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Abstract   We tackle a number of quadratic optimization problems in the behavioural framework, in the spirit of the classical linear quadratic regulator problem for systems in state-space form. The central theme is dissipativity of the system at hand: we interpret our problems in terms of the exchange of energy between the system and the outside world. Using the Kalman-Yakubovich-Popov lemma in the behavioural framework, we arrive at energy and power optimizing control laws in terms of the solutions to a linear matrix inequality in the original data that is used to specify the problems.

Keywords   dissipative systems theory, linear quadratic (LQ) optimal control, H_2 optimal control, behavioural approach, quadratic differential form (QDF), linear matrix inequality (LMI)

1 Introduction

This paper deals with a number of quadratic optimization problems in the spirit of the classical linear quadratic (LQ) and the H_2 optimal control problem for linear systems in state-space description. The LQ optimal control problem was introduced in modern control theory by Kalman [5], as a quadratic optimization problem in the state and input variables of a linear system. In 1971, Willems [8] solved the LQ problem by interpreting it as an energy minimization problem and by using ideas from dissipative systems theory. We use Willems' energy-based perspective to study quadratic optimization, because this point of view has led to a great deal of insight into the structure of the classical LQ optimal control problem.

Loosely speaking, a system is *dissipative* if it dissipates energy at all times, with the flow of energy being defined by means of a *supply rate* that specifies the rate at which energy flows into the system. The notion of dissipativity was introduced by Willems [9], and it may be found in [9] that a system is dissipative if and only if there exists a *storage function*, which may be interpreted as the energy that is being kept in store inside the system at every instant in time. Moreover, the set of all storage functions of a particular dissipative system has a smallest element, called the *available storage*, and a largest element, called the *required supply*. In terms of dissipative systems theory, the LQ problem is to extract as much energy as possible out of the energy storage inside a dissipative system for a given initial value of the storage function, hence the name *energy optimization problem*.

We study the energy optimization problem for *linear differential systems*, that is, systems whose behaviour may be described as the solution set to a linear differential equation with constant coefficients. Because of their generality, linear differential systems are commonly studied in the behavioural approach to linear systems theory, see Willems [10]. We study linear differential systems that are dissipative with respect to a supply rate specified in terms of a *quadratic differential form* (QDF). A QDF is a quadratic function of a signal and some of its higher-order derivatives, and therefore it is particularly apt to describe expressions involving the variables of a linear differential system, see Willems and Trentelman [13]. The current paper is the sequel of our papers Van der Geest and Trentelman [3] and [4], in which we studied dissipativity of linear differential systems in terms of QDFs. The exchange of energy between a system and its environment is related to the behaviour of the external variable of the system; it does not depend on the partition of this external variable into inputs and outputs. Hence, we believe that dissipative systems theory should be studied from a behavioural point of view. This is our motivation for looking at the energy optimization problem in the behavioural framework.

A storage function is related to the supply rate of a dissipative system by means of a *dissipation rate*, which is the rate at which energy is being dissipated by the system, see Willems [9]. Dissipation rates are non-negative by definition, because a dissipative system dissipates energy at all times. The idea behind energy minimizing control is to make the dissipation rate as small as possible, so as to minimize the amount of energy that the system dissipates. It turns out that if the system is *strongly dissipative*, then it is possible to make the dissipation rate corresponding to the available storage equal to zero. As a result, the initial value of the available storage is precisely the amount of energy that may be redeemed from the system (which justifies the name *available storage*). We use the Kalman-Yakubovich-Popov (KYP) lemma for linear differential systems from [3] to characterize the available storage and its corresponding dissipation rate in terms of a linear matrix inequality (LMI) in the original coefficients of the differential equation that is used to specify the system. This is in line with the behavioural philosophy of solving control problems in terms of the original problem data.

Our energy optimization problem in the behavioural framework is a quadratic optimization problem in the spirit

of the classical LQ problem in state-space form. It is not what is known as the *LQ problem in a behavioural framework* that has been studied by Willems [11]. A trajectory is said to be optimal in [11] if every possible deviation with compact support from the trajectory leads to a degradation in cost. Hence, the LQ problem in a behavioural framework is not from a certain initial time onwards and not subject to initial conditions.

If a dissipative system is subject to disturbances containing an infinite amount of energy, then the amount of energy that may be extracted from it could well be infinite. In such a case it may be useful to try and optimize the *average power*, that is, the average amount of energy per unit of time. An example of power optimization is H_2 optimal control of a linear system in state-space description, which aims to minimize the H_2-norm of the transfer matrix from the disturbance input to the to-be-controlled output. The H_2 optimal control problem is studied together with the H_∞ optimal control problem in Zhou *et al.* [14] and the companion paper Doyle *et al.* [1]. It is shown in [14] and [1] that the H_2-norm is the average power of the to-be-controlled output, if the disturbance signal is *white*. Another example of power optimization is the linear quadratic gaussian (LQG) optimal control problem, which is to minimize the variance of the to-be-controlled output of a linear system in state-space description subject to a stochastic gaussian white noise disturbance signal. This variance is again the H_2-norm of the transfer matrix from the disturbance input to the to-be-controlled output.

It turns out that the power optimizing control law for a system under disturbances is – again – the control law that minimizes the energy dissipation rate. By making the dissipation rate equal to zero, we extract a maximal amount of power from the system. Therefore it may not come as a surprise that the power optimizing control law in a behavioural framework may be derived from the KYP lemma in a similar way as the solution to the energy optimization problem.

2 Dissipativity in a behavioural framework

This paper is about the energy characteristics of *linear differential systems*, that is, linear systems whose behaviour consists of the set of all solutions to a linear differential equation with constant coefficients of the form

$$R\left(\frac{d}{dt}\right)w = 0. \tag{1}$$

Here $w : \mathbb{R} \to \mathbb{R}^q$ is the *manifest* or *external* variable of the system, and $R \in \mathbb{R}^{g \times q}[\xi]$ is a g by q polynomial matrix of a certain degree N,

$$R(\xi) = R_0 + R_1\xi + \cdots + R_N\xi^N. \tag{2}$$

Let r_i denote the ith row of R, and let ρ_i denote its *row degree*, that is, the maximum of the degrees of the elements in row i.

The *manifest* or *external behaviour* or simply *behaviour* described by (1) is defined as

$$\mathcal{B} := \left\{ w \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q) \mid w \text{ is a weak solution of (1)} \right\},$$

where $\mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q)$ denotes the set of *locally integrable functions* from $\mathbb{R}$ to $\mathbb{R}^q$. The concept *weak solution* of a differential equation is explained in Polderman and Willems [6]. It may also be found in [6] that the representation (1) is *minimal* if the polynomial matrix R has full row rank, and that the system $\mathcal{B}$ represented by (1) is *controllable* if and only if the rank of the complex-valued matrix $R(\lambda)$ is the same for all $\lambda \in \mathbb{C}$.

We describe quadratic expressions involving the variable of a linear differential system and its derivatives using *quadratic differential forms* (QDFs), like in Willems and Trentelman [13]. Consider a q by q, symmetric, 2-variable polynomial matrix Φ in the commuting indeterminates ζ and η,

$$\Phi(\zeta, \eta) = \sum_{0 \le i,j \le K} \Phi_{ij}\zeta^i\eta^j, \tag{3}$$

where the matrices Φ_{ij} satisfy $\Phi_{ij} = \Phi_{ji}^T$. The number K is called the *degree* of Φ. Consider also signals $w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$, the set of *infinitely often differentiable functions* from $\mathbb{R}$ to $\mathbb{R}^q$. The mapping $Q_\Phi : \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q) \to \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ defined by

$$Q_\Phi(w) := \sum_{0 \le i,j \le K} \left(\frac{d^i w}{dt^i}\right) \Phi_{ij} \left(\frac{d^j w}{dt^j}\right) \tag{4}$$

is the *QDF* associated with Φ. The symbol $\sim$ (tilde) is used to denote the matrix containing the coefficients of a polynomial or a 2-variable polynomial matrix. Thus, the coefficient matrix of Φ is

$$\widetilde{\Phi} := \begin{pmatrix} \Phi_{00} & \Phi_{01} & \cdots & \Phi_{0K} \\ \Phi_{10} & \Phi_{11} & \cdots & \Phi_{1K} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{K0} & \Phi_{K1} & \cdots & \Phi_{KK} \end{pmatrix}. \tag{5}$$

We define the *energy* with respect to the supply rate Q_Φ contained in the signal w as the integral

$$\int_{-\infty}^{\infty} Q_\Phi(w)(t)dt, \tag{6}$$

provided that the integral exists. Note that the energy contained in the signal w is finite if $w \in \mathcal{D}(\mathbb{R}, \mathbb{R}^q)$, the set of $\mathcal{C}^\infty$-functions from $\mathbb{R}$ to $\mathbb{R}^q$ with *compact support*. The following definition states that a system is *dissipative* if the total amount of energy that flows into the system is non-negative along trajectories with compact support.

Definition 2.1 (Dissipativity)
Let $\mathcal{B}$ be a controllable system with manifest variable w : $\mathbb{R} \to \mathbb{R}^q$. Let Φ be a q by q, symmetric, 2-variable polynomial matrix. Then $\mathcal{B}$ is dissipative with respect to the supply rate Q_Φ if

$$\int_{-\infty}^{\infty} Q_\Phi(w)(t)dt \ge 0 \text{ for all } w \in \mathcal{B} \cap \mathcal{D}(\mathbb{R}, \mathbb{R}^q). \tag{7}$$

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A system is dissipative if and only if there exists a storage function, cf. the following result from Willems and Trentelman [13].

Lemma 2.2 (Storage function)
Let $\mathcal{B}$ be a controllable system with manifest variable w : $\mathbb{R} \to \mathbb{R}^q$. Let Φ be a q by q, symmetric, 2-variable polynomial matrix. Then $\mathcal{B}$ is dissipative with respect to the supply rate Q_Φ if and only if there exists a q by q, symmetric, 2-variable polynomial matrix Ψ such that

$$\frac{d}{dt}Q_\Psi(w) \leq Q_\Phi(w) \text{ for all } w \in \mathcal{B} \cap C^\infty(\mathbb{R}, \mathbb{R}^q). \quad (8)$$

The function Q_Ψ is called a *storage function* for the supply rate Q_Φ on the behaviour $\mathcal{B}$. An alternative characterization of dissipativity is by the existence of a *dissipation rate*, cf. the following result from [13].

Lemma 2.3 (Dissipation rate)
Let $\mathcal{B}$ be a controllable system with manifest variable w : $\mathbb{R} \to \mathbb{R}^q$. Let Φ be a q by q, symmetric, 2-variable polynomial matrix. Then $\mathcal{B}$ is dissipative with respect to the supply rate Q_Φ if and only if there exists a q by q, symmetric, 2-variable polynomial matrix Δ such that for all $t \in \mathbb{R}$ and for all $w \in \mathcal{B} \cap C^\infty(\mathbb{R}, \mathbb{R}^q)$,

$$Q_\Delta(w)(t) \geq 0,$$

and such that for all $w \in \mathcal{B} \cap \mathcal{D}(\mathbb{R}, \mathbb{R}^q)$,

$$\int\limits_{-\infty}^{\infty} Q_\Phi(w)(t)dt = \int\limits_{-\infty}^{\infty} Q_\Delta(w)(t)dt.$$

The function Q_Δ is called a *dissipation rate* for the supply rate Q_Φ on the behaviour $\mathcal{B}$.

The energy balance of a dissipative system is given by the *dissipation equality* that describes the relation between the supply rate Q_Φ, a storage function Q_Ψ, and its corresponding dissipation rate Q_Δ. The dissipation equality states that for all $w \in \mathcal{B} \cap C^\infty(\mathbb{R}, \mathbb{R}^q)$ and for all $t_0, t_1 \in \mathbb{R}$,

$$Q_\Psi(w)(t_1) - Q_\Psi(w)(t_0) = \int\limits_{t_0}^{t_1} Q_\Phi(w)(t) - Q_\Delta(w)(t)dt.$$

We define the *extended representation* corresponding to the representation (1) as

$$R^e\left(\frac{d}{dt}\right)w = 0, \quad (9)$$

where R^e denotes the following polynomial matrix containing the rows and some of the derivatives of the rows of R,

$$R^e(\xi) := \begin{pmatrix} R_1^e(\xi) \\ R_2^e(\xi) \\ \vdots \\ R_g^e(\xi) \end{pmatrix}, \text{ where } R_i^e(\xi) := \begin{pmatrix} r_i(\xi) \\ \xi r_i(\xi) \\ \vdots \\ \xi^{N-\rho_i} r_i(\xi) \end{pmatrix}.$$

Let $v \in \mathbb{R}^{(N+1)q}$ denote the vector consisting of w and all its relevant derivatives,

$$v = \begin{pmatrix} w \\ \frac{dw}{dt} \\ \vdots \\ \frac{d^N w}{dt^N} \end{pmatrix}. \quad (10)$$

The reason for introducing extended representations is that

$$\mathcal{B} = \left\{ w \mid \widetilde{R^e}v = 0 \right\}. \quad (11)$$

We now have all the ingredients to formulate the Kalman-Yakubovich-Popov (KYP) lemma that translates dissipativity of a system into the existence of a solution to a linear matrix inequality (LMI).

Lemma 2.4 (KYP lemma)
Assume that the system $\mathcal{B}$ represented by (1) is controllable. Let Φ be a q by q, symmetric, 2-variable polynomial matrix of degree N. Then the following two statements are equivalent

(a) *$\mathcal{B}$ is dissipative with respect to the supply rate Q_Φ.*

(b) *There exists a symmetric matrix $P \in \mathbb{R}^{Nq \times Nq}$ and a number $\tau \geq 0$ such that*

$$\widetilde{\Phi} + \left(\begin{array}{c} 0_q \\ \vdots \\ 0_q \end{array} \boxed{\begin{array}{c} P \end{array}} \right) + \left(\begin{array}{cc} 0_q \cdots 0_q \\ \boxed{\begin{array}{c} P \end{array}} \begin{array}{c} \vdots \\ 0_q \end{array} \end{array} \right) + \tau \widetilde{R^e}^T \widetilde{R^e} \geq 0.$$

Proof: By Finsler's lemma, the result reduces to the KYP lemma for linear differential systems in [3]. □

Remark 2.5 (All storage functions)
The KYP lemma is a systematic way to find all storage functions for a dissipative system. Let S denote the set of solutions $(P, \tau) \in \mathbb{R}^{Nq \times Nq} \times \mathbb{R}$ to the LMI in the KYP lemma. Then there is a one-one relationship between storage functions Q_Ψ for Q_Φ on $\mathcal{B}$ and matrices P such that $(P, \tau) \in S$, characterized by $\widetilde{\Psi} = -P$. This means that for all $w \in \mathcal{B} \cap C^\infty(\mathbb{R}, \mathbb{R}^q)$,

$$Q_\Psi(w) = -v^T \left(\boxed{\begin{array}{c} P \end{array}} \begin{array}{c} 0_q \\ \vdots \\ 0_q \cdots 0_q \end{array} \right) v.$$

Remark 2.6 (All dissipation rates)
The KYP Lemma also characterizes all dissipation rates for a dissipative system. Let $L(P, \tau)$ denote the left-hand side of the LMI in the KYP lemma for a given solution $(P, \tau) \in S$, and let Q_Ψ denote the storage function corresponding to P. Then the dissipation rate Q_Δ corresponding to Q_Ψ is characterized by $\widetilde{\Delta} = L(P, \tau)$, that is to say,

$$Q_\Delta(w) = v^T L(P, \tau)v \text{ for all } w \in \mathcal{B} \cap C^\infty(\mathbb{R}, \mathbb{R}^q).$$

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The set of storage functions of a dissipative system has a smallest and a largest element, called the *available storage* and the *required supply*, respectively, cf. the following result from Trentelman and Willems [7].

Lemma 2.7 (Available storage, required supply)
Let $\mathcal{B}$ be a controllable system with manifest variable w : $\mathbb{R} \to \mathbb{R}^q$. Let Φ be a q by q, symmetric, 2-variable polynomial matrix. Assume that $\mathcal{B}$ is dissipative with respect to the supply rate Q_Φ. Then there exist storage functions $Q_{\Psi-}$ and $Q_{\Psi+}$ for Q_Φ on $\mathcal{B}$ such that for all storage functions Q_Ψ for Q_Φ on $\mathcal{B}$ and for all $w \in \mathcal{B}$,

$$Q_{\Psi-}(w) \leq Q_\Psi(w) \leq Q_{\Psi+}(w). \tag{12}$$

3 Energy minimization

Consider the linear differential system $\mathcal{B}$ represented by (1) and assume that it is controllable. Pick a vector

$$c = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_N \end{pmatrix} \in \mathbb{R}^{(N+1)q}. \tag{13}$$

Let $\mathcal{B}(c)$ denote the set of signals $w \in \mathcal{B}$ that satisfy the initial conditions

$$w(0) = c_0, \frac{dw}{dt}(0) = c_1, \ldots, \frac{d^N w}{dt^N}(0) = c_N. \tag{14}$$

We say that these initial conditions are *admissible* if $\mathcal{B}(c)$ is nonempty, equivalently, if

$$\widetilde{R} \cdot c = 0. \tag{15}$$

A signal $f \in C^\infty((0, \infty), \mathbb{R}^q)$ is a *future* of the system $\mathcal{B}$ with initial conditions (14) if there exists a signal $w \in \mathcal{B}(c)$ such that $w|_{(0,\infty)} = f$. Let $\mathcal{F}(c)$ denote the set of futures of $\mathcal{B}$ with initial conditions (14).

Consider a q by q, symmetric, 2-variable polynomial matrix Φ. The *energy minimization problem* is to minimize the amount of energy with respect to the supply rate Q_Φ that flows into the system $\mathcal{B}$ in the future, given the initial conditions (14),

$$\inf_{f \in \mathcal{F}(c)} \int_0^\infty Q_\Phi(f)(t)dt. \tag{16}$$

It turns out that this infimum is bounded from below if and only if the system is dissipative.

Theorem 3.1 (Energy minimization)
Assume that the system $\mathcal{B}$ represented by (1) is controllable. Let $\mathcal{F}(c)$ be the set of futures of $\mathcal{B}$ with admissible initial conditions (14). Let Φ be a q by q, symmetric, 2-variable polynomial matrix of degree N. Let S be the solution set

to the LMI in the KYP lemma. Assume that $\mathcal{B}$ is dissipative with respect to the supply rate Q_Φ. Then

$$\inf_{w \in \mathcal{F}(c)} \int_0^\infty Q_\Phi(w)(t)dt = \max_{(P,\tau) \in S} c^T \begin{pmatrix} \begin{array}{|c|} \hline P \\ \hline \end{array} & \begin{array}{c} 0_q \\ \vdots \end{array} \\ 0_q \cdots 0_q \end{pmatrix} c. \tag{17}$$

Proof: In terms of the vector v consisting of w and all its relevant derivatives, the energy minimization problem is to minimize

$$\int_0^\infty v^T(t)\widetilde{\Phi}v(t)dt$$

subject to $\widetilde{R^e}v = 0$, $v(0) = c$, and

$$\begin{pmatrix} I_{Nq} & 0_{Nq \times q} \end{pmatrix} \frac{dv}{dt} = \begin{pmatrix} 0_{Nq \times q} & I_{Nq} \end{pmatrix} v. \tag{18}$$

It follows from the dissipation equality and Remark 2.5 that the maximum in (17) is the initial value of the available storage. $\square$

We define the system $\mathcal{B}$ to be *strongly dissipative* with respect to the supply rate Q_Φ if there exists an $\epsilon > 0$ such that for all $w \in \mathcal{B} \cap \mathcal{D}(\mathbb{R}, \mathbb{R}^q)$,

$$\int_{-\infty}^\infty Q_\Phi(w)(t)dt \geq \epsilon \int_{-\infty}^\infty w^T(t)w(t) + \left(\frac{d^N w}{dt^N}(t)\right)^T \frac{d^N w}{dt^N}(t)dt.$$

If the system is strongly dissipative, then the dissipation rate corresponding to the available storage admits a Hurwitz spectral factorization. The following result is an entension of the spectral factorization result in [3].

Lemma 3.2 (Hurwitz spectral factorization)
Assume that the system $\mathcal{B}$ represented by (1) is controllable and that the representation (1) is minimal. Let Φ be a q by q, symmetric, 2-variable polynomial matrix of degree N. Assume that $\mathcal{B}$ is strongly dissipative with respect to the supply rate Q_Φ. Let $Q_{\Delta-}$ denote the dissipation rate corresponding to the available storage $Q_{\Psi-}$. Then there exists a polynomial matrix $F \in \mathbb{R}^{(q-g) \times q}[\xi]$ such that

$$Q_{\Delta-}(w) = \left\| F\left(\frac{d}{dt}\right)w \right\|^2 \text{ for all } w \in \mathcal{B} \cap C^\infty(\mathbb{R}, \mathbb{R}^q),$$

and such that the polynomial matrix $\begin{pmatrix} R \\ F \end{pmatrix}$ is Hurwitz.

It turns out that there exists a unique energy-optimizing future of the system $\mathcal{B}$ with initial conditions (14) if and only if $\mathcal{B}$ is strongly dissipative.

Theorem 3.3 (Energy minimizing control)
Assume that the system $\mathcal{B}$ represented by (1) is controllable. Let $\mathcal{F}(c)$ be the set of futures of $\mathcal{B}$ with admissible initial conditions (14). Let Φ be a q by q, symmetric, 2-variable polynomial matrix of degree N. Let S be the solution set

to the LMI in the KYP lemma, and let $L(P, \tau)$ denote its left-hand side. Assume that $\mathcal{B}$ is strongly dissipative with respect to the supply rate Q_Φ. Pick

$$(P^*, \tau^*) \in \arg \max_{(P, \tau) \in S} c^T \left(\begin{array}{c|c} \boxed{P} & \begin{array}{c} 0_q \\ \vdots \end{array} \\ \hline 0_q \cdots 0_q \end{array} \right) c. \qquad (19)$$

Find a matrix $\widetilde{C} \in \mathbb{R}^{(N+1)q \times (N+1)q}$ such that

$$L(P^*, \tau^*) = \widetilde{C}^T \widetilde{C}. \qquad (20)$$

Let $C \in \mathbb{R}^{(N+1)q \times q}[\xi]$ be the polynomial matrix with coefficient matrix $\widetilde{C}$. Then there exists a unique energy minimizing future $f^* \in \mathcal{F}(c)$, characterized by

$$C\left(\frac{d}{dt}\right) f^* = 0. \qquad (21)$$

Proof: By the dissipation equality and Remark 2.6, the control law (21) makes the dissipation rate corresponding to the available storage equal to zero. It follows from Lemma 3.2, that an alternative characterization of f^* is

$$F\left(\frac{d}{dt}\right) f^* = 0. \qquad (22)$$

We may conclude that (21) is stabilizing, since (22) is stabilizing. To prove that there exists a signal $f^* \in \mathcal{F}(c)$ such that (21) or (22), consider first the case that the matrix R is non-existent. It is clear that there exists a v subject to $v(0) = c$ and (18) such that

$$\widetilde{F} v|_{(0, \infty)} = 0. \qquad (23)$$

To conclude that the result also holds when the matrix R is non-trivial, note that the factorization (20) is defined on the linear subspace in $\mathbb{R}^{(N+1)q}$ characterized by $\widetilde{R^e} v = 0$. Furthermore, the initial conditions (14) are admissible, which means that they are defined on the same subspace. This means that the entire problem is defined on this linear subspace. $\qquad \square$

4 Power optimization

Consider the following linear differential system with manifest variable $w : \mathbb{R} \to \mathbb{R}^q$ subject to a disturbance signal $d : \mathbb{R} \to \mathbb{R}^r$,

$$R\left(\frac{d}{dt}\right) w = D\left(\frac{d}{dt}\right) d, \qquad (24)$$

where R is a $g \times q$ polynomial matrix of degree N and D is a $g \times r$ polynomial matrix. Define the *full behaviour* described by (24) as

$$\mathcal{B}_f := \{(w, d) \in C^\infty(\mathbb{R}, \mathbb{R}^q \times \mathbb{R}^r) \text{ s.t. (24)}\}.$$

The set of all signals w that are possible for a given value of d, is defined as

$$\mathcal{B}(d) := \{w \in C^\infty(\mathbb{R}, \mathbb{R}^q) \text{ s.t. (24)}\}. \qquad (25)$$

The undisturbed counterpart of the system $\mathcal{B}_f$ is the system $\mathcal{B}$ represented by (1).

Consider a q by q, symmetric, 2-variable polynomial matrix Φ. Define the *average power* with respect to the supply rate Q_Φ contained in the signal w as

$$\mathcal{P}_\Phi(w) := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} Q_\Phi(w)(t) dt, \qquad (26)$$

provided that the integral exists for all T and that the limit exists. Define the *autocorrelation matrix* of the signal $d \in C^\infty(\mathbb{R}, \mathbb{R}^r)$ like in Zhou *et al.* [14],

$$R_d(\tau) := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} d(t) d^T(t) dt, \qquad (27)$$

again provided that the integral exists for all T and that the limit exists. We say that $d \in C^\infty(\mathbb{R}, \mathbb{R}^r)$ has *bounded power*, denoted by $d \in \mathcal{L}_2\text{i.m.}(\mathbb{R}, \mathbb{R}^r)$, if

$$\text{trace}(R_d(0)) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} d^T(t) d(t) dt < \infty. \qquad (28)$$

The *spectral density* of d is the Fourier transform of its autocorrelation matrix, that is,

$$S_d(i\omega) := \int_{-\infty}^{\infty} R_d(\tau) e^{-i\omega\tau} d\tau. \qquad (29)$$

We would like to minimize the average power contained in the manifest variable w, given that the disturbance signal d is *white*, which is defined by $S_d(i\omega) = I_r$ for all $\omega \in \mathbb{R}$. However, it is well-known that the primitive of a white signal is a well-defined function, but that a white signal is not, *cf.* the discussion in Zhou *et al.* [14]. Therefore we replace d with a sequence of disturbances of bounded power

$$\{d_k \in C^\infty(\mathbb{R}, \mathbb{R}^r) \cap \mathcal{L}_2\text{i.m.}(\mathbb{R}, \mathbb{R}^r) | k \in \mathbb{N}\} \qquad (30)$$

with the property that

$$\lim_{k \to \infty} S_{d_k}(i\omega) = I_r \text{ for all } \omega \in \mathbb{R}. \qquad (31)$$

We define the *power optimization problem* as to miminize the average power contained in the signal $w \in \mathcal{B}(d_k)$ in the limit as k tends to infinity,

$$\lim_{k \to \infty} \inf_{w \in \mathcal{B}(d_k)} \mathcal{P}_\Phi(w). \qquad (32)$$

The choice that nature makes for the disturbance signal d should be unexpected, so d must not depend on the manifest variable w. Instead, d must be a *free variable*, that is, for all $d \in \mathcal{L}_2\text{i.m.}(\mathbb{R}, \mathbb{R}^r)$ there exists a signal $w \in \mathcal{L}_2\text{i.m.}(\mathbb{R}, \mathbb{R}^q)$ such that $(w, d) \in \mathcal{B}_f$. It is guaranteed that d is free, if the rows of D have lower degrees than the corresponding rows of R.

It turns out that the power optimization problem in a behavioural framework may be solved in terms of the same LMI as in the energy optimization problem.

Theorem 4.1 (Power minimization)

Consider the system $\mathcal{B}_f$ represented by (24) and the sequence $\{d_k\}$ defined by (30) and (31). Assume that the rows of D have lower degrees than the corresponding rows of R. Let Φ be a q by q, symmetric, 2-variable polynomial matrix of degree N. Assume that the undisturbed system $\mathcal{B}$ represented by (1) is controllable and that it is dissipative with respect to the supply rate Q_Φ. Let S be the solution set to the LMI in the KYP lemma. Then

$$\lim_{k\to\infty} \inf_{w\in\mathcal{B}(d_k)} \mathcal{P}_\Phi(w) = \max_{(P,\tau)\in S} \operatorname{trace}\left(\widetilde{D}^T \left(\begin{array}{cc} \boxed{P} & \begin{matrix} 0_q \\ \vdots \end{matrix} \\ 0_q \cdots 0_q & \end{array} \right) \widetilde{D} \right)$$

Proof: The proof is the same as the proof of Theorem 3.1, together with an argument where we take the limit as $k \to \infty$. Details may be found in Van der Geest [2]. □

When we connect a controller to the system, we must leave the disturbance d free. The following result states that there exists a power minimizing control law if and only if the system $\mathcal{B}$ is strongly dissipative.

Theorem 4.2 (Power minimizing control)

Consider the system $\mathcal{B}_f$ represented by (24) and the sequence $\{d_k\}$ defined by (30) and (31). Assume that the rows of D have lower degrees than the corresponding rows of R. Let Φ be a q by q, symmetric, 2-variable polynomial matrix of degree N. Assume that the undisturbed system $\mathcal{B}$ represented by (1) is controllable and that it is strongly dissipative with respect to the supply rate Q_Φ. Let S be the solution set of the LMI in the KYP lemma and let $L(P,\tau)$ denote its left-hand side. Pick

$$(P^*,\tau^*) \in \arg\max_{(P,\tau)\in S} \operatorname{trace}\left(\widetilde{D}^T \left(\begin{array}{cc} \boxed{P} & \begin{matrix} 0_q \\ \vdots \end{matrix} \\ 0_q \cdots 0_q & \end{array} \right) \widetilde{D} \right).$$

Find a matrix $\widetilde{C} \in \mathbb{R}^{(N+1)q\times(N+1)q}$ such that

$$L(P^*,\tau^*) = \widetilde{C}^T\widetilde{C}. \tag{33}$$

Let $C \in \mathbb{R}^{(N+1)q\times q}[\xi]$ be the polynomial matrix with coefficient matrix $\widetilde{C}$. For all $k \in \mathbb{N}$ there exists a unique signal $w_k \in \mathcal{B}(d_k)$ such that

$$C\left(\frac{d}{dt}\right) w_k = 0. \tag{34}$$

Moreover, the sequence $\{w_k\}$ satisfies the property

$$\lim_{k\to\infty} \mathcal{P}_\Phi(w_k) = \max_{(P,\tau)\in S} \operatorname{trace}\left(\widetilde{D}^T \left(\begin{array}{cc} \boxed{P} & \begin{matrix} 0_q \\ \vdots \end{matrix} \\ 0_q \cdots 0_q & \end{array} \right) \widetilde{D} \right).$$

Proof: The proof is the same as the proof of Theorem 3.3, together with an argument where we take the limit as $k \to \infty$. Details may be found in Van der Geest [2]. □

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