Copyright
by
Apurva Arvind Chunodkar
2009

Attitude Dynamics Stabilization with Unknown Delay in Feedback Control Implementation

APPROVED BY

SUPERVISING COMMITTEE:

Maruthi Akella, Supervisor

E. Glenn Lightsey

Attitude Dynamics Stabilization with Unknown Delay in Feedback Control Implementation

by

Apurva Arvind Chunodkar, B.Tech.

THESIS

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

MASTER OF SCIENCE IN ENGINEERING

THE UNIVERSITY OF TEXAS AT AUSTIN

December 2009

To my family.

Acknowledgments

First and foremost, I would like to thank my faculty advisor Dr. Maruthi Akella. He has not only taught me a lot about control theory through his courses, but I have also modeled myself to teach like him. I have learnt and continue learning from him about how to think when conducting research. His unending patience in the face of many mistakes has given me a lot of confidence, which has made this thesis possible. I also acknowledge his effort in supporting an international aerospace engineering graduate student through tough economic times, which has not been easy to do. I look forward to working with him through the course of my doctoral degree.

A special thanks to Dr. Glenn Lightsey for taking the time to be a second reader for this thesis. I have learnt a lot from the hardware oriented "Sensors and Actuators Laboratory" course taught from Dr. Lightsey and would enjoy any opportunity to learn from him in the future. I also acknowledge the support given by the Air Force Research Laboratory's Space Vehicle Directorate, which has played a very important part in making this thesis possible. I look forward to continue working on interesting research problems provided by the wonderful people from AFRL. In addition, I would like to thank the many faculty members of UT Austin, with whom I've had the pleasure of taking some really great courses.

I would like to thank my family back home in India, my father , mother , sister and my dog. Their financial support in hard economic times is but a very small part of what I owe them. Finally, Baba, I can go ahead and get a doctoral degree that you always wanted to get for yourself. Aai and Sonu, thanks very much for supporting my decision to undertake a career in research, I have drawn a lot of courage from our talks over the years. It is said that one cannot choose one's family, but I cannot ask for a better one even if I had the choice.

I am grateful to our research group members Travis, Srikant and Tyler, who with the benefit of experience, have helped me many times in different areas, which are too numerous to list here. Last but not the least, I would like to thank my friends Ganesh, Sowmitra and Aruna, whose exuberant company at the end of the day through the last couple of years enabled me to renew my work afresh every morning. Their help, especially through some difficult times during my Phd. qualifying exams can never be forgotten.

Attitude Dynamics Stabilization with Unknown Delay in Feedback Control Implementation

Apurva Arvind Chunodkar, M.S.E
The University of Texas at Austin, 2009

Supervisor: Maruthi Akella

This work addresses the problem of stabilizing attitude dynamics with an unknown delay in feedback. Two cases are considered: 1) constant time-delay 2) time-varying time-delay. This is to our best knowledge the first result that provides asymptotically stable closed-loop control design for the attitude dynamics problem with an unknown delay in feedback. Strict upper bounds on the unknown delay are assumed to be known. The time-varying delay is assumed to be made of the constant unknown delay with a time-varying perturbation. Upper bounds on the magnitude and rate of the time-varying part of the delay are assumed to be known. A novel modification to the concept of the complete type Lyapunov-Krasovskii (L-K) functional plays a crucial role in this analysis towards ensuring stability robustness to time-delay in the control design. The governing attitude dynamic equations are partitioned to form a nominal system with a perturbation term. Frequency domain analysis is employed in order to construct necessary and sufficient stability conditions

for the nominal system. Consequently, a complete type L-K functional is constructed for stability analysis that includes the perturbation term. As an intermediate step, an analytical solution for the underlying Lyapunov matrix is obtained. Departing from previous approaches, where controller parameter values are arbitrarily chosen to satisfy the sufficient conditions obtained from robustness analysis, a systematic numerical optimization process is employed here to choose control parameters so that the region of attraction is maximized. The estimate of the region of attraction is directly related to the initial angular velocity norm and the closed-loop system is shown to be stable for a large set of initial attitude orientations.

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1 Motivation

The problem of rigid body attitude dynamics and control has been studied extensively over the last few decades, due to its significance with respect to a wide range of applications, ranging from rigid aircraft and spacecraft systems to coordinated robot manipulators [2]. For example, rigid spacecraft applications, in particular, require highly accurate pointing maneuvers. These performance requirements necessitate the spacecraft model to be essentially nonlinear, so that large amplitude angle orientations are accurately stabilized. Several results exist on feedback with attitude dynamics tackling various aspects of the attitude control problem. For instance, it is well known that linear feedback of the states asymptotically stabilizes the closed-loop dynamics [2].

The problem of stabilization of attitude dynamics when feedback is time-delayed is practically motivated and challenging to solve. The time-delay is often unknown and at times time-varying. Time-delay can arise from processing delays in actuator and/or sensor dynamics. For example, consider a rigid spacecraft actuated by gas jet control system with actuators along perpendicular axes. The thrust provided by this actuator system depends on electrical and mechanical delays in the valve circuits and the time for the propellant to flow from the valve to the thruster [13]. Time-delay can also arise in feedback due to communication delays. Analyzing effects of unknown time-delay in feedback for a single spacecraft, which is an open problem, is also a necessary and useful starting point for studying its impact on communication between multiple spacecraft in a formation, an evolving area of research and has applications such as space interferometry, synthetic aperture radar, and on-orbit assembly.

As a result of feedback time-delay, the governing attitude dynamics does not contain a time delay by itself, but is subjected to one in the closed-loop, since current information of states is not available for feedback. The effect of delay in feedback on system stability is an important problem to study because delay is known to generally have a destabilizing effect, which if not accounted for, leads to oscillatory behavior or even loss of stability [3]. Various classical feedback linearization and Lyapunov based control design techniques for nonlinear system stability cannot be employed since the feedback does not contain current values of states. This leads to analyzing the problem from a time delay system (TDS) framework.

1.2 Literature Review

In this section, a brief survey on development in time and frequency domain stability analysis of time-delay systems is presented. In addition, we consider recent progress made in stability of nonlinear time-delay systems and

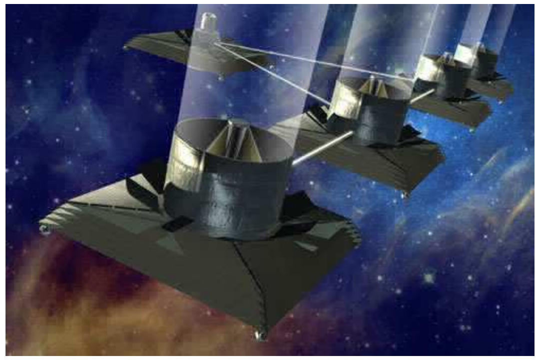


Figure 1.1: Communication delays in spacecraft [1]

stabilizing attitude dynamics with time-delayed feedback.

1.2.1 Time-delay systems: Overview

The study of time-delay systems, although starting at a similar place as delay-free systems, continues along a different path and leads to richer and more complex problems. Over the years, various methods have been studied in order to represent time-delays in mathematical models. A brief description and comparison between these modeling approaches, such as frequency domain or rational approximation of time delay, can be found in Reference [4]. Of these methods, the use of functional differential equations or FDEs to model time-delay effects has proven to be popular of late, because FDEs are able to incorporate the most general forms of delays, and are also generally computationally tractable. In the sequel, the terms "functional differential equations" and "time-delay systems" will be used interchangeably.

The first FDEs were considered by Euler, Bernoulli, Lagrange and others in the 18th century in order to study various geometric problems. Starting in the early 20th century, various practical problems in areas such as viscoelasticity and ship stabilization were modeled using FDEs [5]. Characteristic equations of linear time-invariant FDEs are quasipolynomials in general. Stability analysis through finding zeros of quasipolynomials started with Pontryagin who obtained fundamental results. In the time-domain, properties of FDE solutions have been richly researched since the 1950s, starting with Myshkis in 1949, who for the first time formulated the initial value problem. Several classical results on solution properties of scalar FDEs with constant and time varying delays have since been proposed by Yorke and others [14, 27, 28]. Krasovskii extended Lyapunov's second method to time-delay systems in the 1950s through the construction of so-called Lyapunov-Krasovskii "functionals" for stability analysis, which will be explored in the forthcoming Chapters. An approach using Lyapunov functions instead of functionals was proposed by Razumikhin, who imposed certain restrictions on system trajectories. There are several books covering different aspects of time-delay systems such as Myshkis [5], Bellman and Cooke [6], Hale and Lunel [7], Kharitonov [8].

1.2.2 Stability of nonlinear time-delay systems

Stability analysis of nonlinear time-delay systems presents many more challenges when compared to that of nonlinear systems without delay in general. The problem difficulty depends considerably on the nature of delay present in the system. For instance, consider a nonlinear system where delay is present, in the feedback and also in the system dynamics. Stability of a general class of such systems has been addressed in Reference [9]. For example, consider the system

$$\dot{x}(t) = g(x(t-\tau)) + u(t-\tau) \tag{1.1}$$

In this case, the control can be readily employed to "get rid of the harmful terms" by feedback linearization or backstepping. A more difficult problem arises system where the feedback can be a function of time-delayed states alone, and delay does not arise anywhere else in the system dynamics. For example, consider the following system

$$\dot{x}(t) = f(x(t)) + u(t-\tau) \tag{1.2}$$

Feedback cannot be used to perfectly cancel out the plant dynamics in this case. An example of such a system is rigid body attitude dynamics with feedback being time-delayed as will be shown in subsequent Chapters. A further layer of difficulty is when the delay itself is unknown. Moreover, stability analysis of systems with time-varying delay is more complicated than that of systems with constant time-delay.

Lyapunov based stability methods, which have proven to be a popular tool for control design in nonlinear systems have been extended to TDS. Krasovskii proposed the idea of a Lyapunov functional [10] (i.e. $V(t, x_t)$, $x_t \in [x(t-\tau), x(t)]$, $\tau > 0$), also called a Lyapunov-Krasovskii (L-K) functional, as opposed to a Lyapunov function $V(t, x)$ in order to prove stability of certain classes of TDS. The problem of constructing a L-K functional for any given TDS is comparatively more difficult than constructing a Lyapunov function for its delay free counterpart, as will be shown in later sections of this thesis. Moreover, there is no constructive method to formulate a L-K functional for a particular TDS. Classical or "reduced" L-K methods cannot be applied to systems which are unstable in the absence of delayed terms, i.e. the delayed term is treated as a "perturbation" causing instability to the delay-free "nominal system". As a result, the problem of L-K functional construction in general, has generated considerable interest in TDS research.

During the last few years, Kharitonov and Zhabko [11] proposed a constructive method to formulate a complete-type (i.e. completely quadratic) L-K functional for any given TDS which was known to be exponentially stable. Further, this complete-type L-K functional was employed for robust stability analysis for bounded perturbations to the the exponentially stable nominal system, and estimates on how large the drift term could be were obtained. Niculescu further elaborated this idea as a method to achieve regional stabilization, when the perturbation could be nonlinear and bounded by a linear growth [12]. The concept of the complete type L-K functional and its use in robustness analysis extends naturally from finite dimensional linear time-invariant systems as will be demonstrated in later chapters. In the finite-dimensional case, it is well-known that for a system of the form $\dot{x} = Ax + f(x)$, where $\dot{x} = Ax$ is exponentially stable (i.e. A is Hurwitz) and $\|f(x)\| < \gamma\|x\|$ for some positive γ, an estimate of the attraction region can be derived from Lyapunov's second method using Lyapunov's equation. The estimate depends directly on γ which in turn is is a constant depending on a symmetric matrix P and function $f(x)$ (see, for instance, [13]). In this case, $P > 0$ is the solution of the Lyapunov equation $A^{\mathrm{T}}P + PA = -W$ for any chosen $W > 0$. Further, γ can be maximized by choosing P and W subject to the constraint of the Lyapunov equation. γ is maximized by choosing W to be the identity matrix and P as the solution to the corresponding Lyapunov equation. This method follows from using the Lyapunov function associated with the linear system for stability analysis of the nonlinear system in a neighborhood of the origin.

An important distinction in the construction of the complete-type L-K functional is that no assumptions are imposed on the system in the absence of delayed terms. For example, consider the scalar integrator $\dot{x}(t) = u(t-\tau)$ with delayed feedback $u(t-\tau) = -bx(t-\tau)$. It is well known in literature that this scalar integrator with delayed feedback is exponentially stable if and only if $0 < b < \pi/2\tau$ [14]. Assuming that this condition is met, a complete-type L-K functional can be constructed for this system and used for robustness analysis, when a drift term $f(x(t), x(t-\tau))$ present as in $\dot{x}(t) = f(x(t), x(t-\tau)) - bx(t-\tau)$ with $0 < b < \pi/2\tau$ [12]. The complete type L-K functional has since been applied to a biological problem [15]. However, the estimate on the region of attraction obtained was found to be somewhat conservative [15]. Further, the analysis has been extended to construct a complete type L-K functional which had a cross term in the time derivative [16]. This generalization reduced the conservatism of the estimate.

A notable requirement for this technique in calculating an estimate for the region of attraction of the given system is precise knowledge of the time-delay, which can be restrictive. The analysis in Reference [12] extended the complete type Lyapunov-Krasovskii technique for stability analysis of systems with time-varying delay. The technique treats the terms with time-varying delay as perturbations and employs a "model transformation" method [21, 22] in order to transform the system with a time-varying delay into a system with a constant time-delay with a perturbation which includes the time-varying delay term. As a result of this transformation, certain additional dynamics are introduced, due to which stability of the transformed system implies stability of the actual system, with the converse not necessarily being true. As a result, the stability result obtained is additionally conservative [12]. The technique, in the time-varying delay case also, relies on precise knowledge of the time-delay. Moreover, the sufficient conditions obtained are quite difficult to satisfy because they are not constructive. In certain cases, it has been found that as the magnitude and rate of the time-varying delay become sufficiently large, the sufficient conditions cannot be satisfied.

1.2.3 Attitude Dynamics with time-delay in feedback

Recently, Reference [3] has proposed a velocity-independent time-delay controller for regulating the attitude orientation of a rigid body. Rodrigues Parameters (RPs) are employed to represent the attitude orientation. The control design involves filter construction to avoid velocity measurement. Sufficient conditions for exponential stability of the system inside a region of attraction, whose estimate is calculated, and a measure to evaluate the system rate of convergence of the system to a desired setpoint, are presented. However, the control design requires the delay to be known precisely, which is a restrictive condition. Moreover, the estimate of the region of attraction was found to be quite conservative, especially in terms of initial attitude orientations. Our work will relax the restriction requiring precise knowledge of time-delay and also obtain improved estimates of the region of attraction by making novel modifications in the complete type L-K approach.

1.3 Contributions of this Thesis

This thesis considers the problem of stabilizing attitude dynamics with an unknown delay in the feedback. Two cases are considered: 1) constant time-delay 2) time-varying time-delay. Strict upper bounds on the unknown constant delay are assumed to be known. The time-varying delay is assumed to be made off the constant unknown delay with a time-varying perturbation. Upper bounds on the magnitude and rate of the time-varying part of the delay are assumed to be known. Modified Rodrigues Parameters (MRPs) are employed to represent the attitude orientation. The control input is linear in the delayed states, i.e. MRPs and angular velocities. The following points enumerate the contributions made by this work.

1. A novel modification to the concept of the complete type Lyapunov-Krasovskii (L-K) functional technique for stability analysis of a class

of nonlinear time-delay systems is presented. This modification enables stability robustness to time-delay in the control design i.e. stability holds for all values of time-delay less than the known upper bound.

2. Robust stability analysis with the complete type L-K functional provides sufficient conditions in terms of an estimate for the region of attraction of the nonlinear time-delay system. The estimate of the region of attraction is robust to time-delay.

3. In order to employ the modified theory of complete type L-K functional for robust stability analysis of the attitude problem, the governing dynamics is separated into the form of a nominal dynamics, which is exponentially stable, and a perturbation or drift term as in the case of finite-dimensional systems. The novel separation results in the nominal system being in the form of 3 blocks of double integrators with delayed linear feedback. After finding the range of control gain values for which the nominal (linear) system is exponentially stable by using frequency domain analysis, a complete type L-K functional is constructed.

4. The perturbation term obtained is such that it is a function of the current value of the states alone, which allows specializations to be made in the complete type L-K functional construction. As a result, the extent of conservatism in the region of attraction estimate obtained is reduced.

5. As an intermediate step, an analytical solution for the Lyapunov matrix is obtained by using Kronecker algebra [17] for the governing matrix dif-

ferential equations and associated boundary conditions. Previous to this development, a piecewise linear approximation of the Lyapunov matrix was employed by some authors.

6. Departing from previous approaches, where parameter values are arbitrarily chosen to satisfy the sufficient conditions, numerical optimization is employed in order to choose parameters such that the region of attraction is maximized. The initial angular velocity norm is found to be directly related to the size of the estimate of the region of attraction and the closed-loop system is stable for a large set of initial attitude orientations.

7. The complete type L-K functional approach is extended to the corresponding class of nonlinear systems with unknown time-varying delay. A model transformation is employed to partition the system into a nominal time-invariant time-delay system with a time-varying perturbation. Sufficient conditions on regional stabilization are obtained.

8. Limiting cases arising from the sufficient conditions are studied, along with additional conservatism present in the conditions and comparison is made with corresponding results from the constant delay case.

These are to our best knowledge the first results that provide asymptotically stable closed-loop control designs for the attitude dynamics problem with an unknown delay in feedback.

The rest of the thesis is organized as follows: Chapter 2 offers mathematical preliminaries dealing with time-delay systems, problem statement for attitude dynamics stabilization with unknown delay in feedback and a motivating example for study of effect of delay on stability. Chapter!2 also provides a brief introduction to the Lyapunov-Krasovskii approach, a motivating example as well as certain limitations. Further, the development of a modified complete type L-K functional for a class of nonlinear systems with unknown constant delay as well as unknown time-varying delay is presented in Chapter 3. Chapters 4 and 5 present application of the complete type L-K technique to the attitude stabilization problem with constant and time-varying delay respectively along with simulation results, whereas the summary and discussion are presented in Chapter 6.

Chapter 2

Background

In this chapter, we provide some necessary background on study of time-delay systems, such as mathematical preliminaries and stability analysis with some illustrative examples. We also explain the problem statement of this thesis and provide notation, which will be followed through the remainder of this report.

2.1 Preliminaries

In this section, we introduce some mathematical preliminaries for a prototypical retarded time-delay system. Throughout the remainder of this work, we will employ the use of *time-delay systems* (TDS) interchangeably with *functional differential equations of the retarded type* or *retarded functional differential equations* (RFDEs). For this work, we consider differential equations with a single delay value, i.e., of the form,

$$\dot{x}(t) = g(x(t), x(t - \tau(t))) \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ and $\tau(t) \in \mathbb{R}_+$ denotes the delay and is assumed to be a bounded continuous function $\forall\ t$. For our problem $\tau(t) = \tau_0 + \eta(t)$. Equation 2.1 requires an initial condition for propagation. The initial condition for an ordinary differential equation (ODE) is a point, i.e. $x(t_0) = x_0$, taken at initial time $t = t_0$. However, the initial condition for a functional differential equation is a trajectory in general since the derivative of the state variable $x(t)$ depends on t and $x(\zeta)$ for $t - \tau(t) \leq \zeta \leq t$ [7]. For Equation 2.1, the initial condition can be written as

$$x(t) = \phi(t),\ t \in \Upsilon \tag{2.2}$$

where Υ is given by

$$\Upsilon = \{t \in \mathbb{R} : \ -2\tau_0 \leq t \leq 0\} \tag{2.3}$$

where Equation 2.3 represents a class of trajectories with argument $t \leq 0$ since the time-delay $\tau(t)$ is required to have the construction $\tau_0 = \sup_t |\eta(t)|$, where $t \in \mathbb{R}$. For the special case of a constant delay, i.e. $\tau(t) = \tau_0 \geq 0$, Equation 2.3 reduces to

$$\Upsilon = \{t \in \mathbb{R} :\ t = \theta,\ \ -\tau_0 \leq \theta \leq 0\} \tag{2.4}$$

Referring back to Equation 2.1, $g(u, v)$ represents a Lipschitz function in some neighborhood of the origin, where $u, v \in \mathbb{R}^n$ and

$$\lim_{\|(u,v)\| \to 0} \frac{\|g(u, v)\|}{\|(u, v)\|} = 0 \tag{2.5}$$

From Equation 2.5, it follows that $g(0, 0) = 0$. In order to define a particular solution $x(t, \phi)$ of Equation 2.1, an initial condition trajectory $\phi(t)$, $t \in \Upsilon$, should be given. We assume that ϕ belongs to the space of continuous vector functions $\mathcal{C}$ mapping Υ to $\mathbb{R}^n$. We assign a uniform vector norm $\|\phi\|_c$ to the space $\mathcal{C}$

$$\|\phi\|_c = \sup_{\theta \in \Upsilon} \|\phi(\theta)\| \tag{2.6}$$

Additionally, we denote by $x_t(\phi) = x(t+\theta, \phi)$, $\theta \in \Upsilon$, the propagation of the solution $x(t, \phi)$ on Υ. Throughout this work, we employ the Euclidean norm for vectors i.e. $\|.\| = \|.\|_2$. For any square matrix A, we employ the induced matrix 2-norm i.e. $\|A\| = \sqrt{\lambda_{max}\left(A^{\mathrm{T}}A\right)}$, wherein we denote the maximum and minimum eigenvalues of any symmetric matrix M by $\lambda_{max}(M)$ and $\lambda_{min}(M)$ respectively.

We introduce stability definitions for time-delay systems of the class represented by Equation 2.1 [7]. The definitions mirror those of finite-dimensional systems in general.

Definition 1. The trivial solution $x(t, \phi) = 0$ (or $x(t) = 0$ for simplicity) of (2.1) is stable if for any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that $\|\phi\|_c \leq \delta$ implies $\|x(t, \phi)\| \leq \epsilon$ for all $t \geq 0$.

Definition 2. The trivial solution $x(t) = 0$ of Equation 2.1 is asymptotically stable if it is stable and there exists a $\delta_a > 0$ such that $\|\phi\|_c \leq \delta_a$ implies $x(t, \phi) \rightarrow 0$ as $t \rightarrow \infty$

Definition 3. The trivial solution $x(t) = 0$ of Equation 2.1 is exponentially stable if there exist constants $\delta_e > 0$, $\mu \geq 1$, $\alpha > 0$ such that $\|\phi\|_c \leq \delta_e$ implies $\|x(t, \phi)\| \leq \mu \|\phi\|_c \mathrm{e}^{-\alpha t}$ for all $t \geq 0$.

2.2 Problem Statement

The problem considered in this work is that of stabilizing attitude dynamics of a rigid body with an unknown delay in feedback. We employ Modified Rodrigues Parameters (MRPs) in order to represent the attitude orientation of the rigid body. The system dynamics can be expressed as

$$\dot{\sigma}(t) = \frac{1}{4}B(\sigma(t))\,\omega(t) \tag{2.7}$$

$$J\dot{\omega}(t) = -\omega(t)^{\times}J\omega(t) + u(t-\tau(t)) \tag{2.8}$$

where $\sigma(t) \in \mathbb{R}^3$ is the MRP vector, $J = J^{\mathrm{T}} \in \mathbb{R}^{3\times 3}$ is the positive definite mass inertia matrix and $B(\sigma)$ is defined as

$$B(\sigma) = \left[\left(1 - \sigma^{\mathrm{T}}\sigma\right) I_3 + 2\sigma^{\times} + 2\sigma\sigma^{T}\right] \tag{2.9}$$

where the skew symmetric matrix $\omega(t)^{\times}$ for any $\omega(t) = [\ \omega_1(t)\ ;\ \omega_2(t)\ ;\ \omega_3(t)\ ]^{\mathrm{T}}$ is defined as

$$\omega(t)^{\times} = \begin{bmatrix} 0 & -\omega_3(t) & \omega_1(t) \\ \omega_3(t) & 0 & -\omega_2(t) \\ -\omega_1(t) & \omega_2(t) & 0 \end{bmatrix} \tag{2.10}$$

The MRP vector has the physical interpretation of being

$$\sigma = \hat{e}\tan(\Phi/4) \tag{2.11}$$

where $\hat{e}$ is the three-dimensional unit vector along the principal rotation axis and Φ is the principal rotation angle. The MRP vector $\sigma(t)$ is nonsingular for all rotations up to 360°, i.e., $-\pi < \Phi < \pi$ [18, 30]. If $\sigma(t) \to 0$, then the orientation has returned back to the origin. It is shown in Reference [18] that the singularity at 360° can be avoided by mapping the original MRP vector to its corresponding shadow counterpart σ^s through

$$\sigma^s = -(1/\sigma^2)\sigma \tag{2.12}$$

where $\sigma^2 = \sigma^{\mathrm{T}}\sigma$. By choosing to switch the MRPs whenever $\sigma^2 = 1$, the MRP vector remains bounded within the unit sphere in three dimensions. However, switching of the MRPs leads to discontinuous kinematics as in Equation 2.7, thereby complicating the discussion on existence and uniqueness of solutions. In the problem under consideration, we currently restrict the MRP vector to all rotations represented by $\sigma^2 < 1$ to avoid switching.

We consider two cases: 1) The time-delay $\tau(t)$ is an unknown constant τ with $0 \leq \tau < \tau_{max}$ with τ_{max} being known. This problem is addressed in Chapter 4. 2) The time-delay $\tau(t)$ is an unknown time-varying function with the structure $\tau(t) = \tau_0 + \eta(t)$. This problem is addressed in Chapter 5. Initial condition trajectories for Equations 2.7-2.8 are generated by propagating the same dynamics without control action over the time-interval $-\tau \leq t \leq 0$. Initial conditions $\sigma_0 \in \mathbb{R}^3$, $\omega_0 \in \mathbb{R}^3$ are chosen to initialize this propagation such that they lie within the region of attraction (to be established in the sequel) and moreover, so that state trajectories do not escape from this estimate for the region of attraction during the initial control-free propagation phase. This propagation method for the constant delay case ($\tau(t) = \tau$) is depicted in Figure 2.1. It is assumed that delayed state measurements alone are available for feedback purposes. Equation 2.8 shows that the feedback contains state

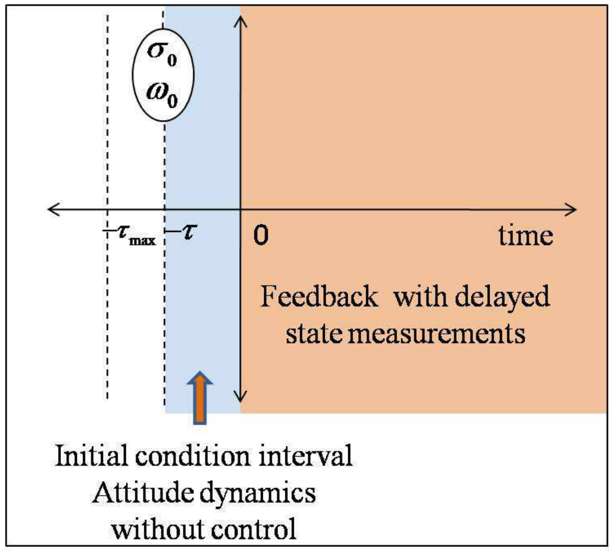


Figure 2.1: Initial condition interval

information at time instant $t - \tau(t)$, where $\tau(t)$ is the time delay present in the feedback and is assumed to be unknown. We assume perfect knowledge of τ_{max}, which is a strict upper bound on the feedback time-delay. In the absence of delay (i.e. with $\tau(t) = 0$) , it is well documented that linear feedback of states $u(t) = -K_1\sigma(t) - K_2\omega(t)$, with K_1 and K_2 being arbitrary positive definite matrices, stabilizes the dynamics in Equations 2.7-2.8 [18]. However, if delay is present and not accounted for in the input, the result is increased closed loop oscillations and even instability [3]. This behavior will be documented in Chapter 4 for the constant delay case and in Chapter 5 for the time-varying delay case. The control objective is to achieve stabilization of the states, i.e. to ensure that $\sigma(t) \to 0$ and $\omega(t) \to 0$ in the presence of unknown constant and time-varying delay in feedback through a complete type L-K approach.

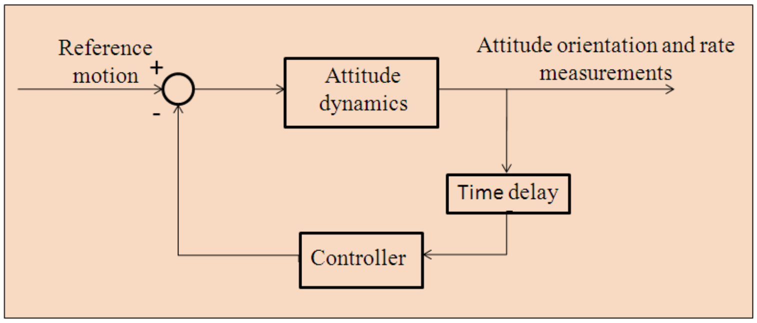


Figure 2.2: Block diagram: Delay in feedback

2.3 Stability Analysis

Ordinary differential equations of the form

$$\dot{x}(t) = g(t, x(t)) \tag{2.13}$$

are the conventional model description for finite-dimensional dynamical systems. However, a fundamental and limiting presumption ascribed to systems modeled by Equation 2.13 is that the future evolution of the system is described completely by the current value of the state variables $x(t) \in \mathbb{R}^n$. In practice, many dynamical systems cannot be modeled satisfactorily by an ODE, the reason being that future evolution of the state variable depends not only on their present values but also on their past values. This aftereffect is an applied problem in general. Reference [5] is an excellent resource for numerous examples of aftereffect or lag appearing in systems in the fields of biology, chemistry, economics, mechanics, viscoelasticity, physics, physiology as well as population dynamics. From the point of view of implementing a feedback control system, lag is introduced from actuator and sensor processes, which require a finite amount of time to complete their functions (process delay). In addition, interconnected or field networks in feedback often introduce such delays (communication or transmission delay). Therefore, it is of importance to understand the sensitivity of the control system with the introduction of small delays in the feedback. For some systems, small delays lead to destabilization, while other systems are robust with respect to small delays.

As a motivating example, consider the following scalar linear time-invariant time-delay system

$$\dot{x}(t) = -ax(t) - bx(t - \tau(t)) \tag{2.14}$$

with a specified initial condition $x(t) = \phi(t)$, where $a > 0$. It is common knowledge that Equation 2.14 is exponentially stable for any $a + b > 0$ in the absence of delay i.e. $\tau(t) = 0$. If the delay is a constant, i.e. $\tau(t) = \tau_0$, it is of interest to investigate the maximum delay τ_{max} such that stability is maintained $\forall\ \tau \in [0, \tau_{max})$. For the case of constant delay, the stability condition is obtained by using frequency sweeping analysis.

The characteristic equation for Equation 2.14 with constant delay, which turns out to be in the form a quasipolynomial is

$$s + a + be^{-\tau_0 s} = 0 \tag{2.15}$$

The system is exponentially stable if and only if the roots of the characteristic equation have negative real parts [7]. It is known that τ_{max} is the minimum delay for which the characteristic function becomes singular for certain $s = j\omega$

[19]. The boundary of the stability region in terms of parameters a, b and the maximum delay τ_{max} can be found by substituting $s = j\omega$ in the characteristic equation and then separating out the real and imaginary terms, which leads to

$$
\begin{aligned}
a &= -b\cos(\omega\tau_{max}) && (2.16) \\
\omega &= b\sin(\omega\tau_{max}) && (2.17)
\end{aligned}
$$

Eliminating ω leads to the following necessary and sufficient condition for τ_{max} in terms of a and b if $a < b$

$$\tau_{max} = \frac{\cos^{-1}(|a|/b)}{\sqrt{b^2 - a^2}} \qquad (2.18)$$

If $a > |b|$, the system is exponentially stable for all delay values since the characteristic equation does not have imaginary roots for any τ_{max}. This motivating example is an important illustration of the frequency sweeping method [8], which will be extensively used for analyzing the stability characteristics of the nominal system in the attitude dynamics application.

2.3.1 Lyapunov-Krasovskii method

As in the study of systems without delay, a popular method for determining the stability of a time-delay system, especially one that contains nonlinear terms, is Lyapunov's second method. For systems without delay, this requires construction of a Lyapunov function $V(t, x(t))$, which in some sense is a potential measure quantifying the deviation of the state $x(t)$ from 0. Since, for a delay free system, $x(t)$ is needed to determine the system's future evolution beyond t, and since, for a time delay system, the "state" required for the same purpose at time t is the value of $x(t)$ in the interval $[t-\tau, t]$ or x_t, it is natural to expect that for a time delay system, the corresponding Lyapunov function be a Lyapunov *functional*, i.e. $V(t, x_t)$ as proposed first by Krasovskii [10], depending on x_t, which should measure the deviation of x_t from 0. Such a functional is called a Lyapunov-Krasovskii (L-K) functional. Consider the following generic time-delay system

$$\dot{x}(t) = f(t, x_t) \tag{2.19}$$

where $x(t) \in \mathbb{R}^n$, $x_t = x(t+\theta)$, $-\tau \leq \theta \leq 0$. The following result is known as the Lyapunov-Krasovskii theorem [19].

Theorem 2.3.1. *The zero solution of the retarded system $\dot{x}(t) = f(t, x_t)$ (Equation 2.19) is asymptotically stable if there exists a continuous functional $V(t, \phi)$ for any ϕ mapping Υ to $\mathbb{R}^n$, which is positive-definite, decreasing, admitting an infinitesimal upper limit ($\exists$ $u(x)$, $v(x)$: $\mathbb{R}^n \to \mathbb{R}^+$, positive-definite, $u(\|\phi(0)\|) \leq V(t, \phi) \leq v(\|\phi\|_c)$) and whose full derivative $V(t, x_t)$ along the trajectories of $x(t)$ is negative definite over a neighborhood of the origin.*

2.3.2 Motivating example

There are limitations in applying the Lyapunov-Krasovskii theorem for stability analysis of a particular time-delay system. The main problem is that there is no constructive method to formulate such a functional for a given system. A common practice is to employ "reduced" or "simple" functionals. To illustrate this method, consider again the following scalar time-delay system from Equation 2.14,

$$\dot{x}(t) = -ax(t) - bx(t-\tau) \tag{2.20}$$

where $a > 0$ and b are constants and the delay τ is a positive constant. This system is exponentially stable in the absence of the delayed term $bx(t-\tau)$. We choose the Lyapunov functional $V(t, x_t)$ to be

$$V(t, x_t) = \frac{1}{2}x^2(t) + \mu \int_{t-\tau}^{t} x^2(\theta)d\theta \tag{2.21}$$

where μ is any positive constant. Differentiating with respect to t

$$\begin{aligned} \dot{V}(t, x_t) &= -ax^2(t) - bx(t)x(t-\tau) + \mu x^2(t) - \mu x^2(t-\tau) \\ &= -\left[\, x(t) \;\; x(t-\tau) \,\right] M \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} \end{aligned} \tag{2.22}$$

$$\text{with } M \doteq \begin{bmatrix} a-\mu & \frac{b}{2} \\ \frac{b}{2} & \mu \end{bmatrix} \tag{2.23}$$

The matrix M is positive definite if and only if $a > \mu$ and $4(a-\mu)\mu > b^2$. The choice of μ that maximizes b is $\mu = \frac{a}{2}$. This gives the (sufficient) condition that if $|b| < a$ then the trivial solution $x(t) = 0$ is uniformly asymptotically stable, which is conservative when compared to the necessary and sufficient conditions obtained from the characteristic equation from Equation 2.18. Note that this condition does not depend on the value of the delay. Lyapunov stability analysis treats the delayed term $bx(t-\tau)$ as a perturbation and obtains stability conditions by dominating it. This approach leads to conservatism in the stability conditions that the parameters must satisfy. Comparison between the frequency approach and the Lyapunov-Krasovskii functional approach shows the inherent conservatism arising from the latter. Moreover, if the system is not stable in the absence of a delayed term (say, $a = 0$), there is no constructive method to formulate a "reduced" (as in not complete) Lyapunov functional for the system. Moreover, there is an inherent assumption that the system without the delayed term is "more stable" than the time delay system since the delayed term is considered to be an unstable perturbation. The L-K method applied here requires that $a > |b|$ in order to achieve stability and hence does not address the case where $a < b$, which the frequency method does as seen in Equation 2.18. These limitations have resulted in great interest in a constructive method to formulate a more completely quadratic functional rather than a "reduced" Lyapunov-Krasovskii functional for stability analysis of a prescribed time-delay system.

Chapter 3

Complete type Lyapunov-Krasovskii Functional approach

It is well known and it has been shown in Chapter 2 that there is considerable difficulty in constructing a L-K functional for stability analysis of any given system [8]. In order to circumvent this obstacle, Kharitonov and his co-workers formulated a completely quadratic L-K functional for any linear time-delay system which was known to be exponentially stable [11]. The same functional was further adopted for robust stability analysis due to nonlinear perturbations, based on the nature of perturbation term present, regional stabilization conditions were obtained [12]. The complete type L-K technique can be viewed as an extension of robustness analysis using the so-called "Lyapunov equation" for delay-free systems. However, a major hypothesis here is that the actual time-delay is assumed to be exactly determined, which can be a restrictive condition. In the following development, we modify the L-K functional concept in order to include robustness to time-delay as well, while still preserving the original features of the complete type L-K approach. Two cases are considered for the delay: 1) Constant unknown delay and 2) Time-varying unknown delay. Sufficient conditions in each case enable us to obtain an estimate for the region of attraction for a class of nonlinear time-delay systems.

3.1 Nonlinear systems with constant time-delay

In this section, we develop a modified complete type L-K functional for robust stability analysis of a class of nonlinear time-delay systems with unknown constant delay. We consider the following class of nonlinear time-delay systems

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) + f(x(t), x(t-\tau)) \tag{3.1}$$

where τ is a positive unknown constant such that $0 \leq \tau < \tau_{max}$ where τ_{max} is perfectly known. The nonlinear system is partitioned to formulate a nominal linear system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) \tag{3.2}$$

with a nonlinear perturbation $f(x(t), x(t-\tau))$. The term $A_1 x(t-\tau)$ represents the contribution of delayed linear state feedback. Having ensured exponential stability of the nominal system, we construct a modified complete type L-K functional and employ it for robust stability analysis in order to obtain a region of attraction estimate.

3.1.1 Nominal system formulation

Consider the linear time-invariant time-delay system represented by Equation 3.2. For any given $\tau_{max} > 0$ such that $\tau \in [\ 0,\ \tau_{max}\ )$, assume that

A_0 and A_1 are such that Equation 3.2 is exponentially stable. This means that $\exists\ \mu > 0$ and $\alpha > 0$ such that

$$\|x(t)\| \leq \mu\|\varphi\|e^{-\alpha t} \tag{3.3}$$

where $\varphi(\theta)$, $-\tau \leq \theta \leq 0$ is the initial condition required for the time delay system. In order to construct a complete-type L-K functional, we start with the observation that given any symmetric positive-definite matrices W_0, W_1, W_2, it follows from employing the Leibnitz rule with substituting $\zeta = t + \theta$ that

$$\frac{d}{dt}\left(\int_{-\tau}^{0} x^{\mathrm{T}}(t+\theta)\left[W_1 + (\tau_{max}+\theta)W_2\right]x(t+\theta)d\theta\right) = x^{\mathrm{T}}(t)(W_1 + \tau_{max}W_2)x(t)$$
$$-x^{\mathrm{T}}(t-\tau)\widetilde{W}_1 x(t-\tau) - \int_{-\tau}^{0} x^{\mathrm{T}}(t+\theta)W_2 x(t+\theta)d\theta \tag{3.4}$$

where $\widetilde{W}_1 \doteq W_1 + (\tau_{max} - \tau)W_2$. So, if there exists a functional $V_0(t, x_t)$ such that $\forall\ t \geq 0$

$$\frac{d}{dt}V_0(t, x_t) \;=\; -w_0(t, x_t) = -x^{\mathrm{T}}(t)\left(W_0 + W_1 + \tau_{max}W_2\right)x(t) \tag{3.5}$$

then the first time derivative of the functional

$$V(t, x_t) = V_0(t, x_t) + \int_{-\tau}^{0} x^{\mathrm{T}}(t+\theta)\left[W_0 + W_1 + (\tau_{max}+\theta)W_2\right]x(t+\theta)d\theta \tag{3.6}$$

is given by

$$\begin{aligned}\frac{d}{dt}V(t, x_t) \;&=\; -x^{\mathrm{T}}(t)W_0x(t) - x^{\mathrm{T}}(t-\tau)\widetilde{W}_1 x(t-\tau) \\ &\quad - \int_{-\tau}^{0} x^{\mathrm{T}}(t+\theta)W_2 x(t+\theta)d\theta\end{aligned} \tag{3.7}$$

If Equation 3.2 is exponentially stable, $\exists\, V_0(t, x_t)$ such that

$$V_0(t, x_t) = \int_0^{\infty} x^{\mathrm{T}}(t)\left(W_0 + W_1 + \tau_{max} W_2\right) x(t) dt \tag{3.8}$$

$V(t, x_t)$ is called the complete type L-K functional associated with Equation 3.2 and is of the form

$$\begin{aligned} V(t, x_t) \;=\; & x^{\mathrm{T}}(t) U(0) x(t) + 2x^{\mathrm{T}}(t) \int_{-\tau}^{0} U(-\tau - \theta) A_1 x(t+\theta) d\theta + \\ & \int_{-\tau}^{0}\int_{-\tau}^{0} x(t+\theta_1) A_1^{\mathrm{T}} U(\theta_1 - \theta_2) A_1 x(t+\theta_2) d\theta_1 d\theta_2 + \\ & \int_{-\tau}^{0} x^{\mathrm{T}}(t+\theta)(W_1 + (\tau_{max} + \theta) W_2) x(t+\theta) d\theta \end{aligned} \tag{3.9}$$

where $U(\theta)$ is called the Lyapunov matrix [11] and is defined as

$$U(\theta) = \int_0^{\infty} K^{\mathrm{T}}(t) \widetilde{W} K(t+\theta) dt \tag{3.10}$$

where $\widetilde{W} = W_0 + W_1 + \tau_{max} W_2$ and $K(t)$ is the unique matrix function that satisfies

$$\begin{aligned} \dot{K}(t) &= A_0 K(t) + A_1 K(t - \tau) \\ K(\theta) &= 0,\ \theta < 0\ ;\ K(0) = I \end{aligned} \tag{3.11}$$

The Lyapunov matrix is well defined because $K(t)$ vanishes for $t < 0$ and approaches zero exponentially as $t \to \infty$, since the nominal system is exponentially stable. The complete type L-K functional from Equation 3.9 can be recovered from Equations 3.6, 3.8 and 3.10 and using the so-called Cauchy formula for the nominal state vector $x(t)$ [6]

$$x(t, \phi) = K(t)\phi(0) + \int_{-\tau}^{0} K(t - \tau - \theta) A_1 \phi(\theta) d\theta, \text{ for t} \geq 0 \tag{3.12}$$

The Lyapunov matrix $U(\theta)$ satisfies the second order matrix differential equation

$$U''(\theta) = U'(\theta)A_0 - A_0^{\mathrm{T}}U'(\theta) + A_0^{\mathrm{T}}U(\theta)A_0 - A_1^{\mathrm{T}}U(\theta)A_1 \tag{3.13}$$

subjected the mixed boundary conditions

$$U'(0) + \left[U'(0)\right]^{\mathrm{T}} = -\widetilde{W} \tag{3.14}$$

$$U'(0) = U(0)A_0 + U^{\mathrm{T}}(\tau)A_1 \tag{3.15}$$

Also, from Reference [8], it follows from Equation 3.10 and using the fact that $\widetilde{W}$ is symmetric, we can write

$$\begin{aligned} U^{\mathrm{T}}(\tau) &= \int_0^\infty K^{\mathrm{T}}(t+\tau)\widetilde{W}K(t)dt \\ &= \int_\tau^\infty K^{\mathrm{T}}(\zeta)\widetilde{W}K(\zeta-\tau)d\zeta \\ &= \int_0^\infty K^{\mathrm{T}}(\zeta)\widetilde{W}K(\zeta-\tau)d\zeta \end{aligned} \tag{3.16}$$

where we have used the fact that $K(\zeta - \tau)$ vanishes when $0 \leq \zeta \leq \tau$. This leads to the useful observation that the Lyapunov matrix $U(\theta)$ is symmetric at $\theta = 0$ and specifically [8]

$$U(\theta) = U^{\mathrm{T}}(-\theta) \tag{3.17}$$

Equation 3.13 together with boundary conditions from Equations 3.14-3.15 and the symmetry property from Equation 3.17 will be employed in the Section 4.2 in order to find an analytical solution for the Lyapunov matrix associated with the nominal system chosen to represent the attitude dynamics.

3.1.2 Robustness analysis

The complete type L-K functional formulated in Equation 3.9 is employed to calculate an estimate for the region of attraction for the nonlinear time-delay system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) + f(x(t), x(t-\tau)) \tag{3.18}$$

where $f(x(t), x(t-\tau))$ satisfies a Lipschitz condition in a certain vicinity of the origin

$$\|f(x(t), x(t-\tau))\| < \gamma \|x(t), x(t-\tau)\| \tag{3.19}$$

where γ is a positive constant. In addition the drift term satisfies

$$\lim_{\|x,y\| \to 0} \frac{\|f(x,y)\|}{\|x,y\|} = 0 \tag{3.20}$$

The following theorem extends results from Reference [12] in order to include robustness to unknown time-delay.

Theorem 3.1.1. *: For any given $\tau_{max} > 0$, let the nominal system (Equation 3.2) be exponentially stable for all $\tau \in [0, \tau_{max})$. Then the nonlinear perturbed system (see Equation 3.18) is asymptotically stable for all $\tau \in [0, \tau_{max})$ if the drift term $f(x(t), x(t-\tau))$ obeys the Lipschitz condition in Equation 3.19 where*

$$0 < \gamma < \min \left\{ \frac{\lambda_{min}(W_0)}{u_0(2 + \|A_1\| \tau_{max})}, \frac{\lambda_{min}(W_1)}{u_0(1 + \|A_1\| \tau_{max})}, \frac{\lambda_{min}(W_2)}{u_0 \|A_1\|} \right\} \tag{3.21}$$

for any selection of $n \times n$ symmetric positive definite matrices W_0, W_1 and W_2 and $u_0 = \sup_{\theta \in [0,\ \tau_{max}]} \|U(\theta)\|$.

Proof. The derivative of Equation 3.9 along the trajectories of the nonlinear system (3.18) is

$$\begin{aligned}\frac{d}{dt}V(t,x_t) = \quad & -w(t,x_t) + 2f^{\mathrm{T}}((x(t),x(t-\tau))\Bigg(U(0)x(t) \\ & - \int_{-\tau}^{0} U(-\tau-\theta)A_1 x(t+\theta)d\theta\Bigg)\end{aligned}$$

where $-w(t,x_t)$ is the derivative of the functional along the trajectories of the nominal system Equation 3.2. From the Lipschitz condition, we have

$$\|f(x(t),x(t-\tau))\| < \gamma\|x(t),x(t-\tau)\|$$

It is easy to see that the following inequality holds

$$\begin{aligned}&\|U(0)x(t) - \int_{-\tau}^{0} U(-\tau-\theta)A_1 x(t+\theta)d\theta\| \\ &\leq u_0\left(\|x(t)\| + \|A_1\|\int_{-\tau}^{0}\|x(t+\theta)\|d\theta\right)\end{aligned} \tag{3.22}$$

It then follows that

$$\begin{aligned}&2f^{\mathrm{T}}((x(t),x(t-\tau))\left(U(0)x(t) - \int_{-\tau}^{0} U(-\tau-\theta)A_1 x(t+\theta)d\theta\right) \\ &\leq \gamma u_0\Big[\left(2+\|A_1\|\tau_{max}\right)\|x(t)\|^2 + \left(1+\|A_1\|\tau_{max}\right)\|x(t-\tau)\|^2 \\ &+\|A_1\|\int_{-\tau}^{0}\|x(t+\theta)\|^2 d\theta\Big]\end{aligned}$$

Considering this inequality is the first time derivative of the complete type L-K functional $V(x_t)$, we arrive at the following inequality

$$\begin{aligned}\frac{d}{dt}V(t,x_t) \leq & -\Big[\lambda_{min}(W_0) - \gamma u_0\left(2+\|A_1\|\tau_{max}\right)\Big]\|x(t)\|^2 \\ & -\Big[\lambda_{min}(W_1) - \gamma u_0\left(1+\|A_1\|\tau_{max}\right)\Big]\|x(t-\tau)\|^2 \\ & -\Big[\lambda_{min}(W_2) - \gamma u_0\|A_1\|\Big]\int_{-\tau}^{0}\|x(t+\theta)\|^2 d\theta\end{aligned} \tag{3.23}$$

If γ satisfies Equation 3.25, then $\dot{V}(t, x_t)$ is negative definite for all trajectories inside the set determined by Equation 3.19, and consequently, the trivial solution $x(t) = 0$ of the nonlinear system in Equation 3.18 is asymptotically stable for all $\tau \in [0, \tau_{max})$. $\square$

The result for estimate of region of attraction can be specialized if the drift term is a function of the current value of the state alone i.e. $f(x(t), x(t-\tau)) = f(x(t))$. Consequently, the conservatism is potentially reduced.

Remark 3.1.1. In the main theorem, we require that the matrices W_j, $j = 0, 1, 2$ be symmetric positive definite. However, the matrices can be positive semi-definite, if the corresponding term to be dominated from the perturbation term is absent, i.e. if the perturbation is a function of the current value of the state alone then, W_1 can be made zero. For example, the case of $W_1 = W_2$ being zero was investigated by Reference [20]. Reference [11] proposed the case of W_j being positive definite in order to achieve robust stability. In our formulation for the attitude dynamics problem, the perturbation turns out to be a function of the current value of the state alone (see Chapter 4), which permits specializations in order to reduce conservatism in region of attraction estimate.

Corollary 3.1.2. *Let the nominal system in Equation 3.2 be exponential stable for all $\tau \in [0, \tau_{max})$. Then the nonlinear system in Equation 3.18 is asymptotically stable for all $\tau \in [0, \tau_{max})$ if the drift term $f(x(t))$ obeys the Lipschitz* *condition*

$$\|f(x(t))\| < \gamma\|x(t)\| \tag{3.24}$$

where

$$0 < \gamma < \min\left\{\frac{\lambda_{min}(W_0)}{u_0(2+\|A_1\|\tau_{max})},\ \frac{\lambda_{min}(W_2)}{u_0\|A_1\|}\right\} \tag{3.25}$$

and $u_0 = \sup_{\theta\in[0,\ \tau_{max}]}\|U(\theta)\|$.

Remark 3.1.2. Due to the modifications made in the construction of the complete type L-K functional from Equation 3.9 as well as the accompanying robustness analysis with its time-derivative along with the Lyapunov matrix evaluation, calculating the size of the estimate of region of attraction γ requires knowledge of a strict upper bound τ_{max} of the time-delay rather than its precise value τ. The actual delay τ is however present in the complete type L-K functional, which is employed for analysis purposes alone.

Remark 3.1.3. In the finite-dimensional case, (see Chapter 1), we can calculate an estimate of a region for a corresponding finite-dimensional nonlinear system by robustness analysis using the Lyapunov equation. In this case, the size of the estimate can be maximized through choice of parameters subject to the Lyapunov equation being satisfied. In the time-delay case, it is not so straightforward to maximize the size of the estimate γ mainly due to increase in number of parameters and the accompanying constraints. However, it is possible to employ numerical optimization in a computing software such as MATLAB in order to maximize γ. This optimization will be performed after application to the attitude stabilization problem in the following Chapter.

3.2 Nonlinear systems with time-varying time-delay

In this section, we extend the concept of the complete type L-K functional for stability robustness with respect to unknown time-varying delay in the system dynamics. This work extends results obtained with respect to unknown constant delay in feedback. In order to simplify some of the accompanying algebra, we choose the generic nonlinear perturbation to be a function of the current value of the state alone. As seen in Chapter 4, this simplification will not restrict application to the attitude stabilization problem.

3.2.1 Nominal system

To begin with complete type L-K functional analysis, we consider the following generic nonlinear time-delay system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau(t)) + f(x(t)) \tag{3.26}$$

where $x \in \mathbb{R}^n$ is the state vector, $A_0, A_1 \in \mathbb{R}^{n \times n}$ are suitably chosen matrices and $f(x(t))$ is a nonlinear function which satisfies the aforementioned Lipschitz condition. The delay $\tau(t) = \tau_0 + \eta(t)$ is the unknown and time varying which is assumed to be differentiable everywhere. We assume perfect knowledge of upper-bounds on the magnitude and rate of $\eta(t)$, as well as strict upper-bound on $\tau(t)$ i.e

$$|\eta(t)| \leq \eta_0,\ |\dot{\eta}(t)| \leq \eta_1 < 1,\ 0 \leq \tau(t) < \tau_{max} \tag{3.27}$$

where η_0, τ_{max}, η_1 are known positive constants. In addition, the delay satisfies the condition $\tau_0 = \sup_t |\eta(t)|$. Equation 3.27 shows that the time variable is scaled without loss of generality so that the rate of change of the time-varying part of the delay is normalized to be strictly less than unity. The initial condition trajectory required to propagate the time-delay system represented by Equation 3.26 is given by

$$x(t) = \phi(t), \ t \in [-2\tau_0, 0] \tag{3.28}$$

We choose the nominal system to be the same as the case with constant unknown delay in feedback from Section 3.1 and to be described by

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau_0) \tag{3.29}$$

We partition the nonlinear time-delay system from Equation 3.26 into the nominal system from Equation 3.29 with a perturbation term. In order to accomplish this step, we use a model-transformation [21, 22] in addition to assuming stability of the nominal system. Model transformation essentially involves using the Newton-Leibnitz formula in order to replace the time-varying delay term $x(t - \tau(t))$ for $t \geq 2\tau_0$ as

$$x(t - \tau(t)) = x(t - \tau_0) - \int_{-\tau(t)}^{-\tau_0} \dot{x}(t + \theta) d\theta \tag{3.30}$$

Substituting for $\dot{x}(t + \theta)$ from Equation 3.26 in the above equation,

$$\begin{aligned} x(t - \tau(t)) &= x(t - \tau_0) - \int_{-\tau(t)}^{-\tau_0} \Big[A_0 x(t + \theta) + A_1 x(t + \theta - \tau(t + \theta)) \\ &+ f(x(t + \theta)) \Big] d\theta \end{aligned} \tag{3.31}$$

Using Equation 3.31, we can write Equation 3.26 as

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A_1 x(t - \tau_0) + A_1 z(t) + f(x(t)) \\ x(t) &= \psi(t), \ t \in [-2\tau_0, 2\tau_0] \end{aligned} \tag{3.32}$$

where $\psi(t)$ is given by

$$\begin{aligned}\psi(t) &= \phi(t),\ t\in[-2\tau_0,0]\,,\\ &= x(t,\phi),\ t\in[0,2\tau_0]\end{aligned}$$

and

$$z(t) = -\int_{-\tau(t)}^{-\tau_0}\Big[A_0x(t+\theta)+A_1x(t+\theta-\tau(t+\theta))+f(x(t+\theta))\Big]d\theta \quad (3.33)$$

Clearly, every solution of Equation 3.32 is a solution of Equation 3.26, and therefore stability of Equation 3.32 implies stability of Equation 3.29. However, the process of model transformation thus used introduces conservatism in the stability analysis. This conservatism is well documented and can be found in detail in References [21] and [22].

Now, we construct a complete type L-K functional associated with Equation 3.29 for the stability of Equation 3.26. For any symmetric positive-definite matrices W_j, $j=0,1,2$, consider the functional

$$\begin{aligned}w(t,\tilde{x}_t) &= x^{\mathrm{T}}(t)W_0x(t)+x^{\mathrm{T}}(t-\tau_0)\widetilde{W_1}x(t-\tau_0)\\ &\quad+\int_{-4\tau_0}^{0}x^{\mathrm{T}}(t+\theta)W_0x(t+\theta)d\theta \qquad (3.34)\end{aligned}$$

where $\tilde{x}_t = x(t+\theta)$, $\theta\in[-\tau_0,0]$ is arbitrary and $\widetilde{W_1} \doteq W_1+(2\tau_{max}-\tau_0)W_2$. If system Equation 3.29 is exponentially stable, then there exists a unique quadratic functional $V(\tilde{x}_t)$, such that

$$\frac{dV(t,\tilde{x}_t)}{dt} = -w(t,x_t) \qquad (3.35)$$

where $\tilde{x}_t = x(t+\theta)$, $\theta \in [-2\tau_0, 0]$, and $x_t = x(t+\theta)$, $\theta \in [-4\tau_0, 0]$. $V(t, \tilde{x}_t)$ is the complete type L-K functional associated with the nominal system in Equation 3.29. The functional is of the form

$$\begin{aligned} V(t, \tilde{x}_t) &= x^{\mathrm{T}}(t)U(0)x(t) - 2x^{\mathrm{T}}(t)\int_{-\tau_0}^{0} U(-\tau_0 - \theta)A_1 x(t+\theta)d\theta \\ &+ \int_{-\tau_0}^{0}\int_{-\tau_0}^{0} x^{\mathrm{T}}(t+\theta_1)A_1^{\mathrm{T}}U(\theta_1 - \theta_2)A_1 x^{\mathrm{T}}(t+\theta_2)d\theta_1 d\theta_2 \\ &+ \int_{-\tau_0}^{0} x^{\mathrm{T}}(t+\theta)(W_1 + (2\tau_{max} + \theta)W_2)x(t+\theta)d\theta \end{aligned} \tag{3.36}$$

where the matrix $U(\theta)$ from Equation 3.10 is defined for $\widetilde{W} = W_0 + W_1 + 2\tau_{max}W_2$, satisfies the second order matrix differential equation from Equation 3.13 and additional conditions Equation 3.14-3.17.

3.2.2 Robustness Analysis

The following theorem extends the results of Reference [12] in order to include robustness to unknown time-varying time-delay.

Theorem 3.2.1. *: Let the nominal system represented by Equation 3.29 be exponential stable. Then the nonlinear system (see Equation 3.26) is asymptotically stable* $\forall\ \tau(t) \in [0, \tau_{max})$, $|\eta(t)| \le \eta_0$, $|\dot{\eta}(t)| \le \eta_1$ *and for any selection of* $n \times n$ *symmetric positive definite matrices* W_0 *and* W_2 *if the drift term*

$f(x(t))$ obeys the Lipschitz condition (see Equation 3.24) where

$$0 < \gamma < \min\left\{ \begin{array}{l} \dfrac{\lambda_{min}(W_0) - u_0\|A_0\|\|A_1\|k_1^{-1}\eta_0 - u_0\|A_1\|^2 k_3^{-1}\eta_0}{u_0(3 + 2\|A_1\|\tau_{max}) + u_0 k_1^{-1}\eta_0}, \\ \dfrac{\lambda_{min}(W_2) - u_0\|A_0\|\|A_1\|^2 k_2^{-1}\eta_0 - u_0\|A_1\|^3 k_4^{-1}\eta_0}{2\mathrm{u}_0\|A_1\| + u_0\|A_1\|^2 k_2^{-1}\eta_0}, \\ \dfrac{\lambda_{min}(W_2) - u_0\|A_0\|\|A_1\|\,(k_1 + k_2\|A_1\|\tau_{max})}{u_0\|A_1\|\,(k_1 + k_2\|A_1\|\tau_{max})} \end{array} \right\} \tag{3.37}$$

and if the following conditions is satisfied for some positive constants k_j, $j =$ $1, 2, 3, 4$

$$\lambda_{min}(W_0) - u_0\|A_0\|\|A_1\|k_1^{-1}\eta_0 - u_0\|A_1\|^2 k_3^{-1}\eta_0 > 0 \tag{3.38}$$

$$\lambda_{min}(W_2) - u_0\|A_0\|\|A_1\|^2 k_2^{-1}\eta_0 - u_0\|A_1\|^3 k_4^{-1}\eta_0 > 0 \tag{3.39}$$

$$\lambda_{min}(W_2) - u_0\|A_0\|\|A_1\|\,(k_1 + k_2\|A_1\|\tau_{max}) > 0 \tag{3.40}$$

$$\lambda_{min}(W_2) > (1-\eta_1)^{-1}\, u_0\|A_1\|^2\,(k_2 + k_3\|A_1\|\tau_{max}) \tag{3.41}$$

where $u_0 = \sup_{\theta\in[0,\ \tau_{max}]}\|U(\theta)\|$.

Proof. The derivative of $V(t, \tilde{x}_t)$ along the closed loop dynamical system in Equation 3.26 for $t \geq 2\tau_0$ is given by

$$\begin{aligned} \frac{dV(t,\tilde{x}_t)}{dt} &= -w(t,x_t) + 2\left[A_1 z_t + f(x(t))\right]^{\mathrm{T}} \Big(U(0)x(t) \\ &\quad - \int_{-\tau_0}^{0} U(\tau_0 - \theta)A_1 x(t+\theta)d\theta \end{aligned} \tag{3.42}$$

The following inequality holds for $t \geq 2\tau_0$

$$\begin{aligned} \|z_t\| &\leq (\|A_0\| + \gamma)\int_{-\tau(t)}^{-\tau_0}\|x(t+\theta)\|d\theta \\ &\quad + \|A_1\|\int_{-\tau(t)}^{-\tau_0}\|x(t+\theta-\tau(t+\theta))\|d\theta \end{aligned} \tag{3.43}$$

where the Lipschitz condition from Equation 3.24 is employed. This leads to

$$\begin{aligned}\|A_1 z_t + f(x(t))\| &< (\|A_0\| + \gamma)\|A_1\| \int_{-\tau(t)}^{-\tau_0} \|x(t+\theta)\| d\theta \\ &+ \|A_1\|^2 \int_{-\tau(t)}^{-\tau_0} \|x(t+\theta-\tau(t+\theta))\| d\theta + \gamma\|x(t)\| \quad (3.44)\end{aligned}$$

Substituting the above expression back into the derivative of $V(x_t)$

$$\begin{aligned}\frac{dV(t,\tilde{x}_t)}{dt} &= -w(t,x_t) + 2u_0\left[\|x(t)\| + \|A_1\|\int_{-\tau_0}^{0}\|x(t+\theta)\|d\theta\right] \times \\ &\left(\gamma\|x(t)\| + \|A_1\|\left(\|A_0\|+\gamma\right)\int_{-\tau(t)}^{-\tau_0}\|x(t+\theta)\|d\theta \right. \\ &\left. + \|A_1\|^2\int_{-\tau(t)}^{-\tau_0}\|x(t+\theta-\tau(t+\theta)\|d\theta\right) \quad (3.45)\end{aligned}$$

It can be easily observed that the following inequalities hold

$$\begin{aligned}&2u_0\gamma\|x(t)\|\left(\|x(t)\| + \|A_1\|\int_{-\tau_0}^{0}\|x(t+\theta)\|d\theta\right) \le u_0\gamma\Big[\left(3+2\|A_1\|\tau_{max}\right)\|x(t)\|^2 \\ &+ 2\|A_1\|\int_{-\tau_0}^{0}\|x(t+\theta)\|^2 d\theta \quad (3.46)\end{aligned}$$

$$\begin{aligned}&2u_0\|A_1\|\left(\|A_0\|+\gamma\right)\int_{-\tau(t)}^{-\tau_0}\left(\|x(t+\theta)\|d\theta\right)\left(\|x(t)\| + \|A_1\|\int_{-\tau_0}^{0}\|x(t+\theta)\|d\theta\right) \\ &\le u_0\|A_1\|\left(\|A_0\|+\gamma\right)\left[k_2^{-1}\|A_1\|\eta_0\int_{-\tau_0}^{0}\|x(t+\theta)\|^2 d\theta + k_1^{-1}\eta_0\|x(t)\|^2 + \right. \\ &\left. \left(k_1 + k_2\|A_1\|\tau_{max}\right)\int_{-\tau(t)}^{-\tau_0}\|x(t+\theta)\|^2 d\theta\right] \quad (3.47)\end{aligned}$$

$$\begin{aligned}&2u_0\|A_1\|^2\left(\|x(t)\| + \|A_1\|\int_{-\tau_0}^{0}\|x(t+\theta)\|d\theta\right)\int_{-\tau(t)}^{-\tau_0}\|x(t+\theta-\tau(t+\theta)\|d\theta \\ &\le u_0\|A_1\|^2\left(k_3^{-1}\eta_0\|x(t)\|^2 + k_4^{-1}\|A_1\|\eta_0\int_{-\tau_0}^{0}\|x(t+\theta)\|^2 d\theta \right. \\ &\left. + \left(k_3 + k_4\|A_1\|\tau_{max}\right)\int_{-\tau(t)}^{-\tau_0}\|x(t+\theta-\tau(t+\theta))\|^2 d\theta\right) \quad (3.48)\end{aligned}$$

Using these inequalities in the derivative of the complete type L-K functional

$$\frac{dV(\tilde{x}_t)}{dt} < -\lambda_{min}(W_0)\|x(t)\|^2 - \lambda_{min}(W_2)\int_{-4\tau_0}^{0} \|x(t+\theta)\|^2 d\theta$$
$$+\lambda_1\|x(t)\|^2 + \lambda_2\int_{-\tau_0}^{0} \|x(t+\theta)\|^2 d\theta + \lambda_3\int_{-\tau(t)}^{-\tau_0} \|x(t+\theta)\|^2 d\theta$$
$$+\lambda_4\int_{-\tau(t)}^{-\tau_0} \|x(t+\theta-\tau(t+\theta))\|^2 d\theta \tag{3.49}$$

where

$$\lambda_1 \doteq u_0\gamma\left(3+2\|A_1\|\tau_{max}\right) + u_0\|A_1\|\left(\|A_0\|+\gamma\right)k_1^{-1}\eta_0$$
$$+u_0\|A_1\|^2 k_3^{-1}\eta_0 \tag{3.50}$$
$$\lambda_2 \doteq 2u_0\gamma\|A_1\| + u_0\|A_1\|^2\left(\|A_0\|+\gamma\right)k_2^{-1}\eta_0 + u_0\|A_1\|^3 k_4^{-1}\eta_0 \tag{3.51}$$
$$\lambda_3 \doteq u_0\|A_1\|\left(\|A_0\|+\gamma\right)\left(k_1+k_2\|A_1\|\tau_{max}\right) \tag{3.52}$$
$$\lambda_4 \doteq u_0\|A_1\|^2\left(k_3+k_4\|A_1\|\tau_{max}\right) \tag{3.53}$$

Observing that

$$\int_{-\tau(t)}^{-\tau_0} \|x(t+\theta)\|^2 d\theta \le \int_{-2\tau_0}^{-\tau_0} \|x(t+\theta)\|^2 d\theta \tag{3.54}$$

and

$$\int_{-\tau(t)}^{-\tau_0} \|x(t+\theta-\tau(t+\theta))\|^2 d\theta \le (1-\eta_1)^{-1}\int_{-4\tau_0}^{-2\tau_0} \|x(t+\theta)\|^2 d\theta \tag{3.55}$$

leads to

$$\frac{dV(t,\tilde{x}_t)}{dt} < -\left(\lambda_{min}(W_0)-\lambda_1\right)\|x(t)\|^2 - \left(\lambda_{min}(W_2)-\lambda_2\right)\int_{-\tau_0}^{0} \|x(t+\theta)\|^2 d\theta$$
$$-\left(\lambda_{min}(W_2)-\lambda_3\right)\int_{-2\tau_0}^{-\tau_0} \|x(t+\theta)\|^2 d\theta$$
$$-\left[\lambda_{min}(W_2)-\lambda_4(1-\eta_1)^{-1}\right]\int_{-4\tau_0}^{-2\tau_0} \|x(t+\theta)\|^2 d\theta < 0 \tag{3.56}$$

where, the inequality 3.56 holds because of the conditions from Equations 3.37 and 3.38-3.41. Hence, $\dot{V}(t, \tilde{x}_t)$ is negative definite for all trajectories inside the set determined by Equation 3.37 and Equation 3.38-3.41, and consequently, the trivial solution $x(t) = 0$ of the nonlinear system (see Equation 3.26) is exponentially stable $\forall\ \tau(t) \in [0, \tau_{max})$, $|\eta(t)| \leq \eta_0$, and $|\dot{\eta}(t)| \leq \eta_1$ $\square$

Remark 3.2.1. Note that in order for γ to be meaningful, we require that Equations 3.38-3.41 be satisfied. The conditions from 3.38-3.41 with respect to the constant delay case arise because the time-varying delay term is treated as a perturbation. It is naturally important to investigate whether these conditions are satisfied for all values of time-varying delay $\tau(t)$, which has not been done in Reference [12] with respect to the development in stability analysis of nonlinear systems with known time-varying delay. Another area of investigation is the conservatism arising due to the model-transformation employed, when compared with results from the constant delay case with $\eta_1 \to 0$ and the constant delay part being the same. We will further explore the aforementioned conditions and conservatism from time-varying delay after application to the attitude stabilization in the following Chapter.

Chapter 4

Attitude stabilization with unknown constant delay in feedback

In this chapter, we apply the complete type L-K functional methodology to the attitude stabilization problem with constant unknown delay in feedback. Modified Rodrigues Parameters (MRPs) are employed to represent the attitude kinematics. The control is chosen to be linear in the delayed states. We express the closed-loop attitude dynamics as a nominal (linear) system with a perturbation term. The linear system consists of 3 blocks of double integrators and the perturbation term is a function of the state at current time alone.

We apply the theoretical development in Section 3.1 to the attitude stabilization problem with constant delay. After providing the problem statement, we formulate the nominal system and obtain an estimate of region of attraction from the perturbed system formulation and analysis over the initial condition interval. Simulations verify the results.

We consider the problem of attitude dynamics with unknown constant delay in feedback with MRP representation for the attitude kinematics

$$\dot{\sigma}(t) = \frac{1}{4} B\left(\sigma(t)\right)\omega(t) \tag{4.1}$$

$$J\dot{\omega}(t) = -\omega(t)^{\times} J\omega(t) + u(t-\tau) \tag{4.2}$$

Initial condition trajectories for Equations 4.1)-(4.2 are generated by propagating the governing attitude dynamics without control action over the time-interval $-\tau \leq t \leq 0$. Initial conditions $\sigma_0 \doteq \sigma(-\tau)$, $\omega_0 \doteq \omega(-\tau)$ are chosen to initialize this propagation such that they lie within the estimated region of attraction and moreover, so that state trajectories do not escape from this estimate region during the initial control-free propagation.

We assume perfect knowledge of τ_{max}, which is a strict upper bound on the feedback time-delay. The control objective is to achieve stabilization of the states, i.e. to ensure that $\sigma(t) \to 0$ and $\omega(t) \to 0$ in the presence of unknown constant delay in feedback through the complete type L-K approach.

4.1 Nominal System

The nominal system for applying this method to attitude dynamics is taken to be a block of 3 double integrators. We can rewrite the attitude dynamics from Equations 4.1-4.2 as

$$\dot{\sigma} = \frac{\omega}{4} + \frac{1}{4}\left[B\left(\sigma\right) - I_{3\times 3}\right]\omega \tag{4.3}$$

$$\dot{\omega} = -J^{-1}\omega^{\times} J\omega + J^{-1}\bar{u} \tag{4.4}$$

where $\bar{u} \doteq u(t-\tau)$. Adding and subtracting $\omega/4$ to the attitude kinematics enables the construction of a perturbation term that satisfies the Lipschitz condition (see Equation 3.19). We write the nominal system as

$$\dot{\sigma}_d = \frac{\omega_d}{4} \tag{4.5}$$

$$\dot{\omega}_d = J^{-1}\bar{u} \tag{4.6}$$

where the subscript d represents the state belonging to the nominal system. Employing the state transformation $q = \omega/4$ and choosing the control $\bar{u}(\bar{\sigma}_d, \bar{q}_d) = -4J\left(K_1\bar{\sigma}_d + K_2\bar{q}_d\right)$, where $K_1 = \omega_n^2$ and $K_2 = 2\xi\omega_n$ ($\omega_n > 0$, $\xi > 0$ representing the natural frequency and the damping coefficient respectively), leads to the nominal system being a block of 3 decoupled double integrators with delayed feedback, as in

$$\dot{\sigma}_d = q_d \tag{4.7}$$

$$\dot{q}_d = -2\xi\omega_n\bar{q}_d - \omega_n^2\bar{\sigma}_d \tag{4.8}$$

On comparison with the generic nominal system (see Equation 3.2), we have the following expressions for A_0 and A_1

$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ A_1 = \begin{bmatrix} 0 & 0 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix} \tag{4.9}$$

The nonlinear perturbation is written as

$$F(\sigma, q) = \begin{bmatrix} \left[B\left(\sigma\right) - I_{3\times3}\right]q \\ -16J^{-1}q^{\times}Jq \end{bmatrix} \tag{4.10}$$

From Equation 4.10, we observe that the nonlinear perturbation is a function of the current value of the states alone. We analyze the double integrator characteristic equation in order to determine the range of parameter values for which the nominal system is stable. Consider the double integrator with delayed feedback:

$$\dot{x}_1 = x_2 \tag{4.11}$$

$$\dot{x}_2 = -\omega_n^2 \bar{x}_1 - 2\xi\omega_n \bar{x}_2 \tag{4.12}$$

The stability of the above system is completely determined by its transcendental characteristic equation [7]:

$$s^2 + \omega_n^2 e^{-\tau s} + 2\xi\omega_n s e^{-\tau s} = 0 \tag{4.13}$$

Specifically, the system is exponentially stable if and only if the characteristic equation has no zero, or root, in the closed right half plane. In order to determine the maximum value of delay the system can tolerate for given control parameters ω_n and ξ, it suffices to determine the critical values of the delay for which the roots of the characteristic equation move from the closed left half plane to the imaginary axis, thus rendering the system unstable [19]. Thus, we wish to find the smallest deviation of the delay from 0, say τ_{max}, such that the characteristic equation has imaginary roots, i.e.

$$(j\omega)^2 + K_1 e^{-\tau_{max} j\omega} + K_2 s e^{-\tau_{max} j\omega} = 0$$

where $j = \sqrt{-1}$. This leads to

$$-\omega^2 + (K_1 + jK_2\omega)\, e^{-j\tau_{max}\omega} = 0$$

Separating the real and imaginary parts leads to

$$-\omega^2 + K_1 \cos \tau_{max}\omega + K_2\omega \sin \tau_{max}\omega = 0 \tag{4.14}$$

$$K_2\omega \cos \tau_{max}\omega - K_1 \sin \tau_{max}\omega = 0 \tag{4.15}$$

Combining Equations 4.14-4.15 leads to $\cos \tau_{max}\omega = K_1/\omega^2$ and $\sin \tau_{max}\omega K_2/\omega$. With some standard algebraic manipulations, an analytical solution for τ_{max} for a given $\omega_n = \sqrt{K_1}$ and $\xi = K_2/(2\omega_n)$ is obtained as

$$\begin{aligned} \tau_{max} &= \frac{1}{\omega_n f} \sin^{-1} \frac{2\xi}{f} \\ f &= \sqrt{2\xi^2 + \sqrt{1 + 4\xi^4}} \end{aligned} \tag{4.16}$$

Equation 4.16 enables us to obtain a maximum delay τ_{max} for given ξ and ω. Equation 4.16 is a necessary and sufficient condition, i.e., the system is critically stable $\tau = \tau_{max}$ and unstable for $\tau > \tau_{max}$. In another context, for a given τ_{max}, we can calculate a set of values that ξ and ω_n can take so that the system is exponentially stable. Choosing a lower ω_n increases τ_{max} for a constant ξ. Hence, for a given τ_{max}, any parameter in the interval $(0, \omega_n]$ results in an exponentially stable system $\forall\ \tau < \tau_{max}$. Next, reducing ξ for a given ω_n results in a higher τ_{max} since the term $\sin^{-1}(2\xi/\sqrt{2\xi^2 + \sqrt{1 + 4\xi^4}})$ is monotonic with respect to ξ. Again, the system is exponential for any parameter in the interval $(0, \xi]\ \forall\ \tau < \tau_{max}$. Concluding, the system is exponentially stable for any parameter in the parameter space $(0, \xi]$, $(0, \omega_n]\ \forall\ \tau < \tau_{max}$. For example, Figure 4.1 shows the ω_n vs ξ curve for $\tau_{max} = 0.2, 0.5, 1$. For stability analysis of the nonlinear system, we enforce the region of attraction condition on the states by obtaining γ from Equation 3.25

$$0 < \gamma < \min \left\{ \frac{\lambda_{min}(W_0)}{u_0(2 + \|A_1\|\tau_{max})}, \frac{\lambda_{min}(W_2)}{u_0\|A_1\|} \right\} \tag{4.17}$$

where A_0 and A_1 are given by Equation 4.9. The analysis to obtain γ does not require knowledge of the structure of the perturbation term added to the

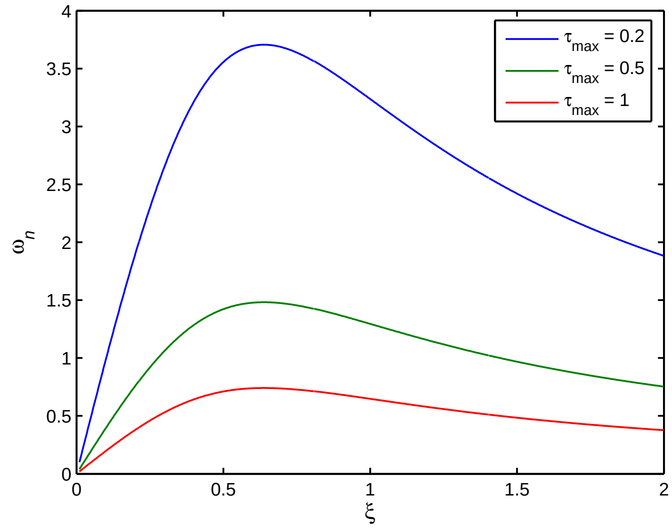


Figure 4.1: ω_n vs ξ for τ_{max}= 0.2,0.5,1

nominal system from Equations 4.7-4.8, since it involves parameters associated with the nominal system and the Lyapunov matrix ODE only. Note that the condition in Equation 4.17 does not contain the W_1 term that Equation 3.25 does, since the drift term does not depend on delayed value of the states (see Equation 4.10). We use direct numerical optimization, choosing parameters $\omega_n = \sqrt{K_1}$, $\xi = K_2/(2\omega_n)$, W_0 and W_2, such that γ is maximized, while keeping the nominal system exponentially stable. An intermediate step involves finding the solution to the matrix differential equation for the Lyapunov matrix represented by Equation 3.13.

4.2 Analytical solution to Lyapunov matrix ODE

We apply kronecker algebra to the second order linear matrix ODE represented by Equation 3.13 in order to obtain an analytical solution to the Lyapunov matrix $U(\theta)$. The ODE can be written as a linear cascade system for the attitude dynamics nominal system with $U_1 \doteq U$ and $U_2 \doteq U'$ as

$$
\begin{aligned}
U_1' &= U_2 && (4.18)\\
U_2' &= U_2 A_0 - A_0^{\mathrm{T}} U_2 + A_0 U_1 A_0 - A_1^{\mathrm{T}} U_1 A_1 && (4.19)
\end{aligned}
$$

where A_0 and A_1 are given by Equation 4.9. The mixed boundary condition in Equation 3.15 can be written as

$$U_2(0) + U_2^{\mathrm{T}}(0) = -W \tag{4.20}$$

$$U_2(0) = U_1(0) A_0 + U_1^{\mathrm{T}}(\tau) A_1 \tag{4.21}$$

Define $Z = \left[U_1^{\mathrm{T}} \ U_2^{\mathrm{T}}\right]^{\mathrm{T}}$. We can rewrite Equations 4.18-4.19 as

$$Z' = P_0 Z Q_0 + P_1 Z Q_1 + M Z \tag{4.22}$$

where

$$
P_0 = \begin{bmatrix} 0_{2\times2} & 0_{2\times2} \\ A_0^{\mathrm{T}} & I_{2\times2} \end{bmatrix}, \ P_1 = \begin{bmatrix} 0_{2\times2} & 0_{2\times2} \\ A_1^{\mathrm{T}} & 0_{2\times2} \end{bmatrix}, \ M = \begin{bmatrix} 0_{2\times2} & I_{2\times2} \\ 0_{2\times2} & -A_0^{\mathrm{T}} \end{bmatrix},
$$
$$
Q_0 = A_0 \ , \ Q_1 = -A_1
$$

We define the transformation $v(X)$ for any $X \in R^{m\times n}$ as

$$v(X) = [\ x_{11} \ x_{12} \ \ldots \ x_{1n} \ x_{21} \ \ldots \ x_{2n} \ \ldots \ x_{mn}]^{\mathrm{T}} \tag{4.23}$$

where x_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$ are the elements of X. From the property of Kronecker products [17], we have

$$v(PXQ) = (P \otimes Q^{\mathrm{T}})v(X) \tag{4.24}$$

Using Equation 4.24 in order to obtain a vector transformation for Equation 4.22 leads to

$$v(Z') = (P_0 \otimes Q_0^{\mathrm{T}} + P_1 \otimes Q_1^{\mathrm{T}} + M \otimes I_{2\times 2})v(Z) \tag{4.25}$$

The nominal system Equation 4.7-4.8 is stable for the range of gain values determined by Equation 4.16 and therefore admits a unique solution for the Lyapunov matrix [11]. The general solution for Equation 4.25 can be written as

$$v(Z(\theta)) = e^{K\theta}v(Z(0)) \tag{4.26}$$

In order to find the particular solution, we need to solve the boundary conditions from Equations 4.20-4.21. Taking the vector transformation, we have

$$v(U_2(0)) + Ev(U_2(0)) = -v(W) \tag{4.27}$$

$$v(U_2(0)) = (I_{2\times 2} \otimes A_0^{\mathrm{T}})v(U_1(0)) + (I_{2\times 2} \otimes A_1^{\mathrm{T}})Ev(U_1(\tau)) \tag{4.28}$$

where E is the permutation matrix [11], which enables us to find the vector transformation for a transposed matrix

$$v(X^{\mathrm{T}}) = Ev(X)$$

Since $Z = \left[U_1^{\mathrm{T}}\ U_2^{\mathrm{T}}\right]^{\mathrm{T}}$, $U_1(\tau)$ can be expressed in terms of $U_1(0)$ and $U_2(0)$ by substituting $\theta = \tau$ in Equation 4.26. As a result of the substitution, Equations 4.27-4.28 have 8 unknowns, namely the elements of $U_1(0)$ and $U_2(0)$, one of which are eliminated since $U_1(0) = U(0)$ is symmetric from Equation 3.17. One of the equations is also eliminated since $U_2(0) + U_2^{\mathrm{T}}(0)$ and W are both symmetric. This leads to 7 equations with 7 unknowns, which can be solved for. Since A_0, A_1 are such that the nominal system is exponentially stable, the differential equation for the Lyapunov matrix $U(\theta)$ admits a unique solution for $\theta \geq 0$. The analytical solution for the Lyapunov matrix will be employed in order to obtain a supremum for $\|U(\theta)\|$ over the interval $\theta \in [0, \tau_{max}]$, to be used in the formula for γ in Equation 3.25.

4.3 Analysis over torque-free interval and regional stabilization

In order to realistically simulate the system, we require that there be no control over the initial condition time interval, i.e. $t \in [\ -\tau,\ 0\ ]$. It is highly important to ensure that during this time evolution, the states do not escape from the estimated domain of attraction as per Equation 4.17. This situation is tackled by calculating upper bound on the states during this interval. Rewriting the system dynamics from Equations 4.1-4.2 with no control,

$$\begin{aligned} \dot{\sigma} &= \frac{1}{4}B(\sigma)\omega \\ J\dot{\omega} &= -\omega^{\times}J\omega \end{aligned}$$

We calculate an upper bound for the angular velocity norm by employing the following positive-definite scalar function $V_\omega = \omega^{\mathrm{T}} J\omega$. The time-derivative over the trajectory is zero. Hence, $\lambda_{min}(J)\|\omega\|^2 \leq \lambda_{max}(J)\|\omega_0\|^2$. Define $\Lambda = \sqrt{\lambda_{max}(J)/\lambda_{min}(J)}$. This leads to an lower-bound for the $\|\omega(t)\|$ over the initial condition interval as

$$\|\omega\| \leq \Lambda\|\omega_0\| \tag{4.29}$$

An upper-bound for $\|\sigma(t)\|$ over the initial condition interval is obtained by calculating upper bounds in terms of σ_0, ω_0 over the initial condition interval. In order to calculate an upper-bound for $\|\sigma\|$, we employ the positive definite scalar function $V_\sigma = 2\log(1+\sigma^{\mathrm{T}}\sigma)$ [23]. The time derivative of V_σ along the state trajectories is calculated to be

$$\begin{aligned} \dot{V}_\sigma &= \sigma^{\mathrm{T}}\omega \leq \|\sigma\|\|\omega\| \\ &\leq \Lambda\|\omega_0\|(e^{V_\sigma/2}-1)^{1/2} \end{aligned} \tag{4.30}$$

wherein we use the substitution $\|\sigma\|^2 = e^{V_\sigma/2}-1$. Next, from $2\sigma\dot{\sigma} = e^{V_\sigma/2}\dot{V}_\sigma/2$, and substituting for $\dot{V}_\sigma$ in Equation 4.30 we now have

$$\frac{4\dot{\sigma}}{(1+\sigma^2)} \leq \Lambda\|\omega_0\| \tag{4.31}$$

Integrating both sides from $-\tau$ to $t \in [-\tau, 0]$ and using the comparison principle lemma [13] leads to

$$\tan^{-1}(\sigma) - \tan^{-1}(\sigma_0) \leq \frac{\Lambda\|\omega_0\|(t+\tau)}{4} \tag{4.32}$$

Note that 4.32 is satisfied if the following holds

$$\|\sigma\| \leq \tan\left(\tan^{-1}(\|\sigma_0\|) + \frac{\Lambda\|\omega_0\|(t+\tau)}{4}\right) \tag{4.33}$$

In order to stabilize the actual system, we formulate the Lipschitz-like condition (See Equation 3.19) for the perturbation term. Rewriting the perturbation term from Equation 4.10

$$F(\sigma, q) = \begin{bmatrix} \left[B(\sigma) - I_{3\times 3}\right] q \\ -16J^{-1}q^{\times}Jq \end{bmatrix} \tag{4.34}$$

Let $F_1(\sigma, q) = \left[B(\sigma) - I_{3\times 3}\right] q$ and $F_2(\sigma, q) = -16J^{-1}q^{\times}Jq$. For the following derivation, we employ the induced 2-norm, i.e. $\|.\| = \|.\|_2$. We upper bound the perturbation term as

$$\|F_1(\sigma, q)\| \leq \| - \sigma^{\mathrm{T}}\sigma I_{3\times 3} + 2\sigma^{\times} + 2\sigma\sigma^{\mathrm{T}}\|\|q\| \tag{4.35}$$

Since, $(\sigma^{\times})^2 = \sigma\sigma^{\mathrm{T}} - \sigma^{\mathrm{T}}\sigma I_{3\times 3}$, we obtain

$$\begin{aligned} \|F_1(\sigma, q)\| &\leq \left(\|(\sigma^{\times})^2\| + 2\|\sigma^{\times}\| + \|\sigma\sigma^{\mathrm{T}}\|\right)\|q\| \\ &\leq 2\|\sigma\|\|q\|\left(1 + \|\sigma\|\right) \end{aligned} \tag{4.36}$$

Let $\|q\| \leq \rho \ \forall \ t$. This leads to

$$\|F_1(\sigma, q)\| \leq 2\rho\|\sigma\|\left(1 + \|\sigma\|\right) \tag{4.37}$$

We have $F_2(\sigma, q) = -16J^{-1}q^{\times}Jq$. The angular dynamics perturbation term is upper bounded as

$$\|F_2(\sigma, q)\| \leq 16\rho\Lambda^2\|q\| \tag{4.38}$$

We wish to enforce the Lipschitz-like condition for the perturbation term $F(\sigma, q)$ as

$$\|F(\sigma, q)\| \leq 16\rho\Lambda^2\|\sigma, q\| \leq \gamma\|\sigma, q\| \tag{4.39}$$

In order to satisfy the above condition, in accordance with Equation (4.37), we enforce the following condition on $\|F_1(\sigma, q)\|$

$$2\rho\|\sigma\|\left(1 + \|\sigma\|\right) \leq 16\rho\Lambda^2\|\sigma\| \tag{4.40}$$

Substituting for $\|\sigma\|$ from Equation 4.33 in Equation 4.40,

$$\tan\left(\tan^{-1}(\|\sigma_0\|) + \frac{\Lambda\|\omega_0\|(t+\tau)}{4}\right) \leq 8\Lambda^2 - 1 \tag{4.41}$$

This leads to an upper bound for $\|\sigma_0\|$ as

$$\|\sigma_0\| < \tan\left(\tan^{-1}(8\Lambda^2 - 1) - \frac{\Lambda\|\omega_0\|(t+\tau)}{4}\right) \tag{4.42}$$

Since $t \in [-\tau, 0]$, and $\tau < \tau_{max}$, the above inequality can be replaced by

$$\|\sigma_0\| \leq \tan\left(\tan^{-1}(8\Lambda^2 - 1) - \frac{\Lambda\|\omega_0\|\tau_{max}}{4}\right) \tag{4.43}$$

Equation 4.43 represents an upper-bound on $\|\sigma_0\|$. This upper-bound is dependent on Λ, $\|\omega\|$ and τ_{max} and is valid only if the following upper-bound on ω_0 holds

$$\|\omega_0\| \leq \frac{4}{\Lambda\tau_{max}}\tan^{-1}(8\Lambda^2 - 1) \tag{4.44}$$

If Equation 4.44 holds, 4.43 can be upper-bounded by

$$\|\sigma_0\| \leq \tan\left(\tan^{-1}(8\Lambda^2 - 1)\right) = 8\Lambda^2 - 1 \tag{4.45}$$

From definition, $\Lambda \geq 1$. The quantity $8\Lambda^2 - 1$ is atleast 7. This corresponds to a minimum permissable principal rotation angle, Φ, of 327.47°. In addition we impose $\sigma^2 < 1$ in order to ensure the MRPs do not pass through a singularity. Comparing Equation 4.39 with Equation 3.19, we obtain the regional stabilization condition for the angular velocity norm ($\|\omega\| \leq 4\rho$) as

$$\|\omega\| \leq \frac{\gamma}{4\Lambda^2} \tag{4.46}$$

Equation 4.46 along with Equation 4.29 leads to an upper bound on the angular velocity norm initial condition $\|\omega_0\|$ as

$$\|\omega_0\| \leq \frac{\gamma}{4\Lambda} \tag{4.47}$$

where γ is obtained using the numerical optimization process. Comparing Equation 4.44 and Equation 4.47 leads to

$$\|\omega_0\| \leq \frac{1}{\Lambda}\min\left\{\frac{\gamma}{4}\ ,\ \frac{4}{\tau_{max}}\tan^{-1}(8\Lambda^2 - 1)\right\} \tag{4.48}$$

Considering Equation 4.48, the second term was found to be typically always dominant over the first term. The norm of initial condition on the angular velocity ω_0 is directly related to the size of the estimate of the region of attraction γ. The closed-loop nonlinear system is stable for all MRP initial conditions σ_0 shown in Equation 4.43. Equations 4.43 and 4.48 together represent the estimate of the region of attraction for the closed-loop system. The initial conditions upper bounds can be obtained provided that γ is calculated using numerical optimization. The condition on ω_0 from Equation 4.48 is evaluated first since it is in terms of γ and τ_{max} alone. Having obtained $\|\omega_0\|$, the condition on σ_0 Equation 4.43 is evaluated. Qualitatively, the larger the initial condition on angular velocity, the less amount of initial rotation is permitted inside the region of attraction estimate.

At this point, we wish to mention an important reason for choosing the MRPs to represent the attitude kinematics rather than the more traditional (and globally non-singular) quaternion parametrization. The kinematics equation with quaternion notation is expressed as [24]

$$\begin{bmatrix} \dot{\epsilon} \\ \dot{\epsilon}_0 \end{bmatrix} = \begin{bmatrix} T(\epsilon) \\ -\epsilon^{\mathrm{T}} \end{bmatrix} \omega \tag{4.49}$$

where $\epsilon \in \mathbb{R}^3$ is the vector part of quaternion and $\epsilon_0 \in \mathbb{R}$ is the scalar part of the quaternion with the norm constraint $\|\epsilon\|^2 + |\epsilon|^2 = 1$. $T(\epsilon) = \left(\epsilon \times + \sqrt{1 - \epsilon^{\mathrm{T}}\epsilon} I_{3\times 3}\right)/2$. $T(\epsilon)$ cannot be made homogenous in the state ϵ by adding and subtracting some $\lambda\omega$, as was done with the MRP representation in Equation 4.3, and consequently, will not lead to definition of a drift term satisfying the Lipschitz-like condition in Equation 3.19. This obstacle can be avoided by using the MRP representation.

4.4 Simulation Results

We implement the control design proposed in the previous section for the attitude dynamics problem with constant unknown delay in feedback. In order to generate realistic trajectories over the initial condition interval, The attitude dynamics is simulated torque-free with initial conditions σ_0, ω_0 over the delay interval $[-\tau, 0]$, which serves as the initial trajectory for the delay problem.

We provide a brief outline of the implementation of the numerical optimization using MATLAB. We employ the inbuilt MATLAB function "fmincon()". The performance index to be maximized is γ subject to constraints on control gains represented by Equation 4.16 for exponential stability of the nominal system represented by 3 blocks of double integrators with delayed feedback. In doing so, we evaluate the analytical solution of the Lyapunov matrix. The parameters to be chosen through optimization are W_0, W_2, ω_n and ξ.

4.4.1 Case I

The inertia matrix J is chosen to be

$$J = \begin{bmatrix} 20 & 2 & 3 \\ 2 & 19 & 2 \\ 3 & 2 & 25 \end{bmatrix} \tag{4.50}$$

The quantity Λ turns out to be 1.2512. $\|\omega_0\|$ was chosen to be 0.00542 which is less than the upper-bound $\gamma/4\Lambda = 0.0056$, and ω_0 was chosen to be $[0.0032\ ,\ 0.0031,\ \ -0.0032\ ]^{\mathrm{T}}$ which satisfies the attraction region condition. σ_0 can be chosen so that $\|\sigma_0\|$ is slightly less than the upper-bound 11.295 obtained from Equation 4.43. The condition on ω_0 ensures that Equation 4.46 is satisfied throughout the initial condition interval $[\ -\tau,\ 0]$. For the case $\tau_{max} = 1$, we obtain $\gamma = 0.02804$ using the numerical optimization. The gain parameters ξ and ω_n obtained using this process turn out to be: $\xi = 0.9112$

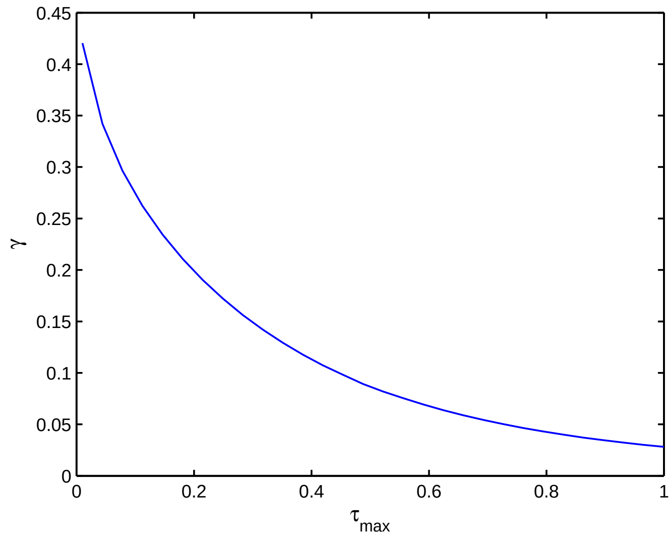


Figure 4.2: γ vs τ_{max} comparison

and $\omega_n = 0.4774$. The W_i parameters turn out to be

$$W_0 = \begin{bmatrix} 0.0755 & 0 \\ 0 & 0.0755 \end{bmatrix},\ W_2 = \begin{bmatrix} 0.0234 & 0 \\ 0 & 0.0234 \end{bmatrix} \tag{4.51}$$

Figure 4.2 shows γ as a function of τ_{max}. However, using the shadow set transformation from Equation 2.12 (i.e. $\sigma^s = -\sigma/\|\sigma\|^2$) leads to the new initial condition MRP norm being significantly small. For instance, if σ_0 is chosen to be $[-5.9\ ,\ -5.1\ ,\ 6.3]^{\mathrm{T}}$. $\|\sigma_0\|$ is 10.0254, which satisfies the upper-bound 11.295 obtained from Equation 4.43 (This value corresponds to a initial principal rotation angle of 337.2151°), the initial MRP condition can be transformed to the corresponding shadow set by employing Equation 2.12 (i.e. $\sigma^s = -\sigma/\|\sigma\|^2$) in order to reduce the control effort required for stabilization, which leads to the new initial MRP vector to be

$$\sigma_0 = [0.0587\ ,\ 0.0507\ ,\ -0.0626]^{\mathrm{T}} \tag{4.52}$$

which corresponds to an initial principal rotation angle of 22.78°. In order to obtain an initial condition with large initial principal rotation angle, we choose σ_0 so that its norm is close to unity, which would lead to a principal rotation close to 180°. In this case, we choose σ_0 to be

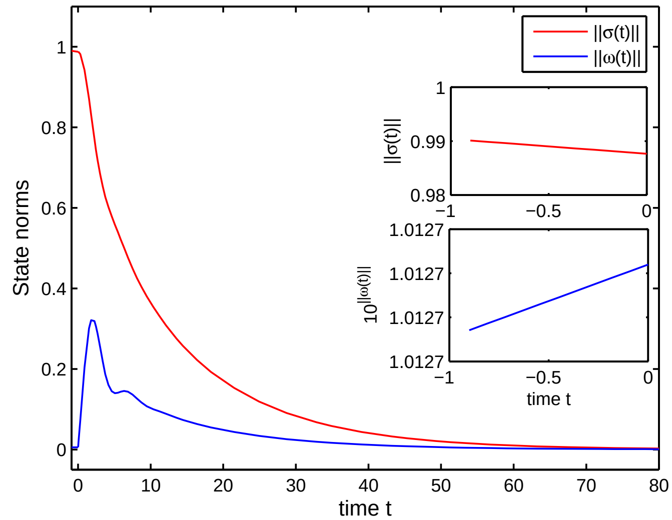
$$\sigma_0 = [0.5831\ ,\ 0.5831\ ,\ -0.5831]^{\mathrm{T}} \tag{4.53}$$

where $\sigma_0 = 1.01$. After employing Equation 2.12 in order to obtain the shadow set, the new initial MRP vector turns out to be

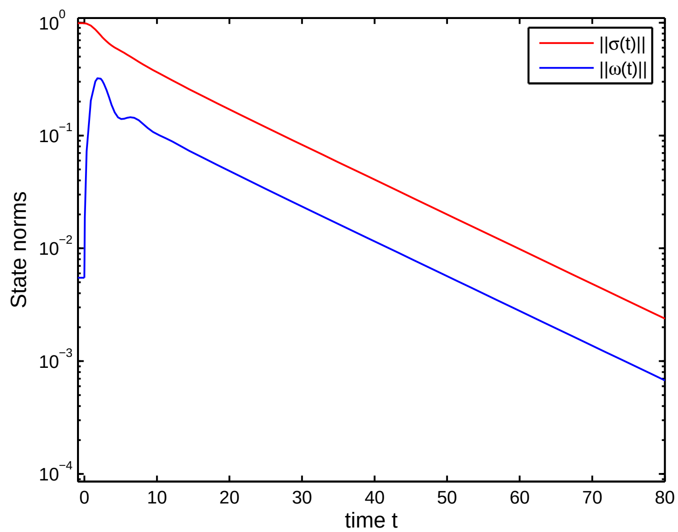
$$\sigma_0 = [-0.5716\ ,\ -0.5716\ ,\ 0.5716]^{\mathrm{T}} \tag{4.54}$$

Figure 4.3(a) shows the trajectories of the state norms as a result of the implementation with feedback gains obtained as $\xi = 0.9112$ and $\omega_n = 0.4774$. The insets in Figure 4.3(a) show the $\|\sigma(t)\|$, $10^{\|\omega(t)\|}$ (in order to emphasize their time-varying nature) over the time interval $-0.9 = -\tau \le t \le 0$. Figure 4.3(b) shows the logarithmic plot of the state norms as a function of time in order to depict the convergence of the states to the origin.

As is observed, the state trajectories converge to the origin. Figure 4.4 shows the control history for the same simulation. Comparing our results with those from Reference [3], the size of the estimate for the region of attraction for a time-delay (which is known) of 0.0125 is 0.0018. This value is an



(a) $\|\sigma(t)\|$, $\|\omega(t)\|$



(b) $\log\|\sigma(t)\|$, $\log\|\omega(t)\|$

Figure 4.3: Case I: $\tau_{max} = 1$, $\tau = 0.9$

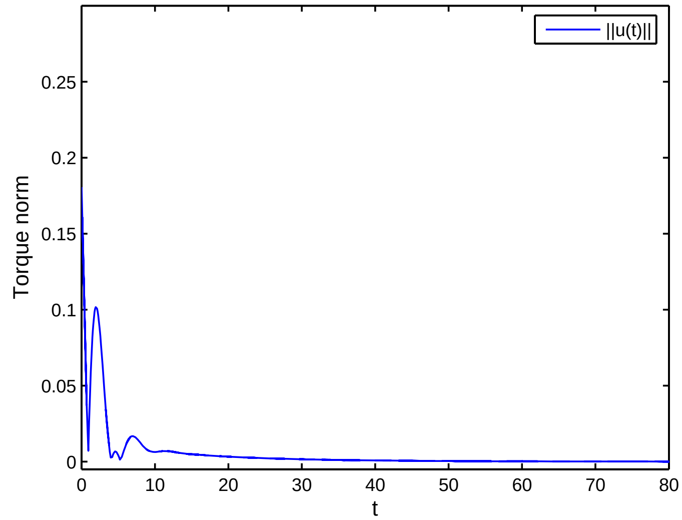


Figure 4.4: Case I: $\|u(t)\|$ for $\tau_{max} = 1$, $\tau = 0.9$

upper-bound on the norm an augmented state vector containing the Rodrigues parameters, angular velocity and the angular velocity filter. The size of the region of attraction considered in our work for a strict upper-bound on the time-delay of 1 is considerably less conservative in comparison to the aforementioned result for a known time-delay of 0.0125 [3]. In passing, we note that the estimate obtained using our approach is still potentially conservative. However, numerical simulations carried out for a time delay $\tau = 1.8$, which is greater than $\tau_{max} = 1$, resulted in the system being unstable. Figure 4.5 shows the first 80 seconds of the simulation.

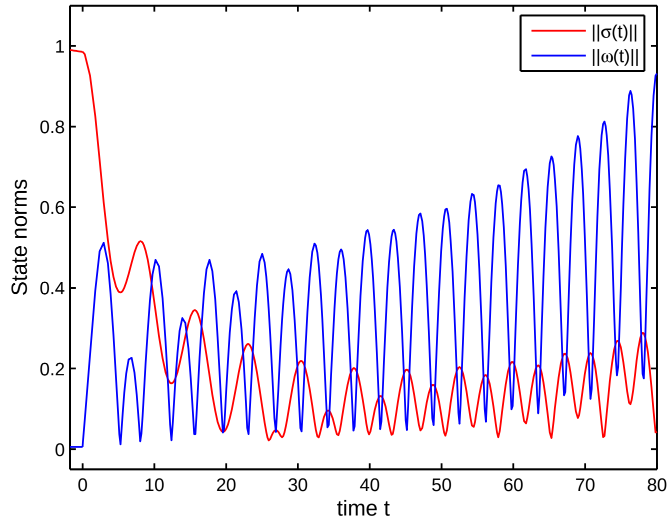


Figure 4.5: Case I: $\|\sigma(t)\|$, $\|\omega(t)\|$ for $\tau_{max} = 1$, $\tau = 1.8$

4.4.2 Case II

The inertia matrix J is chosen to be diag$(1000, 500, 700)$ from Reference [3] for comparison. The maximum delay τ_{max} is chosen to be 0.0125. The Rodrigues parameter vector initial condition is chosen to be $\rho_0 = [0, 0.001, -0.001]^{\mathrm{T}}$, which translates to an initial principal rotation angle of $\phi_0 = 2\tan^{-1}(\|\rho_0\|) = 0.1604$, whereas ω_0 is chosen to be $[0, 0, 0]^{\mathrm{T}}$. From our method, for the chosen J, Λ turns out to be 1.4142. For $\tau_{max} = 0.0125$, γ is calculated through numerical optimization to be 0.411689. The control gains are $\omega_n = 1.5419$ and $\xi = 0.7883$. This leads to upper bounds on σ_0 and ω_0 to be

$$\|\sigma_0\| \leq 10.8036, \;\; \|\omega_0\| \leq 0.0728 \tag{4.55}$$

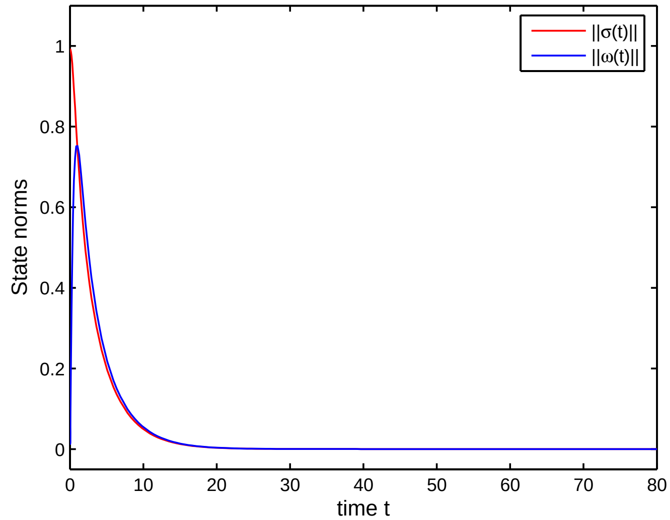


Figure 4.6: Case II: $\|\sigma(t)\|$, $\|\omega(t)\|$ for $\tau_{max} = 0.0125$, $\tau = 0.012$

The condition on σ_0 corresponds to a maximum permissable principal rotation angle $\Phi_0 = 338.84°$, which is two orders of magnitude higher compared to the result in Reference [3]. However, similar to Case I, we choose σ_0 so that its norm is close to unity. σ_0 is chosen to be

$$\sigma_0 = [0.5831\ ,\ 0.5831\ ,\ -0.5831]^{\mathrm{T}} \tag{4.56}$$

where $\sigma_0 = 1.01$, which is the same as Case I. After employing Equation 2.12 in order to obtain the shadow set, the new initial MRP vector turns out to be

$$\sigma_0 = [-0.5716\ ,\ -0.5716\ ,\ 0.5716]^{\mathrm{T}} \tag{4.57}$$

The simulation of the attitude dynamics with the aforementioned initial conditions is depicted in Figure 4.6. We choose initial MRP condition such that

$\|\sigma_0\|$ is close to unity. The initial condition σ_0 is chosen to be

$$\sigma_0 = [0.5831\ ,\ 0.5831\ ,\ -0.5831]^{\mathrm{T}} \tag{4.58}$$

and is transformed to its shadow coordinate by Equation 2.12 in order to reduce the control effort required to achieve stabilization. The new initial MRP vector is calculated to be

$$\sigma_0 = [-0.5716\ ,\ -0.5716\ ,\ 0.5716]^{\mathrm{T}} \tag{4.59}$$

ω_0 is chosen to be

$$\omega_0 = [-0.0420\ ,\ -0.0420\ ,\ 0.0420]^{\mathrm{T}} \tag{4.60}$$

Chapter 5

Attitude Stabilization with unknown time-varying delay in feedback

In this chapter, we discuss the application of the theoretical development in Section 3.2 to the attitude stabilization problem with time-varying delay in feedback. We provide the problem statement, present the separation as a nominal system with a perturbation, discuss implementation of sufficient conditions, provide comparisons of the results with corresponding ones from the constant delay case and present some simulation results.

We denote $\tau(t) = \tau_0 + \eta(t)$ as the unknown time varying time-delay which is assumed to be differentiable everywhere. We assume perfect knowledge of upper-bounds on the magnitude and rate of $\eta(t)$, as well as strict upper-bound on $\tau(t)$ i.e $|\eta(t)| \leq \eta_0$, $|\dot{\eta}(t)| \leq \eta_1 < 1$, $0 \leq \tau(t) < \tau_{max}$. In addition, the delay satisfies the condition $\tau_0 = \sup_t |\eta(t)|$. Initial condition trajectories for Equation 2.7-2.8 are generated by propagating the same dynamics without control action over $-2\tau_0 \leq t \leq 0$. Initial conditions $\sigma_0 \doteq \sigma(-2\tau_0)$, $\omega_0 \doteq \omega(-2\tau_0)$ are chosen to initialize this propagation such that they lie within the estimate and moreover, so that state trajectories do not escape from this estimate during the control-free propagation. The control objective is to achieve stabilization of the states, i.e. to ensure that $\sigma(t) \rightarrow 0$ and $\omega(t) \rightarrow 0$ through a complete type L-K approach.

5.1 System Formulation

The nominal system chosen for the time-varying delay case is the same as in Section 4.1 for the constant delay case (see Equations 4.7-4.8) i.e

$$\dot{\sigma}_d = q_d \tag{5.1}$$

$$\dot{q}_d = \bar{u} \tag{5.2}$$

The Lipschitz condition on the term $f(x(t))$ along with analysis over the initial condition interval $[-2\tau_0, 0]$ leads to the similar upper bounds on the initial condition norms in terms of γ as in Equations 4.47-4.48

$$\|\sigma_0\| \leq \tan\left(\tan^{-1}(8\Lambda^2 - 1) - \frac{\Lambda\|\omega_0\|\tau_{max}}{4}\right) \tag{5.3}$$

$$\|\omega_0\| \leq \frac{1}{\Lambda}\min\left\{\frac{\gamma}{4}\ ,\ \frac{4}{\tau_{max}}\tan^{-1}(8\Lambda^2 - 1)\right\} \tag{5.4}$$

where $\tau_{max} \geq 2\tau_0 \geq \tau(t) \geq 0$, where γ is evaluated from the sufficient condition (see 3.37), provided the inequalities from 3.38-3.41 are satisfied. When applied to the attitude dynamics problem, we have

$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & 0 \\ -\omega_n^2 & -2\omega_n\xi \end{bmatrix} \tag{5.5}$$

which are then substituted in the aforementioned conditions to obtain γ through numerical optimization in MATLAB.

5.2 Implementation of Sufficient Conditions

In order to obtain a meaningful region of attraction estimate, the inequalities from 3.38-3.41 must be satisfied for some positive constants k_j, $j = 1, 2, 3, 4$. Rewriting the inequalities 3.38-3.41 with $x_1 \doteq \lambda_{min}(W_0)/\eta_0 u_0\|A_1\|$, $x_2 \doteq \lambda_{min}(W_2)/\eta_0 u_0\|A_1\|^2$, $x_3 \doteq \lambda_{min}(W_2)/u_0\|A_1\|$, $x_4 \doteq \lambda_{min}(W_2)/u_0\|A_1\|^2(1-\eta_1)$ with $\|A_0\| = 1$ for simplicity leads to

$$x_1 > \frac{1}{k_1} + \frac{\|A_1\|}{k_3} \tag{5.6}$$

$$x_2 > \frac{1}{k_2} + \frac{\|A_1\|}{k_4} \tag{5.7}$$

$$x_3 > k_1 + \|A_1\|\tau_{max}k_2 \tag{5.8}$$

$$x_4 > k_3 + \|A_1\|\tau_{max}k_4 \tag{5.9}$$

Considering the above inequalities $\exists\ \alpha, \beta, \delta, \vartheta \in (0, 1)$ such that

$$k_1 > \frac{1}{\alpha x_1},\ k_3 > \frac{\|A_1\|}{(1-\alpha)x_1} \tag{5.10}$$

$$k_1 < \beta x_3,\ k_2 < \frac{(1-\beta)x_3}{\|A_1\|\tau_{max}} \tag{5.11}$$

$$k_2 > \frac{1}{\vartheta x_2},\ k_4 > \frac{\|A_1\|}{(1-\vartheta)x_2} \tag{5.12}$$

$$k_3 < \delta x_4,\ k_4 < \frac{(1-\delta)x_4}{\|A_1\|\tau_{max}} \tag{5.13}$$

The above inequalities are satisfied only if the following inequalities are satisfied:

$$x_3 < k_1 < \frac{1}{x_1},\ \frac{x_3}{\|A_1\|\tau_{max}} < k_2 < \frac{1}{x_2} \tag{5.14}$$

$$x_4 < k_3 < \frac{\|A_1\|}{x_1},\ \frac{x_4}{\|A_1\|\tau_{max}} < k_4 < \frac{\|A_1\|}{x_2} \tag{5.15}$$

The above conditions are sufficient and constructive in evaluating the positive constants $k_j,\ j = 1, 2, 3, 4$ from Equations 3.38-3.41 because they provide a range in which the constants must lie for given W_0, W_2, ω_n and ξ. An easy way to implement checking of the conditions is to sweep through the ranges while checking for satisfaction of the conditions. As in Chapter 4 with the constant delay case, we employ the inbuilt MATLAB function "fmincon()". The performance index to be maximized is γ subject to constraints on control gains represented by Equation 4.16 for exponential stability of the nominal system represented by 3 blocks of decoupled double integrators with delayed feedback and subject to satisfaction of the conditions 3.38-3.41. In doing so, we evaluate the analytical solution of the Lyapunov matrix. The parameters to be chosen through optimization are W_0, W_2, ω_n and ξ as well as k_1, k_2, k_3 and k_4. We implement the aforementioned sweeping method from Equations 5.14-5.15 in order to verify if there is a feasible solution.

5.3 Simulation results

In this section, we compare region of attraction estimate results with those obtained from constant delay for the same upper bound on delay magnitude τ_{max}, analyze conservatism arising from conditions (see 3.38-3.41) and provide some simulations on attitude stabilization with unknown time-varying delay in feedback.

Table 5.1 provides a comparison in terms of γ and initial conditions between the constant and time-varying delay case for various delay values

Constant Time-delay				Time-varying rime-delay					
τ_{max}	γ	Φ_0	$\|\omega_0\|$	τ_{max}	η_0	η_1	γ	Φ_0	$\|\omega_0\|$
0.03	0.36	340.05°	0.07	0.03	0.01	0.0	0.11	340.12°	0.0257
0.03	0.36	340.05°	0.07	0.03	0.01	0.8	0.07	340.13°	0.0181
0.1	0.27	339.78°	0.05	0.1	0.01	0	0.06	340.05°	0.0133
0.1	0.27	339.78°	0.05	0.1	0.01	0.5	0.03	340.10°	0.0070
0.2	0.19	339.61°	0.03	0.2	0.1	0.1	0.02	340.09°	0.0047

Table 5.1: Comparison between constant and time-varying delay results

τ_{max}, η_1, η_0. Table 5.1 shows that letting $\eta_1 \to 0$ does not recover the region of attraction estimate from the corresponding constant delay case, which was expected because of the additional conservatism. Moreover, for sufficiently large parameter values of delay, the optimization does not converge, indicating the problem may not be feasible. In the constant delay case however, the optimization does converge for the same value of τ_{max}.

As mentioned, the conditions from 3.38-3.41 are not particularly constructive in finding constants k_j, $j = 1, 2, 3, 4$. We observe by using the sweeping method from conditions 5.14-5.15 that as the delay parameters become sufficiently large, the conditions are not satisfied, which is not observed in the constant delay case. We simplify the conditions and fix certain parameters in order to obtain a range of delay parameter values for which a feasible γ can be found. We fix $W_0 = W_2 = I_{2\times2}$ and $W_1 = 0_{2\times2}$ and $\xi = 1$ in order to enforce critical damping for the nominal closed-loop system. The positive constants k_j are eliminated by using their strict upper bounds as obtained in 5.14-5.15. The free parameter remaining is ω_n since $\|A_1\| = \sqrt{\omega_n^4 + 4\omega_n^2}$. As a result, the inequalities 3.38-3.41 can be rewritten as

$$x_1 - \frac{1}{x_3} - \frac{\|A_1\|}{x_4} > 0 \;:\; \text{Inequality 1} \tag{5.16}$$

$$x_2 - \frac{\|A_1\|\tau_{max}}{x_3} - \frac{\|A_1\|^2\tau_{max}}{x_4} > 0 \;:\; \text{Inequality 2} \tag{5.17}$$

$$x_3 - \frac{1}{x_1} - \frac{\|A_1\|\tau_{max}}{x_2} > 0 \;:\; \text{Inequality 3} \tag{5.18}$$

$$x_4 - \frac{\|A_1\|}{x_1} + \frac{\|A_1\|^2\tau_{max}}{x_2} > 0 \;:\; \text{Inequality 4} \tag{5.19}$$

where u_0 is a function of ω_n alone (can be calculated through the analytical approach in Section 4.2 or by polynomial curve fitting), and x_1, x_2, x_3 and x_4 are as defined before with $W_0 = W_2 = I_{2\times2}$, $W_1 = 0_{2\times2}$ and $\xi = 1$. In particular, we wish to find if there is a nonempty set of ω_n values for which the above conditions are simultaneously satisfied. Implementing this idea with MATLAB for various cases leads to

1. $\eta_1 = 0$, $\eta_0 = 0.001$, leads to τ_{max} being a maximum value of 0.23 (see Figure 5.1). The set of ω_n values can be observed in the accompanying figure

2. Keeping η_1 zero and increasing η_0 to 0.01 leads to τ_{max} decreasing to 0.18. Increasing η_1 to 0.5 does not lead to satisfaction of the conditions for τ_{max} as small as 0.001 (see Figure 5.2).

3. Keeping η_0 to 0.01 and increasing η_1 to 0.05 further decreases τ_{max} to 0.15 (see Figure 5.3).

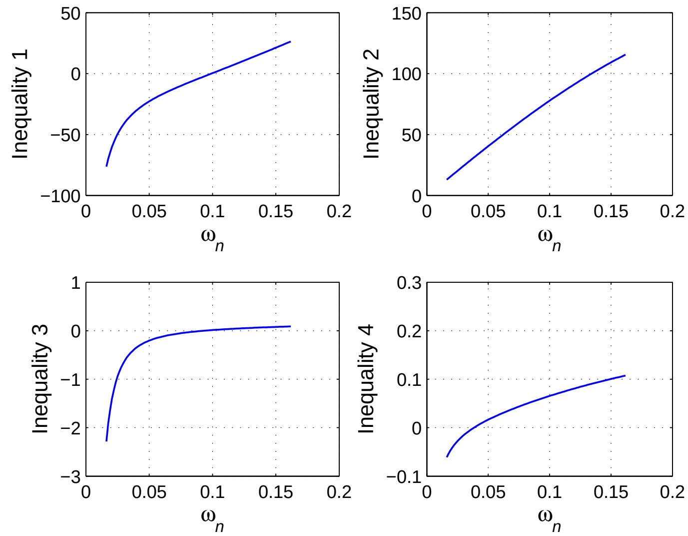


Figure 5.1: Inequalities vs. ω_n for $\eta_1 = 0$, $\eta_0 = 0.001$, $\tau_{max} = 0.23$

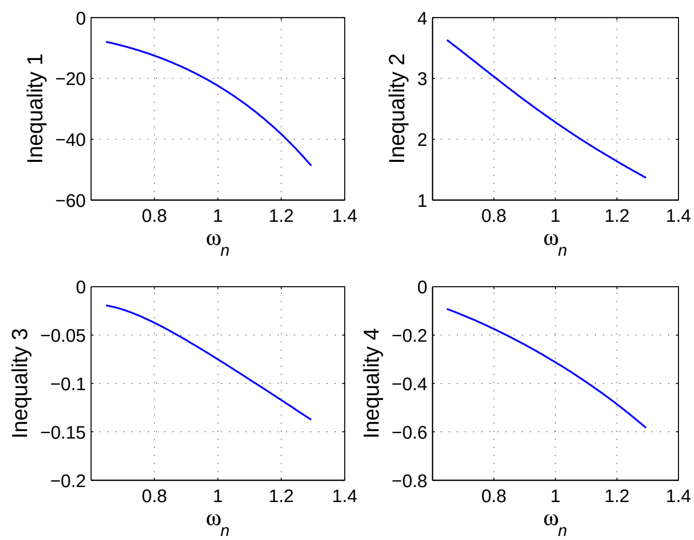


Figure 5.2: Inequalities vs. ω_n for $\eta_1 = 0.5$, $\eta_0 = 0.01$, $\tau_{max} = 0.001$

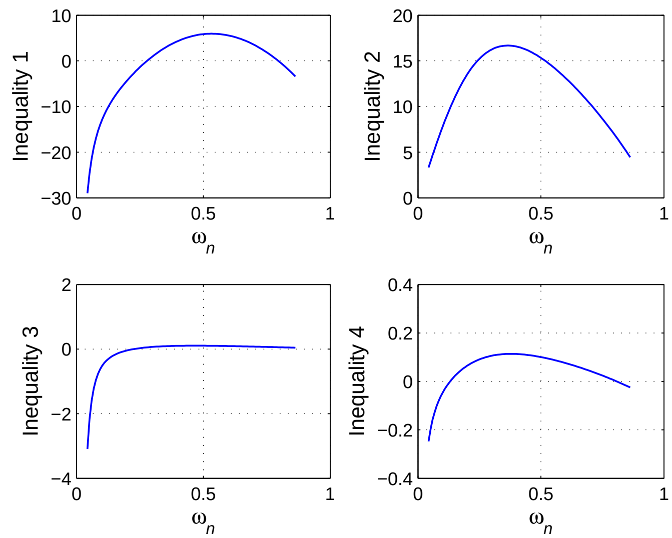


Figure 5.3: Inequalities vs. ω_n for $\eta_1 = 0.05$, $\eta_0 = 0.01$, $\tau_{max} = 0.15$

The above method provides a range of delay parameter values for which there exists a solution. However, this method is only sufficient. For the simulation of attitude dynamics, we choose τ_{max} is chosen to be 0.11, with $\eta_0 = 0.01$ and $\eta_1 = 0.5$. As a result of the attraction estimate conditions, γ is calculated to be 0.087295. We choose $\tau(t) = 0.09 + 0.01\sin(0.5t)$. The inertia matrix is chosen to be the same as Case I in the constant delay simulation i.e.

$$J = \begin{bmatrix} 20 & 2 & 3 \\ 2 & 19 & 2 \\ 3 & 2 & 25 \end{bmatrix} \tag{5.20}$$

Control parameters are calculate through numerical optimization to be $\omega_n = 0.9143$, $\xi = 0.5275$. Initial conditions are chosen as $\omega_0 = [0.01 - 0.01 - 0.01]^{\mathrm{T}}$, $\sigma_0 = [\ -5.9\ \ -5.1\ 6.3]^{\mathrm{T}}$. System simulated for time interval $[-2\tau_0, 0]$ without control. Figures 5.4 and 5.5 show the results of the simulation. In order to

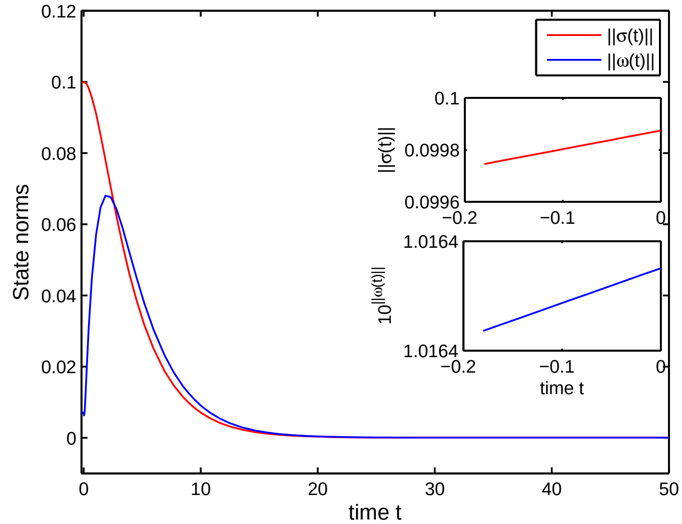


Figure 5.4: Case I: $\|\sigma(t)\|$, $\|\omega(t)\|$ for $\tau(t) = 0.09 + 0.01\sin(0.5t)$

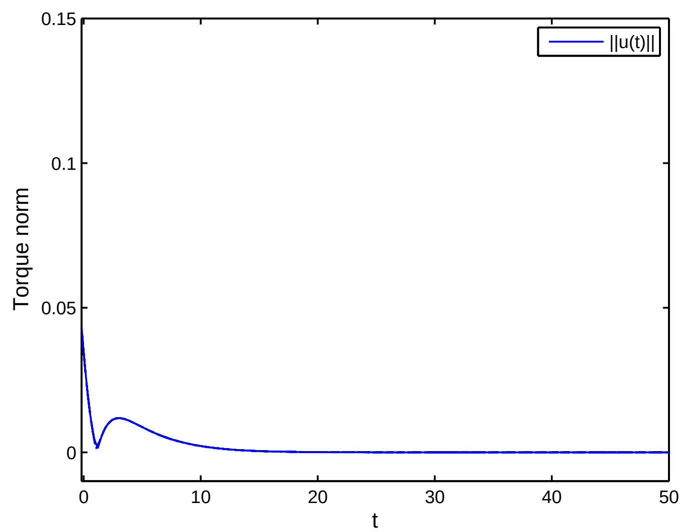


Figure 5.5: Case I: $\|u(t)\|$ for $\tau(t) = 0.09 + 0.01\sin(0.5t)$

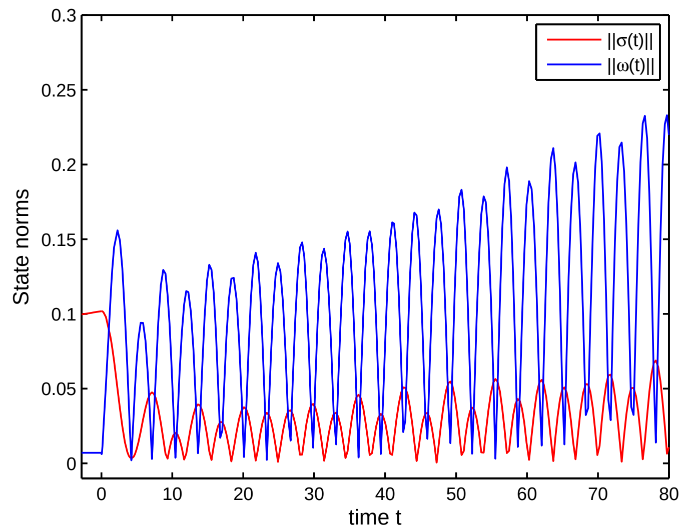


Figure 5.6: Case I: $\|\sigma(t)\|$, $\|\omega(t)\|$ for $\tau(t) = 1.5 + 0.1\sin(0.9t)$

gain some insight into the conservatism of the estimate with respect to time delay, the same system is simulated with time-delay $\tau(t) = 1.5 + 0.1\sin(0.9t)$. Figure 5.6 shows the results of the simulation. The trivial solution $x(t)$ is unstable, however, the conservatism is larger than that of the constant delay case because $\tau(t)$ has to be increased much more from its original value in order to see instability.

Chapter 6

Conclusions and Recommendations

This thesis considered the open problem of finding an estimate of region of attraction for rigid body attitude dynamics with an unknown time-delay in feedback. We considered two cases based on the nature of time-delay: 1) constant and 2) time-varying. In both cases, the actual time-delay was unknown. For the constant delay case, a strict upper bound on the unknown constant delay was known. The time-varying delay was assumed to be made of a constant unknown delay with a time-varying perturbation. Strict upper bounds on the time-varying delay, the magnitude of the time-varying perturbation and the rate of the time-varying perturbation were known.

The concept of the complete type L-K functional was successfully extended in order to investigate stability for a class of nonlinear time-delay systems with unknown time-delay. This extension enabled stability robustness to time-delay in the control design i.e. asymptotic stability held for all values of time-delay less than the known upper bound. The region of attraction estimate was maximized through numerical optimization by choosing the free parameters from the sufficient stability conditions. The results obtained were superior to those from a previous paper where the time-delay was known perfectly. The simulations verified the results obtained, and showed instability occurring if the actual time-delay was higher than the upper bound considered.

In the case of time-varying delay, sufficient conditions on regional stabilization were obtained provided certain inequalities were satisfied which arose from treating the time-varying delay term as a perturbation. The sufficient conditions were tedious and not constructive to evaluate. A constructive method to satisfy the simplified sufficient conditions was presented. Limiting cases arising from the sufficient conditions were studied, along with additional conservatism present in the conditions and comparison was made with corresponding results from the constant delay case. It was shown that allowing the time-varying delay rate go to zero did not recover the estimate that resulted from the constant delay case, which was expected due to the conservatism arising from the model transformation. Moreover, for sufficiently large values of delay, numerical optimization could not converge to a feasible solution indicating that the sufficient conditions in the form of inequalities could not be satisfied. Further work could include a more constructive form in order to evaluate the sufficient conditions. Simulations verified the results obtained, and the actual time-delay had to be increased significantly in order to achieve instability, indicating the increase in conservatism in comparison with the constant delay case.

The control design does not require precise knowledge of the actual time-delay, however, it does require the mass moment of inertia matrix to be known exactly, which can be restrictive in some applications. Future work could include using projection in order to formulate an adaptive controller which would employ an estimate of the mass moment of inertia matrix and still converge to the origin with delayed measurements of the states. The time-delay present in the different actuators is assumed to be the same, which, again is not always physically true. Future work could include using different time-delays in the different control actuators as well as different time-delays in the state measurements, i.e. MRP and angular velocity state vectors delayed by different amounts. Future work could also include extending the control objective to trajectory tracking as well as extending the single spacecraft problem to achieving consensus in formation control with communication delay in feedback, which would present new issues when applying the complete type L-K approach.

Bibliography

[1] NASA Website. http://dst.jpl.nasa.gov/control/.

[2] J. Wen and K. Kreutz-Delgado. The Attitude Control Problem. *IEEE Transactions on Automatic Control*, 36(10):1148–1162, 1991.

[3] A. Ailon, R. Segev, and S. Arogeti. A simple velocity-free controller for attitude regulation of a spacecraft with delayed feedback. *IEEE Transactions on Automatic Control*, 49(1):125–130, 2004.

[4] J. Richard. Time-Delay Systems: An Overview of Some Recent Advances and Open Problems. *Automatica*, 39:1667–1694, 2003.

[5] V. Kolmanovskii and A. Myshkis. *Introduction to the Theory and Applications of Functional Differential Equations.* Mathematics and Its Applications 463, Kluwer Academic Publishers, 1999.

[6] R. Bellman and K. L. Bellman. *Differential-Difference Equations.* The Rand Corporation, 1963.

[7] J. Hale and S. Lunel. *Introduction to Functional Differential Equations.* Applied Mathematical Sciences 99, Springer-Verlag, 1991.

[8] K. Gu, V. Kharitonov, and J. Chen. *Stability of Time-Delay Systems (Control Engineering).* Birkhäuser, 2003.

[9] F. Mazenc and P.-A Bliman. Backstepping design for time-delay nonlinear systems. *IEEE Transactions on Automatic Control*, 51(1):149–154, 2006.

[10] N. Krasovskii. *Stability of Motion: Applications of Lyapunov's Second Method to Differential Systems and Equations With Delay.* Stanford University Press, 1963.

[11] V. Kharitonov and A. Zhabko. Lyapunov-krasovskii approach to robust stability analysis of time-delay systems. *Automatica*, 39(1):15–20, 2003.

[12] D. Melchor-Aguilar and S-I Niculescu. Estimates of attraction region for a class of nonlinear time-delay systems. *IMA Journal of Mathematical Control and Applications*, 24:523–550, 2007.

[13] H. K. Khalil. *Nonlinear Systems.* Prentice Hall: New Jersey, 2002.

[14] J. Yorke. Asymptotic Stability for One Dimensional Differential-Delay Equations. *Journal of Differential Equations*, 7:189–202, 1970.

[15] R. Villafuerte and S. Mondié. Estimate of the region of attraction for a class of nonlinear time delay systems: a leukemia post-transplantation dynamics example. In *Proceedings of the 46th IEEE Conference on Decision and Control*, 2007.

[16] S. Mondié, V. Kharitonov, and O. Santos. Complete type lyapunov-krasovskii functionals with a given cross term in the time derivative. In

Proceeding of the European Control Conference, pages 5060–5064, Seville, Spain, 2005. IEEE.

[17] S. Barnett. Matrix differential equations and kronecker products. *SIAM Journal on Applied Mathematics*, 24(1):1–5, 1973.

[18] S. Marandi and V. Modi. Stereographic Orientation Parameters for Attitude Dynamics: A Generalization of the Rodrigues Parameters. *Journal of Guidance, Control and Dynamics*, 44(4):95–100, 1996.

[19] K. Gu and S-I Niculescu. Survey on Recent Results in the Stability and Control of Time-Delay Systems. *Transactions of the ASME*, 125:158–165, 2003.

[20] E. F. Infante and W. V. Castelan. A lyapunov functional for matrix difference-differential equations. *Journal of Differential Equations*, 29(3):439–451, 1978.

[21] K. Gu and S.-I Niculescu. Additional dynamics in transformed time-delay systems. *IEEE Transactions on Automatic Control*, 45(3):572–575, 2003.

[22] K. Gu and S.-I Niculescu. Further remarks on additional dynamics in various model transformations of linear delay systems. *IEEE Transactions on Automatic Control*, 46(3):497–500, 2003.

[23] P. Tsiotras. Stabilization and Optimality Results for the Attitude Control Problem. *Journal of Guidance, Control and Dynamics*, 19(4):772–779, 1996.

[24] M. Shuster. A Survey of Attitude Representations. *Journal of Astronautical Sciences*, 41(4):439–517, 1993.

[25] J. R. Wertz. *Spacecraft Attitude Determination and Control.* Dordrecht, The Netherlands: Kluwer, 1978.

[26] B. Razumikhin. The Application of Lyapunov's Method to Problems in the Stability of Systems with Delay. *Automation and Remote Control*, 21:740–749, 1960.

[27] T. Yoneyama. On the 3/2 Stability Theorem for One-Dimensional Delay-Differential Equations. *Journal of Mathematical Analysis and Applications*, 125:161–173, 1987.

[28] S. Grossman and J. Yorke. Asymptotic Behavior and Exponential Stability Criteria for Delay Differential Equations. *Journal of Differential Equations*, 12:236–255, 1972.

[29] T. Yoneyama. Uniform Asymptotic Stability for N-Dimensional Delay-Differential Equations. *Funkcialaj Ekvacioj*, 34:495–504, 1991.

[30] H. Schaub, M. Akella, and J. Junkins. Adaptive Control of Nonlinear Attitude Motions Realizing Linear Closed Loop Dynamics. *Journal of Guidance, Control and Dynamics*, 24(1):95–100, 2001.

Vita

Apurva Arvind Chunodkar was born in Mumbai, India on 22 February 1984, the son of Arvind B. Chunodkar and Anuprita A. Chunodkar. He attended school at St. Anne's High School in Mumbai. He received undergraduate schooling and earned the Bachelor of Technology Degree in Aerospace Engineering at the Indian Institute of Technology - Bombay. After his Bachelor degree, he started graduate studies in Aerospace Engineering at the University of Texas at Austin. His research interests include control of time-delay systems and attitude dynamics and control.

Permanent address: 3543 Greystone Drive, Apt. 1067
Austin, Texas 78731

This thesis was typeset with LaTeX† by the author.

†LaTeX is a document preparation system developed by Leslie Lamport as a special version of Donald Knuth's TeX Program.