Stability analysis and controller design for a linear system with Duhem hysteresis nonlinearity

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Stability analysis and controller design for a linear system with Duhem hysteresis nonlinearity

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***Abstract*— In this paper, we investigate the stability of feedback interconnections between a linear system and a Duhem hysteresis operator, where the linear system satisfies either counter-clockwise (CCW) or clockwise (CW) input-output dynamics [1], [13]. More precisely, depending on the input-output dynamics of each system, we present sufficient conditions on the linear system that guarantee the stability of the closed-loop systems. Based on these results we introduce a control design methodology for stabilizing a linear plant with a counterclockwise Duhem hysteresis operator.**

I. Introduction

Hysteresis is a common phenomenon that presents in diverse systems, such as piezo-actuator, ferromagnetic material and mechanical systems. Normally, hysteresis is defined as a nonlinear function with memory, which can not be represented by a single-valued function. For describing hysteretic phenomena in many different physical systems, several hysteresis models have been proposed in the literature, see, for example, [4], [11], [9]. These includes backlash model which is used to describe gear trains; Preisach model for modeling the ferromagnetic systems and elastic-plastic model which is used to study mechanical friction [4], [11]. From the perspective of input-output behavior, the hysteresis phenomena can have counterclockwise (CCW) input-output (I/O) dynamics [1], clockwise (CW) I/O dynamics [13], or even more complex I/O map (such as, butterfly map [3]). For example, backlash model generates CCW hysteresis loops; elastic-plastic model generates CW hysteresis loops and Preisach model can generate CCW, CW or butterfly hysteresis loops depending on the weight of the hysterons which are used in the Preisach model.

In recent work by Angeli [1] counterclockwise (CCW) input-output (I/O) dynamics is characterized by

$$\liminf_{T\to\infty} \int_0^T \dot{y}(t)^T u(t)dt > -\infty,$$

where u is the input signal and y is the corresponding output signal. The integral represents the signed area enclosed by the curve (u, y). Compare with the classical definition of passivity in system theory [17], it can be interpreted as the system is passive from the input u to the time derivative of the corresponding output y.

In our previous results [7], we show that for a certain class of Duhem hysteresis operator $\Phi : AC(\mathbb{R}_+) \times \mathbb{R} \to AC(\mathbb{R}_+)$, there exists a storage function $H_{\circlearrowleft} : \mathbb{R}^2 \to \mathbb{R}_+$ which satisfies

$$\frac{\mathrm{d}H_{\circlearrowleft}(y_\Phi(t), u_\Phi(t))}{\mathrm{d}t} \le \dot{y}_\Phi(t)u_\Phi(t), \qquad (1)$$

where AC is the class of absolutely continuous functions, $u_\Phi \in AC(\mathbb{R}_+)$, $y_\Phi = \Phi(u_\Phi, y_{\Phi_0})$ and $y_{\Phi_0} \in \mathbb{R}$ is the initial condition. This inequality also implies that the Duhem hysteresis operator has CCW input-output dynamics, where we will discuss it in detail in Section II. Here, we use the symbol $\circlearrowleft$ in $H_{\circlearrowleft}$ to indicate the counterclockwise behavior of Φ.

In this paper, we exploit our knowledge on $H_{\circlearrowleft}$ to study the stability of an interconnected system as shown in Figure 1, where $\mathbf{P}$ is a linear system which can be either CW or CCW and Φ is the hysteresis operator. In Theorem 3.1 of this paper, a negative feedback interconnection between $\mathbf{P}$ and Φ is considered and we give sufficient conditions on $\mathbf{P}$ that ensure the stability of the closed-loop system. The conditions are related to the fact that $\mathbf{P}$ should be CW. On the other hand, in Theorem 3.2, we consider a positive feedback interconnection, and sufficient conditions on $\mathbf{P}$ and Φ are given such that the closed-loop system is stable. In this case, the conditions on $\mathbf{P}$ are related to the CCW property of $\mathbf{P}$ and the condition on Φ is related to the sector bound condition on the anhysteresis function of Φ. Based on these results, we present in Section IV a control design methodology that deals with a linear plant and a hysteretic actuator/sensor Φ.

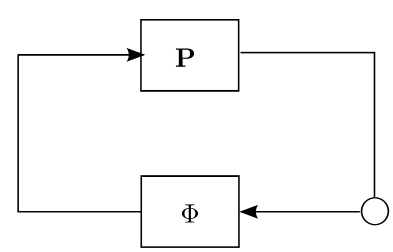

Fig. 1. Feedback interconnection between a linear plant $\mathbf{P}$ and a Duhem operator Φ.

II. Preliminaries

In this section we give the definitions of the CCW and CW dynamics which are based on the work by Padthe [13] and Angeli [1] and give a brief review on the Duhem hysteresis operator and its dissipativity property following our results in [7]. We denote $AC(\mathbb{R}_+, \mathbb{R}^n)$ the space of absolutely continuous function $f : \mathbb{R}_+ \to \mathbb{R}^n$.

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A. Counterclockwise dynamics

Definition 2.1: [1], [13] A (nonlinear) map $G : AC(\mathbb{R}_+, \mathbb{R}^m) \to AC(\mathbb{R}_+, \mathbb{R}^m)$ is *counterclockwise* (CCW) if for every $u \in AC(\mathbb{R}_+, \mathbb{R}^m)$ with the corresponding output map $y := Gu$, the following inequality holds

$$\liminf_{T\to\infty} \int_0^T \langle \dot{y}(t), u(t)\rangle dt > -\infty. \tag{2}$$

For a nonlinear operator G, inequality (2) holds if there exists a function $H : \mathbb{R}^2 \to \mathbb{R}_+$ such that for every input signal u, the inequality

$$\frac{\mathrm{d}H(y(t), u(t))}{\mathrm{d}t} \leq \langle \dot{y}(t), u(t)\rangle, \tag{3}$$

holds for almost every t where the output signal $y := Gu$. Note that the range of G is $AC(\mathbb{R}_+, \mathbb{R}^m)$, thus $\dot{y}$ is measurable.

Definition 2.2: A (nonlinear) map $G : AC(\mathbb{R}_+, \mathbb{R}^m) \to AC(\mathbb{R}_+, \mathbb{R}^m)$ is *strictly-input counterclockwise* (SI-CCW), if for every input $u \in AC(\mathbb{R}_+, \mathbb{R}^m)$, there exists a constant $\epsilon > 0$ such that the inequality

$$\liminf_{T\to\infty} \int_0^T \langle \dot{y}(t),\ u(t)\rangle - \epsilon\|u(t)\|^2 \mathrm{d}t > -\infty, \tag{4}$$

holds where $y := Gu$.

Definition 2.3: A (nonlinear) map $G : AC(\mathbb{R}_+, \mathbb{R}^m) \to AC(\mathbb{R}_+, \mathbb{R}^m)$ is *strictly counterclockwise* (S-CCW) (see also [1]), if for every input $u \in AC(\mathbb{R}_+, \mathbb{R}^m)$, there exists a constant $\delta > 0$ such that the inequality

$$\liminf_{T\to\infty} \int_0^T \langle \dot{y}(t),\ u(t)\rangle - \delta\|\dot{y}(t)\|^2 \mathrm{d}t > -\infty, \tag{5}$$

holds where $y := Gu$.
Note that for a system described by the state space representation as follows:

$$\Sigma : \left.\begin{array}{lll} \dot{x} &= f(x, u), & x(0) = x_0 \\ y &= h(x), & \end{array}\right\} \tag{6}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input and $y \in \mathbb{R}^m$ is the output and f, h are sufficiently smooth functions we could have the following lemma.

Lemma 2.4: Consider the state space system Σ as in (6). If there exists $H : \mathbb{R}^n \to \mathbb{R}_+$, $\epsilon \geq 0$ and $\delta \geq 0$, such that

$$\frac{\partial H(x)}{\partial x} f(x, u) \leq \left\langle \frac{\partial h(x)}{\partial x} f(x, u), u \right\rangle - \epsilon\|u\|^2 - \delta \left\| \frac{\partial h(x)}{\partial x} f(x, u) \right\|^2,$$

holds for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, then Σ is CCW. Moreover if $\epsilon > 0$, it is SI-CCW and if $\delta > 0$, it is S-CCW.

B. Clockwise dynamics

Dual to the concept of counterclockwise I/O dynamics, the notion of clockwise I/O dynamics can be defined as follows.

Definition 2.5: [13] A (nonlinear) map $G : AC(\mathbb{R}_+, \mathbb{R}^m) \to AC(\mathbb{R}_+, \mathbb{R}^m)$ is *clockwise* (CW) if for every input $u \in AC(\mathbb{R}_+, \mathbb{R}^m)$ with the corresponding output map $y := Gu$, the following inequality holds:

$$\liminf_{T\to\infty} \int_0^T y(t)^T \dot{u}(t) \mathrm{d}t > -\infty. \tag{7}$$

For a nonlinear operator G, inequality (7) holds if there exists a function $H : \mathbb{R}^2 \to \mathbb{R}_+$ such that for every input signal $u \in AC(\mathbb{R}_+, \mathbb{R}^m)$, the inequality

$$\frac{\mathrm{d}H(y(t), u(t))}{\mathrm{d}t} \leq \langle y(t), \dot{u}(t)\rangle, \tag{8}$$

holds for a.e. t where the output signal $y := Gu$.

Lemma 2.6: Consider the state space system Σ as in (6). If there exist α, $V : \mathbb{R}^{m+n} \to \mathbb{R}_+$ such that α is positive semi-definite, V is positive definite and proper, and

$$\begin{bmatrix} \frac{\partial V(z,x)}{\partial z} & \frac{\partial V(z,x)}{\partial x} \end{bmatrix} \begin{bmatrix} v \\ f(x, z) \end{bmatrix} \leq \langle h(x), v\rangle - \alpha(x, z), \tag{9}$$

holds for all $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$ and $v \in \mathbb{R}^m$, then Σ is CW.

PROOF. Define the extended state space system (6) as follows

$$\begin{array}{ll} \dot{z} &= v, \\ \dot{x} &= f(x, z), \\ y &= h(x). \end{array} \tag{10}$$

Note that z defines the extended input in (6). It follows from (9) and (10) that

$$\begin{aligned} \dot{V} &\leq \langle h(x), v\rangle - \epsilon\|h(x)\|^2, \\ &= \langle y, \dot{z}\rangle - \epsilon\|y\|^2, \end{aligned}$$

which completes our proof by taking $z = u$ and $\alpha(x, z) = \epsilon\|h(x)\|^2 = \epsilon\|y\|^2$. □

C. Duhem Hysteresis operator

The Duhem operator $\Phi : AC(\mathbb{R}_+) \times \mathbb{R} \to AC(\mathbb{R}_+), (u_\Phi, y_{\Phi_0}) \mapsto \Phi(u_\Phi, y_{\Phi_0}) =: y_\Phi$ is described by ([11], [12], [16])

$$\dot{y}_\Phi(t) = f_1(y_\Phi(t), u_\Phi(t))\dot{u}_{\Phi+}(t) + f_2(y_\Phi(t), u_\Phi(t))\dot{u}_{\Phi-}(t), \quad y_\Phi(0) = y_{\Phi_0}, \tag{11}$$

where $\dot{u}_{\Phi+}(t) := \max\{0, \dot{u}_\Phi(t)\}$, $\dot{u}_{\Phi-}(t) := \min\{0, \dot{u}_\Phi(t)\}$ and $f_1 : \mathbb{R}^2 \to \mathbb{R}$, $f_2 : \mathbb{R}^2 \to \mathbb{R}$ are sufficiently smooth functions.

An equivalent representation of f_1 and f_2 is

$$\left.\begin{array}{ll} f_1(y_\Phi(t), u_\Phi(t)) &= F(y_\Phi(t), u_\Phi(t)) + G(y_\Phi(t), u_\Phi(t)), \\ f_2(y_\Phi(t), u_\Phi(t)) &= -F(y_\Phi(t), u_\Phi(t)) + G(y_\Phi(t), u_\Phi(t)), \end{array}\right\} \tag{12}$$

where $F = \frac{f_1 - f_2}{2}$ and $G = \frac{f_1 + f_2}{2}$. We assume that the implicit function $F(\sigma, \xi) = 0$ can be represented by an explicit function $\sigma = f_{an}(\xi)$ or $\xi = g_{an}(\sigma)$. Such function f_{an} (or g_{an}) is called an anhysteresis function and the corresponding graph $\{(\xi, f_{an}(\xi)) | \xi \in \mathbb{R}\}$ is called an anhysteresis curve. Using f_{an} and the definition of F, it can be checked that $f_1(f_{an}(\xi), \xi) = f_2(f_{an}(\xi), \xi)$ holds.

To show the CCW properties of the Duhem operator, we review our previous results in [7]. In [7], we define a storage function $H_\circlearrowleft : \mathbb{R}^2 \to \mathbb{R}$ for the Duhem operator Φ such that (1) holds (under certain conditions on f_1 and f_2). We also show that $H_\circlearrowleft$ is positive definite if $f_1 > 0$ and $f_2 > 0$.

Before we can define the storage function $H_\circlearrowleft$ for Φ, we need to define a few more functions which depend on f_1 and f_2.

Firstly, we define a function ω_Φ that describes the possible trajectory of Φ when a monotone increasing u_Φ and a monotone decreasing u_Φ is applied to Φ from an initial condition.

For every pair $(y_{\Phi_0}, u_{\Phi_0}) \in \mathbb{R}^2$, let $\omega_{\Phi,1}(\cdot, y_{\Phi_0}, u_{\Phi_0}) : [u_{\Phi_0}, \infty) \to \mathbb{R}$ be the solution of

$$x(\tau) - x(u_{\Phi_0}) = \int_{u_{\Phi_0}}^{\tau} f_1(x(\sigma), \sigma)\ \mathrm{d}\sigma, \qquad x(u_{\Phi_0}) = y_{\Phi_0} \quad \forall \tau \in [u_{\Phi_0}, \infty),$$

and let $\omega_{\Phi,2}(\cdot, y_{\Phi_0}, u_{\Phi_0}) : (-\infty, u_{\Phi_0}] \to \mathbb{R}$ be the solution of

$$x(\tau) - x(u_{\Phi_0}) = \int_{u_{\Phi_0}}^{\tau} f_2(x(\sigma), \sigma)\ \mathrm{d}\sigma, \qquad x(u_{\Phi_0}) = y_{\Phi_0} \quad \forall \tau \in (-\infty, u_{\Phi_0}].$$

Using the above definitions, for every pair $(y_{\Phi_0}, u_{\Phi_0}) \in \mathbb{R}^2$, the function $\omega_\Phi(\cdot, y_{\Phi_0}, u_{\Phi_0}) : \mathbb{R} \to \mathbb{R}$ is defined by the concatenation of $\omega_{\Phi,2}(\cdot, y_{\Phi_0}, u_{\Phi_0})$ and $\omega_{\Phi,1}(\cdot, y_{\Phi_0}, u_{\Phi_0})$:

$$\omega_\Phi(\tau, y_{\Phi_0}, u_{\Phi_0}) = \begin{cases} \omega_{\Phi,2}(\tau, y_{\Phi_0}, u_{\Phi_0}) & \forall \tau \in (-\infty, u_{\Phi_0}) \\ \omega_{\Phi,1}(\tau, y_{\Phi_0}, u_{\Phi_0}) & \forall \tau \in [u_{\Phi_0}, \infty). \end{cases} \tag{13}$$

Again, we remark that the curve $\omega_\Phi(\cdot, y_{\Phi_0}, u_{\Phi_0})$ is the (unique) hysteresis curve where the curve defined in $(-\infty, u_{\Phi_0}]$ is obtained by applying a monotone decreasing $u_\Phi \in AC(\mathbb{R}_+, \mathbb{R}^m)$ to $\Phi(u_\Phi, y_{\Phi_0})$ with $u_\Phi(0) = u_{\Phi_0}$ and $\lim_{t\to\infty} u_\Phi(t) = -\infty$ and, similarly, the curve defined in $[u_{\Phi_0}, \infty)$ is produced by introducing a monotone increasing $u_\Phi \in AC(\mathbb{R}_+, \mathbb{R}^m)$ to $\Phi(u_\Phi, y_{\Phi_0})$ with $u_\Phi(0) = u_{\Phi_0}$ and $\lim_{t\to\infty} u_\Phi(t) = \infty$.

Another function that is needed for defining $H_\circlearrowleft$ is the intersecting function between the anhysteresis function f_{an} and the function ω_Φ as defined above. The function $\Omega : \mathbb{R}^2 \to \mathbb{R}$ is the *intersecting function* if $\omega_\Phi(\Omega(\sigma,\xi), \sigma, \xi) = f_{an}(\Omega(\sigma,\xi))$ for all $(\sigma,\xi) \in \mathbb{R}^2$ and $\Omega(\sigma,\xi) \geq \xi$ whenever $\sigma \geq f_{an}(\xi)$ and $\Omega(\sigma,\xi) < \xi$ otherwise. For simplicity, we assume that Ω is differentiable. In [7, Lemma 3.1] sufficient conditions on f_1 and f_2 such that such Ω exists are f_{an} be monotone increasing and

$$f_1(\sigma,\xi) < \frac{\mathrm{d}f_{an}(\xi)}{\mathrm{d}\xi} - \epsilon \quad \text{whenever} \quad \sigma > f_{an}(\xi) \tag{14}$$

$$f_2(\sigma,\xi) < \frac{\mathrm{d}f_{an}(\xi)}{\mathrm{d}\xi} - \epsilon \quad \text{whenever} \quad \sigma < f_{an}(\xi) \tag{15}$$

hold where $\epsilon > 0$.

Theorem 2.7: Consider the Duhem hysteresis operator Φ defined in (11)-(12) with locally Lipschitz functions $F, G : \mathbb{R}^2 \to \mathbb{R}$, anhysteretic function f_{an} (or g_{an}) and intersecting function Ω. Assume that f_1 and f_2 are positive definite. Suppose that for all (σ,ξ) in $\mathbb{R}^2$

(A) $F(\sigma,\xi) \geq 0$ whenever $\sigma \leq f_{an}(\xi)$,
$F(\sigma,\xi) < 0$ otherwise,

holds. Then Φ is CCW with the storage function $H_\circlearrowleft : \mathbb{R}^2 \to \mathbb{R}_+$ be given by

$$H_\circlearrowleft(\sigma,\xi) = \sigma\xi - \int_0^{\xi} \omega_\Phi(\tau,\sigma,\xi)\ \mathrm{d}\tau + \int_0^{\Omega(\sigma,\xi)} \omega_\Phi(\tau,\sigma,\xi) - f_{an}(\tau)\ \mathrm{d}\tau. \tag{16}$$

PROOF. The proof follows from Theorem 3.3 in [7]. In particular, it is shown in [7] that

$$\frac{\mathrm{d}H_\circlearrowleft(y_\Phi(t), u_\Phi(t))}{\mathrm{d}t} \leq \langle \dot{y}_\Phi(t), u_\Phi(t) \rangle, \tag{17}$$

where $y_\Phi := \Phi(u_\Phi, y_{\Phi_0})$ and $H_\circlearrowleft$ is non-negative. By integrating (17) from 0 to T we have

$$H_\circlearrowleft\big(y_\Phi(T), u_\Phi(T)\big) - H_\circlearrowleft\big(y_\Phi(0), u_\Phi(0)\big) = \int_0^T \dot{y}_\Phi(\tau) u_\Phi(\tau) \mathrm{d}\tau.$$

Since $H_\circlearrowleft$ is nonnegative then

$$\int_0^T \dot{y}_\Phi(\tau) u_\Phi(\tau) \mathrm{d}\tau \geq -H_\circlearrowleft(y_\Phi(0), u_\Phi(0)) > -\infty.$$

□

III. FEEDBACK INTERCONNECTION

In this section we consider either negative or positive feedback interconnection between a linear system and a Duhem hysteresis operator. The stability of the closed-loop system is analyzed by using the CCW or CW properties of the subsystems. The hysteresis operator is represented by the Duhem operator introduced in Section II-C.

Throughout this section, we assume that the functions f_1 and f_2 satisfy the hypotheses in Theorem 2.7, i.e., the Duhem operator Φ has CCW input-output dynamics.

Theorem 3.1: Consider a negative feedback interconnection between a single-input single-output linear system and a Duhem operator Φ satisfying the hypotheses in Theorem 2.7 as follows

$$\left.\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du, \\ \dot{y}_\Phi &= f_1(y_\Phi(t), u_\Phi(t))\dot{u}_{\Phi+}(t) + f_2(y_\Phi(t), u_\Phi(t))\dot{u}_{\Phi-}(t), \\ u &= -y_\Phi,\ u_\Phi = y, \end{aligned}\right\} \tag{18}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$ and $D \in \mathbb{R}$. Assume that there exist $P = P^T > 0$, L and $\epsilon > 0$ such that the following linear matrix inequalities (LMI)

$$P \left[\begin{smallmatrix} 1 \\ 0^{n\times 1} \end{smallmatrix}\right] = \left[\begin{smallmatrix} D \\ C^T \end{smallmatrix}\right], \tag{19}$$

$$\frac{1}{2}\left(P \left[\begin{smallmatrix} 0 & 0^{n\times n} \\ B & A \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 & B^T \\ 0^{n\times n} & A^T \end{smallmatrix}\right] P \right) + \epsilon L^T L \leq 0, \tag{20}$$

hold. Then for every initial conditions, the state trajectories of the closed-loop system (18) is bounded and all state trajectories converges to the largest invariant set in $\{(x, y_\Phi) | L \left[\begin{smallmatrix} -y_\Phi \\ x \end{smallmatrix}\right] = 0\}$.

PROOF. By the assumptions of the theorem, the Duhem operator Φ is CCW with the storage function $H_{\circlearrowleft} : \mathbb{R}^2 \to \mathbb{R}_+$ given in (16).

Define the extended state space of the linear system in (18) by

$$\begin{aligned} \dot{w} &= v, \\ \dot{x} &= Ax + Bw, \\ y &= Cx + Dw, \end{aligned} \tag{21}$$

where $w = u$.

Using $V = \frac{1}{2}[w \;\; x^T]^T P \begin{bmatrix} w \\ x \end{bmatrix}$, a routine computation shows that

$$\dot{V} = \frac{1}{2}[\; w \quad x^T \;] \left(\begin{bmatrix} 0 & B^T \\ 0^{n\times n} & A^T \end{bmatrix} P + P \begin{bmatrix} 0 & 0^{n\times n} \\ B & A \end{bmatrix} \right) \begin{bmatrix} w \\ x \end{bmatrix} + [\; w \quad x^T \;] P \begin{bmatrix} 1 \\ 0^{n\times 1} \end{bmatrix} v.$$

Using (19) and (20),

$$\dot{V} \le \langle y, v\rangle - \epsilon \left\| L \begin{bmatrix} -y_\Phi \\ x \end{bmatrix} \right\|^2. \tag{22}$$

This inequality (22) with $v = \dot{u}$ (by the relation in (21)) implies that the linear system defined in (18) is CW.

Now take $H_{cl}(x, y_\Phi) = H_{\circlearrowleft}(y_\Phi, Cx - Dy_\Phi) + V(x, y_\Phi)$ as the Lyapunov function of the interconnected system (18), where H_{cl} is radially unbounded by the non-negativity of $H_{\circlearrowleft}$ and properness of V. It is straightforward to see that

$$\begin{aligned} \dot{H}_{cl} &= \dot{H}_{\circlearrowleft} + \dot{V}, \\ &\le \langle y, \dot{u}\rangle + \langle \dot{y}_\Phi, u_\Phi\rangle - \epsilon \left\| L \begin{bmatrix} -y_\Phi \\ x \end{bmatrix} \right\|^2, \\ &= -\epsilon \left\| L \begin{bmatrix} -y_\Phi \\ x \end{bmatrix} \right\|^2, \end{aligned} \tag{23}$$

where the last equation is due to the interconnection conditions $u = -y_\Phi$ and $y = u_\Phi$. It follows from (23) and from the radial unboundedness (or properness) of H_{cl}, the signals x and y_Φ are bounded.

Based on the Lasalle's invariance principle [10], the semiflow (x, y_Φ) converges to the largest invariant set contained in $M := \{(x, y_\Phi) \in \mathbb{R}^n \times \mathbb{R} | L \left[\begin{smallmatrix} -y_\Phi \\ x \end{smallmatrix} \right] = 0\}$. □

To illustrate Theorem 3.1, consider the following simple example

$$\begin{aligned} \dot{x} &= -3x + y_\Phi, \\ y &= -2x + y_\Phi, \\ y_\Phi &= -\Phi(y), \end{aligned}$$

where $x \in \mathbb{R}$. By using $P = \left[\begin{smallmatrix} 1 & -2 \\ -2 & 6 \end{smallmatrix} \right]$, we have $V(x, y_\Phi) = \frac{1}{2}(-2x + y_\Phi)^2 + x^2$ which is positive definite and radially unbounded, and

$$\begin{aligned} &\frac{\partial V(x, y_\Phi)}{\partial x}(-3x + y_\Phi) + \frac{\partial V(x, y_\Phi)}{\partial y_\Phi} v \\ &= -2(-3x + y_\Phi)^2 + (-2x + y_\Phi) v \\ &= -2(-3x + y_\Phi)^2 + yv. \end{aligned}$$

Using $H_{cl}(x, y_\Phi) = V(x, y_\Phi) + H_{\circlearrowleft}(-y_\Phi, -2x + y_\Phi)$ as before, routine computation shows that

$$\begin{aligned} \dot{H}_{cl} &\le -2(-3x + y_\Phi)^2 + y\dot{y}_\Phi - \overbrace{\Phi(y)}^{\cdot} y \\ &= -2(-3x + y_\Phi)^2, \end{aligned}$$

and thus, we can conclude that (x, y_Φ) converges to the invariant set where $x = \frac{1}{3} y_\Phi$.

The result in Theorem 3.1 deals with negative feedback interconnection of a linear system and a Duhem hysteresis operator. In the following result, we consider the other case where a positive feedback is used instead. This is motivated by the study of an interconnection between counterclockwise systems as studied in [1] for the general case and in [15] for the linear case.

Theorem 3.2: Consider a positive feedback interconnection between a single-input single-output linear system and a Duhem operator Φ satisfying the hypotheses in Theorem 2.7 as follows

$$\left.\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \\ \dot{y}_\Phi &= f_1(y_\Phi(t), u_\Phi(t))\dot{u}_{\Phi+}(t) + f_2(y_\Phi(t), u_\Phi(t))\dot{u}_{\Phi-}(t), \\ u &= y_\Phi, \;\; u_\Phi = y, \end{aligned}\right\} \tag{24}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$ and $C \in \mathbb{R}^{1\times n}$. Let $\epsilon := (CB)^{-1}$ where we assume $CB > 0$ and there exist $\delta > 0$ and $Q = Q^T > 0$ such that

$$\frac{1}{2}(A^T Q + QA) + \epsilon A^T C^T CA \le 0, \tag{25}$$

$$QB + A^T C^T = 0, \tag{26}$$

$$Q - \delta C^T C > 0, \tag{27}$$

hold and the anhysteresis function f_{an} satisfies $(f_{an}(\xi) - \delta\xi)\xi \le 0$, for all $\xi \in \mathbb{R}$ (i.e. f_{an} belongs to the sector $[0, \delta]$). Then for every initial conditions, the state trajectory of the closed-loop system (24) is bounded and converges to the largest invariant set in $\{(x, y_\Phi) | CAx + CBy_\Phi = 0\}$.

PROOF. Using $V(x) = \frac{1}{2}x^T Qx$ and (25) and (26), it can be checked that

$$\begin{aligned} \dot{V} &= \frac{1}{2}x^T(A^T Q + QA)x + x^T QBu \\ &\le -\epsilon x^T A^T C^T CAx - x^T A^T C^T u \\ &= -\epsilon x^T A^T C^T CAx - 2x^T A^T C^T u - u^T B^T C^T u \\ &\quad + x^T A^T C^T u + u^T B^T C^T u \\ &= -\epsilon x^T A^T C^T CAx - 2\epsilon x^T A^T C^T CBu \\ &\quad - \epsilon u^T B^T C^T CBu + x^T A^T C^T u + u^T B^T C^T u \\ &= (u^T B^T C^T + x^T A^T C^T)u \\ &\quad - \epsilon(CAx + CBu)^T(CAx + CBu) \\ &= \langle \dot{y}, u\rangle - \epsilon \dot{y}^2. \end{aligned}$$

It follows from Lemma 2.4 that the linear system is S-CCW.

By the assumptions of the theorem, the Duhem operator Φ is also CCW with the storage function $H_{\circlearrowleft} : \mathbb{R}^2 \to \mathbb{R}_+$ as given in (16).

Now take $H_{cl}(x, y_\Phi) = V(x) + H_{\circlearrowleft}(y_\Phi, Cx) - Cxy_\Phi$ be the Lyapunov function of the interconnected system (24). We show first that H_{cl} is lower bounded. Substituting the representation of V and $H_{\circlearrowleft}$, we have

$$\begin{aligned} H_{cl} &= \frac{1}{2}x^T Qx + zCx - \int_0^{Cx} \omega_\Phi(\tau, y_\Phi, Cx)\ \mathrm{d}\tau \\ &\quad + \int_0^{\Omega(y_\Phi, Cx)} \omega_\Phi(\tau, y_\Phi, Cx)\mathrm{d}\tau \\ &\quad - \int_0^{\Omega(y_\Phi, Cx)} f_{an}(\tau)\mathrm{d}\tau - Cxy_\Phi \\ &= \frac{1}{2}x^T Qx - \int_0^{Cx} f_{an}(\tau)\mathrm{d}\tau \\ &\quad + \int_{Cx}^{\Omega(y_\Phi, Cx)} \omega_\Phi(\tau, y_\Phi, Cx) - f_{an}(\tau)\mathrm{d}\tau. \end{aligned} \tag{28}$$

Due to the property of the intersecting function Ω (c.f. [7, Lemma 3.1]), the last term on the right hand side of (28) is non-negative. Indeed, by the definition of intersecting function Ω, $\Omega(y_\Phi, Cx) \geq Cx$ whenever $y_\Phi \geq f_{an}(Cx)$ implies that $\omega_\Phi(\tau, y_\Phi, Cx) \geq f_{an}(\tau)$ for all $Cx < \tau < \Omega(y_\Phi, Cx)$. On the other hand $\Omega(y_\Phi, Cx) < Cx$ whenever $y_\Phi < f_{an}(Cx)$ implies that $\omega_\Phi(\tau, y_\Phi, Cx) < f_{an}(\tau)$ for all $\Omega(y_\Phi, Cx) < \tau < Cx$. Thus

$$\begin{aligned} H_{cl} &\geq \frac{1}{2}x^T Qx - \int_0^{Cx} f_{an}(\tau)\mathrm{d}\tau \\ &= \frac{1}{2}x^T Qx - \int_0^{Cx} (f_{an}(\tau) - \delta\tau)\mathrm{d}\tau - \int_0^{Cx} \delta\tau\mathrm{d}\tau \\ &\geq \frac{1}{2}x^T Qx - \frac{\delta}{2}x^T C^T Cx \\ &= \frac{1}{2}x^T (Q - \delta C^T C)x > 0 \qquad \forall x \neq 0, \end{aligned}$$

where the second inequality is due to the sector condition on f_{an} and the last inequality is due to (27). Hence, we can conclude that H_{cl} is positive definite and radially unbounded.

Now computing the time derivative of H_{cl}, we obtain

$$\dot{H}_{cl} = \dot{V} + \dot{H}_{\circlearrowleft} - C\dot{x}y_\Phi - Cx\dot{y}_\Phi \leq -\epsilon\dot{y}^2.$$

Based on the Lasalle's invariance principle, the signals (x, y_Φ) converges to the largest invariant set contained in $M := \{(x, y_\Phi) \in \mathbb{R}^n \times \mathbb{R} | CAx + CBy_\Phi = 0\}$.

□

We illustrate Theorem 3.2 in the following simple example. Consider

$$\begin{aligned} \dot{x} &= -x + y_\Phi,\ y = x, \\ y_\Phi &= \Phi(y), \end{aligned}$$

where $x \in \mathbb{R}$. Using $V(x) = \frac{1}{2}x^2$, where V is positive definite and radially unbounded, and

$$\begin{aligned} \frac{\partial V(x)}{\partial x}(-x + y_\Phi) &= -x^2 + xy_\Phi, \\ &= y_\Phi\dot{y} - (y_\Phi - x)^2, \\ &= y_\Phi\dot{y} - \dot{y}^2. \end{aligned}$$

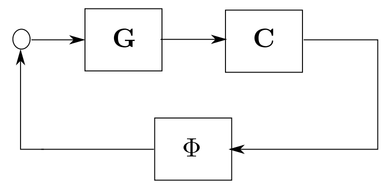

Fig. 2. Feedback interconnection with a linear system $\mathbf{G}$, a controller $\mathbf{C}$ and a hysteresis system Φ.

Using $H_{cl}(x, y_\Phi) = V(x) + H_{\circlearrowleft}(y_\Phi, y) - yy_\Phi$, routine computation shows that

$$\begin{aligned} \dot{V}_{cl} &\leq \dot{y}y_\Phi + \overbrace{\Phi(y)}^{\cdot}y - \dot{y}y_\Phi - y\dot{y}_\Phi - \dot{y}^2 \\ &= -(-x + y_\Phi)^2. \end{aligned}$$

Note that $Q = 1$, $C = 1$, so that (27) holds for $\delta < 1$. This means that the result in Theorem 3.2 holds if the anhysteresis function f_{an} satisfies $(f_{an}(\xi) - \delta\xi)\xi \leq 0$, for all $\xi \in \mathbb{R}$ and $\delta < 1$. In other words, f_{an} belongs to the sector $[0, \delta]$.

IV. CONTROLLER DESIGN

The results in the previous section can be used to design a controller for a linear plant with hysteretic input/actuator. Consider the closed-loop system as shown in Figure 2, where $\mathbf{G}$ and $\mathbf{C}$ are the linear systems of the plant and the controller, respectively, given by

$$\mathbf{G} : \left\{ \begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du, \end{aligned} \right. \quad \mathbf{C} : \left\{ \begin{aligned} \dot{x_c} &= A_c x_c + B_c y, \\ y_c &= C_c x_c + D_c y. \end{aligned} \right. \tag{29}$$

Thus the linear system $\mathbf{CG}$ is given by

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} &= \begin{bmatrix} A & 0 \\ B_c C & A_c \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} + \begin{bmatrix} B \\ B_c D \end{bmatrix} u, \\ y_c &= \begin{bmatrix} D_c C & C_c \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} + D_c D u. \end{aligned} \tag{30}$$

The controller design process can then be carried out by finding $\mathbf{C}$ such that the linear system $\mathbf{CG}$ satisfies either (19)-(20) or (25)-(26) for a known Duhem operator Φ.

Putting (30) into the setting of our main results in Theorem 3.1 and 3.2, the invariant set is characterized by $\{M(x, x_c, y_\Phi) | N \begin{bmatrix} x \\ x_c \\ y_\Phi \end{bmatrix} = 0\}$ for particular N. Thus N can also become a design parameter for determining $\mathbf{C}$.

The following procedure summarizes this control design method:

1) Determine the anhysteresis function f_{an} of the Duhem operator Φ and possibly, the desired N.
2) Find $\mathbf{C}$ such that either (19)-(20) or (25)-(26) holds.
3) If (19)-(20) is solvable, then $\mathbf{C}$ stabilizes the closed-loop system with negative feedback interconnection.
4) If (25)-(26) is solvable, then $\mathbf{C}$ stabilizes the closed-loop system with positive feedback interconnection.

As an example, we consider a mass-damper-spring system with a hysteretic actuator. The mass-damper-spring system is given by

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u,$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x + u. \qquad (31)$$

Assume that the actuator is represented by the Duhem operator (11) where

$$\left.\begin{aligned} f_1(\sigma,\xi) &= -\sigma + 0.475\xi + 0.3, \\ f_2(\sigma,\xi) &= \sigma - 0.475\xi + 0.3, \end{aligned}\right\} \forall(\sigma,\xi) \in \mathbb{R}. \qquad (32)$$

It can be verified that $f_{an}(\xi) = 0.475\xi$.

With $A_c = \left[\begin{smallmatrix} 0 & 1 \\ -2 & -2 \end{smallmatrix}\right]$, $B_c = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$, $C_c = \left[\begin{smallmatrix} -2 & -1 \end{smallmatrix}\right]$ and $D_c = 1$, conditions (19)-(20) are solvable with $P = \left[\begin{smallmatrix} 1 & 1 & 0 & -2 & -1 \\ 1 & 18 & 3 & -19 & -4 \\ 0 & 3 & 5 & -3 & -5 \\ -2 & -19 & -3 & 21 & 5 \\ -1 & -4 & -5 & 5 & 7 \end{smallmatrix}\right]$ and $L = [0\ 0\ 1/4\ 0\ 0]$. Hence the controller $\mathbf{C}$ can stabilize the closed-loop system with negative feedback interconnection. In this case, $N = [0\ 1/4\ 0\ 0\ 0]$. According to Theorem 3.1, the velocity of the mass-damper-spring system converges to zero as $t \to \infty$ and the position of the mass-damper-spring system converges to a constant. The closed-loop system is simulated in Simulink with the initial condition $x(0) = [10, 10]^T$ and the results are shown in Figure 3.

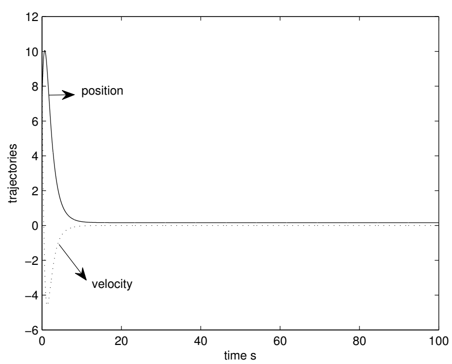

Fig. 3. Simulation results for the negative feedback connection, with initial condition $x(0) = [10\ 10]^T$.

On the other hand, since we have $f_{an}(\xi) = 0.475\xi$, then by taking $A_c = \left[\begin{smallmatrix} 0 & 1 \\ -2 & -4 \end{smallmatrix}\right]$, $B_c = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$, $C_c = \left[\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right]$ and $D_c = 0$, it can be checked that (25)-(26) holds with $\delta = 0.5$ and $Q = \left[\begin{smallmatrix} 6 & 1 & -6 & -2 \\ 1 & 4 & -1 & -4 \\ -6 & -1 & 7 & 3 \\ -2 & -4 & 3 & 7 \end{smallmatrix}\right]$. Moreover, f_{an} belongs to the sector $[0, 0.5]$. It follows from Theorem 3.2 that the closed-loop system with positive feedback interconnection is asymptotically stable to the invariant set $M = \{(x, y_\Phi) | [0\ 0\ 0\ 1\ 0] \left[\begin{smallmatrix} x \\ x_c \\ y_\Phi \end{smallmatrix}\right] = 0\}$. In this invariant set $x_{c1}^* = x_1^* = y_\Phi^*$ and $x_2^* = 0$. The simulation results is shown in Figure 4.

V. Conclusion

In this paper, we studied the feedback interconnection between a linear system and a hysteresis system using the property of counterclockwise (CCW) or clockwise (CW) input-output dynamics of each subsystem. Furthermore, a simple design procedure is also discussed.

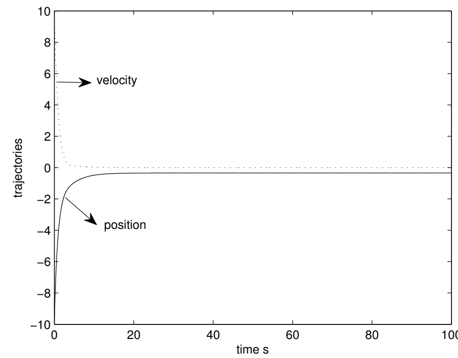

Fig. 4. Simulation results for the positive feedback connection, with initial condition $x(0) = [-10\ 10]^T$.

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