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Claudio De Persis and Bayu Jayawardhana

Abstract— The role of internal model principle is investigated in this paper in the context of collective synchronization and formation control problems. In the collective synchronization problem for nonlinear systems, we propose distributed control laws for passive systems which synchronize to the solution of an incrementally passive exosystem. This generalizes the result from linear systems where the existence of an internal model is required for the output synchronization of networked systems. In our second result, a distributed control law that solves a formation control problem for incrementally passive systems is proposed based also on the internal model principle.

Keywords: Cooperative control, synchronization, nonlinear systems, passivity

I. INTRODUCTION

In this paper, we study state synchronization and formation control for passive nonlinear systems based on the internal model principle. In the synchronization problem, one investigates conditions under which the state variables of all the subsystems asymptotically converge to each other, while in the formation control problem, one studies the distributed control laws which ensure that the position or the velocity of all the subsystems converge to the desired position or velocity. Passivity ([1], [3], [12], [11]), or the weaker notion of semi-passivity ([9], [8], [14]), has been studied in both the synchronization and formation control problems. In the context of output synchronization, the notion of incrementally passive nonlinear systems has been exploited to show that the relative output measurements in a network suffice to ensure output synchronization, i.e., the outputs of all the subsystems asymptotically converge to each other [12]. Moreover, if one assumes a stronger notion of strictly incremental passivity for each subsystem, where the strictness corresponds to the incremental stability of the autonomous systems, then the output synchronization implies the state synchronization. In this paper, we relax the strict incremental passivity assumption of each subsystem by considering controllers which incorporate exosystems which communicate with each other. This is motivated by the recent results in [16] where the existence of an internal model is both necessary and sufficient condition for output synchronization of linear networked systems.

The first main contribution of this paper is to show that the results of [16] can be extended to the nonlinear case. In particular, we propose a distributed control law that uses the local information of the networked exosystems and the relative output measurements of the subsystems and of the exosystems. The resulting control approach enforces, in some sense, a prescribed synchronization on the subsystems via networked exosystems. In particular, the subsystems asymptotically converge to an invariant manifold driven by an autonomous synchronized exosystem.

In the context of formation control, Bai, Arcak and Wen in [2] have discussed the role of passivity for the coordination of networked systems. In that framework we consider a formation control problem in which the followers asymptotically track a leader modeled as an incrementally passive exosystem.

In Section II we provide a few preliminaries about the class of systems under consideration and the synchronization problem we are interested in. In Section III the synchronization problem using incrementally observable internal models is studied. The formation control problem in the presence of incrementally passive leaders is examined in Section IV. Conclusions are drawn in Section V.

II. PRELIMINARIES

Consider N systems connected over an undirected graph $G = (V, E)$, where V is a set of N nodes and $E \subseteq V \times V$ is a set of M edges connecting the nodes. The standing assumption throughout the paper is that the graph G is *connected*. Each system i, with $i = 1, 2, \ldots, N$, is associated to the node i of the graph and the edges connect the nodes or systems which communicate.

A. Description of the networked exosystems

Throughout the paper, we use networked exosystems where each exosystem i is described by

$$W_i : \begin{array}{ll} \dot{w}_i & = s(w_i) + l(w_i)v_i \\ r_i & = c(w_i), \end{array} \tag{1}$$

where the state $w_i \in \mathbb{R}^n$, the input $v_i \in \mathbb{R}^p$, the output $r_i \in \mathbb{R}^p$ and the functions s, l, c are assumed to be locally Lipschitz satisfying $s(\mathbf{0}) = \mathbf{0}$, $l(\mathbf{0}) \neq \mathbf{0}$, $c(\mathbf{0}) = \mathbf{0}$. When $v_i = 0$, the autonomous system above is the familiar description of the exosystem that generates the reference signal in the nonlinear output regulation problem. For each exosystem W_i, we assume the following:

Assumption 1 *There exists a storage function $V_i : \mathbb{R}^n \to \mathbb{R}_+$ which is positive definite and radially unbounded such*

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that for all $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^p$

$$\frac{\partial V_i(w_i)}{\partial w_i}\left(s(w_i) + g(w_i)v_i\right) \leq c(w_i)^T v_i. \qquad (2)$$

Such a system W_i is called a *passive system*.

The autonomous system W_i (with $v_i = 0$) is called *incrementally observable* if for every trajectories w_i, w_j such that $c(w_i) = c(w_j)$, then $w_i = w_j$. The notion of incremental observability is a variation to the standard notion of zero-state observability.

Label one end of each edge in E by a positive sign and the other one by a negative sign. Now, consider the k-th edge in E, with $k \in \{1, 2, \ldots, M\}$, and let i, j be the two nodes connected by the edge. For the synchronization of networked exosystems, which is discussed in Subsection II-C, we need the relative measurements of the signals r_i and r_j. Thus, let z_k describe the difference between the signals r_i and r_j and be defined as follows:

$$z_k = \begin{cases} r_i - r_j & \text{if } i \text{ is the positive end of the edge } k \\ r_j - r_i & \text{if } i \text{ is the negative end of the edge } k . \end{cases}$$

Recall also that the incidence matrix D associated with the graph G is the $N \times M$ matrix such that

$$d_{ik} = \begin{cases} +1 & \text{if node } i \text{ is the positive end of edge } k \\ -1 & \text{if node } i \text{ is the negative end of edge } k \\ 0 & \text{otherwise.} \end{cases}$$

By the definition of D, the variables z can be concisely represented as

$$z = (D^T \otimes I_p)r \qquad (3)$$

where $w = [w_1^T \ldots w_N^T]^T$ and the symbol $\otimes$ denotes the Kronecker product of matrices (see e.g. [1], [11] for some basic properties).

B. Description of the networked systems

For the output synchronization problem and the formation control problem, each subsystem is described by

$$\Sigma_i : \begin{cases} \dot{\xi}_i &= f_i(\xi_i) + g_i(\xi_i)u_i \\ y_i &= h_i(\xi_i), \end{cases} \qquad (4)$$

where $x_i \in \mathbb{R}^{m_i}$, $u_i, y_i \in \mathbb{R}^q$. We assume that for every i the system Σ_i is incrementally passive, i.e.

Assumption 2 *There exists a regular storage function [1] $S_i : \mathbb{R}^{m_i} \times \mathbb{R}^{m_i} \to \mathbb{R}_+$ such that*

$$\frac{\partial S_i(\xi_i, \xi_i')}{\partial \xi_i}\left(f(\xi_i) + g(\xi_i)u_i\right) + \frac{\partial S_i(\xi_i, \xi_i')}{\partial \xi_i'}\left(f(\xi_i') + g(\xi_i')u_{i'}\right)$$
$$\leq \langle h(\xi_i) - h(\xi_i'), u_i - u_{i'}\rangle \quad (5)$$

for all $(\xi_i, \xi_i') \in \mathbb{R}^{m_i} \times \mathbb{R}^{m_i}$ and $(u_i, u_i') \in \mathbb{R}^q \times \mathbb{R}^q$.

Examples of systems satisfying (5) are found in the literature [4].

For the synchronization problem, which is briefly reviewed in

Section III, we need the relative measurements of the signals y_i and y_j. On the other hand, for the coordination problem, which is detailed in Section IV, the relative measurements of the integral form $\int_0^t y_i(\tau)\mathrm{d}\tau$ and $\int_0^t y_j(\tau)\mathrm{d}\tau$ are used. In the coordination problem, the signal y_i can be considered as the velocity measurement and thus the integral form defines the position measurement.

Thus, depending upon specific problems, let ζ_k describe the difference between the signals y_i and y_j (or the difference between the signals $x_i(t) := \int_0^t y_i(\tau)\mathrm{d}\tau + x_i(0)$ and $x_j(t) := \int_0^t y_j(\tau)\mathrm{d}\tau + x_j(0)$ with constants $x_i(0), x_j(0) \in \mathbb{R}$) and be defined as follows:

$$\zeta_k = \begin{cases} y_i - y_j \;\; (\text{or } x_i - x_j) \\ \qquad \text{if } i \text{ is the positive end of the edge } k \\ y_j - y_i \;\; (\text{or } x_j - x_i) \\ \qquad \text{if } i \text{ is the negative end of the edge } k . \end{cases}$$

Using the incidence matrix D, the variables ζ can also be concisely represented as

$$\zeta = (D^T \otimes I_p)y \quad \left(\text{or } \zeta = (D^T \otimes I_p)x\right) \qquad (6)$$

where $y = [y_1^T \ldots y_N^T]^T$ and $x = [x_1^T \ldots x_N^T]^T$.

C. Exosystems synchronization

Let us recall the synchronization for linear systems as discussed in [11]. In the synchronization problem of [11, Theorem 4], each system W_i in (1) is assumed to be linear, identical and passive. For such setting, each (passive) system Σ_i is of the form

$$\begin{aligned} \dot{w}_i &= Sw_i + Bv_i \\ r_i &= C_w w_i \qquad i = 1, 2, \ldots, N \end{aligned} \qquad (7)$$

where $w_i \in \mathbb{R}^n$, $v_i, r_i \in \mathbb{R}^p$ and the passivity of W_i implies that the following assumption holds:

Assumption 3 *There exists an $(n \times n)$ matrix $P = P^T > 0$ such that $S^T P + PS = 0$, $B^T P = C_w$.*

The synchronization problems can then be stated as designing each control law v_i, $i = 1, 2, \ldots, N$, using only the information available to the agent i such that, for every i, $w_i - w_0 \to 0$ where w_0 is the trajectory of the autonomous system $\dot{w}_0 = Aw_0$ which is initialized by the average of the initial states, i.e., $w_0(0) = \frac{1}{N}\sum_i w_i(0)$.

In addition to output synchronization, it is well-known that the *states* of interconnected passive systems synchronize under an observability assumption ([3]). The largest invariant set of the interconnected systems when (C_w, S) is observable, is the set $\{w \in \mathbb{R}^{nN} : w_1 = \ldots = w_N\}$. The exponential synchronization under static output feedback control laws and time-varying graphs has been investigated in [11]. In the following statement, we recall Theorem 4 of [11] specialized to the case of time-invariant undirected graphs:

Theorem 1 *Let Assumption 3 hold and suppose that the pair (C_w, S) is observable. Let the communication graph*

be undirected and connected, and denote $z = (D^T \otimes I_p)r$ as in (3) with $r = [r_1^T \ldots r_N^T]^T$. Then the solutions of

$$\dot{w}_i = Sw_i - B \sum_{k=1}^{M} d_{ik}z_k, \ i = 1, 2, \ldots, N \qquad (8)$$

satisfy

$$\lim_{t \to +\infty} \left\| w_i(t) - \frac{\mathbf{1}_N^T \otimes I_n}{N} w(t) \right\| = 0 \qquad (9)$$

where $w = [w_1^T \ w_2^T \ \cdots \ w_N^T]^T$ and the convergence is exponential. More precisely, the solutions converge exponentially to the solution of $\dot{w}_0 = Sw_0$ initialized to the average of the initial conditions of the systems (8), i.e. $w_0(0) = \mathbf{1}_N^T \otimes I_n w(0)$.

Let $\tilde{w} = w - \frac{\mathbf{1}_N \mathbf{1}_N^T \otimes I_n}{N} w = (\Pi \otimes I_n)w$, with $\Pi = I_N - \frac{\mathbf{1}_N \mathbf{1}_N^T}{N}$, be the disagreement vector. From (8), $\tilde{w}(t)$ obeys the equation

$$\dot{\tilde{w}} = \underbrace{[I_N \otimes S - (I_N \otimes B)(DD^T \otimes I_p)(I_N \otimes C_w)]}_{\tilde{S}} \tilde{w}$$

$$(10)$$

and the convergence result (9) can be restated as $\lim_{t \to +\infty} \|\tilde{w}(t)\| = 0$. The proof of the result rests on showing that the Lyapunov function $V(\tilde{w}) = \tilde{w}^T(I_N \otimes P)\tilde{w}$ along the solutions of (10) satisfies the inequality $\dot{V}(\tilde{w}) \leq -\lambda_2 \|(\Pi \otimes I_p)r\|^2$, where λ_2 is the algebraic connectivity of the graph, i.e. the smallest non-zero eigenvalue of the Laplacian $L = DD^T$. Then the thesis descends from the observability assumption and Theorem 1.5.2 in [10].

The generalization to the nonlinear case is given in the following proposition.

Proposition 1 *Consider the exosystems W_i as in (1). Suppose that for every i, W_i satisfies Assumption 1 and is incrementally observable. Let the communication graph be undirected and connected, and denote $z = (D^T \otimes I_p)r$ as in (3) with $r = [r_1^T \ldots r_N^T]^T$. Then for every initial condition $w(0) \in \mathbb{R}^{nN}$, there exists $\omega_0 \in \mathbb{R}^n$ such that the solution of*

$$\dot{w}_i = s(w_i) - l(w_i) \sum_{k=1}^{M} d_{ik}z_k, \ i = 1, 2, \ldots, N$$

satisfies $\lim_{t \to +\infty} \|w_i(t) - w_0(t)\| = 0$ *where* $w = [w_1^T \ w_2^T \ \cdots \ w_N^T]^T$ *and w_0 is the solution of* $\dot{w}_0 = s(w_0)$, $w_0(0) = \omega_0 \in \mathbb{R}^n$.

Proof: Let $V(w) = \sum_i V_i(w_i)$ be the Lyapunov function of the interconnected systems where V_i satisfies (2). Following the standard passivity argument and the assumptions on the graph, we have that

$$\dot{V} = \sum_i \frac{\partial V_i(w_i)}{\partial w_i} \left(s(w_i) - l(w_i) \sum_{k=1}^{M} d_{ik}z_k \right)$$

$$= -\sum_i c(w_i) \sum_{k=1}^{M} d_{ik}z_k = -r^T(D \otimes I_p)z$$

$$= -r^T(DD^T \otimes I_p)r \leq -\lambda_2 \|(\Pi \otimes I_p)r\|^2$$

where $\Pi = I_N - \frac{\mathbf{1}_N \mathbf{1}_N^T}{N}$. Since V is proper (by the assumption on each V_i), the above inequality implies that the state trajectory w stays in a compact set. By the application of LaSalle invariance principle, w converges to the ω-limit set $\Omega(w(0))$ where we have that $(\Pi \otimes I_p)r = 0$. In the ω-limit set $\Omega(w(0))$, the trajectory of w is a solution to

$$\begin{bmatrix} \dot{\xi}_1 \\ \vdots \\ \dot{\xi}_N \end{bmatrix} = \begin{bmatrix} s(\xi_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & s(\xi_N) \end{bmatrix}, \quad \begin{bmatrix} \xi_1(0) \\ \vdots \\ \xi_N(0) \end{bmatrix} = \xi_0,$$

$$(11)$$

such that $c(\xi_1) = c(\xi_2) = \ldots = c(\xi_N)$ with an initial condition ξ_0. By the incremental observability of W_i, it implies that $\xi_1 = \xi_2 = \ldots = \xi_N$. Therefore, the trajectory of w in the ω-limit set can be described by $\mathbf{1}_N \otimes w_0$ where w_0 is the solution to $\dot{w}_0 = s(w_0)$, $w_0(0) = \omega_0$. $\qquad \square$

Example 1 Here we discuss a class of exosystems satisfying the hypotheses in Proposition 1. Let us consider nonlinear exosystems described by (we drop the dependence on the index i to simplify the notation)

$$\dot{w} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \nabla V^T(w) + \begin{bmatrix} 0 \\ I \end{bmatrix} v \qquad r = \begin{bmatrix} 0 & I \end{bmatrix} \nabla V^T(w),$$

where V is a positive definite and radially unbounded function and $\nabla V(w) := \frac{\partial}{\partial w} V(w)$. Such systems satisfy the passivity condition (2) and belong to the class of port-Hamiltonian systems [15]. Suppose that $w = \begin{bmatrix} w_a \\ w_b \end{bmatrix}$, $w_a, w_b \in \mathbb{R}^n$ and there exists a positive definite function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\left\| \frac{\partial}{\partial w_a} V(\begin{bmatrix} w_a \\ w_b \end{bmatrix}) - \frac{\partial}{\partial \bar{w}_a} V(\begin{bmatrix} \bar{w}_a \\ \bar{w}_b \end{bmatrix}) \right\| \geq \alpha(\|w_a - \bar{w}_a\|) \quad (12)$$

$$\left\| \frac{\partial}{\partial w_b} V(\begin{bmatrix} w_a \\ w_b \end{bmatrix}) - \frac{\partial}{\partial \bar{w}_b} V(\begin{bmatrix} \bar{w}_a \\ \bar{w}_b \end{bmatrix}) \right\| \geq \alpha(\|w_b - \bar{w}_b\|) \quad (13)$$

hold for all $w_a, w_b, \bar{w}_a, \bar{w}_b \in \mathbb{R}^n$. Under such assumptions, the exosystems are incrementally observable. Indeed, given two autonomous exosystems with states $w_1, w_2 \in \mathbb{R}^{2n}$ satisfying

$$\dot{w}_1 = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \nabla V^T(w_1) \qquad r_1 = \begin{bmatrix} 0 & I \end{bmatrix} \nabla V^T(w_1)$$

$$\dot{w}_2 = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \nabla V^T(w_2) \qquad r_2 = \begin{bmatrix} 0 & I \end{bmatrix} \nabla V^T(w_2),$$

if $r_1(t) = r_2(t)$ for all t, then (13) implies that $w_{1b}(t) = w_{2b}(t)$ for all t. Since it also holds that $\dot{w}_{1b} = \dot{w}_{2b}$, (12) implies that $w_{1a}(t) = w_{2a}(t)$ for all t.

Remark 1 The assumption on the symmetry of the graph is not needed. In fact consider the control law $v_i = \sum_j a_{ij}(r_j - r_i)$, where a_{ij} are the entries of the adjacency matrix of a weight-balanced and weakly connected directed graph[2]. Then $\dot{V} = -r^T(L \otimes I_p)r \leq -\tilde{\lambda}_2 \|(\Pi \otimes I_p)r\|^2$, where L is the Laplacian of the graph and $\tilde{\lambda}_2$ is the smallest nonzero

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[2]We recall that a graph is weight-balanced if the out-degree of each node equals its in-degree. It is weakly connected if for each pair of its nodes there is a (not necessarily oriented) path connecting the nodes.

eigenvalue of $(L + L^T)/2$. Then the thesis descends as before. The assumption can be relaxed even further – see e.g. [5],[14].

Remark 2 The result is related with others appeared in recent literature. In [13, Theorem 2], output synchronization is proven with no such an assumption as incremental observability but requiring the systems to be incrementally output feedback passive and the network to satisfy a strong coupling assumption (namely, the algebraic connectivity of the graph should be larger than a certain constant). On the other hand, incremental observability must be assumed to prove state synchronization [4]. Strict incremental passivity for network of systems has been studied in [7] (see also [6]) and used to prove exponential synchronization under integral coupling in an all-to-all graph.

III. OUTPUT SYNCHRONIZATION VIA INTERNAL MODEL

In this section, we discuss the solvability of the output synchronization problem using the internal model approach. This is motivated by the approach that is proposed in [16] where the existence of an internal model is both a necessary and sufficient condition for the output synchronization of linear networked systems. Let us discuss the result of [16] (see also [2], Section 3.6) adapted to passive linear systems.

Given N heterogeneous passive linear systems

$$\begin{aligned} \dot{\xi}_i &= F_i \xi_i + G_i u_i \\ y_i &= H_i \xi_i, \qquad i = 1, 2, \ldots, N \end{aligned} \tag{14}$$

with the storage function $H_i = \xi_i^T P_i \xi_i$, $P_i = P_i^T > 0$ such that $F_i^T P_i + P_i F_i \leq 0$, $P_i G_i = H_i^T$, with (H_i, F_i) detectable and with a graph G (which here, as usual in this paper, we assume static undirected and connected), find a feedback control law u_i for each system i (i) which uses relative measurements concerning only the systems which are connected to the system i via the graph G and (ii) such that output synchronization is achieved, i.e. $\lim_{t \to \infty} ||y_i(t) - y_j(t)|| = 0$ for all $i, j \in \{1, 2, \ldots, N\}$.

Excluding the trivial case in which the closed-loop system has an attractive set of equilibria where the outputs are all zero, the authors of [16] show that the output synchronization problem for N heterogeneous systems is solvable if and only if there exist matrices S, C_w such that $\lim_{t \to \infty} ||y_i(t) - C_w e^{-St} w_0|| = 0$ for each $i \in \{1, 2, \ldots, N\}$, for some w_0. Moreover, provided that $\sigma(S) \subset j\mathbb{R}$, the controllers which solve the regulation problem are

$$u_i = -K_i(y_i - C_w w_i) + \Gamma_i w_i \tag{15}$$

where $K_i > 0$ and $w_i \in \mathbb{R}^p$ are the exosystem states that synchronize via communication channels. The latter are described by

$$\begin{aligned} \dot{w} &= (I_N \otimes S)w - (I_N \otimes B)(D \otimes I_p)z \\ z &= (D^T \otimes I_p)(I_N \otimes C_w)w, \end{aligned} \tag{16}$$

where D is the incidence matrix associated to the graph, the pair (C_w, S) is detectable, the triple (S, B, C_w) satisfies

Assumption 3 and Π_i, Γ_i are matrices which solve the regulator equations

$$\begin{aligned} F_i \Pi_i + G_i \Gamma_i &= \Pi_i S \\ H_i \Pi_i &= C_w. \end{aligned} \tag{17}$$

The controllers (15)–(16) are a modified form of the ones in [16, Eq. (10)] where in the latter, the local controller communicates the entire exosystem state w_i to its connecting nodes and the local controller uses both state-feedback and state-observer.

Proposition 2 *The controllers (15)–(16) solve the output synchronization problem for the N heterogenous passive systems as given in* (14).

In the following, we show that the above result can be extended to the case of nonlinear incrementally passive systems.

Consider N heterogenous nonlinear systems connected over an undirected and connected graph $G = (V, E)$ where each system Σ_i, with $i = 1, 2, \ldots, N$, is described as in (4) and Σ_i is incrementally passive, i.e., there exists a regular function $S_i : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}_+$ such that (5) holds. Suppose that for every $i = 1, \ldots N$, there exists an exosystem W_i as in (1), with state variable w_i and maps s, l and c such that (2) holds. Suppose also that there exist functions $\pi_i(w_i), \alpha_i(w_i)$ such that

$$\frac{\partial \pi_i(w_i)}{\partial w_i} s(w_i) = f_i(\pi_i(w_i)) + g_i(\pi_i(w_i))\alpha_i(w_i) \tag{18}$$

$$g_i(\pi_i(w_i)) = \frac{\partial \pi_i(w_i)}{\partial w_i} l(w_i) \tag{19}$$

hold for all $w_i \in \mathbb{R}^n$. Finally, let the output maps agree in such a way that, for $i \neq j$,

$$h_i(\pi_i(w_i)) = h_j(\pi_j(w_i)) \tag{20}$$

holds for all $w_i \in \mathbb{R}^n$. The following result describes the controllers under which output synchronization is achieved:

Proposition 3 *Let Assumptions 1 and 2 and (18)-(20) hold. Then the response of the interconnected system* (1), (4) *with the controllers*

$$\begin{aligned} v_i &= \sum_{k=1}^M d_{ik} z_k \\ u_i &= \alpha_i(w_i) + v_i - K_i(y_i - h_i(\pi_i(w_i))) \end{aligned} \tag{21}$$

where $K_i > 0$, the local exosystem state w_i satisfies (1), and $z = (D^T \otimes I_p)r$ as in (3) with $r = [r_1^T \ldots r_N^T]^T$, is such that $\lim_{t \to +\infty} ||y_i(t) - y_j(t)|| = 0$ for each $i \neq j$. More precisely, $\lim_{t \to +\infty} ||y_i(t) - h_0(w_0(t))|| = 0$ for each $i = 1, 2, \ldots, N$, where $h_0 = h_i \circ \pi_i$ and w_0 is the solution of $\dot{w}_0 = s(w_0)$, $w(0) = \omega_0 \in \mathbb{R}^n$.

Proof: Denoting $\bar{\xi}_i = \pi_i(w_i)$, we can observe that

$$\begin{aligned} \dot{\bar{\xi}}_i &= f_i(\bar{\xi}_i) + g_i(\bar{\xi}_i)\alpha_i(w_i) + \frac{\partial \pi_i(w_i)}{\partial w_i} l(w_i) v_i \\ &= f_i(\bar{\xi}_i) + g_i(\bar{\xi}_i)(\alpha_i(w_i) - v_i), \end{aligned} \tag{22}$$

where we have used (19) in the last equation. Using S_i and (22), it can be computed that $\dot{S}_i(\xi_i, \bar{\xi}_i) \leq$

$\langle h_i(\xi_i) - h_i(\bar{\xi}_i), u_i - \alpha_i(w_i) - v_i \rangle$.

Thus, by assigning $u_i = \alpha_i(w_i) + v_i - K_i\big(h_i(\xi_i) - h_i(\bar{\xi}_i)\big)$, we arrive at $\dot{S}_i(\xi_i, \bar{\xi}_i) \leq -K_i \|h_i(\xi_i) - h_i(\bar{\xi}_i)\|^2$. Finally, by defining $V(w, \xi) = \sum_i V_i(w_i) + \sum_i S_i(\xi_i, \pi_i(w_i))$, using the above inequality and following the same argument in the proof of Proposition 1 on the derivative of $\sum_i V_i$, we obtain that

$$\dot{V} \leq -\sum_i K_i \left(\|h_i(\xi_i) - h_i(\pi_i(w_i))\|^2 \right) - \lambda_2 \|(\Pi \otimes I_p) r\|^2.$$

Since the trajectories of w are bounded according to Proposition 1 and using the regularity of S_i, this inequality implies that the state trajectories ξ are also bounded and remain in a compact set.

Moreover, by the LaSalle invariance principle, the state trajectory (w, ξ) converges to the ω-limit set $\Omega(w(0), \xi(0))$ where both $\lambda_2 \|(\Pi \otimes I_p) r\|^2$ and $\sum_i \left(\|h_i(\xi_i) - h_i(\pi_i(w_i))\|^2 \right)$ are equal to zero. Similar to the arguments in the proof of Proposition 1, the trajectory of w in the ω-limit set can be described by $\mathbf{1}_N \otimes w_0$ where w_0 is the solution to $\dot{w}_0 = s(w_0)$, $w_0(0) = \omega_0$. Since the output maps agree with each other on the ω-limit set, i.e., $h_i \circ \pi_i = h_j \circ \pi_j$, $i \neq j$, then the synchronization of w_i implies that the outputs of the heterogeneous nonlinear system also synchronize. $\square$

We note that (18) and (20) are the nonlinear counterparts of the *regulator equations* in (17). On the other hand, (19) is a new condition on the exosystem that facilitates the control design. In this case, the function l has to be designed such that (19) holds. Another consequence of (19) is that the dimension of the input v_i is the same as that of u_i. In the case of linear systems (19) becomes $G_i \Pi_i = \Pi_i B$, which has to be added to (17). The corresponding control law which solves the output synchronization for (14) is

$$u_i = -K_i\big(y_i - H_i \Pi w_i)\big) + \Gamma_i w_i + \textstyle\sum_{k=1}^{M} d_{ik} z_k$$
$$= -K_i\big(y_i - C_w w_i)\big) + \Gamma_i w_i + \textstyle\sum_{k=1}^{M} d_{ik} z_k,$$

(c.f., the control law in (15)).

IV. FORMATION CONTROL VIA INTERNAL MODEL

Let us consider again the systems Σ_i, $i = 1, \ldots N$ as in (4) with $h_i(\xi_i) = \xi_i$ and let $x_i(t) := \int_0^t y_i(\tau) \mathrm{d}\tau + x_i(0)$. In this case, each system i can be described by

$$\mathcal{S}_i : \quad \begin{aligned} \dot{x}_i &= \xi_i \\ \dot{\xi}_i &= f_i(\xi_i) + g_i(\xi_i) u_i \\ y_i &= \xi_i. \end{aligned}$$

We assume that the reference velocity $\rho \in \mathbb{R}^p$ is generated by the exosystem

$$\dot{w}_1 = s(w_1), \quad \rho = c(w_1), \tag{23}$$

located at system $\mathcal{S}_1$ where s is continuous, $w_1(0) \in \mathcal{W}$ and $\mathcal{W}$ is a bounded forward invariant set for (23). We define the tracking error as $e_1 = \xi_1 - \rho$. The reference signal ρ is only accessible by system $\mathcal{S}_1$. The other systems generate an estimate r_i of the reference signal via the equations (1)

$$W_i : \quad \begin{aligned} \dot{w}_i &= s(w_i) + l(w_i) v_i \\ r_i &= c(w_i). \end{aligned} \tag{24}$$

For $i \neq 1$, we let $e_i = \xi_i - r_i$. We take the error signals as the new outputs of the system $\mathcal{S}_i$

$$\tilde{\mathcal{S}}_i : \quad \begin{aligned} \dot{x}_i &= \xi_i \\ \dot{\xi}_i &= f_i(\xi_i) + g_i(\xi_i) u_i \\ e_i &= \begin{cases} \xi_1 - \rho & i = 1 \\ \xi_i - r_i & i \neq 1. \end{cases} \end{aligned} \tag{25}$$

We assume that the regulator equations hold, namely, given v_i and the corresponding response r_i of (24), for $i = 2, \ldots, N$, and given ρ, there exist functions $\bar{\xi}_i, \bar{u}_i$ such that

$$\begin{aligned} \dot{\bar{\xi}}_i &= f_i(\bar{\xi}_i) + g_i(\bar{\xi}_i) \bar{u}_i \\ 0 &= \begin{cases} \bar{\xi}_1 - \rho & i = 1 \\ \bar{\xi}_i - r_i & i \neq 1. \end{cases} \end{aligned} \tag{26}$$

Suppose that there exist maps $\pi_i(w_i)$ and $\alpha_i(w_i)$ such that

$$\left. \begin{aligned} \frac{\partial \pi_i(w_i)}{\partial w_i} s(w_i) &= f_i(\pi_i(w_i)) + g_i(\pi_i(w_i)) \alpha_i(w_i) \\ 0 &= \pi_i(w_i) - c(w_i) \end{aligned} \right\} \tag{27}$$

and, for $i \neq 1$,

$$g_i(\pi_i(w_i)) = \frac{\partial \pi_i(w_i)}{\partial w_i}(-l(w_i)). \tag{28}$$

Moreover, let $g_i(\pi_i(w_i))$ be full column-rank for all w_i.

Given the input v_i, define $\bar{\xi}_i = \pi_i(w_i)$, with w_i the solution to (24), for $i \neq 1$, and to (23) for $i = 1$. Then, $\bar{\xi}_i = \pi(w_i)$ satisfies (26) with $\bar{u}_i = \alpha_i(w_i) - v_i$ if $i \neq 1$ and $\bar{u}_1 = \alpha_1(w_1)$.

For each system $\tilde{\mathcal{S}}_i$, $i = 1, 2, \ldots, N$, we assume that a slightly stronger property than (5) holds:

Assumption 4 *There exists a regular storage function $S_i : \mathbb{R}^{m_i} \times \mathbb{R}^{m_i} \to \mathbb{R}_+$ such that*

$$\frac{\partial S_i(\xi_i, \xi_i')}{\partial \xi_i} \left(f_i(\xi_i) + g_i(\xi_i) u_i \right) + \frac{\partial S_i(\xi_i, \xi_i')}{\partial \xi_i'} \left(f_i(\xi_i') + \right.$$
$$\left. g_i(\xi_i') u_i' \right) \leq -W_i(\xi_i - \xi_i') + \langle e_i - e_i', u_i - u_i' \rangle, \tag{29}$$

where W_i is a positive definite function.

Inequality (29) defines incremental output strict-passivity of $\mathcal{S}_i$ from the input u_i to the output ξ_i. Furthermore, for $i = 2, \ldots, N$, we assume the family of internal models (1) to be incrementally passive. Namely, we suppose that:

Assumption 5 *There exists a regular function $H_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ such that for all $(w_i, w_i') \in \mathbb{R}^n \times \mathbb{R}^n$*

$$\frac{\partial H_i(w_i, w_i')}{\partial w_i} \big(s(w_i) + l(w_i) v_i \big) + \frac{\partial H_i(w_i, w_i')}{\partial w_i'} \big(s(w_i') + $$
$$l(w_i') v_i' \big) \leq \langle c(w_i) - c(w_i'), v_i - v_i' \rangle \tag{30}$$

By the incremental passivity property in (29), given u_i, v_i and the solutions $\xi_i, \bar{\xi}_i$ to (25) and (26), respectively, it follows that

$$\frac{\partial S_i(\xi_i, \bar{\xi}_i)}{\partial \xi_i} \left(f_i(\xi_i) + g_i(\xi_i) u_i \right) + \frac{\partial S_i(\xi_i, \bar{\xi}_i)}{\partial \bar{\xi}_i} \left(f_i(\bar{\xi}_i) + \right.$$
$$\left. g_i(\bar{\xi}_i) \bar{u}_i \right) \leq -W_i(\xi_i - \bar{\xi}_i) + e_i^T (u_i - \bar{u}_i).$$

By (30) with $w_i' = w_1$ and $v_i' = 0$, it holds true that for $i \neq 1$

$$\frac{\partial H_i(w_i, w_1)}{\partial w_i}(s(w_i) + l(w_i)v_i) + \frac{\partial H_i(w_i, w_1)}{\partial w_1}s(w_1)$$
$$\leq (r_i - \rho)^T v_i. \tag{31}$$

Set $\tilde{\rho} := r - \mathbf{1}_N \rho$, where $r = (\rho\, r_2 \ldots r_N)$ and $H(w, w_1) = \sum_{i=2}^N H_i(w_i, w_1)$. Observe that $\tilde{\rho}_1 = \mathbf{0}$. Then $\dot{H}(w, w_1) \leq \tilde{\rho}^T v$. Recall from (6) that the relative position measurement between connected systems is given by $\zeta = (D^T \otimes I_p)x$. The variable ζ satisfies $\zeta = (D^T \otimes I_p)(e + \tilde{\rho})$. Hence the function $V(\zeta) = \frac{1}{2}\zeta^T \zeta$ fulfills the equality

$$\dot{V}(\zeta) = \zeta^T(D^T \otimes I_p)(e + \tilde{\rho}).$$

Let $S(\xi, \bar{\xi}) := \sum_{i=1}^N S_i(\xi_i, \bar{\xi}_i)$. Then $\dot{S} = \sum_i \dot{S}_i(\xi_i, \bar{\xi}_i) \leq \sum_i [-W_i(\xi_i - \bar{\xi}_i) + e_i^T(u_i - \bar{u}_i)] = -\sum_i W_i(\xi_i - \bar{\xi}_i) + \sum_i e_i^T(u_i - \bar{u}_i) = -W(\xi - \bar{\xi}) + e^T(u - \bar{u})$.
The function $T(\zeta, w, w_1, \xi, \bar{\xi}) = V(\zeta) + H(w, w_1) + S(\xi, \bar{\xi})$ satisfies

$$\dot{T}(\zeta, w, w_1, \xi, \bar{\xi}) = \zeta^T(D^T \otimes I_p)(e + \tilde{\rho}) + \tilde{\rho}^T v$$
$$-W(\xi - \bar{\xi}) + e^T(u - \bar{u}).$$

The choice

$$v_i = -(D_i \otimes I_p)\zeta, \quad i = 2, \ldots, N$$
$$u = \bar{u} - (D \otimes I_p)\zeta$$

with D_i the row i of the incidence matrix D, gives $\dot{T} = -W(\xi - \bar{\xi})$. Observe that by construction

$$u_i = \begin{cases} \alpha_1(w_1) - (D_1 \otimes I_p)\zeta & \text{if} \quad i = 1 \\ \alpha_i(w_i) & \text{if} \quad i \neq 1. \end{cases} \tag{32}$$

By the regularity of T and the boundedness of w_1, $(\xi, \bar{\xi})$, $(w, \bar{w})$ are defined for all $t \geq 0$, and then one concludes that the solutions of the system converge to the largest invariant set where $\xi - \bar{\xi} = 0$. On such invariant set, $\mathbf{0} = g(\bar{\xi})(u - \bar{u})$, where $g = \text{block.diag}(g_1, \ldots, g_N)$. Since $g_i(\bar{\xi}_i) = g_i(\pi_i(w_i))$ is full-column rank for all i, one concludes that $u = \bar{u}$ and therefore $(D \otimes I_p)\zeta = 0$. The latter, along with $\zeta = (D^T \otimes I_p)x$, implies that $\zeta = 0$. Hence, $\dot{\zeta} = 0$ and this implies that $(D^T \otimes I_p)\tilde{\rho} = 0$. Multiplying the latter on the left by $D \otimes I_p$ shows that $\tilde{\rho}$ must have all the entries equal. As $\tilde{\rho}_1 = 0$, this proves that $\tilde{\rho} = 0$.
We can summarize the above as follows:

Proposition 4 *Consider the systems (25) satisfying Assumption 4 and assume that the reference velocity ρ is generated by the exosystem (23). Consider also the internal models (24), with $i \neq 1$, satisfying Assumption 5. Suppose the regulator equations (27), (28) are satisfied, with $g_i(\pi_i(w_i))$ full-column rank for all i. Let v_i in (24) be given by $v_i = -(D_i \otimes I_p)\zeta$, for $i \neq 1$ and u in (25) be defined by (32) where $\alpha(w)$ is as in (27). Then the solution $(\xi, \bar{\xi})$, (w, w_1) of the closed-loop system is defined for all $t \geq 0$ and converges asymptotically to the set where $\xi = \bar{\xi}$, $\zeta = 0$ and $r = \mathbf{1}_N \rho$.*

Remark 3 Observe that $\xi = \bar{\xi}$ is equivalent to state that $\dot{x}_i = r_i = \rho$, i.e. all the systems of the network have the same velocity ρ. It is immediate to check that $\zeta = 0$ implies that the position variable x for all the systems which are connected via a link takes on the same value. By connectivity of the graph this implies that the position variables of all the systems in the network have the same value.

Remark 4 The controller for the system S_1 is

$$\dot{w}_1 = s(w_1)$$
$$u_1 = \alpha_1(w_1) - (D_1 \otimes I_p)\zeta$$

whereas the controller for system S_i, $i \neq 1$, is

$$\dot{w}_i = s(w_i) - l(w_i)(D_i \otimes I_p)\zeta$$
$$u_i = \alpha_i(w_i)$$

V. Conclusions

We have discussed the role of the internal model principle and of the passivity property in the design of distributed control laws for nonlinear output synchronization and formation control problems for networked nonlinear systems.

References

[1] M. Arcak. Passivity as a design tool for group coordination. *IEEE Transactions on Automatic Control*, 52(8):1380–1390, 2007.

[2] H. Bai, M. Arcak, and J. Wen. *Cooperative Control Design: A Systematic, Passivity-Based Approach*. Communications and Control Engineering. Springer, New York, 2011.

[3] N. Chopra and M. W. Spong. Output synchronization of nonlinear systems with time delay in communication. In *Proceedings of the IEEE Conference on Decision and Control*, pages 4986–4992, 2006.

[4] A. Hamadeh, G-B. Stan, R. Sepulchre, and J. Gonçalves. Global state synchronization in networks of cyclic feedback systems. *IEEE Transactions on Automatic Control*, 57, 2012.

[5] E. Nuño, R. Ortega, L. Basanez, and D. Hill. Synchronization of networks of nonidentical Euler-Lagrange systems with uncertain parameters and communication delays. *IEEE Transactions on Automatic Control*, 56(4):935–941, 2011.

[6] A. Pavlov and L. Marconi. Incremental passivity and output regulation. *Systems and Control Letters*, 57(5):400–409, 2008.

[7] A. Pavlov, E. Steur, and N. Van De Wouw. Controlled synchronization via nonlinear integral coupling. In *Proceedings of the IEEE Conference on Decision and Control*, pages 5263–5268, 2009.

[8] A. Pogromsky and H. Nijmeijer. Cooperative oscillatory behavior of mutually coupled dynamical systems. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 48(2):152–162, 2001.

[9] A. Y. Pogromsky. Passivity based design of synchronizing systems. *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, 8(2):295–319, 1998.

[10] S. Sastry and M. Bodson. *Adaptive Control: Stability, Convergence, and Robustness*. Prentice-Hall Advanced Reference Series (Engineering). Prentice-Hall, 1989.

[11] L. Scardovi and R. Sepulchre. Synchronization in networks of identical linear systems. *Automatica*, 45(11):2557–2562, 2009.

[12] G. Stan and R. Sepulchre. Analysis of interconnected oscillators by dissipativity theory. *IEEE Transactions on Automatic Control*, 52(2):256–270, 2007.

[13] G.-B. Stan, A. Hamadeh, R. Sepulchre, and J. Gonçalves. Output synchronization in networks of cyclic biochemical oscillators. In *Proceedings of the American Control Conference*, pages 3973–3978, 2007.

[14] E. Steur and H. Nijmeijer. Synchronization in networks of diffusively time-delay coupled (semi-)passive systems. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 58(6):1358–1371, 2011.

[15] A.J. van der Schaft. L_2-*Gain and Passivity Techniques in Nonlinear Control*. Springer-Verlag, London, 2000.

[16] P. Wieland, R. Sepulchre, and F. Allgöwer. An internal model principle is necessary and sufficient for linear output synchronization. *Automatica*, 47(5):1068–1074, 2011.