak{B} . In general, if $\mathfrak{B} = \ker(R)$ is a rational kernel representation, then $p(\mathfrak{B}) = \text{rank}(R)$. This follows immediately from the corresponding result for polynomial kernel representations (see [5]).

It is well-known that a behavior $\mathfrak{B} \in \mathcal{L}^w$ is controllable if and only if there exists a real polynomial matrix $M \in \mathbb{R}[\xi]^{w \times \bullet}$ such that

$$\mathfrak{B} = \{w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^w) \mid \exists \ell \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet) \text{ s.t. } w = M\left(\frac{d}{dt}\right)\ell\}. \quad (3)$$

The representation (3) is called a *polynomial image representation* of \mathfrak{B} , and we will write $\mathfrak{B} = \text{im}(M)$. It can be shown that the polynomial matrix M can be chosen of full column rank. Even more, M can be chosen to be right prime over $\mathbb{R}[\xi]$, equivalently, $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. In that case, in (3) the latent variable ℓ is uniquely determined by the manifest variable w , and the image representation is called *observable*.

In [12], the concept of *rational image representation* was introduced. We will give a brief review here. Let $H(\xi)$ be a real rational matrix, and consider the equation

$$w = H\left(\frac{d}{dt}\right)\ell. \quad (4)$$

Of course (4) should be interpreted as

$$\left(I \quad - H\left(\frac{d}{dt}\right) \right) \begin{pmatrix} w \\ \ell \end{pmatrix} = 0,$$

in the context of (2). If $H = D^{-1}N$ is a left coprime factorization over $\mathbb{R}[\xi]$ then $D^{-1}(D \quad - N)$ is a left coprime factorization of $(I \quad - H)$ and therefore (w, ℓ) satisfies (4) if and only if $D\left(\frac{d}{dt}\right)w = N\left(\frac{d}{dt}\right)\ell$. For a given $\mathfrak{B} \in \mathcal{L}^w$, the representation

$$\mathfrak{B} = \{w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^w) \mid \exists \ell \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet) \text{ s.t. } w = H\left(\frac{d}{dt}\right)\ell\}, \quad (5)$$

with $H \in \mathbb{R}(\xi)^{w \times \bullet}$, is called a *rational image representation*. In that case, we write $\mathfrak{B} = \text{im}(H)$. It was shown in [12] that $\mathfrak{B} \in \mathcal{L}^w$ admits a rational image representation if and only if it is controllable.

III. PROBLEM FORMULATION

In this section, we shall state the problems addressed in this paper.

Problem 1: Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}^w$. Let $G_1, G_2 \in \mathbb{R}(\xi)^{\bullet \times w}$ have full row rank. Let $\mathfrak{B}_1 = \ker(G_1)$ and $\mathfrak{B}_2 = \ker(G_2)$. Find necessary and sufficient conditions on G_1 and G_2 so that $\mathfrak{B}_1 = \mathfrak{B}_2$.

Problem 2: Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}^w$ be controllable. Let $H_1, H_2 \in \mathbb{R}(\xi)^{w \times \bullet}$ have full column rank. Let $\mathfrak{B}_1 = \text{im}(H_1)$ and $\mathfrak{B}_2 = \text{im}(H_2)$. Find necessary and sufficient conditions on H_1 and H_2 so that $\mathfrak{B}_1 = \mathfrak{B}_2$.

IV. EQUIVALENCE OF POLYNOMIAL KERNEL REPRESENTATIONS

In this section, we discuss the equivalence of polynomial kernel representations in a slightly different perspective compared to that discussed in [5], and arrive at conditions which we shall use in addressing the issue of equivalence of rational kernel representations.

Before proceeding, we recall the concepts of autonomous behavior and controllable behavior. We state the following definitions from [5]:

Definition 4.1: A behavior \mathfrak{B} is called autonomous if for all $w_1, w_2 \in \mathfrak{B}$, $w_1(t) = w_2(t)$ for $t \leq 0$ implies $w_1(t) = w_2(t)$ for all t .

Definition 4.2: Let $\mathfrak{B} \in \mathcal{L}^w$. It is called controllable if for any two trajectories $w_1, w_2 \in \mathfrak{B}$, there exists a $t_1 \geq 0$ and a trajectory $w \in \mathfrak{B}$ with the property that $w(t) = w_1(t)$ for $t \leq 0$, and $w(t) = w_2(t - t_1)$ for $t \geq t_1$.

We denote the set of all autonomous linear differential systems with w variables by $\mathcal{L}_{\text{aut}}^w$ and the set of all controllable linear differential systems with w variables by $\mathcal{L}_{\text{contr}}^w$.

Given a behavior $\mathfrak{B} \in \mathcal{L}^w$, it can be decomposed into the direct sum of the controllable part $\mathfrak{B}_{\text{contr}}$, and an autonomous part $\mathfrak{B}_{\text{aut}}$, i.e. $\mathfrak{B} = \mathfrak{B}_{\text{contr}} \oplus \mathfrak{B}_{\text{aut}}$. This is dealt with, in detail in [5]. In fact, it is also shown in [5] that, for a given behavior, an autonomous part is not unique. Let

$$\mathcal{A}(\mathfrak{B}) = \{\mathcal{P} \in \mathcal{L}_{\text{aut}}^w \mid \mathcal{P} \oplus \mathfrak{B}_{\text{contr}} = \mathfrak{B}\} \quad (6)$$

denote the set of all autonomous direct summands of $\mathfrak{B}_{\text{contr}}$ in \mathfrak{B} . Similarly, it is also proved in [5] that, for a given behavior the controllable part is unique. The following theorem interprets the equality of behaviors from a set theoretic point of view.

Lemma 4.3: Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}^w$. Then $\mathfrak{B}_1 = \mathfrak{B}_2$ if and only if

- 1) $\mathfrak{B}_{1,\text{contr}} = \mathfrak{B}_{2,\text{contr}}$ and
- 2) $\mathcal{A}(\mathfrak{B}_1) = \mathcal{A}(\mathfrak{B}_2)$.

Proof: (only if):

This part of the proof is obvious.

(if): We have $\mathfrak{B}_1 = \mathcal{P}_1 \oplus \mathfrak{B}_{1,\text{contr}} = \mathcal{P}_1 \oplus \mathfrak{B}_{2,\text{contr}}$, for some $\mathcal{P}_1 \in \mathcal{A}(\mathfrak{B}_1)$. Since $\mathcal{A}(\mathfrak{B}_1) = \mathcal{A}(\mathfrak{B}_2)$, we have $\mathcal{P}_1 \in \mathcal{A}(\mathfrak{B}_2)$, hence $\mathcal{P}_1 \oplus \mathfrak{B}_{2,\text{contr}} = \mathfrak{B}_2$. \square

Kernel representations of the behaviors in $\mathcal{A}(\mathfrak{B}_1)$ and $\mathcal{A}(\mathfrak{B}_2)$ are discussed in [5]. For the sake of completeness, we shall re-state the following lemma, which describes kernel representations of the controllable as well as the autonomous parts of a given behavior.

Lemma 4.4: Let $\mathfrak{B} \in \mathcal{L}^w$. Let $\mathfrak{B} = \ker(R)$ be a minimal polynomial kernel representation, and let $R = U \begin{bmatrix} D & 0 \end{bmatrix} V$ be a Smith form of R , where $D = \text{diag}(z_1, z_2, \dots, z_r)$ and U, V unimodular over $\mathbb{R}[\xi]$. Then we have:

- 1) $\mathfrak{B}_{\text{contr}} = \ker \left(\begin{bmatrix} I & 0 \end{bmatrix} V \right)$ and
- 2) $\mathcal{P} \in \mathcal{A}(\mathfrak{B})$ if and only if $\mathcal{P} = \ker \left(\begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix} WV \right)$, for some W , unimodular over $\mathbb{R}[\xi]$, satisfying $\begin{bmatrix} D & 0 \end{bmatrix} = \begin{bmatrix} D & 0 \end{bmatrix} W$.

The characterization of W is also dealt with in [5], in Exercise 5.6.

Remark 4.5: Let $\mathfrak{B} = \ker(R)$ be a minimal polynomial kernel representation, and let $R = U \begin{bmatrix} D & 0 \end{bmatrix} V$ be a Smith form of R , where $D = \text{diag}(z_1, z_2, \dots, z_r)$ and U, V unimodular over $\mathbb{R}[\xi]$. From the above theorem, it is clear that $\mathcal{P} \in \mathcal{A}(\mathfrak{B})$ if and only if it admits a minimal polynomial kernel representation

$$\ker \left(\begin{bmatrix} D & 0 \\ F & S \end{bmatrix} V \right), \quad (7)$$

where F is an arbitrary polynomial matrix of appropriate dimensions, and S is an arbitrary unimodular matrix over $\mathbb{R}[\xi]$.

Equivalence of polynomial kernel representations has been dealt with in [5] before. We recall the following proposition from [5], Theorem 3.6.2:

Proposition 4.6: Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}^w$. Let $R_1, R_2 \in \mathbb{R}[\xi]^{\bullet \times w}$ be such that $R_1 \left(\frac{d}{dt} \right) w = 0$ and $R_2 \left(\frac{d}{dt} \right) w = 0$ are minimal polynomial kernel representations of \mathfrak{B}_1 and \mathfrak{B}_2 respectively. Then $\mathfrak{B}_1 = \mathfrak{B}_2$ if and only if there exists a unimodular polynomial matrix U such that $R_1 = UR_2$.

In order to proceed, we have the following theorem:

Theorem 4.7: Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}^w$. Let $\mathfrak{B}_1 = \ker(R_1)$ and $\mathfrak{B}_2 = \ker(R_2)$ be minimal polynomial kernel representations of \mathfrak{B}_1 and \mathfrak{B}_2 respectively. Then $\mathfrak{B}_{1,\text{contr}} = \mathfrak{B}_{2,\text{contr}}$ if and only if there exist $M, N \in \mathbb{R}[\xi]^{\bullet \times \bullet}$, square and non-singular, such that $MR_1 = NR_2$.

Proof: We skip the proof due to space limitations. \square

From Lemma 4.3, it is clear that the above theorem gives only a necessary condition in terms of R_1 and R_2 for $\mathfrak{B}_1 = \mathfrak{B}_2$. The following theorem is the main result of this section. It gives additional conditions on the M and N of Theorem 4.7 so that the sets of autonomous parts are also equal.

Theorem 4.8: Let $\mathfrak{B}_i \in \mathcal{L}^w$, $\mathfrak{B}_i = \ker(R_i)$, for $i=1,2$ be minimal polynomial kernel representations. Let $\mathfrak{B}_{1,\text{contr}} = \mathfrak{B}_{2,\text{contr}}$, and let $M, N \in \mathbb{R}[\xi]^{\bullet \times \bullet}$ be square and non-singular such that $MR_1 = NR_2$. Then $\mathcal{A}(\mathfrak{B}_1) = \mathcal{A}(\mathfrak{B}_2)$ if and only if $M^{-1}N$ is unimodular over $\mathbb{R}[\xi]$.

Proof: Let $R_1 = U_1 \begin{bmatrix} D_1 & 0 \end{bmatrix} V_1$ and $R_2 = U_2 \begin{bmatrix} D_2 & 0 \end{bmatrix} V_2$ be Smith forms of R_1 and R_2 respectively. From Lemma 4.4, we have $\mathfrak{B}_{1,\text{contr}} = \ker \left(\begin{bmatrix} I & 0 \end{bmatrix} V_1 \right)$, and $\mathfrak{B}_{2,\text{contr}} = \ker \left(\begin{bmatrix} I & 0 \end{bmatrix} V_2 \right)$.

(only if): Let $MR_1 = NR_2$. Assume $\mathcal{A}(\mathfrak{B}_1) = \mathcal{A}(\mathfrak{B}_2)$. From Lemma 4.3 it is clear that $\mathfrak{B}_1 = \mathfrak{B}_2$, therefore from Proposition 4.6 we have $R_1 = UR_2$, where U is unimodular

over $\mathbb{R}[\xi]$. Further we have $R_1 = M^{-1}NR_2$. Since R_1 and R_2 are minimal kernel representations, it is clear that $U = M^{-1}N$.

(if): This part of proof is more involved. We skip the details due to space limitations. \square

By combining Theorem 4.7 and Theorem 4.8 we now obtain the following corollary:

Corollary 4.9: Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}^w$. Let $\mathfrak{B}_1 = \ker(R_1)$, and $\mathfrak{B}_2 = \ker(R_2)$ be minimal polynomial kernel representations. Then $\mathfrak{B}_1 = \mathfrak{B}_2$ if and only if there exist nonsingular polynomial matrices M, N such that $MR_1 = NR_2$ and $M^{-1}N$ is unimodular over $\mathbb{R}[\xi]$.

Oviously, this is a restatement of Proposition 4.6. However, it shows the origin of the unimodular matrix U . The corollary is derived in two stages. Firstly, it is shown that equality of the controllable parts of a given behavior is equivalent to the existence of square and non-singular matrices M and N . Secondly, unimodularity of $M^{-1}N$ is equivalent to equality of the sets of autonomous parts of the behavior.

V. EQUIVALENCE OF RATIONAL KERNEL REPRESENTATIONS

In this section we address the question of equivalence of minimal rational kernel representations. We will first recall the concepts of polynomial and rational annihilators of a given behavior from [12], Section 7.

Definition 5.1: Let $\mathfrak{B} \in \mathcal{L}^w$.

- 1) $n \in \mathbb{R}[\xi]^{1 \times w}$ is called a *polynomial annihilator* of \mathfrak{B} if $n(\frac{d}{dt})w = 0$ for all $w \in \mathfrak{B}$.
- 2) $n \in \mathbb{R}(\xi)^{1 \times w}$ is called a *rational annihilator* of \mathfrak{B} if $n(\frac{d}{dt})w = 0$ for all $w \in \mathfrak{B}$.

We denote the set of polynomial annihilators of $\mathfrak{B} \in \mathcal{L}^w$ by $\mathfrak{B}^{\perp_{\mathbb{R}[\xi]}}$. The set of rational annihilators of \mathfrak{B} is denoted by $\mathfrak{B}^{\perp_{\mathbb{R}(\xi)}}$. The set of rational annihilators of $\mathfrak{B}_{\text{contr}}$ is denoted by $(\mathfrak{B}_{\text{contr}})^{\perp_{\mathbb{R}(\xi)}}$. It is a well-known result that for $\mathfrak{B} \in \mathcal{L}^w$, $\mathfrak{B}^{\perp_{\mathbb{R}[\xi]}}$ is a finitely generated submodule of the $\mathbb{R}[\xi]$ -module $\mathbb{R}[\xi]^{1 \times w}$. Moreover, if $\mathfrak{B} = \ker(G)$ is a minimal polynomial kernel representation, then this submodule is generated by the rows of G . In the context of rational representations one needs to impose controllability:

Theorem 5.2: Let $\mathfrak{B} \in \mathcal{L}^w$. Then $\mathfrak{B}^{\perp_{\mathbb{R}(\xi)}}$ is a subspace of the $\mathbb{R}(\xi)$ -linear vector space $\mathbb{R}(\xi)^{1 \times w}$ if and only if \mathfrak{B} is controllable. As a consequence, $(\mathfrak{B}_{\text{contr}})^{\perp_{\mathbb{R}(\xi)}}$ is a subspace of the $\mathbb{R}(\xi)$ -linear vectorspace $\mathbb{R}(\xi)^{1 \times w}$. If $G(\frac{d}{dt})w = 0$ is a minimal rational kernel representation of \mathfrak{B} , then the rows of G form a basis of $(\mathfrak{B}_{\text{contr}})^{\perp_{\mathbb{R}(\xi)}}$.

Proof: We skip the proof due to space limitations. \square

The following theorem is an immediate consequence of the above Theorem:

Theorem 5.3: Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}^w$. Let $\mathfrak{B}_1 = \ker(G_1)$ and $\mathfrak{B}_2 = \ker(G_2)$ be minimal rational kernel representations. Then the following statements are equivalent:

- 1) $\mathfrak{B}_{1,\text{contr}} = \mathfrak{B}_{2,\text{contr}}$.

- 2) There exists a nonsingular rational matrix W such that $G_1 = WG_2$.

- 3) There exist nonsingular polynomial matrices M and N such that $MG_1 = NG_2$.

Proof: The equivalence of (2) and (3) is obvious. We first prove the implication (1) \Rightarrow (2). As $\mathfrak{B}_{1,\text{contr}} = \mathfrak{B}_{2,\text{contr}}$ we have $(\mathfrak{B}_{1,\text{contr}})^{\perp_{\mathbb{R}(\xi)}} = (\mathfrak{B}_{2,\text{contr}})^{\perp_{\mathbb{R}(\xi)}} =: \mathcal{L}$. From Lemma 5.2, the rows of G_1 and G_2 both form a basis for the subspace \mathcal{L} of $\mathbb{R}(\xi)^{1 \times w}$. Then, from basic linear algebra, there exists a square, nonsingular rational matrix W such that $G_1 = WG_2$.

Conversely, assume $G_1 = WG_2$. Let $G_1 = P_1^{-1}Q_1$ and $G_2 = P_2^{-1}Q_2$ be left coprime factorizations over $\mathbb{R}[\xi]$ of G_1 and G_2 . Let $W = M^{-1}N$ be a left coprime factorization over $\mathbb{R}[\xi]$ of W . Then both M and N are nonsingular. By definition we have $\mathfrak{B}_1 = \ker(Q_1)$ and $\mathfrak{B}_2 = \ker(Q_2)$. Then,

$$\begin{aligned} G_1 = WG_2 &\iff P_1^{-1}Q_1 = M^{-1}NP_2^{-1}Q_2 \\ &\iff Q_1 = P_1M^{-1}NP_2^{-1}Q_2 \end{aligned}$$

Now factorize $P_1M^{-1}NP_2^{-1} = \tilde{M}^{-1}\tilde{N}$. Then we have $\tilde{M}Q_1 = \tilde{N}Q_2$. From Theorem 4.7, (1) follows. \square

Evidently the above Theorem only gives a necessary condition on G_1 and G_2 for their behaviors to be equal. However, we would like to obtain conditions that are necessary and sufficient.

In case of polynomial kernel representations, statement 3 of the above Theorem 5.3 together with unimodularity of $M^{-1}N$ serves the purpose. Hence, a first guess is to check whether this also holds true for rational representations. However, the following simple counter example shows this is not the case.

Example 5.4: $G_1(\xi) = 1$ and $G_2(\xi) = \frac{1}{\xi}$. These are equivalent representations since they both represent the $\{0\}$ -behavior. For all M, N such that $MG_1 = NG_2$, we have $M^{-1}N = \frac{1}{\xi}$, which is not even a polynomial.

In order to proceed we need following definition:

Definition 5.5: A *greatest common left divisor (gclid)* of two matrices $P, Q \in \mathbb{R}[\xi]^{m \times \bullet}$ is any square polynomial matrix D such that $P = DP_1$ and $Q = DQ_1$, and with the property that for all square polynomial matrices D_1 such that $P = D_1P_1$ and $Q = D_1Q_1$, there exists F such that $D = D_1F$.

If $\begin{bmatrix} P & Q \end{bmatrix}$ has full row rank, then their gclid is a nonsingular polynomial matrix. In that case any two gclids are related by post-multiplication with a unimodular matrix over $\mathbb{R}[\xi]$.

Now, the following Theorem is the first main result of this paper. The Theorem states that the additional conditions on M and N so that the autonomous parts of $\ker(G_1)$ and $\ker(G_2)$ are also equal involves the greatest common left divisor matrices $\text{gclid}(M, MG_1)$ and $\text{gclid}(N, NG_2)$. More precisely:

Theorem 5.6: Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}^w$. Let $\mathfrak{B}_1 = \ker(G_1)$ and $\mathfrak{B}_2 = \ker(G_2)$ be minimal rational kernel representations. Assume $\mathfrak{B}_{1,\text{contr}} = \mathfrak{B}_{2,\text{contr}}$, and let $M, N \in \mathbb{R}[\xi]^{\bullet \times \bullet}$ be square and nonsingular such that $MG_1 = NG_2$. Then we have $\mathcal{A}(\mathfrak{B}_1) = \mathcal{A}(\mathfrak{B}_2)$ if

and only if MG_1 and NG_2 are polynomial matrices, and $\text{gcd}(M, MG_1)^{-1}\text{gcd}(N, NG_2)$ is unimodular over $\mathbb{R}[\xi]$.

Proof: (only if): Let $G_i = U_i \Pi_i^{-1} \begin{bmatrix} D_i & 0 \\ F_i & S_i \end{bmatrix} V_i$, for $i=1,2$ be Smith-McMillan forms of G_1 and G_2 respectively, where U_1, U_2, V_1, V_2 are unimodular matrices over $\mathbb{R}[\xi]$, $D_i = \text{diag}(z_{1i}, z_{2i}, \dots, z_{ri})$ and $\Pi_i = \text{diag}(\pi_{1i}, \pi_{2i}, \dots, \pi_{ri})$ for $i=1,2$. Let $MG_1 = NG_2$. Assume $\mathcal{A}(\mathfrak{B}_1) = \mathcal{A}(\mathfrak{B}_2)$. Then from Remark 4.5, $\mathcal{P} \in \mathcal{A}(\mathfrak{B}_1)$ admits a polynomial kernel representation $\ker \left(\begin{bmatrix} D_1 & 0 \\ F_1 & S_1 \end{bmatrix} V_1 \right)$, similarly it also admits a polynomial kernel representation $\ker \left(\begin{bmatrix} D_2 & 0 \\ F_2 & S_2 \end{bmatrix} V_2 \right)$, where F_1, F_2 are arbitrary polynomial matrices of appropriate dimensions and S_1, S_2 are unimodular matrices over $\mathbb{R}[\xi]$. From Proposition 4.6, there exists a U , unimodular over $\mathbb{R}[\xi]$, such that

$$\begin{bmatrix} D_1 & 0 \\ F_1 & S_1 \end{bmatrix} V_1 = U \begin{bmatrix} D_2 & 0 \\ F_2 & S_2 \end{bmatrix} V_2 \text{ holds.}$$

It is easily verified that U must be of form $U = \begin{bmatrix} U_{11} & 0 \\ U_{21} & U_{22} \end{bmatrix}$, where U_{11} and U_{22} are unimodular over $\mathbb{R}[\xi]$. Therefore we have

$$\begin{bmatrix} D_1 & 0 \\ F_1 & S_1 \end{bmatrix} V_1 = \begin{bmatrix} U_{11} & 0 \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} D_2 & 0 \\ F_2 & S_2 \end{bmatrix} V_2,$$

which implies

$$\begin{aligned} & \begin{bmatrix} \Pi_1 U_1^{-1} U_1 \Pi_1^{-1} \begin{bmatrix} D_1 & 0 \\ F_1 & S_1 \end{bmatrix} V_1 \\ \begin{bmatrix} D_1 & 0 \\ F_1 & S_1 \end{bmatrix} V_1 \end{bmatrix} \\ &= \begin{bmatrix} U_{11} & 0 \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} \Pi_2 U_2^{-1} U_2 \Pi_2^{-1} \begin{bmatrix} D_2 & 0 \\ F_2 & S_2 \end{bmatrix} V_2 \\ \begin{bmatrix} D_2 & 0 \\ F_2 & S_2 \end{bmatrix} V_2 \end{bmatrix} \end{aligned}$$

Define $M := \Pi_1 U_1^{-1}$ and $N := U_{11} \Pi_2 U_2^{-1}$. Then we have

$$\begin{aligned} & \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} G_1 \\ [F_1 \ S_1] V_1 \end{bmatrix} \\ &= \begin{bmatrix} N & 0 \\ U_{21} \Pi_2 U_2^{-1} & U_{22} \end{bmatrix} \begin{bmatrix} G_2 \\ [F_2 \ S_2] V_2 \end{bmatrix}. \end{aligned}$$

It is evident from the above equation that MG_1 and NG_2 are polynomial matrices such that $MG_1 = NG_2$ holds. Define $L = MG_1 = NG_2$. Then we have $\text{gcd}(M, L) = I =: R_1$, and similarly $\text{gcd}(N, L) = U_{11} =: R_2$. Hence, it is evident that $R_1^{-1} R_2 = U_{11}$ is unimodular over $\mathbb{R}[\xi]$.

(if): Let $\text{gcd}(M, L) = R_1$ and $\text{gcd}(N, L) = R_2$. Let $G_1 = P_1^{-1} Q_1$ and $G_2 = P_2^{-1} Q_2$ be left coprime factorizations of G_1 and G_2 over $\mathbb{R}[\xi]$. Let $MG_1 = NG_2$ be a polynomial matrix. Denote it by L . It can be shown that $L = \tilde{R}_1 Q_1$, and similarly $L = \tilde{R}_2 Q_2$, where $\tilde{R}_1, \tilde{R}_2 \in \mathbb{R}[\xi]^{\bullet \times \bullet}$ are square and nonsingular. Further it can be verified easily that \tilde{R}_1 and \tilde{R}_2 are the gcds of $\begin{bmatrix} M & L \end{bmatrix}$ and $\begin{bmatrix} N & L \end{bmatrix}$ respectively. Also, since M and N are square and nonsingular, $\begin{bmatrix} M & L \end{bmatrix}$ and $\begin{bmatrix} N & L \end{bmatrix}$ have full row rank. Consequently, we have R_1 and R_2 nonsingular. Therefore there exists polynomial unimodular matrices U_1 and U_2 such that $R_1 = \tilde{R}_1 U_1$ and $R_2 = \tilde{R}_2 U_2$. Define $\tilde{M} := R_1^{-1} U_1$, $\tilde{N} := R_2^{-1} U_2$. Then we have, $\tilde{M} Q_1 = \tilde{N} Q_2$, and $\tilde{M}^{-1} \tilde{N} =$

U , which is unimodular over $\mathbb{R}[\xi]$. Therefore, from Lemma 4.8, we have $\mathcal{A}(\mathfrak{B}_1) = \mathcal{A}(\mathfrak{B}_2)$. \square

The following Theorem is our second main result. It gives necessary and sufficient conditions on the rational matrices G_1 and G_2 for $\ker(G_1)$ and $\ker(G_2)$ to be equal. In fact by combining Theorems 5.3 and 5.6 we obtain:

Theorem 5.7: Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}^w$. Let $\mathfrak{B}_1 = \ker(G_1)$ and $\mathfrak{B}_2 = \ker(G_2)$ be minimal rational kernel representations. Then $\mathfrak{B}_1 = \mathfrak{B}_2$ if and only if there exist $M, N \in \mathbb{R}[\xi]^{\bullet \times \bullet}$, square and non-singular such that

- 1) $MG_1 = NG_2$ is a polynomial matrix and
- 2) $\text{gcd}(M, MG_1)^{-1}\text{gcd}(N, NG_2)$ is unimodular over $\mathbb{R}[\xi]$.

Theorem 5.7 is illustrated below in the following examples.

Example 1: $G_1(\xi) = 1$, $G_2(\xi) = \frac{1}{\xi}$ represent the same behavior:

- 1) $MG_1 = NG_2$ with $N(\xi) = \xi$, $M(\xi) = 1$ nonsingular polynomial,
- 2) $MG_1 = NG_2 = 1$ is polynomial and $\text{gcd}(N, NG_2) = \text{gcd}(\xi, 1) = 1$, $\text{gcd}(M, MG_1) = \text{gcd}(1, 1) = 1$

Example 2: $G_1(\xi) = (\xi \ \xi)$, $G_2(\xi) = (\frac{1}{\xi} \ \frac{1}{\xi})$ do not represent the same behavior:

- 1) their controllable parts are the same: $MG_1 = NG_2$ with $N(\xi) = \xi^2$, $M(\xi) = 1$ nonsingular polynomial,
- 2) for any M, N such that $MG_1 = NG_2$ we must have $N(\xi) = \xi^2 M(\xi)$. Hence $\text{gcd}(M, MG_1) = \text{gcd}(M, \xi M, \xi M) = M$, while $\text{gcd}(N, NG_2) = \text{gcd}(\xi^2 M, \xi M, \xi M) = \xi M$.

Remark 5.8: We note that, in the case that G_1 and G_2 are polynomial matrices, Theorem 5.7 immediately yields Corollary 4.9. Indeed, in that case $\text{gcd}(M, MG_1) = M$ and $\text{gcd}(N, NG_2) = N$ so condition (2) becomes: $M^{-1}N$ is unimodular over $\mathbb{R}[\xi]$.

VI. EQUIVALENCE OF RATIONAL IMAGE REPRESENTATIONS

A given behavior $\mathfrak{B} \in \mathcal{L}^w$ admits a polynomial image representation if and only if it is controllable. In fact, we quote Theorem 9 from [12]:

Theorem 6.1: Let $\mathfrak{B} \in \mathcal{L}^w$. Then the following statements are equivalent:

- 1) \mathfrak{B} is controllable,
- 2) \mathfrak{B} admits a polynomial image representation,
- 3) \mathfrak{B} admits a polynomial image representation $\mathfrak{B} = \text{im}(M)$ with $M \in \mathbb{R}[\xi]^{\bullet \times \bullet}$ right prime over $\mathbb{R}[\xi]$,
- 4) \mathfrak{B} admits a rational image representation,

We will now study the problem of equivalence of image representations. For this, the following result will be useful. The result states that right coprime factorization of a rational image representation leads to a polynomial image representation.

Lemma 6.2: Let $\mathfrak{B} \in \mathcal{L}_{\text{contr}}^w$. Let $H \in R(\xi)^{\bullet \times \bullet}$ be such that $\mathfrak{B} = \text{im}(H)$. Let $H = MP^{-1}$ be a right coprime factorization over $\mathbb{R}[\xi]$. Then $\mathfrak{B} = \text{im}(M)$.

Proof: We skip the proof due to space limitations. \square

We will now first study the question under which conditions two polynomial image representations are equivalent, i. e. represent the same behavior.

Theorem 6.3: 1) Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{contr}}^w$. Let $M_1, M_2 \in \mathbb{R}[\xi]^{w \times \bullet}$ have full column rank, and be such that $\mathfrak{B}_1 = \text{im}(M_1)$ and $\mathfrak{B}_2 = \text{im}(M_2)$. Then $\mathfrak{B}_1 = \mathfrak{B}_2$ if and only if there exists a nonsingular rational matrix R such that $M_2 = M_1 R$.

2) Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{contr}}^w$. Let $M_1, M_2 \in \mathbb{R}[\xi]^{w \times \bullet}$ be right prime over $\mathbb{R}[\xi]$, and such that $\mathfrak{B}_1 = \text{im}(M_1)$ and $\mathfrak{B}_2 = \text{im}(M_2)$. Then $\mathfrak{B}_1 = \mathfrak{B}_2$ if and only if there exists a unimodular polynomial matrix U such that $M_2 = M_1 U$.

Proof: We first prove the 'only if' part of statement 2. By right primeness, both $M_1(\lambda)$ and $M_2(\lambda)$ have full column rank for all $\lambda \in \mathbb{C}$, so correspond to observable image representations. From $\mathfrak{B}_1 = \mathfrak{B}_2$ it follows that also the orthogonal complements coincide, i. e. $\mathfrak{B}_1^\perp = \mathfrak{B}_2^\perp$ (see [11]). By observability we have $\mathfrak{B}_i^\perp = \ker(M_i^\sim)$, where $M_i^\sim(\xi) := M_i^\top(-\xi)$ ($i = 1, 2$). By Proposition 4.6 there exists a unimodular polynomial matrix V such that $M_2^\sim = V M_1^\sim$. This implies $M_2 = M_1 U$, with $U := V^\sim$ again unimodular.

Next, we prove the 'only if' part of statement 1. Both M_1 and M_2 have full column rank. Hence, we can factorize $M_i = \overline{M}_i R_i$, with \overline{M}_i right prime over $\mathbb{R}[\xi]$ and R_i a nonsingular polynomial matrix ($i = 1, 2$). By nonsingularity, $R_i(\frac{d}{dt})$ is surjective, and therefore $\text{im}(M_i) = \text{im}(\overline{M}_i)$ ($i = 1, 2$). Consequently, $\mathfrak{B}_1 = \mathfrak{B}_2$ implies $\text{im}(\overline{M}_1) = \text{im}(\overline{M}_2)$. Then, by the 'only if' part of statement 2, there exists a unimodular polynomial matrix U such that $\overline{M}_2 = \overline{M}_1 U$. This implies $M_2 = M_1 R$, with $R := R_1^{-1} U R_2$.

Finally we prove the 'if' part of statement 1. Assume that $M_2 = M_1 R$ with R a nonsingular rational matrix. Let $R = K L^{-1}$ be a right coprime factorization of R over $\mathbb{R}[\xi]$. Then we have $M_2 L = M_1 K$, with K and L nonsingular polynomial matrices. Again by surjectivity of $L(\frac{d}{dt})$ and $K(\frac{d}{dt})$, we obtain $\mathfrak{B}_1 = \text{im}(M_1) = \text{im}(M_1 K) = \text{im}(M_2 L) = \text{im}(M_2) = \mathfrak{B}_2$. This also proves the 'if' part of statement 2. \square

Next, we consider controllable behaviors represented by rational image representations.

Theorem 6.4: Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{contr}}^w$. Let $H_1, H_2 \in \mathbb{R}(\xi)^{w \times \bullet}$ have full column rank, and be such that $\mathfrak{B}_1 = \text{im}(H_1)$ and $\mathfrak{B}_2 = \text{im}(H_2)$. Then $\mathfrak{B}_1 = \mathfrak{B}_2$ if and only if there exists a nonsingular rational matrix R such that $H_2 = H_1 R$.

Proof: Let $H_i = M_i P_i^{-1}$ be a right coprime factorization over $\mathbb{R}[\xi]$. Then by Lemma 6.2, $\mathfrak{B}_i = \text{im}(M_i)$ ($i = 1, 2$). By Theorem 6.3, $\mathfrak{B}_1 = \mathfrak{B}_2$ implies that there exists a nonsingular rational matrix \overline{R} such that $M_2 = M_1 \overline{R}$. Thus $H_2 = H_1 R$, with $R := P_1 \overline{R} P_2^{-1}$ nonsingular. Conversely,

if $H_2 = H_1 R$ then $M_2 = M_1 P_1^{-1} R P_2$. Then, by Theorem 6.3, $\text{im}(M_1) = \text{im}(M_2)$ so $\mathfrak{B}_1 = \mathfrak{B}_2$. \square

VII. CONCLUSION

In this paper we have addressed the question of equivalence of rational representations of a given behavior. We have obtained necessary and sufficient conditions for the equivalence of rational kernel representations of controllable as well as uncontrollable behaviors. We also have derived new conditions for the equivalence of polynomial kernel representations of a given behavior in terms of the controllable parts and the sets of autonomous parts of the behaviors associated with the two kernel representations. Further we have obtained necessary and sufficient conditions for the equivalence of image representations in the context of both polynomial and rational representations.

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